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HARMONIC DIMENSION OF COVERING SURFACES AND MINIMAL FINE NEIGHBORHOOD

Dedicated to Professor Yukio Kusunoki on his seventieth birthday

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Introduction

Consider an *end* Ω in the sense of Heins [7]: Ω is a relatively noncompact subregion of an open Riemann surface R, which is of null boundary and has a single ideal boundary component, and the relative boundary $\partial\Omega(\neq \emptyset)$ of Ω consists of finitely many analytic closed Jordan curves. We denote by $\mathcal{P}(\Omega)$ the class of nonnegative harmonic functions on Ω with vanishing boundary values on $\partial\Omega$. The minimum number of elements of $\mathcal{P}(\Omega)$ generating $\mathcal{P}(\Omega)$ provided that such a finite set exists, otherwise ∞, is referred to as the *harmonic dimension* of Ω, $\dim\mathcal{P}(\Omega)$ in notation. In terms of Martin compactification, it is known that $\dim\mathcal{P}(\Omega)$ coincides with the number of minimal boundary points in the Martin compactification of R (cf. e.g. [4]), and hence $\dim\mathcal{P}(\Omega)=\dim\mathcal{P}(\Omega')$ for any pair (Ω,Ω') of ends of R.

Denote by D the punctured unit disc $\{0<|z|<1\}$ and let W be a p-sheeted $(1<p<\infty)$ unlimited covering surface of D such that the projection of branch points of W accumulates only at $z=0$. Then W is naturally considered as a subregion of an open Riemann surface R which is a p-sheeted unlimited covering surface of $\{0<|z|\leq\infty\}$. If R has a single ideal boundary component, it is seen that W is an end. We denote by $\mathcal{E}_p$ the class of ends W of this kind. In this paper we are especially concerned with ends belonging to $\mathcal{E}_p$. First of all, it is noted that $1\leq\dim\mathcal{P}(W)\leq p$ for each $W\in\mathcal{E}_p$ (cf. Heins [7]). Roughly speaking, if each sheet of W is closely connected with any of the other sheets, then $\dim\mathcal{P}(W)=1$ and if each sheet of W is faintly connected with the other sheets, then $\dim\mathcal{P}(W)=p$. This intuition is realized as follows. Consider two positive decreasing sequences $\{a_n\}$ and $\{b_n\}$ satisfying $b_{n+1}<a_n<b_n<1$ and $\lim_{n\to\infty}a_n=0$. Set $G=\{0<|z|<1\}-I$, where $I=\cup_{n=1}^{\infty}I_n$ and $I_n=[a_n,b_n]$. We take p copies $G_1,\cdots,G_p$ of G and join the upper edge of I_n on G_j with the lower edge of I_n on G_{j+1} (j mod p) for every n. Then we obtain a p-sheeted covering surface W_1 of D which belongs to $\mathcal{E}_p$. For this end W_1 we have showed the following (cf. [9], [11] and [14]).

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Theorem A. (i) *If I is thin at $z = 0$, in the sense that $z = 0$ is an irregular boundary point of G with respect to Dirichlet problem, then* $\dim \mathcal{P}(W_1) = p$.
(ii) *If I is thick (= not thin) at $z = 0$, then* $\dim \mathcal{P}(W_1) = 1$.

The proof of the above theorem in [9] and [11] essentially relies on symmetry of G relative to the real line and the fact that the cover transformation group of W_1 acts transitively and cyclically on each fiber. Main purpose of this paper is to show the following theorem which characterizes harmonic dimension in terms of fine topology and gives generalization of Theorem A.

Main Theorem. *Let $\mathcal{M}$ be the class of open connected subsets M of D such that $M \cup \{0\}$ is a fine neighborhood of $z = 0$. For every $W \in \mathcal{E}_p$, it holds that*

$$\dim \mathcal{P}(W) = \max_{M \in \mathcal{M}} n_W(M),$$

where $n_W(M)$ is the number of connected components of $\pi^{-1}(M)$ and π is the projection of W onto D.

After Preliminaries (§1), the proof of Main Theorem will be given in §2. In §3 from Main Theorem we shall derive Proposition 3.1 and Theorems 3.1 and 3.2, which include Theorem A above. Applying Main Theorem, we shall also show that, for an arbitrary given integer q with $1 \leq q \leq p$, there exists a $W \in \mathcal{E}_p$ such that $\dim \mathcal{P}(W) = q$.

Consider the cover transformation group $\mathcal{G}_W$ of $W \in \mathcal{E}_p$. In this paper we say that W is *normal* if there always exists a $\tau \in \mathcal{G}_W$ which carries a given point w into a prescribed point w' with same projection (cf. [6]). In §4 we show that the *harmonic dimension of W divides p if W is normal.* We also say that W is *cyclic* if W is normal and $\mathcal{G}_W$ is cyclic. The end W_1 in Theorem A is a typical example of cyclic covering surfaces. So it might be interesting whether the range of harmonic dimensions of cyclic covering surfaces in $\mathcal{E}_p$ is $\{1, p\}$. Second purpose of §4 is to answer this question negatively. In fact, for each divisor q of p, we shall give an example $W \in \mathcal{E}_p$ such that W is cyclic and $\dim \mathcal{P}(W) = q$.

