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SURFACES WITH CANONICAL MAP OF DEGREE THREE AND $K^2 = 3p_g - 5$

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Introduction

The canonical map of a nonsingular variety X of dimension n, $X \xrightarrow{\phi_{K_X}} \boldsymbol{P}^{p_g-1}$, is the rational map given by $x \mapsto (s_1(x), \cdots, s_{p_g}(x))$ where $\langle s_i \rangle_{i=1,\cdots,p_g}$ is a basis of $H^0(X, \mathcal{O}_X(K_X))$ and where K_X, the so called canonical divisor, is a divisor such that $\mathcal{O}_X(K_X)$ is the sheaf of holomorphic n-forms. Let $\phi_{K_X}(X) = \Sigma$ be the image. If we assume $\dim \Sigma = n$ then there is a natural number $d = \deg \phi_{K_X}$ associated to K_X.

If $n = 1$ then d can only be 1 or 2 and $d = 1$ is the general case. The special case $d = 2$ occurs, and, by this feature, admits a very explicit description: in fact $d = 2$ if and only if X is a hyperelliptic curve.

If $n = 2$ Castelnuovo proved that if $K_X^2 < 3p_g - 7$ then $d = 2$ and Σ is a ruled surface, while if $K_X^2 = 3p_g - 7$ then $d = 1$ or $d = 2$ and Σ is a ruled surface. He also classified surfaces with $K_X^2 = 3p_g - 7$ and $d = 1$, (see [1] for a modern reference). Since then the theory of the canonical map of surfaces has been extensively studied by several authors; here we can quote [20], [18], [9], [3], [21], [15]. However, the case $d = 3$ is not yet well understood. The initial idea, due to Castelnuovo, to study the case $d = 3$ was to consider a fibration of X on a smooth curve B, $f : X \to B$, such that the canonical linear system $\mid K_X \mid$ induces a g_3^1 on the fibers of f. In fact, following this idea, Pompilj proved that if $d = 3$ and $K_X^2 = 3p_g - 6$ then $q = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X(K_X)) = 0$ and $(p_g, K_X^2) = (3,3), (4,6)$ or $(5,9)$. He also classified these surfaces completely ([18]). In the seventies Horikawa rediscovered these surfaces except for the case $p_g = 5$ ([12], [13]). In [14] Konno gives a detailed classification of surfaces with $K_X^2 = 3p_g - 6$. In particular he considers the case $d = 3$ and $p_g = 5$. We also know that if $d = 3$ and $q > 0$ then $K_X^2 \geq 3p_g - 4$ [7, Proposition 5.1]. Moreover by [21, Theorem 2] we know that $p_g \leq 9$ if $K^2 = 3p_g - 5$ and $d = 3$. Thus the problem of classifying surfaces with $K_X^2 = 3p_g - 5$ and $d = 3$ arises very naturally. In this paper we show that the line $K^2 = 3p_g - 5$ gives rise to two families which we completely described. One of them ($K^2 = 7$, $p_g = 4$, $\deg\phi_{|K_X|} = 3$) is of a certain interest for two reasons:

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(i) it was considered by F. Enriques in [8, Cap.VIII, p.280] who claimed its non existence, (ii) the result of [5, Theorem 5.19], our main theorem and a forthcoming article by I. Bauer show the non trivial result that the moduli space of surfaces with $K^2 = 7$, $p_g = 4$, $q = 0$ is irreducible and unirational. In fact we can consider the 3-fold $\boldsymbol{P} = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(4))$ and let T be a tautological divisor on $\boldsymbol{P}$, Π a fiber of the natural projection $\boldsymbol{P} \xrightarrow{\pi} \boldsymbol{P}^1$, $X_0 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - \Pi))$, $X_1 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - 2\Pi))$, $X_2 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - 4\Pi))$ sections which give a projective coordinate system on Π, $\langle t_0, t_1 \rangle$ a basis of $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(\Pi))$, y the fibre coordinate of the line bundle $[2T - 6\Pi]$ on $\boldsymbol{P}$ then we have:

Main Theorem

S is a minimal surface with $p_g = 4$, $K_S^2 = 7$ and $\deg\phi_{|K_S|} = 3$ if and only if there exists a sublinear system $| F |$ in $| K_S |$ which is a rational pencil of non-hyperelliptic curves of genus 3 with a simple base point P' and such that the relative canonical model of the fibration induced on the blowing up S' of S in P' is the complete intersection in the total space of $[2T - 6\Pi]$ of the following two hypersurfaces:

$$\begin{cases} t_0 y = X_0 X_2 \\ \alpha y^2 + Qy + c_1 X_1^4 + X_2 P = 0; \end{cases}$$

where $\alpha \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(4\Pi))$, $\alpha_{|t_0=0} \neq 0$, $c_1 \in \mathbb{C} \setminus \{0\}$. Moreover $Q \in | 2T - 2\Pi |$ and $Q = c_0 X_0^2 + \alpha_1 X_0 X_1 + \alpha_2 X_1^2 + \alpha_4 X_1 X_2 + \alpha_6 X_2^2$ where $c_0 \in \mathbb{C} \setminus \{0\}$, $\alpha_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(i\Pi))$; $P \in | 3T - 4\Pi |$, $P = \beta_1 X_1^3 + \beta_2 X_1^2 X_2 + \beta_3 X_1 X_2^2 + \beta_4 X_2^3$ where $\beta_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(2i\Pi))$.

For lack of reference we include the classification of surfaces with $p_g = 3$, $K^2 = 4$ and $d = 3$. Let T be a tautological divisor on $\boldsymbol{P} = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(2) \oplus \mathcal{O}_{\boldsymbol{P}^1}(3))$, Π a fiber of the natural projection $\boldsymbol{P} \xrightarrow{\pi} \boldsymbol{P}^1$, $X_0 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - \Pi))$, $X_1 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - 2\Pi))$, $X_2 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - 3\Pi))$, $\langle t_0, t_1 \rangle$ a basis of $H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(\Pi))$, y the fibre coordinate of the line bundle $[2T - 5\Pi]$ on $\boldsymbol{P}$ then we have:

Theorem 1. *S is a minimal surface with $p_g = 3$, $K_S^2 = 4$ and $\deg\phi_{|KS|} = 3$ if and only if there exists a sublinear system $| F |$ in $| K_S |$ which is a rational pencil of non-hyperelliptic curves of genus 3 with a simple base point P' and such that the relative canonical model of the fibration induced on the blowing up S' of S in P' is the complete intersection in the total space of $[2T - 5\Pi]$ of the following two hypersurfaces:*

$$\begin{cases} t_0 y = X_0 X_2 \\ \alpha y^2 + Qy + \beta_1 X_1^4 + c_1 X_0 X_1^3 + X_2 P = 0; \end{cases}$$

where $\alpha \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(3\Pi))$, $\alpha_{|t_0=0} \neq 0$, $\beta_1 \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(\Pi))$ and $c_1 \in \mathbb{C}$. Moreover $Q \in | 2T-2\Pi |$ and $Q = c_0 X_0^2 + \alpha_1 X_0 X_1 + \alpha_2 X_1^2 + \alpha_3 X_1 X_2 + \alpha_4 X_2^2$ with $c_0 \in \mathbb{C} \setminus \{0\}$,

$\alpha_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(i\Pi))$; $P \in \mid 3T - 4\Pi \mid$, $P = \beta_2 X_1^3 + \beta_3 X_1^2 X_2 + \beta_4 X_1 X_2^2 + \beta_5 X_2^3$ *with* $\beta_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(i\Pi))$.

These surfaces are probably known (see added in proof [13, §2 p.110]). However we can classify them by the same technics used in the case $p_g = 4$.

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CONVENTIONS AND GENERAL REMARKS

Let R R' be divisors on a nonsingular variety X. Then $[R]$ is the line bundle associated to R, $\mathcal{O}_X(R)$ the sheaf of sections of $[R]$, $h^i(X, R)$ is the dimension of the i-th cohomological space;

$R \equiv R'$ denotes rational equivalence of divisors,

$R \sim R'$ denotes numerical equivalence of divisors,

$R' \prec R$ denotes that R' is a subdivisor of R,

$\mid R \mid$ is the projective space of divisors $R' \equiv R$,

$\phi_{|R|} : X - \to \boldsymbol{P}^{h^0(X,R)-1}$ is the rational map associated to $\mid R \mid$.

If K_X is a canonical divisor $0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + R) \to \mathcal{O}_R(K_R) \to 0$ is called adjunction sequence for R; if $\dim X = 2$ then $2\rho_a - 2 = R^2 + RK_X$ is called adjunction formula, where $\rho_a = 1 - \chi(\mathcal{O}_R)$ is the arithmetical genus of R.

