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COEFFECTIVE-DOLBEAULT COHOMOLOGY OF COMPACT INDEFINETE KÄHLER MANIFOLDS[1]

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1. Introduction

In this paper we consider indefinite Kähler manifolds, that is, complex manifolds with a compatible indefinite metric such that the associated Kähler form is closed. The class of indefinite Kähler manifolds is a particular class of symplectic manifolds containing the class of Kähler manifolds. There exists some similarities of the theory of indefinite Kähler manifolds with the theory of Kähler manifolds, in particular, the formalism associated with the covariant derivative and the curvature operator [1, 3], but there exists also some known differences:

1. The minimal model of a compact Kähler manifold is formal [12], but there are examples of non-formal compact indefinite Kähler manifolds [6, 13, 14].
2. Any compact Kähler manifold satisfies the Hard Lefschetz theorem [17], but this is in general false for compact indefinite Kähler manifolds [5].
3. The Frölicher spectral sequence of a compact Kähler manifold always collapses at the E_1 term [17], but examples of indefinite Kähler manifolds which Frölicher spectral sequence may not collapse even at E_2 are known [7, 8, 9].

Notice that the known examples of indefinite Kähler manifolds with no Kähler structure or not satisfying Kähler properties are compact nilmanifolds or solvmanifolds. These classes of compact homogeneous manifolds have proved to be very useful in producing a rich and wide variety of examples of compact manifolds with special properties (see [2, 6, 11, 16]).

In [4] T. Bouché defines a differential subcomplex of de Rham complex on a symplectic manifold, and he obtains some results on the cohomology of this complex: the coeffective cohomology. In particular, he proves that the coeffective cohomology is related to the de Rham cohomology for compact Kähler manifolds, but this is not true in general for any compact symplectic manifold (see [2, 15]).

The aim of this paper is to introduce for indefinite Kähler manifolds a differential subcomplex of Dolbeault complex defined analogously to the above subcomplex for symplectic manifolds. More precisely, in Section 2, for an indefinite Kähler manifold M with Kähler form ω, we study the complex

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$$(1) \qquad \cdots \longrightarrow \mathcal{A}^{p,q-1}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}(M) \longrightarrow \cdots$$

where $\overline{\partial}$ denotes the Dolbeault operator obtained in the decomposition of the exterior diferential $d = \partial + \overline{\partial}$, and $\mathcal{A}^{p,q}(M)$ is defined by

$$\mathcal{A}^{p,q}(M) = \{\alpha \in \Lambda^{p,q}(M) \mid \alpha \wedge \omega = 0\}.$$

In Section 3 we show that for compact Kähler manifolds the cohomology of the complex (1) (the coeffective-Dolbeault cohomology) is related to the Dolbeault cohomology. This property gives a new difference between the indefinite Kähler and Kähler theories, because it is not satisfied in general for any compact indefinite Kähler manifold. To show this we need to prove, in Section 4, a Nomizu-type theorem for the coeffective-Dolbeault cohomology groups of a compact indefinite Kähler nilmanifold, which permits us to calculate such cohomology groups at the Lie algebra level. Then, in Section 5, we construct an example of a compact nilmanifold with an indefinite Kähler structure for which the Kähler property relating the coeffective-Dolbeault cohomology and the Dolbeault cohomology is not satisfied.

Moreover, in Section 3, we prove a Hodge decomposition theorem [17] for the coeffective cohomology of a compact Kähler manifold, relating this cohomology with the coeffective-Dolbeault cohomology. But, in Section 5 we show an example of compact nilmanifold with an indefinite Kähler structure not satisfying such a property.

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2. Coeffective-Dolbeault cohomology

Let M be any real differentiable manifold. We denote by $\mathfrak{F}_{\mathbb{C}}(M)$ the algebra of C^∞ complex-valued functions on M and $\mathfrak{X}_{\mathbb{C}}(M)$ the Lie algebra of derivations of $\mathfrak{F}_{\mathbb{C}}(M)$ that can be regarded as the complex C^∞ vector fields on M.

Now assume that M has an almost complex structure, that is, a real tensor J of type $(1,1)$ on M satisfying $J^2 = -I$. Then, it is posible to decompose $\mathfrak{X}_{\mathbb{C}}(M)$ as $\mathfrak{X}_{\mathbb{C}}(M) = \mathfrak{X}_{1,0}(M) \oplus \mathfrak{X}_{0,1}(M)$ where

$$\begin{aligned} \mathfrak{X}_{1,0}(M) &= \{X \in \mathfrak{X}_{\mathbb{C}}(M) | JX = \sqrt{-1}X\}, \\ \mathfrak{X}_{0,1}(M) &= \{X \in \mathfrak{X}_{\mathbb{C}}(M) | JX = -\sqrt{-1}X\}. \end{aligned}$$

Notice that $\overline{\mathfrak{X}_{1,0}(M)} = \mathfrak{X}_{0,1}(M)$.

Next let M be a complex manifold of complex dimension n. This means that in a neighborhood of each point of M it is posible to introduce a system of local complex coordinates $(z_1, \cdots, z_n)$ such that the transition functions between any two systems of local complex coordinates are holomorphic. Every complex manifold has a canonically associated almost complex structure J such that for any coordinate system

$$J\left(\frac{\partial}{\partial z_j}\right) = \sqrt{-1}\frac{\partial}{\partial z_j}, \quad J\left(\frac{\partial}{\partial \bar{z}_j}\right) = -\sqrt{-1}\frac{\partial}{\partial \bar{z}_j},$$

for $j = 1, \cdots, n$. Therefore, near each point of M a vector field X of bidegree $(1,0)$ can be expressed as

$$X = \sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j},$$

where the f_j's are C^∞ functions. The same holds for a vector field $\overline{X}$ of bidegree $(0,1)$,

$$\overline{X} = \sum_{j=1}^{n} g_j \frac{\partial}{\partial \bar{z}_j}.$$

REMARK 2.1. It is known that an almost complex structure J on a manifold M is the almost complex structure associated to a complex structure iff J is integrable, that is, the Nijenhuis tensor N_J of J vanishes [1, 20], where

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY],$$

for X and Y vector fields on M.

Let M be a real differentiable manifold and $\Lambda^*(M)$ denote the $\mathfrak{F}_{\mathbb{C}}(M)$-module of complex differential forms. If M is an almost complex manifold it is possible to define the submodule $\Lambda^{p,q}(M)$ of differential forms of bidegree (p,q).