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1. Preliminaries from potential theory

1.1 We begin with recalling the definition of balayage. Consider an open Riemann surface F possessing a Green's function. Denote by $\mathcal{S} = \mathcal{S}(F)$ the class of all nonnegative superharmonic functions on F. Let E be a subset of F and s

belong to $\mathcal{S}$. Then the *balayage* $\widehat{\mathrm{R}}_s^E = {}^F\widehat{\mathrm{R}}_s^E$ of s relative to E on F is defined by

$$\widehat{\mathrm{R}}_s^E(w) = \liminf_{x \to w} \inf\{u(x) : u \in \mathcal{S}, \ u \geq s \text{ on} E\}$$

(cf. e.g. [2]). We here review fundamental properties of balayage (cf. [2], [3], [8], etc).

Proposition 1.1. (i) *If* $E_1 \subset E_2$, *then* $\widehat{\mathrm{R}}_s^{E_1} \leq \widehat{\mathrm{R}}_s^{E_2}$;
(ii) $\widehat{\mathrm{R}}_s^{E_1 \cup E_2} \leq \widehat{\mathrm{R}}_s^{E_1} + \widehat{\mathrm{R}}_s^{E_2}$;
(iii) *if* $u, v \in \mathcal{S}$ *and* $s, t > 0$, *then* $\widehat{\mathrm{R}}_{su+tv}^E = s\widehat{\mathrm{R}}_u^E + t\widehat{\mathrm{R}}_v^E$;
(iv) *if* E_1 *and* E_2 *are closed subsets of* F *and* N *is a common connected component of both* $F - E_1$ *and* $F - E_2$, *then* $\widehat{\mathrm{R}}_s^{E_1} = \widehat{\mathrm{R}}_s^{E_2}$ *on* N.

The following lemma gives us a relation between balayage on F and that on a covering surface of F (cf. [11]).

Lemma 1.1. *Let* $\tilde{F}$ *be an unlimited covering surface of* F *with the canonical projection* π *from* $\tilde{F}$ *onto* F, E *a subset of* F *and* s *belong to* $\mathcal{S}$. *Then it holds that*

$${}^{\tilde{F}}\widehat{\mathrm{R}}_{s \circ \pi}^{\pi^{-1}(E)} = {}^F\widehat{\mathrm{R}}_s^E \circ \pi$$

on $\tilde{F}$.

1.2 We next state definitions of thinness and fine neighborhood (cf. [3]).

DEFINITION 1.1. Let a be a point in $\mathbf{C}$ and set $F = \{|z - a| < r\}$ and $g_a(z) = \log(r/|z - a|)$. We say that a subset E of $\mathbf{C}$ is *thin* at a if ${}^F\widehat{\mathrm{R}}_{g_a}^{E \cap F} \neq g_a$.

It is easily seen that the above definition does not depend on a choice of $r > 0$. If E is closed and a belongs to E in the above, it is well-known that E is thin at a if and only if a is an irregular boundary point of $F - E$ with respect to Dirichlet problem (cf. e.g. [3]). We also say that E is *thick* at $z = 0$ if E is not thin at $z = 0$.

DEFINITION 1.2. A subset U in $\mathbf{C}$ which contains a is said to be a *fine neighborhood* of a if $\mathbf{C} - U$ is thin at a.

The notion of fine neighborhood is originally defined in the category of *fine topology*. However, since it is well-known that the above definition of fine neighborhood coincides with the original definition, we adopt the above for convenience.

As for thinness of circular projection of closed set in D, the following proposition is applied in §3 (cf. [8]).

Proposition 1.2. *Let E be a closed subset of D and put $E' = \{|z| : z \in E\}$. If E is thin at $z = 0$, then E' is thin at $z = 0$.*

In §3, we are also in need of the following proposition (cf. [5]).

Proposition 1.3. *Let U be a fine neighborhood of $z = 0$. Then there exists a polar subset Z of $\{|z| = 1\}$ satisfying the following property ; for every ray $l_\theta = \{\arg z = \theta\}$ with $l_\theta \cap Z = \emptyset$, there exists a positive number ρ such that $l_\theta \cap \{|z| < \rho\} \subset U$.*

1.3 Let F be an open Riemann surface possessing a Green's function. We denote by Δ the Martin boundary of F and by k_ζ the Martin kernel on F with pole at ζ. The *minimal boundary* of F, Δ_1 in notation, is defined as the set of all minimal points in Δ. Similarly as thinness and fine neighborhood, *minimal* thinness and *minimal* fine neighborhood are defined as follows (cf. [3]).

DEFINITION 1.3. Let ζ be a point in Δ_1 and E a subset of F. We say that E is *minimally thin* at ζ if ${}^F\widehat{\mathrm{R}}^E_{k_\zeta} \neq k_\zeta$.

DEFINITION 1.4. Let ζ be a point in Δ_1 and U a subset of F. We say that $U \cup \{\zeta\}$ is a *minimal fine neighborhood* of ζ if $F - U$ is minimally thin at ζ.

The following proposition plays a fundamental role in the study of (minimal) thinness and (minimal) fine neighborhood (cf. [12]).