If $n \in \mathbb{Z}$ is positive we put $\boldsymbol{F}_n = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(n))$, Δ and Γ are respectively a section with $\Delta^2 = n$ and a fiber of the natural projection $\boldsymbol{F}_n \to \boldsymbol{P}^1$; $\widehat{\boldsymbol{F}}_{n-1} \subset \boldsymbol{P}^n$ is the cone on the normal rational curve of degree $n-1$ in $\boldsymbol{P}^{n-1}$.

We recall that if $\mathcal{E}$ is a locally free sheaf of rank r on X, $\boldsymbol{P}(\mathcal{E}) \xrightarrow{\pi} X$ is the associated projective bundle and T is the tautological divisor then $K_{\boldsymbol{P}(\mathcal{E})} = \mathcal{O}_{\boldsymbol{P}(\mathcal{E})}(-rT + \pi^\star(\det(\mathcal{E}) + K_X))$ and $\mathrm{Pic}(\mathbf{P}(\mathcal{E})) = \pi^\star(\mathrm{Pic(X)}) \oplus \mathrm{T}\mathbb{Z}$.

If S is a non singular projective surface over $\mathbb{C}$, then $p_g = h^0(S, K_S)$ is the geometric genus and $q = h^1(S, K_S)$ is the irregularity. If $f : S \to B$ is a fibration on a smooth curve B then $K_{S|B} = K_S - f^\star(K_B)$ and $\Delta(f) = \deg f_\star(K_{S|B})$. The relative canonical algebra is $\mathcal{R}(S/B) = \bigoplus_{n \geq 0} \mathcal{R}_n$ where $\mathcal{R}_n = f_\star(K_{S/B}^{\otimes n})$, $\mathrm{Proj}_B \mathcal{R}(S/B)$ is the relative canonical model and the image of S in $\boldsymbol{P}(f_\star(K_{S/B}))$ is the relative canonical image.

1. $K^2 = 3p_g - 5, d = 3 \implies p_g = 3$ or $p_g = 4$

We know by [21, Theorem 2] that $p_g \leq 9$ where $K^2 = 3p_g - 5$ and $d = 3$; more precisely we can show:

Proposition 1.1. *If S is a minimal surface with $K^2 = 3p_g - 5$ and $d = 3$ then $q(S) = 0$, and $p_g = 3$ or $p_g = 4$. Moreover if $p_g = 4$ then the canonical image $\widehat{\boldsymbol{F}}_2 \subset \boldsymbol{P}^3$ is the cone on the non singular conic of $\boldsymbol{P}^2$.*

Proof. By [7, Proposition 5.1] we have $q(S)=0$.
Let M and Z be respectively the mobile part and the fixed part of $\mid K_S \mid$, in particular $K_S \equiv M+Z$ and $\phi_{|K_S|}=\phi_{|M|}$. Since $\Sigma \subset \boldsymbol{P}^{p_g-1}$ is an nondegenerate irreducible 2-dimensional variety and $\deg(\phi_{|\mathrm{M}|})=3$ then $M^2 \geq 3\deg\Sigma \geq 3(p_g-2)$. If $Z \neq 0$, by [4, lemma 1], $MZ \geq 2$ and by

$$3p_g-5=K^2=M^2+MZ+KZ \geq 3(p_g-2)+2+KZ$$

we obtain $KZ \leq -1$: a contradiction since S is minimal of general type. Let $\tilde{S} \xrightarrow{\sigma} S$ be a minimal composition of quadratic transformations among those with the property that the variable part $\mid L \mid$ of $\mid \sigma^{\star}K \mid$ is free from base points. Since $K=M$, $M^2 \geq L^2$ and $3p_g-5=K^2 \geq L^2=3\deg\Sigma \geq 3(p_g-2)$ we get $L^2=3(p_g-2)$ which implies: $\deg\Sigma=p_g-2$, $\mid K_S \mid$ has an unique base point P and σ is the blowing up of P. Now by Del Pezzo's theorem, for a modern reference see [17], these Σ are well known:

Del Pezzo's Theorem. If $\Sigma \subset \boldsymbol{P}^n$ is a nondegenerate surface of degree $n-1$ then Σ is one of the following surfaces:
i) $\boldsymbol{P}^2$, $n=2$,
ii) The Veronese surface in $\boldsymbol{P}^5$, $n=5$,
iii) $\boldsymbol{F}_d$ immersed in $\boldsymbol{P}^n$ by $\mid \Delta_0+\frac{n-3-d}{2}\Gamma \mid$ with $n-3-d \geq 0$,
iv) The image $\widehat{\boldsymbol{F}}_{n-1} \subset \boldsymbol{P}^n$ of $\boldsymbol{F}_{n-1}$ by $\mid \Delta \mid$.

We put $n=p_g-1$, and we consider the four cases separately.

i) This case is mentioned in [13, §2 pg.109–110]. See also the last section of this paper.

To deal with the remaining cases we will use that if $E=\sigma^{-1}(P)$ is the exceptional curve in $\tilde{S}$ then $LE=1$.

ii) If $\Sigma=(\boldsymbol{P}^2,\mathcal{O}(2))$ there exists C such that $L=2C$: a contradiction.

iii) If Σ is $\boldsymbol{F}_d$ immersed by $\mid \Delta_0+\frac{n-3-d}{2}\Gamma \mid$ there exist two divisors C and F on $\tilde{S}$ such that $L \equiv C+\frac{n-3-d}{2}F$. Moreover, since $\deg\phi_{|\mathrm{L}|}=3$, F is irreducible and by adjunction formula we have the following contradiction: $2g(F)-2=F(K_{\tilde{S}}+F)=FK_{\tilde{S}}=F(L+2E)=3+2FE$.

iv) In this case Σ is the cone on the rational normal curve of degree $n-1$. By [12, Lemma1] we have $L \equiv (n-1)F+G$ where $\mid F \mid$ is a rational pencil, the generic F is irreducible and G is the divisor associated to the ideal sheaf generated by the pull-back of the ideal of the vertex of the cone. In particular $LG=0$, $LF=3$ and $FG \geq 0$. Now by adjunction we have $2\rho_a(F)-2=3+2EF+F^2$, then $F^2 \geq 1$ and F^2 is odd. Since $3=LF=(n-1)F^2+FG \geq n-1$ there are only two possibilities: I) $p_g=5$, $F^2=1$ and $FG=0$ or II) $p_g=4$, $F^2=1$ and $FG=1$. The case I) is impossible. In fact $LG=FG=0$ implies $G^2=0$, thus, by Hodge index theorem, $G \sim 0$ and then we get $1=LE=3FE$. □

We now collect some facts which easily follow by the proof of 1.1.

Lemma 1.2. *If S is a minimal surface with $K^2 = 7$, $p_g = 4$ and $\deg\phi_{|K_S|} = 3$ then $q(S) = 0$, $\mid K_S \mid$ is without fixed part and it has an unique base point P. If $\tilde{S} \xrightarrow{\sigma} S$ is the blowing up of P, $E = \sigma^{-1}(P)$ and L is the mobile part of $\mid K_{\tilde{S}} \mid$ then $L \equiv 2F + G$ where:*
j) $\mid F \mid$ is a rational pencil of curves of genus 3 with a simple base point Q,
jj) G is an effective divisor with $EG = 1$, $\rho_a(G) = 1$, $E \not\prec G$ and $G \not\prec F$ for the generic $F \in\mid F \mid$. Moreover the following identities are true: $FG = 1$, $FE = 0$, $G^2 = -2$. In particular $\sigma(Q) = P' \neq P$.

2. Surfaces with $K^2 = 7$, $p_g = 4$ and $d = 3$

In this section we will prove the main theorem (see the Introduction). Firstly a remark on the form of the equation; we call elementary a monomial of the form $X_0^i X_1^j X_2^k$. Looking at Q and P in the statement of the theorem we see that in Q the elementary monomial $X_0^2 X_2^2$ does not appear and X_0 does not occur in P, nevertheless we will say that Q and P are generic if the coefficients α_i and β_i are generic in the usual sense and consequently we will say that the surface is generic if Q and P are generic.

We briefly outline the proof. By 1.2 S has a rational pencil of curves of genus 3 with a transversal point P'. We blow-up P', $\sigma' : S' \to S$ and we get a relatively minimal fibration $f : S' \to \boldsymbol{P}^1$ with $K^2_{S'|\boldsymbol{P}^1} = 3\Delta(f) + 1$. By [16] we know that f has a special fiber F_0'. We will describe the structure of F_0' which gives useful informations on $\mid K_{S'} \mid$. Then we will be able to write down the equation of the relative canonical model and of the relative canonical image. In particular we will see that in the generic case S' is isomorphic to its relative canonical model.

