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FLEXIBLE BOUNDARIES IN DEFORMATIONS OF HYPERBOLIC 3-MANIFOLDS

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0. Introduction

Let M be a cusped hyperbolic 3-manifold with non-empty geodesic boundary. A small Dehn filling deformation of M on the cusps can be performed so that the boundary is kept to be geodesic. Then assigning to each deformation a hyperbolic structure on the boundary, we get a map B_M from the space of such deformations to the Teichmüller space of ∂M. More precise argument for this fact will be given in §1.

Motivated by the conjectures posed in Cooper-Long [1] and Kapovich [5], Neumann and Reid [7] discovered many examples of M such that B_M is a constant map. Fujii [3] also obtained another concrete family of small deformations for some M such that B_M maps his family to a constant structure. These examples at first contrasted with our naive intuition that the deformation of hyperbolic structure affects everywhere. But the fact itself would not be too surprising once we realized that the dimension of the source can be bigger than that of the target and in that case B_M can never be injective. A more pertinent problem to set up for the moment would be what the map B_M looks like. In fact, we have known only a little about it so far.

Under the circumstances above, it would be worth finding examples for which we can convert this rather difficult problem to something we can do by hand. In this paper, we will construct infinitely many one-cusped examples of M so that B_M is a local embedding at the complete structure. The polyhedral construction will be discussed in §2. Then by using its polyhedral structure, we will compute the derivative of B_M at the complete structure by hand in the later sections.

The authors are indebted to Professor T. Soma for the argument in §1, and we would like to thank him.

1. The map B_M

We briefly review Dehn filling deformations of cusped hyperbolic 3-manifolds. Let N be a noncompact, orientable, complete hyperbolic 3-manifold of finite volume, and $\overline{\rho}_0 : \pi_1(N) \to \mathrm{PSL}_2(\mathbf{C})$ its holonomy representation. According to

Thurston [9] (cf. [2]), $\overline{\rho}_0$ has a lift $\rho_0 : \pi_1(N) \to \mathrm{SL}_2(\mathbf{C})$. Since $\mathrm{SL}_2(\mathbf{C})$ is an algebraic set, the space of representations $\mathrm{Hom}(\pi_1(N), \mathrm{SL}_2(\mathbf{C}))$ is also an algebraic set. To each representation ρ, associated is its character χ_ρ. Culler and Shalen [2] showed that the irreducible component of $\mathrm{Hom}(\pi_1(N), \mathrm{SL}_2(\mathbf{C}))$ containing ρ_0 is mapped by this correspondence onto the space of characters X, which is an affine variety. The preimage of a character χ_ρ near χ_{ρ_0} consists of representations conjugate to ρ. Thus a small neighborhood of χ_{ρ_0} in X is bijectively identified with the set of conjugacy classes of $\mathrm{SL}_2(\mathbf{C})$-representations near the conjugacy class of ρ_0.

This small neighborhood is also identified with the set of conjugacy classes of $\mathrm{PSL}_2(\mathbf{C})$-representations near the conjugacy class of $\overline{\rho}_0$. This is because any path in $\mathrm{Hom}(\pi_1(N), \mathrm{PSL}_2(\mathbf{C}))$ based at $\overline{\rho}_0$ lifts to a path in $\mathrm{Hom}(\pi_1(N), \mathrm{SL}_2(\mathbf{C}))$ based at ρ_0 and hence the covering projection $\mathrm{SL}_2(\mathbf{C}) \to \mathrm{PSL}_2(\mathbf{C})$ induces a local homeomorphism of X at χ_{ρ_0}. Through this correspondence, we will identify the set of equivalence classes of $\mathrm{PSL}_2(\mathbf{C})$-representations near the conjugacy class of $\overline{\rho}_0$ with a small neighborhood of χ_{ρ_0} in X.

It has been known by Thurston [9] and Neumann-Zagier [8] that the complex dimension of X is equal to the number of cusps of N and that the character of ρ_0 is a smooth point. If we choose a set of meridional elements $\{m_j\}$ for all cusps of N, then the traces of these elements turn out to be a local coordinate of X at the conjugacy class of ρ_0.