For a complex manifold M a differential form of bidegree (p,q) can be expressed in any local complex coordinate system of M as

$$\sum f^{i_1 \cdots i_p}_{j_1 \cdots j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

Moreover, if d is the exterior differential,

$$d\left(\Lambda^{p,q}(M)\right) \subseteq \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M).$$

Thus, for a complex manifold we get a decomposition of d as $d = \partial + \overline{\partial}$, where

$$\partial\left(\Lambda^{p,q}(M)\right) \subseteq \Lambda^{p+1,q}(M) \quad \text{and} \quad \overline{\partial}\left(\Lambda^{p,q}(M)\right) \subseteq \Lambda^{p,q+1}(M).$$

From this decomposition of d and since $d^2 = 0$, we obtain that $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$ and the known Dolbeault complex for complex manifolds:

$$\cdots \longrightarrow \Lambda^{p,q-1}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q+1}(M) \longrightarrow \cdots. \tag{2}$$

The cohomology groups of (2) are the so-called *Dolbeault cohomology groups* and they are denoted by $H^{p,q}_{\overline{\partial}}(M)$.

An almost complex manifold M of real dimension $2n$ is said *almost Hermitian* if there exists a real indefinite metric g on M which is compatible with the almost complex structure J of M, that is, $g(JX, JY) = g(X, Y)$, for X and Y vector fields on M. The *Kähler form* (or *fundamental* 2*-form*) of an almost Hermitian manifold M is defined by $\omega(X, Y) = g(JX, Y)$. The Kähler form always has maximal rank, that is, $\omega^n \neq 0$, it is real and of bidegree $(1, 1)$ with respect to the bigraduation.

Moreover, an almost Hermitian manifold M is said

i) *indefinite Kähler* iff J is integrable and ω is closed,

ii) *Kähler* iff it is indefinite Kähler and g is a Riemannian (or positive definite) metric.

Notice that indefinite Kähler manifolds are in particular complex and symplectic manifolds.

REMARK 2.2. A *Lorentzian* metric may not be a compatible metric with an almost complex structure, because the signature of such a metric is $(2n-1, 1)$ and for the almost Hermitian case the signature of the metric is of the form $(2n-2p, 2p)$ (see [1]).

From now on, we suppose that M is an indefinite Kähler manifold of real dimension $2n$ with integrable almost complex structure J, indefinite metric g and Kähler form ω. Then, we have the symplectic operator $L : \Lambda^{k-2}(M) \longrightarrow \Lambda^k(M)$ defined by $L\alpha = \alpha \wedge \omega$, for $\alpha \in \Lambda^{k-2}(M)$. This operator is real since ω is a real 2-form and from $\omega \in \Lambda^{1,1}(M)$ it is expressed with respect to the bigraduation as

$$L : \Lambda^{p-1,q-1}(M) \longrightarrow \Lambda^{p,q}(M). \tag{3}$$

Lemma 2.3. *The operator* L *given by* (3) *is surjective for* $p + q \geq n + 1$ *and injective for* $p + q \leq n + 1$.

Proof. It follows inmediately since the symplectic operator $L : \Lambda^{k-2}(M) \longrightarrow \Lambda^k(M)$ is surjective for $k \geq n + 1$ and injective for $k \leq n + 1$ [4], and from the decomposition [17, 20]

$$\Lambda^k(M) = \bigoplus_{p+q=k} \Lambda^{p,q}(M). \tag{4}$$

□

Next we introduce the subspace $\mathcal{A}^{p,q}(M)$ of $\Lambda^{p,q}(M)$ defined by

$$\begin{aligned}\mathcal{A}^{p,q}(M) &= \{\alpha \in \Lambda^{p,q}(M) \mid \alpha \wedge \omega = 0\}\\ &= \operatorname{Ker}\{L : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q+1}(M)\}.\end{aligned}$$

A differential form $\alpha \in \mathcal{A}^{p,q}(M)$ is said to be a *coeffective* (*bigraduate*) *form of* bidegree(p, q).

From the decomposition $d = \partial + \overline{\partial}$ and that $\omega \in \Lambda^{1,1}(M)$, then $\overline{\partial}\omega = 0$ and the operators L and $\overline{\partial}$ commute. Therefore, it may be considered the subcomplex of Dolbeault complex

$$\cdots \longrightarrow \mathcal{A}^{p,q-1}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q}(M) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}(M) \longrightarrow \cdots \tag{5}$$

for $0 \leq p \leq n$; called the *coeffective-Dolbeault complex.* The cohomology groups of the complex (5) are called *coeffective-Dolbeault cohomology groups* and they are denoted by $H^{p,q}_{\overline{\partial}}(\mathcal{A}(M))$.

As a consequence of Lemma 2.3 we obtain that $\mathcal{A}^{p,q}(M) = \{0\}$ for $p+q \leq n-1$, therefore

$$H^{p,q}_{\overline{\partial}}(\mathcal{A}(M)) = \{0\}, \quad \text{for} \quad p+q \leq n-1. \tag{6}$$

Proposition 2.4. *For* $0 \leq p \leq n$, *the coeffective-Dolbeault complex* $(\mathcal{A}^{p,*}(M), \overline{\partial})$ *is elliptic in* degree q *if* $p+q \geq n+1$.

Proof. The complex $(\mathcal{A}^{p,*}(M), \overline{\partial})$ is elliptic [4, 20] in degree q if for each point $x \in M$, the following complex

$$\cdots \longrightarrow \mathcal{A}^{p,q-1}_x(M) \xrightarrow{i\theta_{0,1}\wedge\cdot} \mathcal{A}^{p,q}_x(M) \xrightarrow{i\theta_{0,1}\wedge\cdot} \mathcal{A}^{p,q+1}_x(M) \longrightarrow \cdots$$

is exact in degree q, for every element of the cotangent bundle $\theta \in T^*_x(M) - \{0\}$, with $\theta = \theta_{1,0} + \theta_{0,1}$; where the space $\mathcal{A}^{p,q}_x(M)$ is $\operatorname{Ker} L \cap \Lambda^{p,q}T^*_x(M)$.

For each $x \in M$, it is posible to consider a local complex coordinate system $(z_1, \cdots, z_n)$ such that $\omega = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ and $\theta_{0,1} = d\bar{z}_1$. Therefore, the problem is reduced to the study of the exactness of the complex

$$\cdots \longrightarrow \mathcal{A}^{p,q-1}_x(M) \xrightarrow{d\bar{z}_1\wedge\cdot} \mathcal{A}^{p,q}_x(M) \xrightarrow{d\bar{z}_1\wedge\cdot} \mathcal{A}^{p,q+1}_x(M) \longrightarrow \cdots$$

Then, we consider a non-zero $u \in \mathcal{A}^{p,q}_x(M) \cap \operatorname{Ker}(d\bar{z}_1 \wedge .)$, with $p+q \geq n+1$, that is, u verifies:

i) $\omega \wedge u = 0$,

ii) $d\bar{z}_1 \wedge u = 0$.

