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STRUCTURE OF A CLASS OF POLYNOMIAL MAPS WITH INVARIANT FACTORS

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Let $\mathbf{R}[x_1, \cdots, x_n]$ be a ring of polynomials of n $(n \geq 3)$ indeterminates with coefficients in $\mathbf{R}$.

Let $\phi = (\phi_1, \ldots, \phi_n) \in (\mathbf{R}[x_1, \cdots, x_n])^n$ be a polynomial map from $\mathbf{R}^n$ to $\mathbf{R}^n$. Let $x = (x_1, x_2, \ldots, x_n)$ and define

$$\lambda_{n,a}(x) := \sum_{i=1}^{n} x_i^2 - a \prod_{i=1}^{n} x_i \in \mathbf{R}[x_1, x_2, \ldots, x_n].$$

where a $(\neq 0) \in \mathbf{R}$. We will write λ in stead of $\lambda_{n,a}$ if no confusion happens. $\lambda_{n,a}$ is called an invariant factor of ϕ if

$$(1) \qquad \lambda_{n,a} \circ \phi = \lambda_{n,a}.$$

Now let

$$G_{n,a} = \{\phi;\ \phi \in (\mathbf{R}[x_1, \ldots, x_n])^n,\ \lambda_{n,a} \circ \phi = \lambda_{n,a}\},$$

that is, $G_{n,a}$ is the set of polynomial maps of which invariant factor is $\lambda_{n,a}$. The main aim of this note is to determine the structure of $G_{n,a}$.

Let $\Omega_{n,a} = \{x \in \mathbf{R};\ \lambda_{n,a}(x) = 0\}$. Then by the equality (1), for any $n \in \mathbf{N}$,

$$\phi^n(\Omega_{n,a}) \subset \Omega_{n,a},$$

that is, $\Omega_{n,a}$ is an invariant variety of ϕ^n, where ϕ^n denotes $n-$th iteration of ϕ (see [3]). By using this property, we may investigate the asymptotic dynamical behaviours of iterations of ϕ ([1, 2, 3]). We are led naturally to study the structure of $G_{n,a}$. In fact, we will prove first that $G_{n,a}$ is a group, then we will determine the generators of the group.

In the case $n = 3$, we have showed the following:

Theorem 1 ([2]). *With the notations above,* $G_{3,1} = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$ *is a group generated by* $\tau_1(x, y, z) = (y, x, z)$, $\tau_2(x, y, z) = (z, y, x)$, $\tau_3(x, y, z) = (-x, -y, z)$, $\tau_4(x, y, z) = (x, y, xy - z)$.

The proof of Theorem 1 depends strongly on the reducibility of polynomial u^2+v^2-auv, but when $n \geq 4$, the corresponding polynomial that we have to treat is irreducible, thus the method for $n=3$ is failed.

Let $p \in \mathbf{R}[x_1, \ldots, x_n]$, $\phi \in (\mathbf{R}[x_1, \ldots, x_n])^n$, we denote by $\deg p$ the degree of the polynomial p , and define the degree of ϕ as $\deg \phi = \sum_{i=1}^n \deg \phi_i$.

Let S_n be the symmetric group on n letters, we have

$$\mathcal{S}_n := \{(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)});\ \pi \in S_n\} \simeq S_n.$$

So we can denote by π the permutation $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$.

Lemma 1. *Let $n \geq 3$, $a \neq 0$, c is a constant, then $\lambda_{n,a}+c$ is irreducible.*

Proof. If the conclusion of the lemma is not true, then $\lambda_{n,a}+c$ is reducible, i.e. we have the non-trivial factorization of $\lambda_{n,a}+c$

$$(*) \qquad \lambda_{n,a}+c = p_1 p_2.$$

Thus if we consider $\lambda_{n,a}+c$ as a polynomial of x_n with degree 2, then we have either

$$\begin{aligned} p_1 &= f_1(x_1, \ldots, x_{n-1})x_n + g_1(x_1, \ldots, x_{n-1}), \\ p_2 &= f_2(x_1, \ldots, x_{n-1})x_n + g_2(x_1, \ldots, x_{n-1}) \end{aligned}$$

or

$$\begin{aligned} p_1 &= f_1(x_1, \ldots, x_{n-1})x_n^2 + g_1(x_1, \ldots, x_{n-1})x_n + h_2(x_1, \ldots, x_{n-1}), \\ p_2 &= f_2(x_1, \ldots, x_{n-1}). \end{aligned}$$