Thurston originally introduced another parameter which is the set of complex lengths of the $\rho(m_j)$'s. The complex length of $\rho(m_j)$ is well-defined with sign by orienting the axis of $\rho(m_j)$. Their squares turn out to be a local coordinate at χ_{ρ_0} in X. To each representation ρ near ρ_0, he assigned a hyperbolic manifold N_ρ, see [9]. It can be interpreted as a small deformation of N and is called a Dehn filling deformation of N. The topological type of the deformation may be different from the original. Thurston's parameter can be converted to the set of pairs of real numbers which is a generalization of Dehn surgery coefficients in the classical knot theory.

Now, suppose that M is an orientable complete hyperbolic 3-manifold of finite volume with both cusps and compact geodesic boundaries. Let DM be the double of M along the boundary and ρ_0 be a holonomy representation of DM. The manifold DM admits an obvious involution τ switching the sides.

DEFINITION. Fix a set of meridians m_j closed under τ, and choose a small neighborhood U of χ_{ρ_0} so that the traces of the m_j's become a local coordinate of the space of characters at χ_{ρ_0}. Let $\mathcal{D}_M$ be a diagonal set in U fixed by an involution on U which is induced by τ. It is a smooth submanifold of real dimension $= \#\{\text{cusps of } DM\}$.

Lemma 1. *The restriction of a representation ρ near ρ_0 whose conjugacy class* *is in $\mathcal{D}_M$ to $\pi_1(\partial_0 M)$ is fuchsian (i.e., a discrete faithful representation to $\mathrm{SL}_2(\mathbf{R})$), where $\partial_0 M$ is a component of the boundary ∂M.*

Proof. Choose ρ as above, and denote the associated Dehn filling deformation by DM_ρ. There is also a topological involution τ_ρ on DM_ρ switching sides. Then since ρ is close to ρ_0, τ_ρ induces a nearby deformation of DM and the induced structure is the same as DM_ρ by the local parametrization around the complete structure ρ_0 (see [8] [9]). Hence τ_ρ can be deformed to an isometry T_ρ by a tiny isotopy. We want to show that T_ρ is an involution fixing a geodesic surface isotopic to ∂M.

There is a dense subset Y in $\mathcal{D}_M$ near $[\rho_0]$, corresponding to a set of rational rays in the Dehn filling coefficient space, so that the deformation DM_y for $y \in Y$ is a cone manifold with cone angle more than 2π. Then by changing the metric near the cusps as in [4], DM_y can be modified to a compact negatively curved manifold $\overline{DM}_y$ with obvious involutive symmetry so that T_y is also modified near the cusps to an isometry $\overline{T}_y$ of $\overline{DM}_y$. Then $\overline{T}_y^2$ is an isometry on the compact negatively curved manifold which is isotopic to the identity. Thus $\overline{T}_y^2$ must be the identity and $\overline{T}_y$ is an isometric involution. Since $\overline{T}_y$ and T_y are the same map away from neighborhoods of the cusps, the fact implies that T_y itself is an involution.

Since T_y is deformed to τ_y by a tiny isotopy, ∂M admits an equivariant collar neighborhood with respect to τ_y which separates DM_y into two parts. We may choose one of them A such that $T_y(A) \cap A = \phi$ and $DM_y - (T_y(A) \cup A)$ is homeomorphic to $\partial M \times I$. This implies by Kim and Tollefson [6] that T_y fixes a surface isotopic to ∂M. Since T_y is an isometry, the fixed surface must be totally geodesic.

Let ρ_y be a corresponding representation. Then since ∂M is realized as the geodesic surface, the restriction of ρ_y to $\pi_1(\partial_0 M)$ is fuchsian. Since Y is dense in $\mathcal{D}_M$, and since the set of fuchsian representations is closed, we are done. □

Assigning the hyperbolic structure of the boundary to such a deformation DM_ρ where $\rho \in \mathcal{D}_M$, we get a map

$$B_M : \mathcal{D}_M \to \mathcal{T}(\partial M)$$

where $\mathcal{T}$ is the Teichmüller space of ∂M.