Proposition 1.4. *Let ζ be a point in Δ_1 and E a closed subset of F. If E is minimally thin at ζ, then ${}^F\widehat{\mathrm{R}}^E_{k_\zeta}$ is a potential and there exists a unique connected component U of $F - E$ such that ${}^F\widehat{\mathrm{R}}^E_{k_\zeta} < k_\zeta$ on U. Moreover, under the condition of Definition* 1.1, *if E is a closed subset of F and thin at a, then ${}^F\widehat{\mathrm{R}}^E_{g_a}$ is a potential and there exists a unique connected component U of $F - E$ such that ${}^F\widehat{\mathrm{R}}^E_{g_a} < g_a$ on U.*

We close Preliminaries by stating the following, which is easily verified from the above proposition (cf. [12]).

Proposition 1.5. *Let ζ be a point in Δ_1 and U an open subset of F such that $U \cup \{\zeta\}$ is a minimal fine neighborhood of ζ. Then there exists a unique connected component V of U such that $V \cup \{\zeta\}$ is a minimal fine neighborhood of ζ. Moreover, let a be a point in $\mathbf{C}$ and U an open subset of $\mathbf{C}$ such that $U \cup \{a\}$ is a fine neighborhood of a. Then there exists a connected component V of U such that*

$V \cup \{a\}$ is a fine neighborhood of a.

2. Proof of Main Theorem

2.1 Throughout this section, let W belong to $\mathcal{E}_p$, where $\mathcal{E}_p$ is defined in Introduction. Denote by ∂W the relative boundary of W and by $\pi = \pi_W$ the projection of $\overline{W} = W \cup \partial W$ onto $\{0 < |z| \leq 1\}$. Note that $\pi(\partial W) = \{|z| = 1\}$. We consider the Martin compactification W^* of W. Then W^* takes a form $W^* = W \cup \partial W \cup \Delta^W$, where Δ^W is the *Martin ideal boundary* of a bordered surface $\overline{W}$. We also denote by Δ_1^W the set of minimal points in Δ^W. It is well-known that $\dim \mathcal{P}(W)$ coincides with the number of points in Δ_1^W (cf. e.g. [4]). We again note that

$$(2.1) \qquad 1 \leq \dim \mathcal{P}(W) \leq p$$

for every $W \in \mathcal{E}_p$ (cf. [7]). For simplicity of notation, here and hereafter denote by $\widehat{\mathrm{R}}_f^E$ the balayage ${}^W\widehat{\mathrm{R}}_f^E$ on W and set $g(z) = \log(1/|z|)$. We first maintain the following

Lemma 2.1. *Let ζ belong to Δ_1^W and N be an open subset of W. Suppose that $N \cup \{\zeta\}$ is a minimal fine neighborhood of ζ. Then $\pi(N) \cup \{0\}$ is a fine neighborhood of $z = 0$.*

Proof. By (2.1), we can put $\Delta_1^W = \{\zeta_1, \cdots, \zeta_n\}$ $(n \leq p)$, where $\zeta_1 = \zeta$. Let k_i be the Martin kernel with pole at ζ_i $(i = 1, \cdots, n)$. In view of Proposition 1.5, we may assume that N is connected. By Proposition 1.4,

$$(2.2) \qquad \widehat{\mathrm{R}}_{k_1}^{W-N} < k_1$$

on N. It is easily seen that $g \circ \pi \geq c k_i$ with $c > 0$ $(i = 1, \cdots, n)$. Therefore the Martin representation theorem (cf. [4], [8]) implies that there exist positive numbers c_i satisfying

$$(2.3) \qquad g \circ \pi = \sum_{i=1}^{n} c_i k_i.$$

From this with Lemma 1.1 and Proposition 1.1 it follows that

$${}^D\widehat{\mathrm{R}}_g^{D-\pi(N)} \circ \pi = \widehat{\mathrm{R}}_{g\circ\pi}^{W-\pi^{-1}(\pi(N))} \leq \widehat{\mathrm{R}}_{g\circ\pi}^{W-N} = \sum_{i=1}^{n} c_i \widehat{\mathrm{R}}_{k_i}^{W-N} \leq \sum_{i=1}^{n} c_i k_i.$$

Hence, in view of (2.2) and (2.3), we have

$$ {}^{D}\widehat{\mathrm{R}}_g^{D-\pi(N)} \circ \pi < \sum_{i=1}^{n} c_i k_i = g \circ \pi $$

on N, i.e. ${}^{D}\widehat{\mathrm{R}}_g^{D-\pi(N)} < g$ on $\pi(N)$. Since ${}^{D}\widehat{\mathrm{R}}_g^{E} = {}^{D_0}\widehat{\mathrm{R}}_g^{E}$ in general, where $D_0 = \{|z| < 1\}$, this means that $\pi(N) \cup \{0\}$ is a fine neighborhood of $z = 0$. □

2.2 As stated in Main Theorem, let $\mathcal{M}$ be the class of open connected subsets M of D such that $M \cup \{0\}$ is a fine neighborhood of $z = 0$, or equivalently, $D - M$ is thin at $z = 0$. We next claim

Lemma 2.2. *Let M belong to $\mathcal{M}$ and N be an arbitrary connected component of $\pi^{-1}(M)$, where $\pi = \pi_W$. Then there exists a point $\zeta \in \Delta_1^W$ such that $N \cup \{\zeta\}$ is a minimal fine neighborhood of ζ.*