Then, we shall show the existence of $v \in \mathcal{A}_x^{p,q-1}(M)$ such that $u = d\bar{z}_1 \wedge v$.

Denote by ω' the 2-form $\omega - dz_1 \wedge d\bar{z}_1 = \sum_{j=2}^n dz_j \wedge d\bar{z}_j$. From ii) we have that there exists $v_1 \in \Lambda^{p,q-1}T_x^*(M)$ such that $d\bar{z}_1 \wedge v_1 = u$ and v_1 does not contain the term $d\bar{z}_1$. Moreover, i) implies that

$$\omega' \wedge d\bar{z}_1 \wedge v_1 = \omega \wedge d\bar{z}_1 \wedge v_1 = \omega \wedge u = 0.$$

Consequently, $\omega' \wedge v_1 = 0$. Now, we distinguish two cases:

1. $dz_1 \wedge v_1 = 0$. Then $\omega \wedge v_1 = 0$ and we finish the proof.
2. $dz_1 \wedge v_1 \neq 0$. Then $v_1 = v_2 + dz_1 \wedge v_3$, where v_2 and v_3 contain neither dz_1 nor $d\bar{z}_1$, in their coordinate expressions. Taking into account that we work in a point of M, the form v_2 can be considered as a form of bidegree $(p, q-1)$ on a complex manifold of complex dimension $(n-1)$ and with Kähler form ω' (notice that we have now the coordinates $(z_2, \cdots, z_n)$). From Lemma 2.3, the operator L' associated to ω' is surjective in bidegree $(p, q-1)$ if $p + (q-1) \geq (n-1) + 1$, that is, $p + q \geq n + 1$. Thus, there exists a form v_4 of bidegree $(p-1, q-2)$ such that $\omega' \wedge v_4 = v_2$.

Consider the form v (in $x \in M$) given by $v = v_1 - dz_1 \wedge d\bar{z}_1 \wedge v_4$. Since $v_1 - v_2 = dz_1 \wedge v_3$, the form v verifies

$$\begin{aligned}
\omega \wedge v &= dz_1 \wedge d\bar{z}_1 \wedge v_1 - \omega \wedge dz_1 \wedge d\bar{z}_1 \wedge v_4 \\
&= dz_1 \wedge d\bar{z}_1 \wedge v_1 - \omega' \wedge dz_1 \wedge d\bar{z}_1 \wedge v_4 \\
&= dz_1 \wedge d\bar{z}_1 \wedge v_1 - dz_1 \wedge d\bar{z}_1 \wedge v_2 \\
&= dz_1 \wedge d\bar{z}_1 \wedge (v_1 - v_2) = 0.
\end{aligned}$$

Moreover,

$$d\bar{z}_1 \wedge v = u;$$

and we conclude the proposition. □

As a consequence [20], we obtain

Theorem 2.5. *For a compact indefinite Kähler manifold of real dimension $2n$, the cohomology groups $H_{\bar{\partial}}^{p,q}(\mathcal{A}(M))$ have finite dimension for $p + q \geq n + 1$.*

Since $\overline{\partial}\omega = 0$, we have that $[\omega] \in H_{\bar{\partial}}^{1,1}(M)$ and we consider the subspace of $H_{\bar{\partial}}^{p,q}(M)$ given by the Dolbeault cohomology classes truncated by the class of the Kähler form $[\omega]$, that is,

$$\widetilde{H}_{\bar{\partial}}^{p,q}(M) = \{a \in H_{\bar{\partial}}^{p,q}(M) \mid a \wedge [\omega] = 0\}, \tag{7}$$

where $[\alpha]$ denotes the cohomology class of a form α in $H^{p,q}_{\overline{\partial}}(M)$.

Problem. Is there any relation between the coeffective-Dolbeault cohomology groups and the subspaces of the Dolbeault cohomology groups given by (7) ?

Next we define the mapping $\psi_{p,q} : H^{p,q}_{\overline{\partial}}(\mathcal{A}(M)) \longrightarrow \widetilde{H}^{p,q}_{\overline{\partial}}(M)$ by

$$\psi_{p,q}(\{\alpha\}) = [\alpha], \tag{8}$$

where $\{\alpha\}$ denotes the cohomology class of a form α in $H^{p,q}_{\overline{\partial}}(\mathcal{A}(M))$. This mapping permits us to give a first answer to the above problem for any indefinite Kähler manifold.

Proposition 2.6. *For an indefinite Kähler manifold of real dimension $2n$, the mapping $\psi_{p,q}$ defined by* (8) *is surjective for $p+q \geq n$.*

Proof. Let $a \in \widetilde{H}^{p,q}_{\overline{\partial}}(M)$, that is, $a \in H^{p,q}_{\overline{\partial}}(M)$ and $a \wedge [\omega] = 0$ in $H^{p+1,q+1}_{\overline{\partial}}(M)$. Consider a representative α of a and suppose that $\alpha \notin \mathcal{A}^{p,q}(M)$ (notice that if $\alpha \in \mathcal{A}^{p,q}(M)$, then α defines a cohomology class in $H^{p,q}_{\overline{\partial}}(\mathcal{A}(M))$ such that $\psi_{p,q}(\{\alpha\}) = a$).

Since $a \wedge [\omega] = 0$, there exists $\sigma \in \Lambda^{p+1,q}(M)$ such that $\alpha \wedge \omega = \overline{\partial}\sigma$. Then, from Lemma 2.3, there exists $\gamma \in \Lambda^{p,q-1}(M)$ such that $L\gamma = \sigma$. Thus, $L(\alpha - \overline{\partial}\gamma) = 0$ and $\overline{\partial}(\alpha - \overline{\partial}\gamma) = 0$; therefore, $\alpha - \overline{\partial}\gamma$ defines a cohomology class in $H^{p,q}_{\overline{\partial}}(\mathcal{A}(M))$ such that $\psi_{p,q}(\{\alpha - \overline{\partial}\gamma\}) = a$. $\square$

3. Coeffective-Dolbeault cohomology for Kähler manifolds

The purpose of this section is to answer the above problem for compact Kähler manifolds and to prove a coeffective version of the Hodge decomposition theorem. From the remainder of this section we consider M a Kähler manifold of real dimension $2n$ with integrable almost complex structure J, Riemannian metric g and Kähler form ω.

Defined on M we have the symplectic operator L, the differentials d, ∂, $\overline{\partial}$ and the Hodge star operator $\star$ associated to the Riemannian metric. Then, we consider the codifferential $d^* : \Lambda^{k+1}(M) \longrightarrow \Lambda^k(M)$ given by $d^* = - \star d \star$, and the dual operators of ∂ and $\overline{\partial}$ given by $\partial^* = - \star \overline{\partial} \star$ and $\overline{\partial}^* = - \star \partial \star$, respectively, where $\star : \Lambda^{p,q}(M) \longrightarrow \Lambda^{n-q,n-p}(M)$ is the Hodge star operator on $\Lambda^{*,*}(M)$ (see [20]).