2. Examples

Consider the Whitehead link $L = K_1 \cup K_2$ in S^3. Removing a thin tubular neighborhood of K_2 from the complement of L, we obtain a manifold W with one compact toral boundary and one toral end. Choose an arc Σ connecting two points on ∂W as in Figure 1.

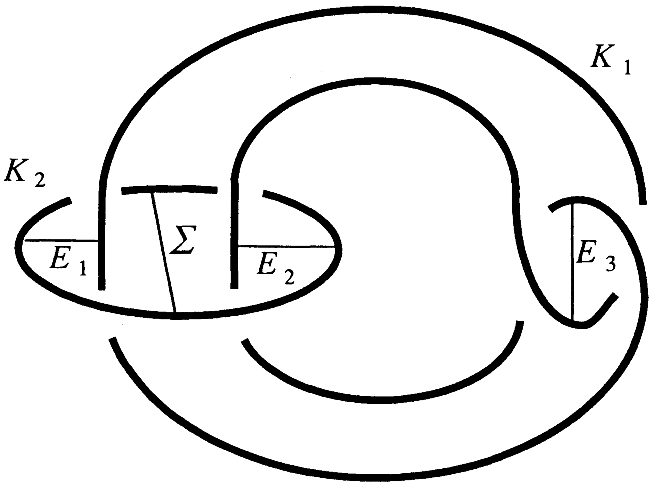

Fig. 1.

To give hyperbolic orbifold structures O_n on W with the singular set Σ indexed by natural numbers $n \geq 2$, we recall the fact, for instance in [9], that the regular ideal octahedron is a fundamental domain of the hyperbolic manifold homeomorphic to the Whitehead link complement. Replace the regular ideal octahedron by the truncated octahedron as in Figure 2, where the dihedral angle along each edge connecting truncated faces is $\pi/2n$ and that of each edge through ∞ is $\pi/2$. Then the faces topologically identified to create the Whitehead link complement are still isometric and the identification gives a hyperbolic orbifold O_n whose underlying space is W where the singular set is Σ with rotation angle $2\pi/n$.

Decompose the truncated octahedron into 4 truncated tetrahedra and we will deform them to obtain deformations of O_n in the next section. The edge lengths of the truncated tetrahedra determine their shapes. As we will see later, they are subject to the relations required by the gluing consistency to give nonsingular hyperbolic orbifold structures with geodesic boundary on W. Thus we obtain a space $\mathcal{W}$ of the deformations of O_n parameterized by the edge lengths of the truncated tetrahedra.

Let $\pi_1^{\text{orb}}(W)$ be the fundamental group of the orbifold O_n and ρ_0 a lift of the holonomy representation of $\pi_1^{\text{orb}}(W)$ in $\mathrm{SL}_2(\mathbf{C})$. Take a meridional element m for the cusp K_1. To each deformation of O_n represented by an element $w \in \mathcal{W}$, we have two data. One is a representation ρ near ρ_0 and the other is a canonical direction of the axis of $\rho(m)$ to which the end of truncated tetrahedra spiral. Assigning to each $w \in \mathcal{W}$ the complex length of $\rho(m)$ with respect to this orientation, we get a map $G : \mathcal{W} \to \mathbf{C}$. It will be shown in Lemma 3 that G is a local diffeomorphism at O_n.