Proof. Let $\Delta_1^W = \{\zeta_1, \cdots, \zeta_n\}$ and k_i be the Martin kernel with pole at ζ_i $(i = 1, \cdots, n)$. In the same way as in the proof of the preceding lemma (see (2.3)), there exist positive numbers c_i such that

$$ g \circ \pi = \sum_{i=1}^{n} c_i k_i. \tag{2.4} $$

By definition and Proposition 1.4, we have ${}^{D}\widehat{\mathrm{R}}_g^{D-M} < g$ on M. Therefore Proposition 1.1, (2.4) and Lemma 1.1 imply that

$$ \sum_{i=1}^{n} c_i \widehat{\mathrm{R}}_{k_i}^{W-\pi^{-1}(M)} = \widehat{\mathrm{R}}_{g\circ\pi}^{W-\pi^{-1}(M)} = {}^{D}\widehat{\mathrm{R}}_g^{D-M} \circ \pi < g \circ \pi = \sum_{i=1}^{n} c_i k_i $$

on $\pi^{-1}(M)$. Hence there exists a $\nu \in \{1, \cdots, n\}$ such that $\widehat{\mathrm{R}}_{k_\nu}^{W-\pi^{-1}(M)} \neq k_\nu$ on N. On the other hand, in view of (iv) of Proposition 1.1, it is easily seen that $\widehat{\mathrm{R}}_{k_\nu}^{W-\pi^{-1}(M)} = \widehat{\mathrm{R}}_{k_\nu}^{W-N}$ on N. Therefore we conclude that $\widehat{\mathrm{R}}_{k_\nu}^{W-N} \neq k_\nu$ on N, i.e. $N \cup \{\zeta_\nu\}$ is a minimal fine neighborhood of ζ_ν. □

2.3 In addition to Lemmas 2.1 and 2.2, we need the following lemma (cf. [10]) for the proof of Main Theorem.

Lemma 2.3. *Let E be a subset of W. If E is minimally thin at every $\zeta \in \Delta_1^W$, then $\pi(E)$ is thin at $z = 0$.*

Before starting with the proof of Main Theorem, recall the definition of $n_W(M)$. For each $M \in \mathcal{M}$, let $n_W(M)$ be the number of connected components of $\pi^{-1}(M) = \pi_W^{-1}(M)$.

Proof of Main Theorem. Set $\dim \mathcal{P}(W) = n$ and $\Delta_1^W = \{\zeta_1, \cdots, \zeta_n\}$. We first show that there exists an $M \in \mathcal{M}$ such that $n \leq n_W(M)$, which implies that $\dim \mathcal{P}(W) \leq \max_{M \in \mathcal{M}} n_W(M)$. Let $\{N_1, \cdots, N_n\}$ be mutually disjoint open connected subsets of W such that each $N_i \cup \{\zeta_i\}$ is a minimal fine neighborhood of ζ_i. Set $S = \cap_{i=1}^n \pi(N_i)$ and $E = \cap_{i=1}^n N_i^c$. Since each $\pi(N_i) \cup \{0\}$ is a fine neighborhood of $z = 0$ by means of Lemma 2.1, $S \cup \{0\}$ is also a fine neighborhood of $z = 0$. Since each N_i^c is minimally thin at ζ_i by definition, E is minimaly thin at each ζ_i. Hence $\pi(E)$ is thin at $z = 0$ by means of Lemma 2.3. Therefore $(S - \pi(E)) \cup \{0\}$ is a fine neighborhood of $z = 0$. Then, by Proposition 1.5, there exists a connected component M of $S - \pi(E)$ belonging to $\mathcal{M}$. Let w be a point in $N_i \cap \pi^{-1}(M)$ and C be an arbitrary curve in W such that w is the initial point of C and $\pi(C)$ is contained in M. Then it is seen that the end point of C belongs to N_i. By this reasoning, for each $i = 1, \cdots, n$, we can take a connected component O_i of $\pi^{-1}(M)$ such that $O_i \subset N_i$. It is evident that O_i's are mutually disjoint. Consequently we conclude that $n_W(M) \geq n$.

We next show that $n \geq n_W(M)$ for every $M \in \mathcal{M}$, which implies that $\dim \mathcal{P}(W) \geq \max_{M \in \mathcal{M}} n_W(M)$. Set $m = n_W(M)$ and let $\{N_1, \cdots, N_m\}$ be the totality of connected components of $\pi^{-1}(M)$. For each N_i, by virtue of Lemma 2.2, there exists an $\eta_i \in \Delta_1^W$ such that $N_i \cup \{\eta_i\}$ is a minimal fine neighborhood of η_i. Since $N_i \cap N_j = \emptyset$ if $i \neq j$, we see that $\eta_i \neq \eta_j$ if $i \neq j$. Therefore we obtain that $\{\eta_1, \cdots, \eta_m\} \subset \Delta_1^W$ or $m \leq \dim \mathcal{P}(W)$.

The proof is herewith complete. □

By virtue of Main Theorem and (2.1), the following corollaries are instantly verified.

Corollary 2.1. *If there exists an $M \in \mathcal{M}$ such that $n_W(M) = p$, then* $\dim \mathcal{P}(W) = p$.

Corollary 2.2. *If $n_W(M) = 1$ for every $M \in \mathcal{M}$, then* $\dim \mathcal{P}(W) = 1$.