Therefore, we have the Laplacians: $\Delta = dd^* + d^*d$, $\Box = \partial\partial^* + \partial^*\partial$ and $\overline{\Box} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. For a Kähler manifold it is well known [20] that $\Box$ and $\overline{\Box}$ are real operators,

$$\Delta = 2\Box = 2\overline{\Box}, \quad \text{and} \quad L\Delta = \Delta L. \tag{9}$$

Denote by $\mathcal{H}^k(M)$ the space of harmonic k-forms on M and $\mathcal{H}^{p,q}(M)$ the space of harmonic forms of bidegree (p,q) on M.

Lemma 3.1. *The operator $L : \mathcal{H}^{p-1,q-1}(M) \longrightarrow \mathcal{H}^{p,q}(M)$ given by* (3) *is surjective for $p+q \geq n+1$.*

Proof. Since the operator $L : \mathcal{H}^{k-2}(M) \longrightarrow \mathcal{H}^k(M)$ is surjective for $k \geq n+1$ (see [4]), from (9) and that $\overline{\Box}$ preserves the bigraduation, the result follows easily. $\square$

As Dolbeault complex is elliptic it is known [20] that

$$H^{p,q}_{\overline{\partial}}(M) \cong \mathcal{H}^{p,q}(M). \tag{10}$$

Theorem 3.2. *For a compact Kähler manifold M of real dimension $2n$,*

$$H^{p,q}_{\overline{\partial}}(\mathcal{A}(M)) \cong \widetilde{H}^{p,q}_{\overline{\partial}}(M), \tag{11}$$

for $p,q \geq 0$ and $p+q \neq n$.

Proof.
Part 1: $p+q \leq n-1$.
From (6) we know that $H^{p,q}_{\overline{\partial}}(\mathcal{A}(M)) = \{0\}$ for $p+q \leq n-1$. Moreover, form (9) and (10),

$$\begin{aligned} \widetilde{H}^{p,q}_{\overline{\partial}}(M) &\cong \{\alpha \in \mathcal{H}^{p,q}(M) \,|\, \alpha \wedge \omega \in \overline{\partial}\left(\Lambda^{p+1,q}(M)\right)\} \\ &\cong \{\alpha \in \mathcal{H}^{p,q}(M) \,|\, \alpha \wedge \omega = 0\}. \end{aligned} \tag{12}$$

Thus, from Lemma 2.3 we conclude that $\widetilde{H}^{p,q}_{\overline{\partial}}(M) = \{0\}$ for $p+q \leq n-1$. This finishes the proof for $p+q \leq n-1$.
Part 2: $p+q \geq n+1$.
We shall see that the mapping $\psi_{p,q}$ given by (8) is an isomorphism for $p+q \geq n+1$. From Proposition 2.6, it is sufficient to show the injection.
Let $a \in H^{p,q}_{\overline{\partial}}(\mathcal{A}(M))$ such that $\psi(a) = 0$ in $\widetilde{H}^{p,q}_{\overline{\partial}}(M)$ and suppose that α is a representative of a. Since $\psi(a) = \psi(\{\alpha\}) = [\alpha] = 0$ in $\widetilde{H}^{p,q}_{\overline{\partial}}(M)$, there exists $\beta \in \Lambda^{p,q-1}(M)$ such that

$$\alpha = \overline{\partial}\beta.$$

Suppose $\beta \notin \mathcal{A}^{p,q-1}(M)$ (notice that if $\beta \in \mathcal{A}^{p,q-1}(M)$, then $a = 0$ and we conclude the proof). Since L and $\overline{\partial}$ commute, then $\overline{\partial}(L\beta) = L(\overline{\partial}\beta) = L\alpha = 0$;

therefore $L\beta$ defines a Dolbeault cohomology class $[L\beta] \in H^{p+1,q}_{\overline{\partial}}(M)$. From (10),

$$L\beta = h + \overline{\partial}\gamma,$$

for $h \in \mathcal{H}^{p+1,q}(M)$ and $\gamma \in \Lambda^{p+1,q-1}(M)$. By Lemma 3.1 there exists $v \in \mathcal{H}^{p,q-1}(M)$ such that $Lv = h$ and by Lemma 2.3, there exists $\sigma \in \Lambda^{p,q-2}(M)$ such that $L\sigma = \gamma$. Thus,

$$L(\beta - v - \overline{\partial}\sigma) = 0, \quad \text{and} \quad \overline{\partial}(\beta - v - \overline{\partial}\sigma) = \alpha.$$

Then, $a = \{\alpha\}$ is the zero class in $H^{p,q}(\mathcal{A}(M))$ and this finishes the proof. □

Taking into account that the Hodge decomposition theorem [17] relates the de Rham cohomology of a compact Kähler manifold to the Dolbeault cohomology, we shall prove a coeffective version of this result. Remember [4] that for a symplectic manifold M, $H^k(\mathcal{A}(M))$ denotes the coeffective cohomology group of degree k and $\widetilde{H}^k(M)$ the subspace of $H^k(M)$ containing the de Rham cohomology classes truncated by the class of the Kähler form $[\omega]$.

Theorem 3.3 (coeffective Hodge decomposition theorem). *For a compact Kähler manifold M of real dimension $2n$,*

i) $\widetilde{H}^k(M) \cong \bigoplus_{p+q=k} \widetilde{H}^{p,q}_{\overline{\partial}}(M)$.

ii) *For $k \geq n+1$,*

$$H^k(\mathcal{A}(M)) \cong \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(\mathcal{A}(M)). \tag{13}$$

Proof. Let $a \in \widetilde{H}^k(M)$ and α a representative of a. There is no loss of generality in assuming that α is harmonic and $\alpha \wedge \omega = 0$. From (4)

$$\alpha = \alpha_{k,0} + \cdots + \alpha_{p,q} + \cdots + \alpha_{0,k},$$

and from (9) $\overline{\Box}\alpha = \Delta\alpha = 0$ and since $\overline{\Box}$ preserves the bigraduation, we have

$$\overline{\Box}\alpha_{k,0} = \cdots = \overline{\Box}\alpha_{p,q} = \cdots = \overline{\Box}\alpha_{0,k} = 0.$$

Moreover, since ω is of bidegree $(1,1)$ and $\alpha \wedge \omega = 0$, then

$$\alpha_{k,0} \wedge \omega = \cdots = \alpha_{p,q} \wedge \omega = \cdots = \alpha_{0,k} \wedge \omega = 0.$$

Thus, part i) follows from (12).