Proof of the Theorem. The proof is divided into two parts. In the first one we will construct the relative canonical model of S' which is a complete intersection of two hypersurfaces. In the second one we will show that the minimal model of the complete intersection is a surface S with $q(S) = 0$, $p_g = 4$, $K_S^2 = 7$ and $d = 3$.

First part.

Let S be a surface with $p_g = 4$, $K^2 = 7$, $q = 0$ and $d = 3$. We use the notations of 1.2.

Lemma 2.1. i) $Q \in \mathrm{supp}(G)$
ii) *$\exists F_0 \in\mid F \mid$ such that $G \prec F_0$.*

Proof. i) If $\mathrm{supp}(G)$ is irreducible then, by 1.2 jj), G is also reduced and

$FG = 1$. If $Q \notin$ supp(G) then the rational map $\tilde{S} \to \boldsymbol{P}^1$ induced by $\mid F \mid$ gives an isomorphism $G \to \boldsymbol{P}^1$: a contradiction since $\rho_a(G) = 1$. We suppose now that supp(G) is reducible and $Q \notin$ supp(G). We decompose $G = G_0 + G_1$ where $FG_0 = 1$, $G_1 \neq 0$ and $FG_1 = 0$; in particular $G_0 \not\prec F$. Since $LG_0 = 0$ by Hodge index theorem we have $G_0^2 < 0$ and then by adjunction we have $0 \leq \rho_a(G_0) \leq 1$. As above we exclude that $\rho_a(G_0) = 1$. If $\rho_a(G_0) = 0$ by $LG_0 = LG_1 = 0$ we have the following relations:

$$\ell) \quad \begin{cases} G_0^2 + 2EG_0 = -2 \\ G_0^2 + G_1 G_0 = -2 \\ G_0 G_1 + G_1^2 = 0. \end{cases}$$

If $EG_0 = 0$ then $G_0^2 = -2$ and $G_1^2 = G_1 G_0 = 0$. In particular by Hodge index theorem we have $G_1 \sim 0$. This is impossible because $G_1 E = 1$. If $EG_0 = 1$ then $EG_1 = 0$, $G_1 G_0 = 2$ and G_1 is a chain of (-2)-rational curves. Moreover if G_1 is decomposable then it is 1-connected. In fact let G_2 and G_3 be two non zero effective divisors such that $G_1 = G_2 + G_3$ and $G_2 G_3 = 0$. Since $G_0 G_1 = 2$ we can also suppose that $G_0 G_3 \leq 1$. Now we put $M_2 = \sigma_\star(2F + G_0 + G_2)$ and $M_3 = \sigma_\star(G_3)$. Since $K_S \equiv \sigma_\star L \equiv \sigma_\star(2F + G_0 + G_2 + G_3)$ we have $K_S \equiv M_2 + M_3$ with $M_2 M_3 \leq 1$ contradicting the 2-connectedness of K_S.

Claim: There exists $F_0 \in \mid F \mid$ such that $G_1 \prec F_0$.
In fact since G_1 is 1-connected then by (cf.[2, Corollary 12.3]) $H^0(G_1, \mathcal{O}_{G_1}) = \mathbb{C}$ and by duality $H^1(G_1, \omega_{G_1}) = \mathbb{C}$. In particular since $q(\tilde{S}) = 0$ by the adjunction sequence for G_1 we have $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + G_1)) = 0$. The claim is now an easy consequence of the cohomology of the following sequence:

$$0 \to \mathcal{O}_{\tilde{S}}(-G_1) \to \mathcal{O}_{\tilde{S}}(F - G_1) \to \mathcal{O}_F(Q) \to 0.$$

We now put $D = F_0 - G_1$. Since $G_0 \not\prec F$ then $DG_0 \geq 0$. Since $G_1 G_0 = 2$ we obtain the desired absurd: $1 = FG_0 = (D + G_1)G_0 \geq 2$.

ii) By 1.2 in the generic F there is not any component of G. Since $FG = 1$ there exists an unique irreducible reduced component $G_0 \prec G$ such that $FG_0 = 1$ and we can decompose G: $G = G_0 + G_1$ with $FG_1 = 0$. In particular by (i) $Q \in G_0$ and by 1.2 (j), $\exists F_0 \in \mid F \mid$ with $G_0 \prec F_0$. We will show that $G \prec F_0$. We are reduced to prove $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(F - G)) = 1$. Since $F_{|F} = Q$, $G_{|F} = G_{0|F} = Q$ then $\mathcal{O}_F(F - G) = \mathcal{O}_F$, and by uppersemicontinuity (cf.[10, Proposition 12.8]) we have $h^0(F, \mathcal{O}_F(F - G)) \geq 1$ $\forall F$. By the cohomology of the following sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-G) \to \mathcal{O}_{\tilde{S}}(F - G) \to \mathcal{O}_F \to 0$$

and by Serre's duality it remains to prove:

Claim: $H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + G)) = 0$.
We notice that this is rather obvious if $G_1 = 0$. In fact, in this case, $G_0 = G$ then

G is a reduced irreducible curve and by 1.2 jj) we have $\rho_a(G) = 1$. Since $q(\tilde{S}) = 0$ our claim easily follows by the exact sequence of adjunction for G: $0 \to K_{\tilde{S}} \to K_{\tilde{S}} + G \to K_G \to 0$. Suppose that $G_1 \neq 0$.
We can prove as in (i) that $G_0^2 \leq -1$; and we can use the two last lines of l). Moreover by 1.2 jj) $E \not< G_0$, $E \not< G_1$ and since $EG = 1$ we have $0 \leq EG_0 \leq 1$. In particular by adjunction formula we have: $\rho_a(G_0) \leq 1$. For reader's convenience we collect the previous results in the following table:

$$\ell\ell) \quad \begin{cases} G_0^2 + 2G_0E + 2 = 2\rho_a(G_0) \\ G_0^2 \leq -1 \text{ and } 0 \leq \mathrm{EG}_0 \leq 1. \end{cases}$$

We can distinguish the two cases: $\rho_a(G_0) = 1, 0$.

If $\rho_a(G_0) = 1$ then, by ll), $G_0^2 = -2$ and $G_0E = 1$. Thus by l) we have $G_1^2 = G_0G_1 = 0$ and from $EG = 1$ we have $G_1E = 0$. By the 2-connectedness of K_S we have $G_1 = 0$: a contradiction.

If $\rho_a(G_0) = 0$ and $EG_0 = 0$ we obtain a contradiction as in the analogous case of (i). If $\rho_a(G_0) = 0$ and $EG_0 = 1$, by ll) we have $G_0^2 = -4$. Then by l) $G_0G_1 = 2$, $G_1^2 = -2$. Since $EG_1 = 0$ then supp($\mathrm{G_1}$) is a union of (-2)-rational curves. Moreover if G_1 is decomposable then as in (i) it is 1-connected and then $h^1(\omega_{G_1}) = 1$. From the cohomology of the adjunction sequence for G_1 we easily obtain that $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + G_1) = 0$. Finally, since G_0 is a smooth rational curve, the cohomology of the sequence

$$0 \to \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + G_1) \to \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + G_1 + G_0) \to \mathcal{O}_{G_0} \to 0$$

proves the claim and the lemma follows. $\square$

We put $\sigma(Q) = P' \in S$.

Lemma 2.2. *We use the notations of* 1.2.
Let $\sigma' : S' \to S$ be the blowing up of P', $E' = \sigma'^{-1}(P')$ and $\sigma'^*(K_S) = L'$. In S' the following relations hold:
(i) *$\sigma'^*\sigma_\star(F) = F' + E'$, $F'E' = 1$ and the rational pencil $\mid F' \mid$ induces a non-hyperelliptic genus-3 fibration $f : S' \to \boldsymbol{P}^1$.*
(ii) *$\sigma'^*\sigma_\star(G) = G' + E'$ where $G'^2 = -2$, $E' \not< G'$, $E'G' = 1$ and $\rho_a(G') = 1$.*
(iii) *$K_{S'} \equiv 2F' + G' + 4E'$. Moreover the restriction map $H^0(S', \mathcal{O}_{S'}(K_{S'} + 2F')) \to H^0(F', \mathcal{O}_{F'}(K_{F'}))$ is surjective.*
(iv) *There exists $F_0' \in \mid F' \mid$ such that $F_0' = G' + H$ where $\rho_a(H) = 1$, $HG' = 2$, $H^2 = -2$ and $HE' = 0$.*

Proof. (i) is a straight consequence of 1.2 (j). (ii) is a straight consequence of 1.2 (jj) and 2.1 (i).