3. Applications of Main Theorem

3.1 In this section, we are concerned with application of Main Theorem. Let $\{I_n\}_{n=1}^{\infty}$ and $\{J_n\}_{n=1}^{\nu}$ $(0 \leq \nu \leq \infty)$ be sequences of closed segments in D accumulating only at $z = 0$ such that $I_m \cap I_n = J_m \cap J_n = \emptyset$ $(m \neq n)$ and $I_m \cap J_n = \emptyset$. We also assume that $I_n \subset \{\arg z = \theta_n\}$ for a sequence $\{\theta_n\}$. Set

$$I = \bigcup_{n=1}^{\infty} I_n, \quad J = \bigcup_{n=1}^{\nu} J_n \quad \text{and} \quad S = S(I, J) = D - I - J.$$

We consider a subclass $\mathcal{F}_p(I,J)$ of $\mathcal{E}_p$ which consists of ends $W \in \mathcal{E}_p$ satisfying the following three conditions, where we denote by π_W the projection of W onto D:

(i) $\pi_W^{-1}(S)$ consists of p connected components $S_1, \cdots, S_p$, where each S_i is a copy of S,

(ii) every branch point of W lies over a point in the set of end points of $\{I_n\}$ and $\{J_n\}$,

(iii) for every end point z of $\{I_n\}$, there exists a branch point of W of order $p-1$ (i.e. of multiplicity p) which lies over z.

We first maintain

Proposition 3.1. *Let W be in $\mathcal{F}_p(I,J)$. If $I \cup J$ is thin at $z=0$, then* $\dim \mathcal{P}(W) = p$.

The above result was originally proved in [11]. We here give an alternative and very short proof of the above by applying Corollary 2.1.

Proof. By assumption, S belongs to $\mathcal{M}$. It follows from the above condition (i) that $n_W(S) = p$. Therefore, by virtue of Corollary 2.1, we have $\dim \mathcal{P}(W) = p$. □

3.2 Here and hereafter, we set $E' = \{|z| : z \in E\}$ for a subset E of D.

Theorem 3.1. *Let W be in $\mathcal{F}_p(I,J)$. Suppose that $\{I'_n\}$ are mutually disjoint. If I' is thick at $z=0$ and J' is thin at $z=0$, then* $\dim \mathcal{P}(W) = 1$.

Proof. It suffices to show that $n_W(M) = 1$ for every $M \in \mathcal{M}$. For this purpose, set $E = D - M$. Since E is thin at $z=0$, it follows from Proposition 1.2 that E' is thin at $z=0$. Denoting by K the totality of end points of $\{I_n\}$, we see that $E' \cup J' \cup K'$ is thin at $z=0$, and therefore $I' - (E' \cup J' \cup K')$ is thick at $z=0$. Hence there exist an r with $0 < r < 1$ and a positive integer m such that r belongs to $I'_m - (E' \cup J' \cup K')$. Setting $C_1 = \{|z| = r\}$, we deduce that $C_1 \subset M$, $C_1 \cap J = \emptyset$ and $C_1 \cap I$ consists of a single point a which contained in $I_m - K$. We shall show that $\pi_W^{-1}(C_1)$ consists of a single closed Jordan curve. Consider a closed Jordan curve C_2 passing through a such that $C_2 - \{a\}$ is contained in S, C_2 surrounds only one end point of I_m and the inside of C_2 is contained in $S \cup I_m$. We give C_i $(i=1,2)$ anti-clockwise orientation and parametrization $z = z_i(t)$ $(0 \le t \le 1)$ with $z_i(0) = z_i(1) = a$. Take an arbitrary point $b \in \pi_W^{-1}(a)$ and denote by $\Gamma_i : w = w_i(t)$ $(0 \le t \le 1)$ the lift of C_i by π_W^{-1} satisfying $w_i(0) = b$. Then, since $C_1 - \{a\}$ and $C_2 - \{a\}$ are contained in S, we see that $w_1(1) = w_2(1)$. On the other hand, it follows from the conditions (ii) and (iii) in no. 3.1 that $\pi_W^{-1}(C_2)$ consists of a single closed Jordan curve. Consequently, we obtain that

$\pi_W^{-1}(C_1)$ also consists of a single closed Jordan curve. Hence, in view of $C_1 \subset M$, it is not difficult to see that $\pi_W^{-1}(M)$ is connected, which completes the proof. □

It is easily seen that Proposition 3.1 and Theorem 3.1 yield Theorem A in Introduction.

3.3 Applying the same argument as in the above proof, we can prove the following, which gives another sufficient condition in order that $\dim \mathcal{P}(W) = 1$.

Theorem 3.2. *Let W be in $\mathcal{F}_p(I, J)$ and $l_\theta = \{\arg z = \theta\}$. Suppose that $I \subset l_\theta$ and there exists a positive number ε such that $J \subset D - \{|\arg z - \theta| < \varepsilon\}$. If both of I and $l_\theta - I$ are thick at $z = 0$, then* $\dim \mathcal{P}(W) = 1$.