Now, from part i), Bouche's result [4] and Theorem 3.2,

$$H^k(\mathcal{A}(M)) \cong \widetilde{H}^k(M) \cong \bigoplus_{p+q=k} \widetilde{H}^{p,q}_{\bar\partial}(M) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(\mathcal{A}(M)),$$

and it follows part ii). □

Denote $c_k(M)$ the dimension of the coeffective cohomology group $H^k(\mathcal{A}(M))$ and $c^{p,q}(M)$ the dimension of the coeffective-Dolbeault cohomology group $H^{p,q}_{\bar\partial}(\mathcal{A}(M))$.

Corollary 3.4. *For a compact Kähler manifold M of real dimension $2n$,*

$$c_k(M) = \sum_{p+q=k} c^{p,q}(M), \quad \textit{for} \quad k \geq n+1.$$

REMARK 3.5. The author have proved in [18] that for a compact Kähler manifold,

$$c_k(M) = b_k(M) - b_{k+2}(M),$$

where $b_k(M)$ is the k^{th} Betti number, that is, $c_k(M)$ depends only on the topology of M. Now, it may be proved in a similar way that

$$c^{p,q}(M) = h^{p,q}(M) - h^{p+1,q+1}(M),$$

where $h^{p,q}(M)$ denotes the dimension of the Dolbeault cohomology group $H^{p,q}_{\bar\partial}(M)$; then $c^{p,q}(M)$ depends only on the complex structure of M.

4. Compact indefinite Kähler nilmanifolds

The main problem to construct an example of compact indefinite Kähler manifold not satisfying the isomorphism (11) or the isomorphism (13) is the difficulty to compute the coeffective-Dolbeault cohomology of an indefinite Kähler manifold. In this section we prove a Nomizu-type theorem which reduces the calculation of such cohomology of a compact indefinite Kähler nilmanifold to the calculation at the Lie algebra level.

Let $M = \Gamma\backslash G$ be a compact nilmanifold of dimension $2n$, where the Lie group G posseses a left invariant integrable almost complex structure J^*, so that $\Gamma\backslash G$ inherits an integrable almost complex structure J from that of G by passing to the quotient. If, moreover, there is a complex basis $\{\omega_i; 1 \leq i \leq n\}$ of forms of type $(1,0)$, such that satisfy the equations

$$d\omega_i = \sum_{j<k\leq i} A_{ijk}\, \omega_j \wedge \omega_k + \sum_{j,k\leq i} B_{ijk}\, \omega_j \wedge \bar\omega_k \quad (1 \leq i \leq n),$$

where A_{ijk} and B_{ijk} are complex numbers, Cordero, Fernández, Gray and Ugarte have proved in [10] that there exists a canonical isomorphism

$$H^{p,q}_{\overline{\partial}}(\Gamma\backslash G) \cong H^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}}), \tag{14}$$

where $H^{*,*}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ denotes the cohomology ring of the differential bigraded algebra $\Lambda^{*,*}(\mathfrak{g}^{\mathbb{C}})^*$, associated to the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$, with respect to the operator $\overline{\partial}$ in the canonical decomposition $d = \partial + \overline{\partial}$ of the Chevalley-Eilenberg operator in $\Lambda^*(\mathfrak{g}^{\mathbb{C}})^*$.

Moreover, suppose that G posseses a left invariant indefinite metric g^* compatible with J^* and ω^* is the associated left invariant Kähler form, so that $\Gamma\backslash G$ inherits an indefinite metric g compatible with J and the Kähler form ω from those of G. If ω^* is closed then $\Gamma\backslash G$ is an indefinite Kähler nilmanifold. Under this conditions we consider the subspace $\widetilde{H}^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ of $H^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ defined by

$$\widetilde{H}^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}}) = \{a^* \in H^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}}) \mid a^* \wedge [\omega^*] = 0\}.$$

Now from (14) it is easy to see that there exists a canonical isomorphism

$$\widetilde{H}^{p,q}_{\overline{\partial}}(\Gamma\backslash G) \cong \widetilde{H}^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}}). \tag{15}$$

Next we consider the subspace $\mathcal{A}^{p,q}(\mathfrak{g}^*)$ of $\Lambda^{p,q}(g^{\mathbb{C}})^*$ defined by

$$\mathcal{A}^{p,q}(\mathfrak{g}^*) = \{\alpha^* \in \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^* \mid \alpha^* \wedge \omega^* = 0\}.$$

Then, as ω^* is $\overline{\partial}$-closed, we have the complex

$$\cdots \longrightarrow \mathcal{A}^{p,q-1}(\mathfrak{g}^*) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q}(\mathfrak{g}^*) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}(\mathfrak{g}^*) \longrightarrow \cdots \tag{16}$$

and we denote by $H^{p,q}_{\overline{\partial}}(\mathcal{A}(\mathfrak{g}^*))$ its cohomology groups.

On account of Lemma 2.3 and Proposition 2.6 at the Lie algebra level, we have the following result.

Lemma 4.1.

i) *The mapping* $L^* : \Lambda^{p-1,q-1}(\mathfrak{g}^{\mathbb{C}})^* \longrightarrow \Lambda^{p,q}(\mathfrak{g}^{\mathbb{C}})^*$ *defined by*

$$L^*(\alpha^*) = \alpha^* \wedge \omega^*,$$

for $\alpha^* \in \Lambda^{p-1,q-1}(\mathfrak{g}^{\mathbb{C}})^*$, *is surjective for* $p + q \geq n + 1$.

ii) *The mapping* $\psi^*_{p,q} : H^{p,q}_{\overline{\partial}}(\mathcal{A}(\mathfrak{g}^*)) \longrightarrow \widetilde{H}^{p,q}_{\overline{\partial}}(\mathfrak{g}^{\mathbb{C}})$ *defined by*

$$\psi^*_{p,q}(\{\alpha^*\}) = [\alpha^*],$$

for $\{\alpha^*\} \in H^{p,q}_{\overline{\partial}}(\mathcal{A}(\mathfrak{g}^*))$, *is surjective for* $p + q \geq n$.

Theorem 4.2. *There exists a canonical isomorphism*

$$H^{p,q}_{\overline{\partial}}(\mathcal{A}(\Gamma\backslash G)) \cong H^{p,q}_{\overline{\partial}}(\mathcal{A}(\mathfrak{g}^*)),$$

for $p+q \geq n+1$.

Proof. A similar proof to the given in [15] for the coeffective cohomology groups of a compact symplectic nilmanifold still goes for the coeffective-Dolbeault cohomology groups of a compact indefinite Kähler nilmanifold when we consider the Nomizu-type theorem for the Dolbeault cohomology given in (14) and the Lemma 4.1. □

5. Counterexamples

This section is devoted to prove that the isomorphisms (11) and (13) does not hold for arbitrary compact indefinite Kähler manifolds by constructing counterexamples.