(iii) The first assertion follows easily from (i), (ii) and 1.2. Since $q(S^{'}) = 0$ by the adjunction sequence for $F^{'}$ we have $h^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + F^{'})) = 7$, $h^1(S^{'}, \mathcal{O}_{S'}(K_{S'} + F^{'})) = 0$. Thus we have $h^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + 2F^{'})) = 10$ and the surjectivity of the restriction map.
(iv) By 2.1 (ii) $H = \sigma^{'*}\sigma_\star(F_0 - G)$ is an effective divisor and by (i) we have $E^{'} \not< H$ and $F^{'}H = 0$. Since $0 = F^{'2} = F^{'}(G^{'} + H) = G^{'2} + HG^{'} + F^{'}H = -2 + HG^{'}$ then $HG^{'} = 2$. Now by $F^{'}H = 0$ we have $H^2 = -2$. Since $1 = F^{'}E^{'} = G^{'}E^{'} + HE^{'}$ we have $HE^{'} = 0$. Then by adjunction and (iii) we obtain $\rho_a(H) = 1$. □

In the next lemma we construct a basis of $H^0(S^{'}, \mathcal{O}_{S'}(K_{S'|\boldsymbol{P}^1}))$.

Lemma 2.3. *We use the notations of* 2.2. *If* $\zeta \in H^0(S^{'}, \mathcal{O}_{S'}(E^{'}))$, $g \in H^0(S^{'}, \mathcal{O}_{S'}(G^{'}))$, $h \in H^0(S^{'}, \mathcal{O}_{S'}(H))$ *and if* $\langle t_0, t_1\rangle$ *is a basis of* $H^0(S^{'}, \mathcal{O}_{S'}(F^{'}))$ *with* $t_0 = hg$ *then there exist* $x \in H^0(S^{'}, \mathcal{O}_{S'}(L^{'}))$ *and* $\eta \in H^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'}))$ *such that*:

$$\langle t_0^4 x_0\zeta, t_0^3 t_1 x_0\zeta, t_0^2 t_1^2 x_0\zeta, t_0 t_1^3 x_0\zeta, t_1^4 x_0\zeta, t_0^2 x\zeta, t_0 t_1 x\zeta, t_1^2 x\zeta, t_0 h\eta, t_1 h\eta\rangle$$

is a basis of $H^0(S^{'}, \mathcal{O}_{S'}(K_{S'|\boldsymbol{P}^1}))$ *where* $x_0 = g\zeta^3$. *Moreover if* $R_1 = \mathrm{div(x)}$, $R_0 = \mathrm{div}(\eta)$ *then* $E^{'}, F^{'} \not< R_1$ *for every* $F^{'}$; $E^{'}, G^{'} \not< R_0$ *and* $E^{'}R_1 = 0$, $E^{'}R_0 = 0$, $G^{'}R_0 = 0$.

Proof. Since $P \neq P^{'}$ and $L^{'} = \sigma^{'*}K_S$ then $\exists x \in H^0(S^{'}, \mathcal{O}_{S'}(L^{'}))$ such that $E^{'} \not< \mathrm{div(x)} = \mathrm{R}_1$ and $E^{'}R_1 = 0$. In particular by 2.2 (i) we have $F^{'} \not< R_1$. Now we split $F_0^{'}$ in its two component $G^{'}$, H and by 2.2 (iii) we have $K_{S'} + F^{'} - H \equiv K_{S'} + G^{'} \equiv L^{'} + E^{'} + G^{'}$.

Claim: $h^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'})) = 5$.
In fact by 2.2 (ii) we have $L^{'} + E^{'} + G^{'} \equiv \sigma^{'*}(K_S + \sigma_\star(G))$ and by 1.2 (jj) $\sigma^\star(K_S + \sigma_\star(G)) \equiv K_{\tilde{S}} + G$. Then by the second claim in the proof of 2.1, we have:

$$5 = h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(K_{\tilde{S}} + G)) = h^0(S, \mathcal{O}_S(K_S + \sigma_\star(G))) = h^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'})).$$

Since $h^0(S^{'}, \mathcal{O}_{S'}(L^{'})) = 4$ from the inclusion $H^0(S^{'}, \mathcal{O}_{S'}(L^{'})) \xrightarrow{\otimes g\zeta} H^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'}))$ we see that there exists $\eta \in H^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'}))$ with $E^{'}, G^{'} \not< \mathrm{div}(\eta) = \mathrm{R}_0$ such that:

$$\langle t_0^2 g x_0\zeta, t_0 t_1 g x_0\zeta, t_1^2 g x_0\zeta, g\zeta x, \eta\rangle$$

is a basis of $H^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'}))$. It also easy to check that $E^{'}R_0 = 0$, $G^{'}R_0 = 0$. By the proof of 2.2 (iii) we know that $h^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + F^{'})) = 7$ then since $t_0 = gh$ by the inclusion $H^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + G^{'})) \xrightarrow{\otimes h} H^0(S^{'}, \mathcal{O}_{S'}(K_{S'} + F^{'}))$, we have that

$$\langle t_0^3 x_0\zeta, t_0^2 t_1 x_0\zeta, t_0 t_1^2 x_0\zeta, t_1^3 x_0\zeta, t_0\zeta x, t_1\zeta x, h\eta\rangle$$

is a basis of $H^0(S', \mathcal{O}_{S'}(K_{S'} + F'))$. Since $h^0(S', \mathcal{O}_{S'}(K_{S'} + 2F')) = 10$ by the inclusion $H^0(S', \mathcal{O}_{S'}(K_{S'} + F')) \overset{\otimes t_0}{\to} H^0(S', \mathcal{O}_{S'}(K_{S'} + 2F'))$ the lemma follows. □

Corollary 2.4. $\phi_{|K_{S'}+2F'|}$ *is a birational morphism.*

Proof. Since $K^2_{S'|\boldsymbol{P}^1} = 22$ and $\deg f_\star K_{S'|\boldsymbol{P}^1} = 7$ it is a special case of [16, Theorem 3.2] where it is shown that the relative canonical map is a morphism if $f : S \to B$ is a non-hyperelliptic fibration of genus 3 with $K^2_{S|B} = 3\Delta(f)+1$. □

We conclude the proof of the first part. By [16, Theorem 3.2] and [19, p.6] we know that the relative canonical algebra is generated in degrees ≤ 2.

Put $\xi_0 = h\eta$, $\xi_1 = x\zeta$, $\xi_2 = g\zeta^4$ and $\tilde{\eta} = \eta\zeta^4$. Then $\{\xi_0, \xi_1, \xi_2\}$ induces a basis of $H^0(F', \mathcal{O}_{F'}(K_{F'}))$ for any F' and $\tilde{\eta} \in H^0(S', \mathcal{O}_{S'}(2K_{S'} - 2F'))$. Since $h^0(S', K^{\otimes 2}_{S'|\boldsymbol{P}^1}) = 35$ it is easy to see that the 34 products of t_i, ξ_j' and $t_1^6\eta\zeta^4$ are a basis of $H^0(S', K^{\otimes 2}_{S'|\boldsymbol{P}^1})$. In particular it easily follows that $\xi_0, \xi_1, \xi_2, \tilde{\eta}$ are generators of the relative canonical algebra. Furthermore, we have a relation:

$$t_0\tilde{\eta} = \xi_0\xi_2.$$

In $H^0(S', \mathcal{O}_{S'}(4K_{S'}))$, which is 47-dimensional, we can find 41 products of t_i, ξ_j' s and 6 elements of the form (quadrics in the ξ_i) $\tilde{\eta}$ modulo the above relation. It is easy to see that these are independent. Therefore, $t_1^4\tilde{\eta}^2$ can be expressed as a linear combination of them, that is, we get another relation:

$$\alpha\tilde{\eta}^2 + Q\tilde{\eta} + c_1\xi_1^4 + \xi_2 P = 0.$$

Obviously we have no further relations. Let y be the fibre coordinate of $[2T - 6\Pi]$ on $\boldsymbol{P}$. By 2.4 the relative canonical map ($X_i = \xi_i$, $i = 0, 1, 2$) is a birational morphism and it can be lifted to a holomorphic map into $[2T - 6\Pi]$ by putting $y = \tilde{\eta}$, and the image is nothing but the relative canonical model:

$$(\star) \qquad \begin{cases} t_0 y = X_0 X_2 \\ \alpha y^2 + Qy + c_1 X_1^4 + X_2 P = 0. \end{cases}$$

By eliminating y we obtain the equation of the relative canonical image Y. It is now easy to see that Y has a double locus along $t_0 = X_0X_2 = 0$ and that α, Q, c_1 and P are as in the statement of the main theorem. Moreover it is an easy computation (see 2.5) that for generic Q and P the relative canonical model is smooth, that is, it is isomorphic to S'.