On the other hand, each representation corresponds to a pair of complex lengths

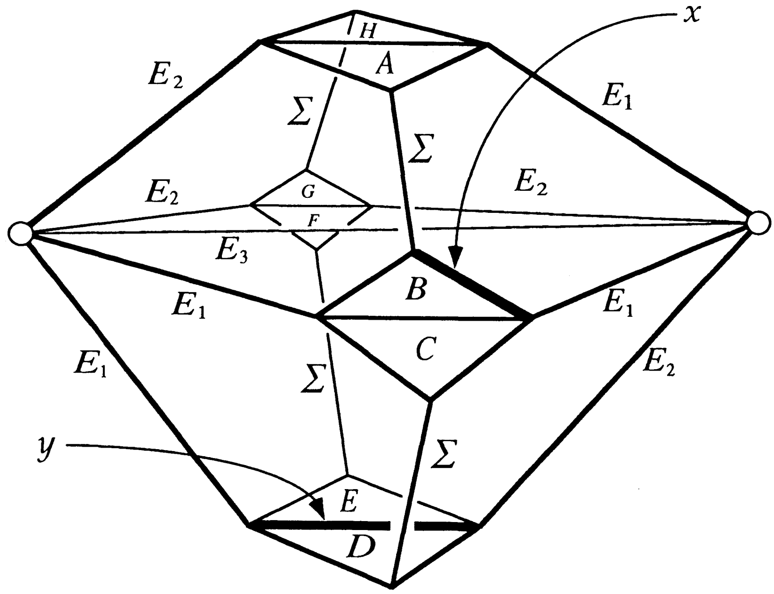

Fig. 2.

which differ only in sign. Hence by assigning to each $w \in \mathcal{W}$ the character χ_ρ of ρ, we get a map π of $\mathcal{W}$ to the space of characters of the representations of $\pi_1^{\mathrm{orb}}(W)$ in $\mathrm{SL}_2(\mathbf{C})$. The space of characters at χ_{ρ_0} has complex dimension > 1 since W has a boundary with negative Euler characteristic, and π doubly covers the image $\mathcal{D}_{O_n} = \pi(\mathcal{W})$ of complex dimension $= 1$ branched at χ_{ρ_0}.

The orbifold O_n has a toral boundary with two cone points of rotation angle $2\pi/n$. Let $\mathcal{T}(\partial O_n)$ be the Teichmüller space of the orbifold ∂O_n. It is homeomorphic to $\mathbf{R}^4$. Assigning the hyperbolic structure of the boundary to each χ_ρ, we get a map

$$B_{O_n} : \mathcal{D}_{O_n} \to \mathcal{T}(\partial O_n).$$

Our goal is

Theorem. *The derivative of the map B_{O_n} at χ_{ρ_0} has* rank 2.

Now let P_n be an n-fold cyclic covering of W branched along Σ. The manifold P_n supports a complete hyperbolic structure M_n which covers O_n. Pulling back the hyperbolic structure of the geodesic boundary associated to each $w \in \mathcal{W}$ by the covering : $P_n \to W$, we get an embedding of $\mathcal{T}(\partial O_n)$ into $\mathcal{T}(\partial M_n)$. Also the covering induces an injective homomorphism : $\pi_1(P_n) \to \pi_1^{\mathrm{orb}}(W)$, and hence a map : $\mathcal{D}_{O_n} \to \mathcal{D}_{M_n}$ on the characters by the restriction. It is a local diffeomorphism onto the image at χ_{ρ_0} because m^n lies in $\pi_1(P_n)$, and actually a local diffeomorphism

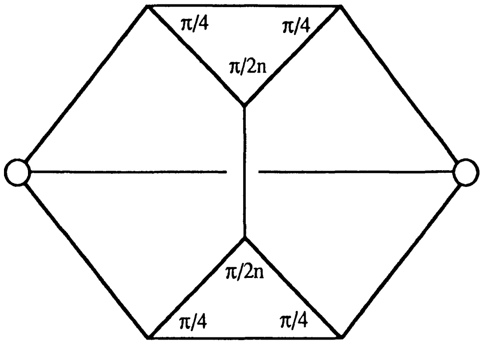

Fig. 3.

by dimension count. Therefore, by Theorem, $B_{M_n} : \mathcal{D}_{M_n} \to \mathcal{T}(\partial M_n)$ is a local embedding at M_n. Varying n, we have

Corollary. *There are infinitely many hyperbolic 3-manifolds M with both a cusp and a boundary such that the map $B_M : \mathcal{D}_M \to \mathcal{T}(\partial M)$ is a local embedding near the complete structure.*

3. Truncated Tetrahedra and Gluing Consistency

The truncated octahedron to create O_n is decomposed into four congruent truncated tetrahedra as in Figure 3. We will parametrize the deformations of O_n in terms of the deformations of the shapes of these blocks.