5.1. The compact nilmanifold R^6

Consider the 6-dimensional compact nilmanifold $R^6 = \Gamma\backslash G$ (see [2, 15]), where G is a simply connected nilpotent Lie group of dimension 6 defined by the left invariant 1-forms $\{\alpha_i \mid 1 \leq i \leq 6\}$ such that

$$(17) \qquad \begin{cases} d\alpha_i = 0, & 1 \leq i \leq 3, \\ d\alpha_4 = -\alpha_1 \wedge \alpha_2, \\ d\alpha_5 = -\alpha_1 \wedge \alpha_3, \\ d\alpha_6 = -\alpha_1 \wedge \alpha_4, \end{cases}$$

and Γ is a discrete and uniform subgroup of G. The manifold R^6 can be alternatively described as a $\mathbb{T}^4$-bundle over $\mathbb{T}^2$ (see [15]). In [2] it has been proved that R^6 has no Kähler structures.

Let $\{X_i \mid 1 \leq i \leq 6\}$ be the basis of vector fields dual to the basis of 1-forms $\{\alpha_i \mid 1 \leq i \leq 6\}$, then

$$[X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_1, X_4] = X_6,$$

and the others zero. Define the almost complex structure J on R^6 by

$$(18) \qquad \begin{cases} JX_1 = -X_2 + X_3 - X_4, & JX_2 = X_1 + X_4, \\ JX_3 = -X_3 + 2X_4, & JX_4 = -X_3 + X_4, \\ JX_5 = -X_5 + 2X_6, & JX_6 = -X_5 + X_6. \end{cases}$$

It is easy to see that J is integrable, that is, it defines a complex structure on R^6. Moreover, the indefinite metric

$$g = \frac{1}{2}\alpha_1\#\alpha_1 - \alpha_1\#\alpha_3 - \alpha_1\#\alpha_4 + \alpha_1\#\alpha_5 - \alpha_2\#\alpha_5 - \alpha_2\#\alpha_6 + \alpha_3\#\alpha_3 + \alpha_3\#\alpha_4 + \frac{1}{2}\alpha_4\#\alpha_4, \tag{19}$$

where $\#$ denotes the symmetric product, is compatible with J and its Kähler form is given by

$$\omega = \alpha_1 \wedge \alpha_5 + \alpha_1 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_4 + \alpha_1 \wedge \alpha_3. \tag{20}$$

Since $d\omega = 0$, then J, g and ω define an indefinite Kähler structure on R^6.

Lemma 5.1. *On the compact nilmanifold R^6 we consider the almost complex structure J defined by* (18). *Then,*

i) *a basis of vector fields of bidegree* $(1,0)$ *is given by* $\{U_1, U_2, U_3\}$, *where*

$$U_1 = \frac{1}{2}(X_1 + X_4) + \frac{\sqrt{-1}}{2}X_2,$$

$$U_2 = \frac{1}{2}\left(-\frac{1}{2}X_5 + X_6\right) + \frac{\sqrt{-1}}{4}X_5,$$

$$U_3 = \frac{1}{2}(X_3 - X_4) - \frac{\sqrt{-1}}{2}X_4,$$

ii) *the basis of* 1*-forms of bidegree* $(1,0)$ $\{\mu_1, \mu_2, \mu_3\}$ *dual to* $\{U_1, U_2, U_3\}$ *is given by*

$$\mu_1 = \alpha_1 - \sqrt{-1}\alpha_2,$$

$$\mu_2 = \alpha_6 + \sqrt{-1}(-2\alpha_5 - \alpha_6),$$

$$\mu_3 = \alpha_3 + \sqrt{-1}(-\alpha_1 + \alpha_3 + \alpha_4).$$

Proof. From (18) it follows that

$$\begin{aligned} &J(X_1 + X_4) = -X_2, \\ &J(-\frac{1}{2}X_5 + X_6) = -\frac{1}{2}X_5, \\ &J(X_3 - X_4) = X_4. \end{aligned} \tag{21}$$

Then, (21) permits us to prove the lemma. □

Now, the Kähler form ω, defined in (20), is expressed in this new basis $\{\mu_1, \mu_2, \mu_3\}$ by:

$$\omega = \frac{1}{4}\left(1-\sqrt{-1}\right)\mu_1 \wedge \overline{\mu}_2 - \frac{1}{4}\left(1+\sqrt{-1}\right)\mu_2 \wedge \overline{\mu}_1 + \frac{\sqrt{-1}}{2}\mu_3 \wedge \overline{\mu}_3. \tag{22}$$

Moreover, from (17) and the Lemma 5.1, we obtain that the 1-forms $\{\mu_1, \mu_2, \mu_3\}$ verify:

$$\begin{cases} d\mu_1 = 0, \\ d\mu_2 = \frac{1}{2}(1+\sqrt{-1})(\mu_1 \wedge \mu_3 - \mu_3 \wedge \overline{\mu}_1), \\ d\mu_3 = -\frac{1}{2}\mu_1 \wedge \overline{\mu}_1. \end{cases} \tag{23}$$

Now from the isomorphism (14) we calculate the Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(R^6)$. They are:

$$\begin{aligned}
H^{1,0}_{\overline{\partial}}(R^6) &= \{[\mu_1]\}, \\
H^{0,1}_{\overline{\partial}}(R^6) &= \{[\overline{\mu}_1], [\overline{\mu}_3]\}, \\
H^{2,0}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \mu_3]\}, \\
H^{1,1}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \overline{\mu}_3], [\mu_2 \wedge \overline{\mu}_1], [\mu_1 \wedge \overline{\mu}_2 - (1-\sqrt{-1})\mu_3 \wedge \overline{\mu}_3]\}, \\
H^{0,2}_{\overline{\partial}}(R^6) &= \{[\overline{\mu}_1 \wedge \overline{\mu}_2], [\overline{\mu}_2 \wedge \overline{\mu}_3]\}, \\
H^{3,0}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \mu_2 \wedge \mu_3]\}, \\
H^{2,1}_{\overline{\partial}}(R^6) &= \{[\mu_2 \wedge \mu_3 \wedge \overline{\mu}_1], [\mu_1 \wedge \mu_3 \wedge \overline{\mu}_3], \\
&\qquad [(1-\sqrt{-1})\mu_1 \wedge \mu_2 \wedge \overline{\mu}_3 - (1+\sqrt{-1})\mu_1 \wedge \mu_3 \wedge \overline{\mu}_2]\}, \\
H^{1,2}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3], [\mu_2 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2], [\mu_2 \wedge \overline{\mu}_1 \wedge \overline{\mu}_3]\}, \\
H^{0,3}_{\overline{\partial}}(R^6) &= \{[\overline{\mu}_1 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3]\}, \\
H^{3,1}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \overline{\mu}_1], [\mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \overline{\mu}_3]\}, \\
H^{2,2}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \mu_2 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2], [\mu_1 \wedge \mu_3 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3], [\mu_2 \wedge \mu_3 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2]\}, \\
H^{1,3}_{\overline{\partial}}(R^6) &= \{[\mu_2 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3]\}, \\
H^{3,2}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2], [\mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3]\}, \\
H^{2,3}_{\overline{\partial}}(R^6) &= \{[\mu_2 \wedge \mu_3 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3]\}, \\
H^{3,3}_{\overline{\partial}}(R^6) &= \{[\mu_1 \wedge \mu_2 \wedge \mu_3 \wedge \overline{\mu}_1 \wedge \overline{\mu}_2 \wedge \overline{\mu}_3]\}.
\end{aligned}$$