From the hypothesis $n \geq 3$ and by comparing the degree of two sides of the equality $(*)$, it is easy to see that the factorizations above are impossible for both cases. This proves the lemma. □

Now define $\psi = (x_1, x_2, \ldots, x_{n-1}, a\prod_{i=1}^{n-1} x_i - x_n)$, which will play an important role in the studies of this note. By a direct calculation, we see immediately $\psi \in G_{n,a}$.

Lemma 2. *Let $\phi \in G_{n,a}$. If $\deg \phi > n$, then there exists $\pi \in \mathcal{S}_n$, such that* $\deg(\psi \circ \pi \circ \phi) < \deg \phi$.

Proof. Because we can find a permutation π, such that

$$\deg \phi_{\pi(1)} \leq \deg \phi_{\pi(2)} \leq \cdots \leq \deg \phi_{\pi(n)}.$$

we can assume that

$$\deg \phi_n \geq \deg \phi_{n-1} \geq \cdots \geq \deg \phi_1. \tag{2}$$

Since $\deg \phi > n$, we have $\deg \phi_n \geq 2$. From the equality (1),

$$\left(\phi_n - a\prod_{i=1}^{n-1} \phi_i\right)\phi_n + \sum_{i=1}^{n-1} \phi_i^2 = \sum_{i=1}^{n} x_i^2 - a\prod_{i=1}^{n} x_i. \tag{3}$$

□

1. If $\deg \phi_n \neq \deg \left(a\prod_{i=1}^{n-1} \phi_i\right)$, then

$$\deg\left(\phi_n - a\prod_{i=1}^{n-1} \phi_i\right) = \sup\left\{\deg \phi_n,\ \deg\left(a\prod_{i=1}^{n-1} \phi_i\right)\right\}.$$

Thus by (2) and (3), we have

$$\deg \lambda_{n,a} = n = \sup\left\{\deg \phi_n,\ \deg(a\prod_{i=1}^{n-1} \phi_i)\right\} + \deg \phi_n \geq \sum_{i=1}^{n} \deg \phi_i = \deg \phi > n,$$

This contradiction follows that

$$\deg \phi_n = \deg(\phi_1 \cdots \phi_{n-1}). \tag{4}$$

2. If $\deg \phi_n = \deg \phi_{n-1}$, then by (4), we have $\deg \phi_i = 0$, $1 \leq i \leq n-2$. Thus from the equality (3), there exists constants c_1 and c_2, such that

$$\phi_n^2 + \phi_{n-1}^2 - ac_1\phi_n\phi_{n-1} = \sum_{i=1}^{n} x_i^2 - a\prod_{i=1}^{n} x_i + c_2.$$

Notice that the left member of the equality above is reducible, but by Lemma 1, the right member of the equality above is irreducible, this contraction yields that

$$\deg \phi_n > \deg \phi_{n-1}. \tag{5}$$

3. If $\deg \left(a\prod_{i=1}^{n-1} \phi_i - \phi_n\right) = \deg \phi_n$, then

$$\deg\left(\prod_{i=1}^{n-1} \phi_i\right) \leq \deg \phi_n. \tag{6}$$

Using (5), (6) and using the analyses similar to the case 1, we have

$$n = \deg \lambda = \deg \lambda \circ \phi = 2 \deg \phi_n \geq \sum_{i=1}^{n} \deg \phi_i = \deg \phi > n.$$

This contradiction implies that $\deg \left(a \prod_{i=1}^{n-1} \phi_i - \phi_n \right) \neq \deg \phi_n$, thus from (4), we have $\deg \left(a \prod_{i=1}^{n-1} \phi_i - \phi_n \right) < \deg \phi_n$. By the definition of ψ, we obtain finally

$$\deg(\psi \circ \pi \circ \phi) < \deg \phi.$$

Now, define $\rho = (-x_1, -x_2, x_3, \ldots, x_n)$.