First of all, label the triangular faces by A, B, and their edges by A_i, B_i ($i = 1, 2, 3$) as in Figure 4. We call each of these edges *an external edge*, and denote the length of A_j and B_j by a_j and b_j respectively. These lengths are subject to two equations.

One is the following. If we let l be the length of the edge shared by two pentagonal faces, then regarding it as the bottom of the left pentagon, we obtain an expression of l in terms of a_1 and b_2,

$$\cosh l = \frac{\cosh a_1 \cosh b_2 + 1}{\sinh a_1 \sinh b_2}.$$

Simultaneously, if we regard it as the bottom of the right pentagon, we obtain an expression of l in terms of a_2 and b_1,

$$\cosh l = \frac{\cosh a_2 \cosh b_1 + 1}{\sinh a_2 \sinh b_1}.$$

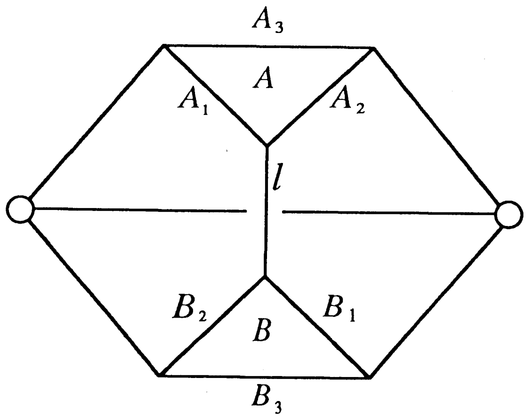

Fig. 4.

Then since these two are the same quantity, we obtain one identity involving edge lengths, which we call the relation of type (1).

The other is concerned with angles. By the hyperbolic cosine rule for the top triangle, we have

$$\cos\theta_{\text{top}} = \frac{\cosh a_1\ \cosh a_2 - \cosh a_3}{\sinh a_1\ \sinh a_2},$$

where θ_{top} is the angle between A_1 and A_2. If we look at the bottom triangle, then the corresponding angle θ_{bottom} has an expression in terms of the b_j's. They represent the same dihedral angle, and we obtain another relation,

$$(2) \qquad \theta_{\text{top}} = \theta_{\text{bottom}}.$$

It is not hard to verify that the set of six length variables subject to the relations (1) and (2) parametrizes isometry classes of labelled truncated tetrahedra.

To create a nonsingular but not necessarily complete hyperbolic orbifold structure on W, it is sufficient to verify gluing consistency which consists of the isometricity conditions for faces to be identified, and the cone angle conditions along edges. We will see when these are satisfied.

If the external edges to be identified have the same length, then the isometricity condition for face identification is satisfied. Since there are twelve such pairs, there are twelve simple identities in $\{a_j, b_j\}$ we must obviously require. For simplicity, we just assign the same variable to each pair to be identified from the beginning and reduce the number of the variables to the half.

Then the relations of type (1) and (2) for the four truncated tetrahedra become dependent after gluing. In fact, reading off the lengths of the bottom edges of the pentagonal faces in order, we can see that one of the four equations of type (1) becomes a consequence of the other three.

To compute the cone angle conditions along edges, we label them by E_1, E_2, E_3 and Σ as in Figure 1. The dihedral angle of each edge is described in terms of the lengths of external edges as the above expression of θ_{top}. To obtain a nonsingular orbifold structure, the total sum of dihedral angles around the first three edges must be 2π and the last $2\pi/n$. These constraints give four identities. The last one is independent from the others, however one of the first three identities is a consequence of the other two. To see this, recall that a toral section of the end always admits a similarity structure. Then the total sum of angles of triangles appeared in the horospherical triangulation is $4 \times 2\pi$. It is equal to the sum of the total sum of dihedral angles along E_1 and E_2 and the double of that of E_3.