Second part.

We now prove that the minimal model of the surface given by $(\star)$ has $K^2 = 7$, $p_g = 4$ and $d = 3$. In the proof the following rational curves: $L_0 = \{x \in \boldsymbol{P} \mid t_0(x) = X_0(x) = 0\}$, $L_2 = \{x \in \boldsymbol{P} \mid t_0(x) = X_2(x) = 0\}$, $L_{12} = \{x \in \boldsymbol{P} \mid X_1(x) = X_2(x) = 0\}$ and the relative quartic:

$$Y = \{x \in \boldsymbol{P} \mid \alpha X_0^2 X_2^2 + Q t_0 X_0 X_2 + (c_1 X_1^4 + X_2 P) t_0^2 = 0\},$$

will play the central role. In fact $S^{'}$ lives in the 3-fold obtained by the blowing up of $\boldsymbol{P}$ with center $L_0 \cup L_2$ and it is the proper transform of Y, while $E^{'}$ is the proper transform of L_{12}. We consider the fiber $\Pi_0 = \{x \in \boldsymbol{P} \mid t_0(x) = 0\}$ and let Q_0 be the singular conic with support on $L_0 \cup L_2$. It is easy to see that Y is singular on Q_0. We can say more:

REMARK 2.5. Let $\mathcal{A} \subset H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(4T - 6\Pi))$ be the sublinear system of relative quartics having Q_0 as a double conic and $Y \in \mathcal{A}$ a generic element. If $\mathrm{Sing}(Y)$ is the support of the singular locus of Y then $\mathrm{Sing}(Y) = L_0 \cup L_2$ and Y has equation as above.

Proof. We need only to produce an element of $\mathcal{A}$ which satisfies the assertion. Now in the equation defining Y we put $Q = X_0^2$ and $P = \beta X_2^3$ where $\beta \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(8\Pi))$ is without multiple roots; an easy computation shows that for these elements our assertion is true. $\square$

We attain the proof of the second part through the resolution of the singularities of Y. We need some more notations. We put $V = [2T - 6\Pi]$. Let H be the tautological divisor of the 4-fold $\boldsymbol{P}(V) = \boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}} \oplus \mathcal{O}_{\boldsymbol{P}}(2T - 6\Pi))$, $\mu : \boldsymbol{P}(V) \to \boldsymbol{P}$ the canonical projection, $y_0 \in H^0(\boldsymbol{P}(V), \mathcal{O}_{\boldsymbol{P}(V)}(H))$, $y_\infty \in H^0(\boldsymbol{P}(V), \mathcal{O}_{\boldsymbol{P}(V)}(H - \mu^\star(2T - 6\Pi)))$. Obviously $V = \{y_\infty = 1\}$ and $y = y_{0|\{y_\infty = 1\}}$. We define $\boldsymbol{P}'$ to be the singular 3-fold in $\mid H + \mu^\star \Pi \mid$ given by the equation:

$$\boldsymbol{P}' = \{x \in \boldsymbol{P}(V) \mid \mu^\star(t_0)(x) y_0(x) - \mu^\star(X_0 X_2)(x) y_\infty(x) = 0\}.$$

We denote $\mu^{-1}(L_0) =_{\text{def}} \Sigma_0^{'}$, $\mu^{-1}(L_2) =_{\text{def}} \Sigma_2^{'}$ and let $S^{'}$ be the proper transform of Y. It is easy to see that $S^{'}$ has equation given by $(\star)$. Let $\mu^{'} : \boldsymbol{P}^{''} \to \boldsymbol{P}'$ be the blowing up of $\boldsymbol{P}'$ in its singular point, $\nu =_{\text{def}} \mu_{|\boldsymbol{P}'} \circ \mu^{'}$, Σ the exceptional locus of $\mu^{'}$, Σ_i the proper transform of $\Sigma_i^{'}$ where $i = 0, 2$. Since the singular point of $\boldsymbol{P}'$ is not on $S^{'}$ we will not distinguish between $S^{'} \subset \boldsymbol{P}'$ and $S^{'} \subset \boldsymbol{P}^{''}$. In particular $S^{'} \cap \Sigma = \emptyset$. We remark that on $S^{'}$ we have the fibration $f = \pi \circ \nu_{|S'} = \pi \circ \mu_{|S'}$.

Key Lemma

The following conditions hold:

a) *S' is a smooth surface.*
b) $H^0(S', \mathcal{O}_{S'}(K_{S'})) \simeq H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - 2\Pi))$, $q(S') = 0$, $p_g(S') = 4$ *and* $K^2_{S'} = 6$.
c) $\mid K_{S'} \mid \equiv L' + E'$ *where L' is the mobile part and E' is a (-1)-rational curve.*

Proof.

a) By abuse of notation we put $\mu^\star(x) = x$ for each variable on $\boldsymbol{P}$. By 2.5 we know that $S' \cap \{t_0 = 1\}$ is smooth. We put $t = \frac{t_0}{t_1}$ and, by abuse of notation, $x_i = X_{i|\{X_j \neq 0\}}$ for $i \neq j$, $i = 0, 1, 2$ and $j = 0$ or $j = 2$. A simple computation on $\mathbb{C}_{t,x_0,x_1,y}$ and on $\mathbb{C}_{t,x_1,x_2,y}$ shows that if α, Q and P are generic then the system $(\star)$ gives a nonsingular surface on each affine chart.

b) To show that $H^0(S', \mathcal{O}_{S'}(K_{S'})) \simeq H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(T - 2\Pi))$ we need the following

Lemma 2.6. *If Π_0'' is the ν-proper transform of Π_0 then*
(i) $\mathcal{O}_{\Pi_0''}(\nu^\star(T)) = \mathcal{O}_{\Pi_0''}(1)$,
(ii) $\mathcal{O}_{\Pi_0''}(\Pi_0'') = \mathcal{O}_{\Pi_0''}(-2)$,
(iii) $\Pi_0'' \cap S' = \emptyset$.

Proof. (i) is obvious. Let Π_0' be the μ-proper transform of Π_0. In particular $\Pi_0' = \{x \in \boldsymbol{P} \mid y_\infty(x) = 0\}$. Then $\Pi_0'' = \mu'^{-1}(\Pi_0')$ and $\Pi_0'' \cap \Sigma = \emptyset$. Now (iii) is obvious and (ii) is a direct consequence of the following relations: $\nu^\star(\Pi)_{|\Pi_0''} \equiv 0$, $\nu^\star(\Pi_0) = \Pi_0'' + \Sigma_0 + \Sigma_2 + 2\Sigma$ and $\mathcal{O}_{\Pi_0''}(\Sigma_i) = \mathcal{O}_{\Pi_0''}(1)$ for $i = 0, 2$. □

We consider $K_{S'}$. Since $K_{\boldsymbol{P}''} \equiv \nu^\star(K_{\boldsymbol{P}}) + \Sigma_0 + \Sigma_2 + 3\Sigma$ and $S' \equiv \nu^\star(4T - 6\Pi) - 2\Sigma_0 - 2\Sigma_2 - 4\Sigma$ then, by adjunction $K_{S'} \equiv (\nu^\star(K_{\boldsymbol{P}} + Y) - \Sigma_0 - \Sigma_2 - \Sigma)_{|S'} \equiv (\nu^\star(T - \Pi) - \Sigma_0 - \Sigma_2 - \Sigma)_{|S'}$. Then by the proof of 2.6 (ii) we have $K_{S'} \equiv (\nu^\star(T - 2\Pi) + \Pi_0'' + \Sigma)_{|S'}$ and by 2.6 (iii) $K_{S'} \equiv (\nu^\star(T - 2\Pi))_{|S'}$. In particular, since the fundamental relation in $\mathrm{Pic}(\boldsymbol{P})$ is $T^2 = 7\Pi T$, then $K^2_{S'} = (\nu^\star(T - 2\Pi))^2(\nu^\star(Y) - 2\Sigma_0 - 2\Sigma_2 - 4\Sigma) = (T - 2\Pi)^2(4T - 6\Pi) = 6$. Moreover

$$0 \to \mathcal{O}_{\boldsymbol{P}''}(\nu^\star(T - 2\Pi) + \Pi_0'') \to \mathcal{O}_{\boldsymbol{P}''}(\nu^\star(T - 2\Pi) + \Pi_0'' + \Sigma) \to \mathcal{O}_\Sigma(\Sigma) \to 0.$$

By 2.6 (i), (ii) we have

$$0 \to \mathcal{O}_{\boldsymbol{P}''}(\nu^\star(T - 2\Pi)) \to \mathcal{O}_{\boldsymbol{P}''}(\nu^\star(T - 2\Pi) + \Pi_0'') \to \mathcal{O}_{\Pi_0''}(-1) \to 0.$$

The cohomology of these sequences combined with that of the adjunction sequence for $S' \subset \boldsymbol{P}''$ and the previous result, that is $\mathcal{O}_{\boldsymbol{P}''}(K_{\boldsymbol{P}''} + S') = \mathcal{O}_{\boldsymbol{P}''}(\nu^\star(T - 2\Pi) + \Pi_0'' + \Sigma)$, implies $p_g(S') = 4$, $q(S') = 0$ and $H^0(S', K_{S'}) \simeq H^0(\boldsymbol{P}, T - 2\Pi)$.

c) We now show that $K_{S'} \equiv L' + E'$ with $(E')^2 = -1$. From now on we consider $S' \subset \boldsymbol{P}'$.