Lemma 3. *Let $\mathcal{L}_n = \{\phi \in G_n;\ \deg \phi_i = 1,\ 1 \leq i \leq n\}$, then $\mathcal{L}_n$ is a group generated by S_n and ρ.*

Proof. Since $\deg \phi_i = 1$, we can write $\phi_i := \phi_i(x_1, \ldots, x_n) = h_i(x_1, \ldots, x_n) + c_i$, where h_i are homogeneous linear polynomials of $x_1, \ldots, x_n$, and $c_i \in \mathbf{R}$ are constants. By the equality (1),

$$\sum_{i=1}^{n} (h_i + c_i)^2 - a \prod_{i=1}^{n} (h_i + c_i) = \sum_{i=1}^{n} x_i^2 - a \prod_{i=1}^{n} x_i. \tag{7}$$

By comparing the coefficients of the terms of degree n of the two sides of (7), we have $h_i = d_i x_{\pi(i)}$, where $d_i \in \mathbf{R}$, $\pi \in S_n$. By comparing the coefficients of the terms of degree $n-1$, we have $c_i = 0$. By comparing the coefficients of the square terms, we have $d_i^2 = 1$, $1 \leq i \leq n$. Finally notice that $|\{i;\ d_i = -1,\ 1 \leq i \leq n\}| \in 2\mathbf{N}$ and notice the role of the action of ρ, we obtain this lemma. □

Lemma 4. *Let $\phi \in G_{n,a}$. Then for any i, $1 \leq i \leq n$, we have $\deg \phi_i \geq 1$. Moreover, there exists $\varphi \in \langle \psi, S_n \rangle$, such that*

$$\deg(\varphi \circ \phi)_1 = \cdots = \deg(\varphi \circ \phi)_n = 1.$$

Proof. We prove the lemma by induction. By Theorem 4 of §3 of [1], the lemma holds for $n = 3$. Now suppose that the conclusions of the lemma are true for the positive integers less than n $(n \geq 4)$.

If $\deg \phi > n$, by using Lemma 2 repeatedly, we can decrease the degree of ϕ by using ψ and a suitable $\pi \in S_n$, and the degree of each component of ϕ does not increase. Thus we can assume that $\deg \phi \leq n$.

If the conclusion of the lemma is not true, then there exists some ϕ_i, being ϕ_n without loosing generality, such that $\phi_n \equiv c$, where c is a constant. If $c = 0$, by the equality (1),

$$\phi_1^2 + \cdots + \phi_{n-1}^2 = x_1^2 + \cdots + x_n^2 - a x_1 x_2 \cdots x_n. \tag{8}$$

Notice that the left member of the equality (8) is always non-negative. But for $n \geq 3$, we can choose $x_1, \ldots, x_n$, such that the right member of the equality (8) is strictly negative. thus $c \neq 0$.

Now let $\phi = (\phi_1, \ldots, \phi_{n-1}, c)$, $c \neq 0$. Define

$$\begin{aligned}
\phi_i^{(j)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) &:= \\
\phi_i(x_1, \ldots, x_{j-1}, c, x_{j+1}, \ldots, x_n) &\in \mathbf{R}[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n], \\
\phi^{(j)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) &:= \\
(\phi_1^{(j)}, \ldots, \phi_{n-1}^{(j)}) &\in (\mathbf{R}[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n])^{n-1}.
\end{aligned}$$

where $1 \leq j \leq n$.

From (1) and $\phi_n \equiv c$, we check directly for any j, $1 \leq j \leq n$

$$\lambda_{n-1,ca} \circ \phi^{(j)} = \lambda_{n-1,ca}. \tag{9}$$

Since $\deg \phi \leq n$, we have $\deg \phi^{(j)} \leq n$, $1 \leq j \leq n$.