From (15) we calculate $\widetilde{H}^{p,q}_{\overline{\partial}}(R^6)$:

$$
\begin{aligned}
\widetilde{H}^{1,0}_{\overline{\partial}}(R^6) &= \{0\},\\
\widetilde{H}^{0,1}_{\overline{\partial}}(R^6) &= \{[\overline{\mu}_1]\},\\
\widetilde{H}^{2,0}_{\overline{\partial}}(R^6) &= \{0\},\\
\widetilde{H}^{1,1}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{0,2}_{\overline{\partial}}(R^6) &= \{[\overline{\mu}_1\wedge\overline{\mu}_2]\},\\
\widetilde{H}^{3,0}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\mu_2\wedge\mu_3]\},\\
\widetilde{H}^{2,1}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\mu_3\wedge\overline{\mu}_3],[(1-\sqrt{-1})\mu_2\wedge\mu_3\wedge\overline{\mu}_1\\
&\qquad +(1-\sqrt{-1})\mu_1\wedge\mu_2\wedge\overline{\mu}_3-(1+\sqrt{-1})\mu_1\wedge\mu_3\wedge\overline{\mu}_2]\},\\
\widetilde{H}^{1,2}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3],[\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{0,3}_{\overline{\partial}}(R^6) &= \{[\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{3,1}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_1],[\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{2,2}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_2-(1+\sqrt{-1})\mu_1\wedge\mu_3\wedge\overline{\mu}_2\wedge\overline{\mu}_3],\\
&\qquad [\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2]\},\\
\widetilde{H}^{1,3}_{\overline{\partial}}(R^6) &= \{[\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{3,2}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2],[\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_2\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{2,3}_{\overline{\partial}}(R^6) &= \{[\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3]\},\\
\widetilde{H}^{3,3}_{\overline{\partial}}(R^6) &= \{[\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3]\}.
\end{aligned}
$$

Moreover, the coeffecive-Dolbeault cohomology groups $H^{p,q}_{\overline{\partial}}(\mathcal{A}(R^6))$ may be calculated by means of Theorem 4.2:

$$
\begin{aligned}
H^{3,0}_{\overline{\partial}}(\mathcal{A}(R^6)) &\supseteq \{\{\mu_1\wedge\mu_2\wedge\mu_3\}\},\\
H^{2,1}_{\overline{\partial}}(\mathcal{A}(R^6)) &\supseteq \{\{\mu_1\wedge\mu_3\wedge\overline{\mu}_1\},\{(1+\sqrt{-1})\mu_2\wedge\mu_3\wedge\overline{\mu}_1+\\
&\qquad (1+\sqrt{-1})\mu_1\wedge\mu_2\wedge\overline{\mu}_3+(1-\sqrt{-1})\mu_1\wedge\mu_3\wedge\overline{\mu}_2\},\\
&\qquad \{\mu_1\wedge\mu_2\wedge\overline{\mu}_1-(1+\sqrt{-1})\mu_1\wedge\mu_3\wedge\overline{\mu}_3\}\},\\
H^{1,2}_{\overline{\partial}}(\mathcal{A}(R^6)) &\supseteq \{\{\mu_1\wedge\overline{\mu}_1\wedge\overline{\mu}_3\},\{\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\},\{\mu_1\wedge\overline{\mu}_1\wedge\overline{\mu}_2+\\
&\qquad (1-\sqrt{-1})\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_3\},\{(1-\sqrt{-1})\mu_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3+\\
&\qquad (1+\sqrt{-1})\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_3\}\},\\
H^{0,3}_{\overline{\partial}}(\mathcal{A}(R^6)) &\supseteq \{\{\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3\}\},\\
H^{3,1}_{\overline{\partial}}(\mathcal{A}(R^6)) &= \{\{\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_1\},\{\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_3\}\},
\end{aligned}
$$

$$
\begin{aligned}
H^{2,2}_{\overline{\partial}}(\mathcal{A}(R^6)) &= \{\{\mu_1\wedge\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_3\},\{\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\},\\
&\qquad \{\mu_1\wedge\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_2-(1+\sqrt{-1})\mu_1\wedge\mu_3\wedge\overline{\mu}_2\wedge\overline{\mu}_3\}\},\\
H^{1,3}_{\overline{\partial}}(\mathcal{A}(R^6)) &= \{\{\mu_2\wedge\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3\}\},\\
H^{3,2}_{\overline{\partial}}(\mathcal{A}(R^6)) &= \{\{\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\},\{\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_2\wedge\overline{\mu}_3\}\},\\
H^{2,3}_{\overline{\partial}}(\mathcal{A}(R^6)) &= \{\{\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3\}\},\\
H^{3,3}_{\overline{\partial}}(\mathcal{A}(R^6)) &= \{\{\mu_1\wedge\mu_2\wedge\mu_3\wedge\overline{\mu}_1\wedge\overline{\mu}_2\wedge\overline{\mu}_3\}\}.
\end{aligned}
$$

Theorem 5.2. *For the compact nilmanifold R^6 with the described indefinite Kähler structure, the isomorphism* (11) *is not satisfied.*

Proof.
Part 1: $p+q\leq n-1=2$,

$$H^{p,q}_{\overline{\partial}}(\mathcal{A}(R^6))\not\cong\widetilde{H}^{p,q}_{\overline{\partial}}(R^6),\quad \text{for}\quad (p,q)=(0,1),\,(1,1),\,(0,2).$$

Part 2: $p+q\geq n+1=4$,

$$H^{2,2}_{\overline{\partial}}(\mathcal{A}(R^6))\not\cong\widetilde{H}^{2,2}_{\overline{\partial}}(R^6).$$

□

5.2. The Iwasawa manifold I_3

The Iwasawa manifold can be realized as the compact quotient $I_3=\Gamma\backslash G$ where G is the complex Heisenberg group of matrices of the form

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}$$

where x, y, z are complex numbers and Γ is the subgroup of G consisting of those matrices whose entries are Gaussian integers. It is easy to see that I_3 posseses a basis of holomorphic 1-forms $\{\alpha,\beta,\gamma\}$ such that

$$d\alpha=d\beta=0,\quad d\gamma=-\alpha\wedge\beta.$$

In [13] it is proved that I_3 has no Kähler structures. There is no such strong statement for indefinite Kähler structures on I_3, but at least we can say that there is not indefinite Kähler structures with respect to the natural complex structure on I_3, because no closed form of bidegree $(1,1)$ can have maximal rank.