Lemma 2.7. *We consider $T_i = \{x \in \boldsymbol{P} \mid X_i(x) = 0\}$ with $i = 0,1,2$, $\Pi_0 \cap T_j = L_j$ with $j = 0,2$ and $L_{12} = T_1 \cap T_2$. Let $\mu : \boldsymbol{P}' \to \boldsymbol{P}$ be the blowing up with center $L_0 \cup L_2$, let S', Σ_2', T_2' be respectively the proper transform of Y, L_2 and T_2. We put $T_1' = \mu^\star(T_1)$, $\Sigma'_{2_{|S'}} = G'$, $T'_{1_{|S'}} = H_1$ and $T'_{2_{|S'}} = H_2$ and let F_0', F_1' be two different fibers of $\pi_{|Y} \circ \mu_{|S'} = f : S' \to \boldsymbol{P}^1$. Then*

$$\langle\, H_1,\, 2F_0' + H_2 + G',\, F_0' + F_1' + H_2 + G',\, 2F_1' + H_2 + G' \,\rangle$$

represents a basis of $H^0(S', K_{S'})$. Moreover let E' be the μ-proper transform of L_{12}, then $E' \prec H_1$, $E' \prec H_2$ and the following identities hold:
(i) $H_2 = 4E'$
(ii) *G' is, generically, a smooth elliptic curve and $G'E' = 1$.*
(iii) *$H_1 = E' + R_1$ where $F' \not\prec R_1$, $\forall F' \in \mid F_0 \mid$, $E' \not\prec R_1$ and $E'R_1 = 0$. Moreover R_1 and G' do not have any common component and $R_1 G' = 1$.*

Proof. By b) of the Key-lemma the first part is obvious.

(i) It is easy to see that $\mu^{-1}(L_{12}) = E' + f_2^1$ where $f_2^1 = \{x \in \Sigma_2' \mid X_1 = 0\}$. We note that f_2^1 is not contained in S'. Moreover since

$$E' = \{x \in \boldsymbol{P}(V) \mid y_0 = X_1 = X_2 = 0\}$$

and $T_2' = \{x \in \boldsymbol{P}(V) \mid y_0 = X_2 = 0\}$ then $H_2 = \operatorname{div}(X_1^4)_{|S'}$.

(ii) Since $G' = \{x \in \Sigma_2' \mid \alpha(0)y_0^2 + Qy_0y_\infty + c_1X_1^4y_\infty^2 = 0\}$ we easily see that G' is smooth and it can be realized as a double cover of L_2 branched on the four points given by $Q^2_{|L_2} - 4\alpha(0)c_1X_1^4 = 0$. Then $\rho_a(G') = 1$. By the proof of (i) we have $G'E' = 1$.

(iii) By definition $E' \prec H_1$. Put $R_1 = H_1 - E'$ and $\rho = \mu_{|T_1'}$. It is easy to see that $\rho : T_1' \to T_1$ is the blowing-up of the two points $P_0 = \{t_0 = X_0 = 0\}$ and $P_2 = \{t_0 = X_2 = 0\}$. On P_2, ρ is given by $t_0y_0 = X_2y_\infty$ and in the affine chart $W_{t_0,y_0} = \{x \in T_1' \mid y_\infty = t_1 = X_0 = 1\}$ we have:

$$R_1 \cap W_{t_0,y_0} = \{x \in W_{t_0,y_0} \mid c_0 + \alpha y_0 + \alpha_6 t_0^2 y_0^2 + \beta_4 t_0^3 y_0^3 = 0\}.$$

Since $W_{t_0,y_0} \cap E' = \{x \in W_{t_0,y_0} \mid y_0 = 0\}$ we easily see that $E' \not\prec R_1$ and $E'R_1 = 0$. Finally since R_1 is not contained in Σ_2' while $G' \subset \Sigma_2'$ it is obvious that they does not have common components. Furthermore they have an ordinary intersection in the point $a \in W_{t_0,y_0}$ given by $t_0 = 0$ and $y_0 = c_0/\alpha(0)$. □

Now we can prove c). By b) of the Key-lemma and 2.7 we have $K_{S'} \equiv E' + R_1$. Now by adjunction and 2.7 (iii) we have $E'^2 = -1$. Morever by 2.7 (iii) there is not any other fixed component. This completes the proof of the Key lemma. □

End of the proof of the main theorem.
Let $\sigma' : S' \to S$ be the contraction of E'. By the Key-lemma we only have to show that $\deg\phi_{|K|} = 3$; but $\phi_{|K|}$ is the map induced on S by $\phi_{|T-2\Pi|_{|Y}} : Y \to \widehat{\boldsymbol{F}}_2 \subset \boldsymbol{P}^3$, which is of degree 3. In fact when restricted to the generic plane quartic it is the projection from the point of the quartic given by: $X_2 = X_1 = 0$.

Moduli of surfaces with $K^2 = 7$, $p_g = 4$ and $d = 3$

We end this section with an easy computation of the number $\mathcal{M}^3_{4,7}$ of moduli of surfaces with $K^2 = 7$, $p_g = 4$ and $d = 3$. By [5, Theorem 5.19] we know that regular surfaces with $K^2 = 7$, $p_g = 4$ and $| K |$ free from base points form an irreducible unirational open set of their moduli space. By our theorem we easily see that the locus $M^3_{4,7}$ of surfaces with $K^2 = 7$, $p_g = 4$ and $d = 3$ (in this case $| K |$ has a base point) is irreducible and unirational. We can say more:

Corollary 2.8. $\mathcal{M}^3_{4,7} = 35$.

Proof. We have seen in the proof of the theorem that the family of all surfaces with $K^2 = 7$, $p_g = 4$ and $d = 3$ is parametrized by an open set $U \subset \boldsymbol{P}^{46}$. Let S_1, S_2 be two minimal surfaces with $K^2 = 7$, $p_g = 4$, $d = 3$ and $Y_1, Y_2 \in U$ be respectively their non normal models in $\boldsymbol{P}$. Since S_1, S_2 are minimal of general type then S_1, S_2 are isomorphic if and only if Y_1, Y_2 are isomorphic. Since two nonsingular plane quartics are isomorphic if and only if they differ by an automorphism of $\boldsymbol{P}^2$ and the fibration on Y_i induced by the canonical projection of $\boldsymbol{P}$ is not isotrivial we see that there exists a morphism $U \to M^3_{4,7}$, whose fibers are images of the group of the following transformations of $\boldsymbol{P}$: $X_0 = a_0X_0' + b_1(t)X_1' + b_3(t)X_2'$, $X_1 = a_1X_1' + b_2(t)X_2'$, $X_2 = a_2X_2'$ where $a_i \in \mathbb{C}^\star$ and $b_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(i\Pi))$ for $i = 1, 2, 3$. Since the vector space of all this transformations has dimension 12 we have $\mathcal{M}^3_{4,7} = 46 - 11 = 35$. □

3. Surfaces with $K^2 = 4$, $p_g = 3$ and $d = 3$

Surfaces with $p_g = 3$, $K^2 = 4$ and $d = 3$ are probably known (see added in proof [13, §2 p.110]). However for lack of reference we include their complete classification. The proof of the theorem 1 in the introduction is similar to that of the main theorem. In particular the desingularization process is a verbatim translation of the previous one and given the relative canonical image Y we obtain S in the same way as before. We only show how to reconstruct Y by S. Also in this case the strategy of the proof is to find a rational pencil of genus-3 non-hyperelliptic curves with a simple base point.