We thus have obtained ten relations with twelve variables from gluing consistency. These relations define a map

$$f : \mathbf{R}^{12} \longrightarrow \mathbf{R}^{10},$$

such that its zero set $f^{-1}(0)$ consists of the points in $\mathbf{R}^{12}$ satisfying the gluing consistency.

Let $\boldsymbol{w}_0 \in \mathbf{R}^{12}$ be the point corresponding to the complete hyperbolic structure. Denote by x and y the two variables indicated in Figure 2, and by $z_1, \cdots z_{10}$ the other 10 variables. It is not hard to see that the rank of the matrix $(\partial f_i/\partial z_j(\boldsymbol{w}_0))$ is 10 and also find the unique solutions of the following two linear equations in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively:

$$\left(\frac{\partial f_i}{\partial z_j}(\boldsymbol{w}_0)\right)\boldsymbol{u} = -\left(\frac{\partial f_i}{\partial x}(\boldsymbol{w}_0)\right),$$
$$\left(\frac{\partial f_i}{\partial z_j}(\boldsymbol{w}_0)\right)\boldsymbol{v} = -\left(\frac{\partial f_i}{\partial y}(\boldsymbol{w}_0)\right).$$

Denote the unique solutions by $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ respectively. Then we have obtained

Lemma 2. *A neighborhood $\mathcal{W}$ of $\boldsymbol{w}_0$ in $f^{-1}(0)$ is a 2-dimensional smooth manifold and we have two paths on $\mathcal{W} \subset \mathbf{R}^{12}$,*

$$\xi(t) = \boldsymbol{w}_0 + \boldsymbol{x}t + (\text{higher order}),$$
$$\eta(t) = \boldsymbol{w}_0 + \boldsymbol{y}t + (\text{higher order}),$$

such that

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ \boldsymbol{u}_0 \end{pmatrix}, \quad \boldsymbol{y} = \begin{pmatrix} 0 \\ 1 \\ \boldsymbol{v}_0 \end{pmatrix},$$

where the y-component of $\xi(t)$ and the x-component of $\eta(t)$ are constant, and the x-component of $\xi(t)$ and the y-component of $\eta(t)$ have no terms of degrees n $(n \geq 2)$.

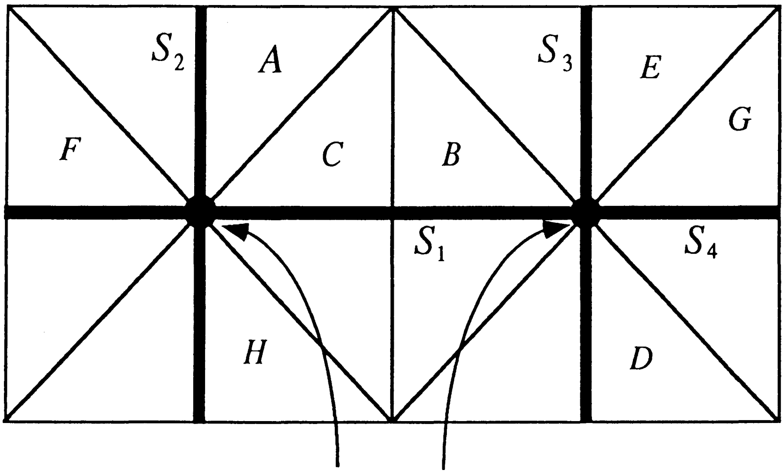

Fig. 5.

4. Dehn Filling Space and Computation

Lemma 3. *The map $G : \mathcal{W} \longrightarrow \mathbf{C}$ defined in §2 is a local diffeomorphism at $\boldsymbol{w}_0 \in \mathcal{W}$.*

Proof. The complex length of a meridional element can be read off from the triangulation by horospherical sections as in [8]. By direct computation using triangulation induced from the polyhedral decomposition of W, we can verify that the rank of the Jacobian of G is 2. □

Recall that $\pi : \mathcal{W} \to \mathcal{D}_{O_n}$ is a 2-fold covering branched at $\pi(\boldsymbol{w}_0)$. In fact π is a map which locally looks like a square function : $z \to z^2$.