However, there are other complex structures on I_3 that do posseses indefinite Kähler structure. Let $\{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$ be the real vector fields dual to $\{\mathrm{Re}\alpha, \mathrm{Im}\alpha, \mathrm{Re}\beta, \mathrm{Im}\beta, \mathrm{Re}\gamma, \mathrm{Im}\gamma\}$. Then in [13] it is proved that the almost complex structure J given by

$$JX_1 = Y_1, \quad JX_2 = Y_2, \quad JZ_1 = Z_2$$

is integrable. Put $\omega_1 = (1/2)(\mathrm{Re}\alpha + \sqrt{-1}\mathrm{Re}\beta)$, $\omega_2 = (1/2)(\mathrm{Im}\alpha + \sqrt{-1}\mathrm{Im}\beta)$, $\omega_3 = (1/2)(\gamma)$. Then

$$g = \omega_1\#\overline{\omega_3} + \overline{\omega_1}\#\omega_3 - \sqrt{-1}\,(\omega_1\#\overline{\omega_2} - \overline{\omega_1}\#\omega_2 + \omega_2\#\overline{\omega_3} - \overline{\omega_2}\#\omega_3),$$

is an indefinite metric compatible with J and its Kähler form is

$$\omega = \omega_1 \wedge \overline{\omega_2} + \overline{\omega_1} \wedge \omega_2 + \omega_2 \wedge \overline{\omega_3} + \overline{\omega_2} \wedge \omega_3 + \sqrt{-1}\,(\omega_1 \wedge \overline{\omega_3} - \overline{\omega_1} \wedge \omega_3).$$

Since $d\omega = 0$, J, g and ω define a indefinite Kähler structure on I_3.
Notice that

$$\begin{cases} d\omega_1 = d\omega_2 = 0, \\ d\omega_3 = \omega_1 \wedge \overline{\omega_2} + \omega_2 \wedge \overline{\omega_1} - \sqrt{-1}\,(\omega_1 \wedge \overline{\omega_1} - \omega_2 \wedge \overline{\omega_2}). \end{cases}$$

From Nomizu's theorem [19] and a Nomizu-type theorem for the coeffective cohomology groups [15], we have:

$$\begin{aligned} H^4(I_3) = \{&[\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_1}], [\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_3}], [\omega_1 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \wedge \overline{\omega_3}], \\ &[\omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \wedge \overline{\omega_3}], [\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3}], [\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2}], \\ &[\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_2 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3}], \\ &[\omega_1 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3} - \omega_2 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3}]\}. \end{aligned}$$

and

$$\begin{aligned} H^4(\mathcal{A}(I_3)) = \widetilde{H}^4(I_3) = \{&[\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_1}], [\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_3}], \\ &[\omega_1 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \wedge \overline{\omega_3}], [\omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \wedge \overline{\omega_3}], \\ &[\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_2 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3}], \\ &[\omega_1 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3} - \omega_2 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3}], \\ &[\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2}]\}. \end{aligned}$$

Moreover, from (14), (15) and Theorem 4.2, we have

$$
\begin{aligned}
H^{3,1}_{\overline{\partial}}(I_3) &= \{[\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_1}], [\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_2}], [\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_3}]\},\\
H^{2,2}_{\overline{\partial}}(I_3) &= \{[\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3}], [\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2}], [\omega_2 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2}],\\
&\qquad [-\sqrt{-1}\,\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_1 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3}],\\
&\qquad [\omega_1 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3} - \omega_2 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3}],\\
&\qquad [\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_2 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3}]\},\\
H^{1,3}_{\overline{\partial}}(I_3) &= \{[\omega_1 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \wedge \overline{\omega_3}], [\omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2} \wedge \overline{\omega_3}]\};
\end{aligned}
$$

and

$$
\begin{aligned}
H^{3,1}_{\overline{\partial}}(\mathcal{A}(I_3)) &= \widetilde{H}^{3,1}_{\overline{\partial}}(I_3) = H^{3,1}_{\overline{\partial}}(I_3),\\
H^{2,2}_{\overline{\partial}}(\mathcal{A}(I_3)) &= \widetilde{H}^{2,2}_{\overline{\partial}}(I_3) = \{[\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_2 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3}],\\
&\quad [\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2}],\\
&\quad [\sqrt{-1}\,\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3} - \omega_2 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_2}],\\
&\quad [\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3} + \sqrt{-1}\,\omega_1 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3} - \omega_1 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3}],\\
&\quad [\omega_2 \wedge \omega_3 \wedge \overline{\omega_2} \wedge \overline{\omega_3} - \sqrt{-1}\,\omega_1 \wedge \omega_2 \wedge \overline{\omega_1} \wedge \overline{\omega_3} - \sqrt{-1}\,\omega_2 \wedge \omega_3 \wedge \overline{\omega_1} \wedge \overline{\omega_3}]\},\\
H^{1,3}_{\overline{\partial}}(\mathcal{A}(I_3)) &= \widetilde{H}^{1,3}_{\overline{\partial}}(I_3) = H^{1,3}_{\overline{\partial}}(I_3).
\end{aligned}
$$

Therefore,

Theorem 5.3. *For the Iwasawa manifold I_3 with the above indefinite Kähler structure, we have that the isomorphisms given in Theorem* 3.3 *are not satisfied, concretely,*

$$
\begin{aligned}
\widetilde{H}^4(I_3) &\ncong \widetilde{H}^{3,1}_{\overline{\partial}}(I_3) \oplus \widetilde{H}^{2,2}_{\overline{\partial}}(I_3) \oplus \widetilde{H}^{1,3}_{\overline{\partial}}(I_3);\\
H^4(\mathcal{A}(I_3)) &\ncong H^{3,1}_{\overline{\partial}}(\mathcal{A}(I_3)) \oplus H^{2,2}_{\overline{\partial}}(\mathcal{A}(I_3)) \oplus H^{1,3}_{\overline{\partial}}(\mathcal{A}(I_3)).
\end{aligned}
$$

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