Lemma 3.1. *Let S be a minimal surface with $p_g = 3$, $K_S^2 = 4$ and $\deg\phi_{|K_S|} = 3$. Then $q(S) = 0$, $| K_S |$ has not fixed part and it has an unique base point P. Let*

$\sigma : \tilde{S} \to S$ be the blowing up of P, $E = \sigma^{-1}(P)$ and L the mobile part of $\mid K_{\tilde{S}} \mid$. Then the morphism $\phi_{|L|} : \tilde{S} \to \boldsymbol{P}^2$ is not finite. Moreover there exists $x \in \boldsymbol{P}^2$ such that the sublinear system $\Lambda \subset H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(L))$ induced on $\tilde{S}$ by the lines which pass on x are of the following form: $L_x = \tilde{G} + \tilde{F}$, where $\tilde{G}$ is the fixed part of Λ and $\tilde{F}$ is a rational pencil of genus-3 curves with a simple base point $\tilde{Q} \notin E$. In particular the following numerical identities hold:

(i) $L\tilde{G} = 0$, $L\tilde{F} = 3$, $\tilde{F}\tilde{G} = 2$, $\tilde{F}^2 = 1$ *and* $\tilde{G}^2 = -2$.

(ii) $E\tilde{F} = 0$ $E\tilde{G} = 1$.

Proof. The first part is shown in 1.1. If we suppose that there exists an effective divisor $\tilde{G}$ on $\tilde{S}$ such that $E\tilde{G} > 0$ and $\phi_{|L|}(\tilde{G})$ is a point of $\boldsymbol{P}^2$ then the lemma is an easy consequence of the index theorem of Hodge. We now prove, by contradiction, the existence of such divisor $\tilde{G}$. Since $d = 3$ the generic L is a nonhyperelliptic curve of genus 5 (cf.[13, p.109]). Since $LE = 1$ we can put $P_L = L \cap E$. By adjunction $\omega_L = (2L + 2E)_{|L} = 2L_{|L} + 2P_L$ then by the sequence $0 \to \mathcal{O}_{\tilde{S}}(L) \to \mathcal{O}_{\tilde{S}}(2L) \to \mathcal{O}_L(\omega_L - 2P_L) \to 0$ we have: $h^0(S, \mathcal{O}_{\tilde{S}}(2L)) = 6$ and $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = 1$. Since $\chi(\mathcal{O}_{\tilde{S}}(3L)) = 10$ by the cohomology of $0 \to \mathcal{O}_{\tilde{S}}(2L) \to \mathcal{O}_{\tilde{S}}(3L) \to \mathcal{O}_L(3L) \to 0$ we have $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) = 10 \,\text{or}\, 11$.

Claim: $h^0(S, \mathcal{O}_{\tilde{S}}(3L)) = 10$.

By contradiction we suppose that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) = 11$ and $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) = 1$. By [6, Theorem 4.1] we know that the bicanonical linear system on S is without base points: in our case this implies that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = 7$. By (cf.[11, p.45]) we know that $\phi_{|L|}(E)$ is a line in $\boldsymbol{P}^2$ then we can put $(\phi_{|L|})^\star(\phi_{|L|}(E)) = E + C_0 \equiv L$. Since $\mid K_S \mid$ is 2-connected then C_0 is a 1-connected effective divisor. Moreover since $1 = LE$ and $L^2 = 3$ then $C_0E = 2$ and $C_0^2 = 0$. By adjunction $3L_{|C_0} = (2L + E + C_0)_{|C_0} = (L+2E+2C_0)_{|C_0} = \omega_{C_0} + C_{0_{|C_0}}$. Now by the cohomology of $0 \to \mathcal{O}_{\tilde{S}}(2L + E) \to \mathcal{O}_{\tilde{S}}(3L) \to \omega_{C_0} + C_{0_{|C_0}} \to 0$ we have: $h^1(C_0, \omega_{C_0} + C_{0_{|C_0}}) = 1$ that is, by the duality of Serre, $h^1(C_0, -C_{0_{|C_0}}) = 1$. By the assumptions that there is not a divisor $\tilde{G}$ on $\tilde{S}$ such that $E\tilde{G} > 0$, that $\phi_{|L|}(\tilde{G})$ is a point of $\boldsymbol{P}^2$ and by an easy analysis on the possible form of C_0 we easily see that $C_0H = 0$ for each irreducible component of C_0. In particular by (cf.[2, Proposition 12.2]) we have $C_{0_{|C_0}} = \mathcal{O}_{C_0}$. Since $q(\tilde{S}) = 0$ this is a contradiction with the cohomology of the following exact sequence: $0 \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{S}}(C_0) \to \mathcal{O}_{C_0}(C_0) \to 0$. We can finish now the proof of the lemma. By the claim we have $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) = (\phi_{|L|})^\star H^0(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(3))$. Let $\zeta \in H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(E))$ and $c_0 \in H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(C_0))$. Since $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = 6$ and $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = 7$ there exists ψ such that $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = \zeta(\phi_{|L|})^\star H^0(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(2)) \oplus \psi\mathbb{C}$. Thus by the inclusion $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) \overset{\otimes c_0}{\to} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L))$ we have

a) $$c_0\psi = \sum \alpha_{ijk} x_0^i x_1^j x_2^k$$

where $i + j + k = 3$ and $\langle x_0, x_1, x_2 \rangle$ is a basis of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(L))$. If we now suppose that there exists no component $\tilde{G} \prec C_0$ such that $= \phi_{|L|}(\tilde{G})$ is a point and $E\tilde{G} = 1$

then there exists $H \prec C_0$ such that $\phi_{|L|}(H) = \phi_{|L|}(E)$ and there exists $x \in E \cap H$ such that $\langle x_0, x_1 \rangle$ is a basis of the sublinear system of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(L))$ induced on $\tilde{S}$ by the lines passing on $\phi_{|L|}(x)$. In particular $x_0 = c_0\zeta$ and $x_2(x) \neq 0$. Since $h \mid c_0$ and $h(x) = 0$ by $a)$ we obtain $\alpha_{003} = 0$. Since h does not divide x_1 we can repeat the above argument and we obtain $\alpha_{0ij} = 0$ if $i + j = 3$. Thus by $a)$ we obtain:

b) $$c_0\psi = c_0\zeta \sum \alpha_{ijk} x_0^{i-1} x_1^j x_2^k$$

where $i + j + k = 3$ that is $\zeta \mid \psi$: a contradiction. □

Lemma 3.2. *We use the notation of* 3.1. *We put* $G = \sigma_\star(\tilde{G})$, $F = \sigma_\star(\tilde{F})$. *Then:*

(j) $F^2 = 1$, $FG = 2$, $G^2 = -1$ *and* $P \in \operatorname{supp}(G)$.

(jj) $\mid F \mid$ *is a genus-*3 *rational pencil with a simple base point* $P' = \sigma_\star(\tilde{Q})$. *Moreover* $P' \neq P$.

(jjj) $G = G_0 + G_1$ *where* $FG_0 = FG_1 = 1$, G_1 *is a chain of* (-2)*-rational curves and* $P' \in \operatorname{supp}(G_0)$.

Proof. (j) and (jj) follow immediately by (i) and (ii) of 3.1.

(jjj) We first prove that G is reducible. By contradiction we suppose that G is irreducible. By 3.1 we know that $\rho_a(G) = 1$; in particular if G is a rational curve with a node then by 3.1 P is not the node. Let $g \in H^0(S, \mathcal{O}_S(G))$ and $\langle t_0, t_1 \rangle$ be a basis of $H^0(S, \mathcal{O}_S(F))$ where $P \in \operatorname{supp}(t_0)$. Since $p_g = 3$, $\exists z_2$ such that $\langle z_0, z_1, z_2 \rangle$ is a basis of $H^0(S, \mathcal{O}_S(K_S))$ where $z_0 = t_0 g$, $z_1 = t_1 g$. Since $q(S) = 0$ then by adjunction sequence for G we obtain $h^0(S, \mathcal{O}_S(K_S + G)) = 4$. Thus by the inclusion $H^0(S, \mathcal{O}_S(K_S)) \overset{\otimes g}{\to} H^0(S, \mathcal{O}_S(K_S + G))$, we obtain that $\exists u$ such that $\langle t_0 g^2, t_1 g^2, z_2 g, u \rangle$ is a basis of $H^0(S, \mathcal{O}_S(K_S + G))$. Since $P \in \operatorname{supp}(G)$ and $(K_S + G)G = 0$ then $\{u = 0\} \cap G = \emptyset$. We need to show that $P' \in G$. Since $\chi(2G) = 1$ then $H^1(S, \mathcal{O}_S(2G)) = \emptyset$. Thus by the cohomology of the sequence: $0 \to \mathcal{O}_S(2G) \to \mathcal{O}_S(K_S + G) \to \mathcal{O}_F(K_S + G) \to 0$ we obtain that $\mid K_S + G \mid$ cuts on F a complete linear series and since $\deg \mathcal{O}_F(K_S + G) = 5$ we see that $\phi_F = \phi_{|K_S+G|_{|F}}$ is a birational morphism. On the other hand $\mid K_S + G \mid_{|F} \equiv \omega_F - P' + G_{|F}$, and if $P' \notin G$ then $\mid K_S + G \mid_{|F}$ is without base points. Thus $G \cap F = P_F^1 + P_F^2$ and ϕ_F contracts the three points P', P_F^1, P_F^2. In particular $\phi_F(F)$ is a plane quintic with a triple point; that is F is an hyperelliptic curve: a contradiction since $\omega_F - P'$ is a g_3^1 without base points. Since $P' \in G$ and $FG = 2$ then $G \cap F = P' + P_F$. We distinguish two cases.