Proof. Without loss of generality, we may assume that $\theta = 0$. As well as Proof of the preceding theorem, we have only to show that $n_W(M) = 1$ for every $M \in \mathcal{M}$. By means of Proposition 1.3, there exist two rays l_α and l_β with $-\varepsilon < \alpha < 0 < \beta < \varepsilon$ and a positive number ρ such that $\{re^{i\alpha} : 0 < r < \rho\}$ and $\{re^{i\beta} : 0 < r < \rho\}$ are contained in M. Since $(D - M)'$ is thin at $z = 0$ by Proposition 1.2 and I is thick at $z = 0$, there exists a positive number s with $s < \rho$ such that $C_s = \{se^{i\gamma} : \alpha \leq \gamma \leq \beta\} \subset M$ and $s \in I - K$, where K is the totality of end points of $\{I_n\}$. By similar reasoning, there exists a positive number t with $t < s$ such that $C_t = \{te^{i\gamma} : \alpha \leq \gamma \leq \beta\} \subset M$ and $t \in l_0 - I$. Joining C_t, $\{re^{i\alpha} : t \leq r \leq s\}$, C_s and $\{re^{i\beta} : t \leq r \leq s\}$ in order, we obtain a closed curve C in M. Since $C - \{s\} \subset S$ and $s \in I - K$, applying the same argument as in the preceding proof, we can conclude that $\pi_W^{-1}(C)$ is connected. Hence, by the fact $C \subset M$, we see that $n_W(M) = 1$ and $\dim \mathcal{P}(W) = 1$. This completes the proof. □

We here remark that, in Theorem 3.2, the assumption that $J \subset D - \{|\arg z - \theta| < \varepsilon\}$ for $\varepsilon > 0$ can not be removed. Let I be the same as in Theorem 3.2 and set $I_n = \{te^{i\theta} : a_n \leq t \leq b_n\}$. Take a decreasing sequence $\{\theta_n\}$ with $\lim_{n\to\infty} \theta_n = \theta$. Put $J_n = \{te^{i\theta_n} : a_n \leq t \leq b_n\}$ and $J = \cup_{n=1}^{\infty} J_n$. For this pair (I, J), choose a $W \in \mathcal{F}_2(I, J)$ such that the projection of the totality of branch points of W coincides with the totality of end points of $\{I_n\}$ and $\{J_n\}$. Denote by L_{2n} (resp. L_{2n-1}) the closed segment $[a_n e^{i\theta}, a_n e^{i\theta_n}]$ (resp. $[b_n e^{i\theta}, b_n e^{i\theta_n}]$). Making the convergence $\theta_n \to \theta$ be sufficiently rapid, we can suppose that $L = \cup_{n=1}^{\infty} L_n$ is thin at $z = 0$. Then it holds that $\dim \mathcal{P}(W) = 2$. In fact, $M = D - L$ belongs to $\mathcal{M}$ and $\pi_W^{-1}(M)$ consists of two components. Hence, by Corollary 2.1, we have that $\dim \mathcal{P}(W) = 2$.

We also remark that, in Theorem 3.2, the assumption that $l_\theta - I$ is thick at $z = 0$

can not be removed (cf. [11] and see also no. 4.4).

3.4 Although it seems to be well-known that, for an arbitrarily given integer q with $1 \leq q \leq p$, there exists an end $W \in \mathcal{E}_p$ such that $\dim \mathcal{P}(W) = q$, we can find no references to show this explicitly. For the sake of completeness, applying Main Theorem, we shall give an example $W_q \in \mathcal{E}_p$ with $\dim \mathcal{P}(W_q) = q$ for every $q \in \{1, \cdots, p\}$.

If q is 1 or p, then there exists a $W \in \mathcal{E}_p$ with $\dim \mathcal{P}(W) = q$ by means of Theorem A in Introduction. Assume that $1 < q < p$. For convenience' sake, we use the notation $\mathcal{E}_n$ for $n = 1$: $\mathcal{E}_1 = \{D\}$, where D is considered to be an end of $\{0 < |z| \leq \infty\}$. By Theorem A and the fact that $\dim \mathcal{P}(D) = 1$, there exists an end $V_1 \in \mathcal{E}_{q-1}$ (resp. $V_2 \in \mathcal{E}_{p-q+1}$) such that $\dim \mathcal{P}(V_1) = q - 1$ (resp. $\dim \mathcal{P}(V_2) = 1$). Denote by π_1 (resp. π_2) the projection of V_1 (resp. V_2) onto D. Consider a sequence $\{K_n\}_{n=1}^{\infty}$ of mutually disjoint closed segments in D such that K is thin at $z = 0$ and there exist no branch points of V_1 (resp. V_2) on $\pi_1^{-1}(K)$ (resp. $\pi_2^{-1}(K)$), where $K = \cup_{n=1}^{\infty} K_n$. For every n, let κ_{1n} (resp. κ_{2n}) be a closed segment in V_1 (resp. V_2) with $\pi_1(\kappa_{1n}) = K_n$ (resp. $\pi_2(\kappa_{2n}) = K_n$). Joining V_1 with V_2 crosswise along slits κ_{1n} and κ_{2n} for every n, we obtain an end W_q which belongs to $\mathcal{E}_p$. To prove that $\dim \mathcal{P}(W_q) = q$, we denote by π the projection of W_q onto D. By virtue of Main Theorem, we can find an $M \in \mathcal{M}$ with $n_{V_1}(M) = q - 1$. Let M' be the connected component of $M - K$ such that $z = 0$ is a boundary point of M'. Then M' also belongs to $\mathcal{M}$, because K is thin at $z = 0$. Since $n_{V_1}(M') = q - 1$ and $n_{V_2}(M') \geq 1$, it is easily seen that $n_{W_q}(M') \geq q$, and hence $\dim \mathcal{P}(W_q) \geq q$ by Main Theorem. On the other hand, by Main Theorem, we have that $n_{V_2}(M) = 1$ for every $M \in \mathcal{M}$. From this and the fact that $n_{V_1}(M) \leq q - 1$ for every $M \in \mathcal{M}$, it follows that $n_{W_q}(M) \leq q$ for every $M \in \mathcal{M}$, and hence $\dim \mathcal{P}(W_q) \leq q$ by means of Main Theorem. The proof of $\dim \mathcal{P}(W_q) = q$ is herewith complete.