1. Suppose that $\deg \phi^{(j)} = n$.

By Lemma 2, there exists $\pi \in \mathcal{S}_{n-1}\{1, \ldots, j-1, j+1, \ldots, n\}$ such that

$$\deg \psi^{(j)} \circ \pi \circ \phi^{(j)} < \deg \phi^{(j)} = n, \tag{10}$$

where

$$\begin{aligned}
&\psi^{(j)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \\
&= (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n-1},\ acx_1 \cdots x_{j-1} x_{j+1} \ldots x_{n-1} - x_n).
\end{aligned}$$

From (9) and the induction hypothesis, we have

$$\psi^{(j)} \circ \pi \circ \phi^{(j)} = (\varepsilon_1^{(j)} x_{\tau^{(j)}(1)}, \ldots, \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)}, \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)}, \ldots, \varepsilon_n^{(j)} x_{\tau^{(j)}(n)}),$$

where $\tau^{(j)} \in \mathcal{S}_{n-1}\{1, \ldots, j-1, j+1, \ldots, n\}$, $\varepsilon_i^{(j)} = \pm 1$.

Since $(\psi^{(j)})^2 = id$, we have

$$\begin{aligned}
\phi^{(j)} = \pi^{-1} \circ (&\varepsilon_1^{(j)} x_{\tau^{(j)}(1)}, \ldots, \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)}, \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)}, \ldots, \\
&ac\varepsilon_1^{(j)} x_{\tau^{(j)}(1)} \cdots \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)} \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)} \cdots \varepsilon_{n-1}^{(j)} x_{\tau^{(j)}(n-1)} - \varepsilon_n^{(j)} x_{\tau^{(j)}(n)}).
\end{aligned} \tag{11}$$

Since $\phi^{(j)} = (\phi_1, \ldots, \phi_{n-1})\big|_{x_j=c}$ for any c and ϕ is a polynomial in $x_1, \ldots, x_n$, it follows from (11) that

$$(\phi_1, \ldots, \phi_{n-1}) = \pi^{-1} \circ (\varepsilon_1^{(j)} x_{\tau^{(j)}(1)}, \ldots, \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)}, \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)}, \ldots,$$
$$a x_j \varepsilon_1^{(j)} x_{\tau^{(j)}(1)} \cdots \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)} \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)} \cdots \varepsilon_{n-1}^{(j)} x_{\tau^{(j)}(n-1)} - \varepsilon_n^{(j)} x_{\tau^{(j)}(n)}).$$

for some π and τ. Therefore,

$$n \geq \deg \phi = 2n - 3,$$

which contradicts with $n \geq 4$.

2. Now suppose that $\deg \phi^{(j)} \leq n - 1$ for $j = 1, \ldots, n$.

From (9) and by the induction hypothesis, we have

$$(12) \quad \phi^{(j)} = (\varepsilon_1^{(j)} x_{\tau^{(j)}(1)}, \ldots, \varepsilon_{j-1}^{(j)} x_{\tau^{(j)}(j-1)}, \varepsilon_{j+1}^{(j)} x_{\tau^{(j)}(j+1)}, \ldots, \varepsilon_n^{(j)} x_{\tau^{(j)}(n)}).$$

for any c, where $\tau^{(j)} \in \mathcal{S}_{n-1}\{1, \ldots, j-1, j+1, \ldots, n\}$, $\varepsilon_i^{(j)} = \pm 1$ and they may depend on c. Since $\phi^{(j)} = (\phi_1, \ldots, \phi_{n-1})\big|_{x_j=c}$ for any c and ϕ is a polynomial in $x_1, \ldots, x_n$, it follows from (12) that $(\phi_1, \ldots, \phi_{n-1})$ is independent of x_j for any $j = 1, \ldots, n$. Thus, $\deg \phi = 0$, which is absurd since $\phi \in G_{n,a}$.

These contradictions come from the hypothesis that $\phi_i \equiv c$ for some i, so we have $\deg \phi_i \geq 1$ for $i = 1, \ldots, n$. Since $\deg \phi \leq n$, this implies that

$$\deg \phi_1 = \cdots = \deg \phi_n = 1,$$

which completes the proof of Lemma 4. □

Corollary 1. *Suppose that $\phi \in G_{n,a}$. Then there exists $\varphi \in G_{n,a}$, such that*

$$\varphi \circ \phi = (d_1 x_{\pi(1)}, \ldots, d_n x_{\pi(n)}).$$

Proof. It follows immediately from Lemma 3 and Lemma 4. □

The foregoing results complete the proof of the following

Theorem 2. *$G_{n,a}$ is a group generated by $\mathcal{S}_n, \rho$ and ψ.*

References

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