The lengths L_i of geodesic segments S_i $(i = 1, \dots, 4)$ which are illustrated by thick lines in Figure 5 give rise to a quadruple (L_1, L_2, L_3, L_4) which defines a global coordinate of $\mathcal{T}(\partial O_n)$.

Let $\widetilde{B}$ be a map assigning to the element of $\mathcal{W}$ the corresponding hyperbolic structure of the boundary. Then its induced map from $\mathcal{D}_{O_n}$ is B_{O_n}. Let $\widetilde{B}_i$ (resp. B_i) be the composition of $\widetilde{B}$ (resp. B_{O_n}) with L_i.

$$\begin{array}{ccccc} \mathcal{W} & \overset{\widetilde{B}}{\searrow} & & & \\ \pi \downarrow & & & & \\ \mathcal{D}_{O_n} & \xrightarrow[B_{O_n}]{} & \mathcal{T}(\partial O_n) & \xrightarrow[(L_1, L_2, L_3, L_4)]{} & \mathbf{R}^4. \end{array}$$

Now consider a quadrilateral in general. If the lengths of four sides and one of diagonals are known, then the length of the other diagonal can be expressed in terms of them by hyperbolic trigonometry. Applying this to the quadrilateral in Figure 5 which is made of two triangular faces, we have an expression of $\widetilde{B}_i$ as a function of our length parameters.

Because of the local picture of π, letting $\overline{\xi}(t) = \pi \circ \xi(\sqrt{t})$ and $\overline{\eta}(t) = \pi \circ \eta(\sqrt{t})$, we obtain smooth paths on $\mathcal{D}_{O_n}$ such that its tangent vectors

$$\begin{aligned} v &= \frac{d}{dt}\overline{\xi}(t)|_{t=0}, \\ w &= \frac{d}{dt}\overline{\eta}(t)|_{t=0} \end{aligned}$$

are nontrivial. The images of these vectors by the derivative dB_i are now expressed by

$$\begin{aligned} dB_i(v) &= \left.\frac{dB_i(\overline{\xi}(t))}{dt}\right|_{t=0} = \left.\frac{d\widetilde{B}_i(\xi(\sqrt{t}))}{dt}\right|_{t=0}, \\ dB_i(w) &= \left.\frac{dB_i(\overline{\eta}(t))}{dt}\right|_{t=0} = \left.\frac{d\widetilde{B}_i(\eta(\sqrt{t}))}{dt}\right|_{t=0}. \end{aligned}$$

To carry out the actual computation of the right hand sides, we used the Taylor expansions of $\xi(t)$ and $\eta(t)$ up to the second degree, which can be derived from the formula,

$$\frac{d^2 f_i(\xi)}{dt^2}(0) = \sum_{j,k} \frac{\partial^2 f_i}{\partial z_j \partial z_k}(\boldsymbol{w}_0)\frac{d\xi_k}{dt}(0)\frac{d\xi_j}{dt}(0) + \sum_j \frac{\partial f_i}{\partial z_j}(\boldsymbol{w}_0)\frac{d^2\xi_j}{dt^2}(0).$$

By performing this rather lengthy but direct computations by hand, we verified the following:

Lemma 4.

$$dB_1(v) = -\frac{1}{\sqrt{c}} \quad (<0), \qquad dB_2(v) = \frac{1}{\sqrt{c}} \quad (>0),$$

$$dB_1(w) = \frac{s^2(1-c)}{4\sqrt{c}} \quad (>0), \quad dB_2(w) = \frac{s+1}{8s\sqrt{c}} \quad (>0),$$

where $c = \cos\dfrac{\pi}{2n}$ *and* $s = \sin\dfrac{\pi}{2n}$.

Lemma 4 shows that these tangent vectors on $\mathcal{D}_{O_n}$ go to a linearly independent pair in the tangent space of $\mathcal{T}(\partial O_n)$ at the original structure and we complete the proof of Theorem.

Finally, we would like to mention that our naive method works equally well to verify the vanishing of the derivative of B_M for Neumann and Reid's example. The details were worked out by T. Hayakawa at Tokyo Institute of Technology.

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