4. Normal and cyclic covering surfaces

4.1 We start with fixing terminology and notation. Let F be an open Riemann surface and $\tilde{F}$ an *unlimited* covering surface of F. Denote by $\pi = \pi_{\tilde{F}}$ the canonical projection of $\tilde{F}$ onto F. A conformal mapping τ of $\tilde{F}$ onto itself is said to be a *cover transformation* of $\tilde{F}$ if $\pi(\tau(w)) = \pi(w)$ for every $w \in \tilde{F}$. Denote by $\mathcal{G}_{\tilde{F}}$ the group of cover transformations of $\tilde{F}$. We proceed to the statement of definitions of normal and cyclic covering surfaces. Although normality is usually defined only for unramified covering surfaces(cf. [1]), we adopt the following definition, in which normality is defined even for ramified covering surfaces (cf. [6]).

DEFINITION 4.1. We say that $\tilde{F}$ is *normal* if $\mathcal{G}_{\tilde{F}}$ is transitive on $\pi^{-1}(z)$ for every $z \in F$, *that is*, for every pair (w_1, w_2) of $\pi^{-1}(z)$, there exists a $\tau \in \mathcal{G}_{\tilde{F}}$ such that $\tau(w_1) = w_2$.

DEFINITION 4.2. We say that $\tilde{F}$ is *cyclic* if $\tilde{F}$ is normal and $\mathcal{G}_{\tilde{F}}$ is cyclic.

If $\tilde{F}$ is p-sheeted $(1 < p < \infty)$, it is not difficult to see that the order of $\mathcal{G}_{\tilde{F}}$ is a divisor of p and that $\tilde{F}$ is normal if and only if $\mathcal{G}_{\tilde{F}}$ is exactly of order p (cf. e.g. [1]).

4.2 Assuming that $W \in \mathcal{E}_p$ is normal, we are interested in relation between the harmonic dimension of W and the number p of sheets of W. Our first assertion of this section is the following

Theorem 4.1. *If $W \in \mathcal{E}_p$ is normal, then the harmonic dimension of W divides p.*

Proof. We first put $\dim \mathcal{P}(W) = m$. By means of Main Theorem, there exists an $M \in \mathcal{M}$ such that $n_W(M) = m$. We denote by $\{N_1, \cdots, N_m\}$ the totality of connected components of $\pi^{-1}(M)$ and set $H = \{\tau \in \mathcal{G}_W : \tau(N_1) = N_1\}$. It is evident that H is a subgroup of $\mathcal{G}_W$. Let r be the order of H and put $q = p/r$. Consider the decomposition of $\mathcal{G}_W$ by left cosets of H:

$$\mathcal{G}_W = \tau_1 H + \cdots + \tau_q H, \tag{4.1}$$

where $\tau_1 H = H$.

Observe that every $\tau \in \mathcal{G}_W$ induces a permutation on $\{N_1, \cdots, N_m\}$. Therefore we can consider a mapping $\varphi : \tau_i \mapsto N_\nu$ from $\{\tau_1, \cdots, \tau_q\}$ to $\{N_1, \cdots, N_m\}$ such that $\tau_i(N_1) = N_\nu$. Suppose that $\varphi(\tau_i) = \varphi(\tau_j)$. Then we have that $\tau_j^{-1}\tau_i(N_1) = N_1$ or $\tau_j^{-1}\tau_i \in H$, and hence $\tau_i = \tau_j$ by (4.1). This implies that φ is injective. On the other hand, for each N_ν, there exists a $\tau \in \mathcal{G}_W$ satisfying $\tau(N_1) = N_\nu$ by assumption. Putting $\tau = \tau_i\sigma$ for a $\sigma \in H$, we see that $\tau_i(N_1) = \tau_i\sigma(N_1) = \tau(N_1) = N_\nu$. This implies that φ is also surjective. Consequently we obtain that φ is bijective, and hence $m = q$. This completes the proof. □

4.3 Put $\mathcal{H}_p = \{\dim \mathcal{P}(W) : W \in \mathcal{E}_p,\ W \text{ is cyclic}\}$ and denote by $\mathcal{I}_p$ the totality of divisors of p. Our second assertion of this section is that $\mathcal{I}_p \subset \mathcal{H}_p$. In fact, for an arbitrarily given $q \in \mathcal{I}_p$, we shall construct a $W \in \mathcal{E}_p$ such that W is cyclic and $\dim \mathcal{P}(W) = q$. On the other hand, by Theorem 4.1, we see that $\mathcal{H}_p \subset \mathcal{I}_p$. Consequently we obtain

Theorem 4.2. *There exists a cyclic covering surface $W \in \mathcal{E}_p$ with* $\dim \mathcal{P}(W) = q$ *if and only if q is a divisor of p.*

Suppose $q \in \mathcal{I}_p$ and put $p = qr$. Consider a sequence $\{a_n\}_{n=1}^{\infty}$ with $2^{-n} < a_n < 2^{-n+1}$. Making each a_n draw near 2^{-n+1}, we can assume that $\cup_{n=1}^{\infty}[a_n, 2^{-n+1}]$ is thin at $z = 0$ (cf. e.g. [16]) and

$$\sum_{n=1}^{\infty} \log \frac{a_n}{2^{-n}} = \infty. \tag{4.2}$$

For each $j = 1, \cdots, q$ and each $n = 1, 2, \cdots$, set

$$I_n^j = \{te^{i\frac{2j\pi}{q}} : 2^{-n} \leq t \leq a_n\} \qquad \text{and} \qquad G = D - \cup_{j=1}^{q} I^j,$$

where $I^j = \cup_{n=1}^{\infty} I_n^j$. For convenience' sake, we denote by I_n^{j+} (resp. I_n^{j-}) the left (resp. right) side edge of the slit I_n^j in D with respect to the ray $\{\arg z = 2j\pi/q\}$. Take p copies $G_1, \cdots, G_p$ of G. Joining I_n^{j+} on G_i with I_n^{j-} on G_{i+1} (i mod p) for every $j = 1, \cdots, q$ and every $n = 1, 2, \cdots$, we obtain a covering surface W_{pq} of D belonging to $\mathcal{E}_p$. It is clear that W_{pq} is cyclic. Our goal is to show that $\dim \mathcal{P}(W_{pq}) = q$. The proof is accomplished in no. 4.4.

4.4 We first show that $\dim \mathcal{P}(W_{pq}) \leq q$. For this purpose, we need the following (cf. [13] and [15])

Proposition 4.1. *Let Ω be an end in the sense of Heins and $\{A_n\}_{n=1}^{\infty}$ a sequence of mutually disjoint subset of Ω such that each A_n consists of at most μ mutually disjoint annuli $\{A_{nm}\}_{m=1}^{\mu_n}$ ($\mu_n \leq \mu$) and A_{n+1} separates A_n from the ideal boundary of Ω for every n. Suppose that $\sum_{n=1}^{\infty} \mathrm{mod}\, A_n = \infty$, where $\mathrm{mod}\, A_{nm}$ are moduli of A_{nm} and $(\mathrm{mod}\, A_n)^{-1} = (\mathrm{mod}\, A_{n1})^{-1} + \cdots + (\mathrm{mod}\, A_{n\mu_n})^{-1}$. Then $\dim \mathcal{P}(\Omega)$ is at most μ.*

Set $A_n = \pi^{-1}(\{2^{-n} < |z| < a_n\})$ ($n = 1, 2, \cdots$), where π is the projection of W_{pq} onto D. By construction of W_{pq}, it is not difficult to see that each A_n consists of q disjoint annuli $A_{n1}, \cdots, A_{nq}$ which are conformally equivalent to $\{1 < |z| < \sqrt[r]{2^n a_n}\}$, where $r = p/q$. This yields that

$$(\mathrm{mod}\, A_n)^{-1} = (\mathrm{mod}\, A_{n1})^{-1} + \cdots + (\mathrm{mod}\, A_{nq})^{-1} = qr(\log(2^n a_n))^{-1}.$$

Hence, by (4.2), we have that

$$\sum_{n=1}^{\infty} \mathrm{mod}\, A_n = \frac{1}{p} \sum_{n=1}^{\infty} \log(2^n a_n) = \infty.$$

It is evident that A_{n+1} separates A_n from the ideal boundary of W_{pq}. Consequently, by means of Proposition 4.1, we see that $\dim \mathcal{P}(W_{pq}) \leq q$.

We next show that $\dim \mathcal{P}(W_{pq}) \geq q$. Set

$$J^j = D \cap \{\arg z = \frac{2j\pi}{q}\} - I^j, \quad E = \bigcup_{j=1}^{q} \overline{J^j} \quad \text{and} \quad S = D - E,$$

where $\overline{J^j}$ is the closure of J^j. Since each $\overline{J^j}$ is thin at $z = 0$ by assumption, it follows that E is thin at $z = 0$, and hence S belongs to $\mathcal{M}$ defined in no. 2.2. It is not difficult to see that $\pi^{-1}(S)$ consists of q components as well as A_n, i.e. $n_{W_{pq}}(S) = q$. Therefore, by virtue of Main Theorem, we conclude that $\dim \mathcal{P}(W_{pq}) \geq q$. The proof of $\dim \mathcal{P}(W_{pq}) = q$ is herewith complete.

REMARK. Set $I = I^1$ and $J = \cup_{j=2}^{q} I^j$ for $\{I^j\}_{j=1}^{q}$ in no. 3.1. Then the above W_{pq} for $q > 1$ shows that, in Theorem 3.2, the assumption that $l_\theta - I$ is thick at $z = 0$ can not be removed.

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