i) If G is smooth then $P' = P_F$ $\forall F$. In particular if $F_1, F_2 \in \mid F \mid$ then F_1, F_2 are tangent in P': a contradiction since $1 = F_1 F_2$.

ii) If G is singular then P' is the node thus $\exists F_0 \in \mid F \mid$ such that $G \prec F_0$: a contradiction since $1 = F^2 = F(G + (F_0 - G)) \geq 2$. This shows that G is reducible.

Since G is reducible then $\mid K_S + G \mid$ has a fixed part. In fact in the opposite case we can show as before that $P' \in \mathrm{supp}(G)$. Let G_0 be an irreducible reduced component such that $P' \in G_0$. By 3.1 we have $0 \leq K_S G_0 \leq 1$ and by adjunction $-2 \leq G_0^2 \leq -1$. As in the previous case we obtain that $(K_S + G)G_0 = 0$ thus by the Euler-Poincaré formula we see that $h^1(S, \mathcal{O}_S(G+G_0)) = 0$. Since $\omega_F \leq (K_S+G_0)_{|F}$ by the cohomology of $0 \to \mathcal{O}_S(G+G_0) \to \mathcal{O}_S(K_S+G_0) \to \mathcal{O}_F(K_S+G_0) \to 0$ we obtain that $h^0(S, \mathcal{O}_S(K_S+G_0)) = 4$ that is $G - G_0$ is in the fixed part. Moreover G cannot be the fixed part of $\mid K_S + G \mid$. In fact in this case we have $3 = h^0(S, \mathcal{O}_S(K_S)) = h^0(S, \mathcal{O}_S(K_S+G)) = 4$. Thus there exists a non trivial proper component $G_1 \prec G$ such that G_1 is the fixed part of $\mid K_S + G \mid$ and $G = G_0 + G_1$. In particular by $K_S G = 1$ we have two possibilities: $K_S G_0 = 1$ and $K_S G_1 = 0$ or $K_S G_0 = 0$ and $K_S G_1 = 1$. Now we consider $\phi_{|K_S+G_0|}$. Obviously $\mid K_S + G_0 \mid$ is without fixed part, we will show that it is without base point and from this we will obtain easily the assert. Let $g_0 \in H^0(S, \mathcal{O}_S(G_0))$, $g_1 \in H^0(S, \mathcal{O}_S(G_1))$ and $u = g_1 v$. Then $\langle t_0 g g_0, t_1 g g_0, z_2 g_0, v\rangle$ is a basis of $H^0(S, \mathcal{O}_S(K_S+G_0))$. Since $FG_0 \geq 0$ and $G_1 G_0 \geq 1$ by $K_S G_0 \leq 1$ we obtain $G_0^2 < 0$. Since $(K_S + G_0)G_0 \geq 0$ we then have $K_S G_0 = 1$, $G_0^2 = -1$. In particular $K_S G_1 = 0$ and G_1 is a chain of (-2)-rational curves. Since $(K_S + G_0)G_0 = 0$ then $\mathrm{div}(v)G_0 = 0$, thus $\mathrm{supp}(\mathrm{div}(v)) \cap G_0 = \emptyset$. From this fact it follows easily that if P_1 is a base point then $P_1 = P$; in particular $P \in \mathrm{supp}(\mathrm{div}(v))$ thus $P \in \mathrm{supp}(G_1)$: a contradiction since $K_S G_1 = 0$. Thus there are not base points. Since $K_S \equiv F + G_0 + G_1$ by $1 = K_S G_0$, $0 = K_S G_1$ and $G_0^2 = -1$ we obtain:

$$\begin{cases} FG_0 + G_1 G_0 = 2 \\ FG_1 + G_0 G_1 + G_1^2 = 0. \end{cases}$$

Since $FG_0 \geq 0$, $FG_1 \geq 0$, $G_0 G_1 \geq 1$ and $FG = 2$, if $FG_0 = 0$ then $(K_S + G_0)_{|F} \equiv \omega_F - P'$; thus $\phi_{|K_S+G_0|}(F)$ is a straight line: a contradiction. Then $FG_0 = 1$, $G_0 G_1 = 1$, $FG_1 = 1$ and $G_1^2 = -2$. If $G_{0_{|F}} \neq P'$ we have a contradiction as above. Then $P' \in \mathrm{supp}(G_0)$. □

We then have the analogous of 2.2. In the following lemma G_0 plays the role of G in 2.2.

Lemma 3.3. *We use the notations of* 3.2.
Let $\sigma' : S' \to S$ be the blowing up of P', $E' = \sigma'^{-1}(P')$, $\sigma'^*(K_S) = L'$ and F' the proper transform of F. In S' the following relations hold:

(i) *$\sigma'^*(F) = F' + E'$, $F'E' = 1$ and the rational pencil $\mid F' \mid$ induces a non-hyperelliptic genus-3 fibration $f : S' \to \boldsymbol{P}^1$.*

(ii) *$\sigma'^*(G_0) = G_0' + E'$ where $G_0'^2 = -2$, $E' \not\prec G_0'$, $E'G_0' = 1$, $F'G_0' = 0$ and $\rho_a(G_0') = 1$. Moreover if we put $\sigma'^*(G_1) = G_1'$ then $G_1'^2 = -2$, $F'G_1' = 1$.*

(iii) *$K_{S'} \equiv F' + G_0' + G_1' + 3E'$, and the restriction map $H^0(S', K_{S'} + 2F') \to H^0(F', \mathcal{O}_{F'}(K_{F'}))$ is surjective.*

(iv) *There exists* $F_0' \in | F' |$ *such that* $F_0' = G_0' + H$ *where* $HG_0' = 2$, *and* $HE' = 0$.

Proof. The same as 2.2. □

Now we have the analogous of 2.3.

Lemma 3.4. *We use the notation of* 3.3. *If* $\zeta \in H^0(S', \mathcal{O}_{S'}(E'))$, $g_0 \in H^0(S', \mathcal{O}_{S'}(G_0'))$, $g_1 \in H^0(S', \mathcal{O}_{S'}(G_1'))$, $h \in H^0(S', \mathcal{O}_{S'}(H))$ *and* $\langle t_0, t_1 \rangle$ *is a basis of* $H^0(S', \mathcal{O}_{S'}(F'))$ *where* $t_0 = hg_0$, *then there exist* $x \in H^0(S', \mathcal{O}_{S'}(L'))$ *and* $\eta \in H^0(S', \mathcal{O}_{S'}(K_{S'} + G_0'))$ *such that:*

$$\langle t_0^3 x_0 \zeta, t_0^2 t_1 x_0 \zeta, t_0 t_1^2 x_0 \zeta, t_1^3 x_0 \zeta, t_0^2 x \zeta, t_0 t_1 x \zeta, t_1^2 x \zeta, t_0 h \eta, t_1 h \eta \rangle$$

is a basis of $H^0(S', \mathcal{O}_{S'}(K_{S'} + 2F'))$ *where* $x_0 = g_0 g_1 \zeta^2$. *Moreover if we put* $R_1 = \mathrm{div(x)}$, $R_0 = \mathrm{div}(\eta)$ *then* $E', F' \not\prec R_1$ *for every* F'; $E', C \not\prec R_0$ *where* $C \prec G_0'$ *is any component;* $E'R_1 = 0$, $E'R_0 = 0$, $G_0'R_0 = 0$.

Proof. It is equal to the proof of 2.3. □

We now conclude the proof of theorem 1 in a slight different way with respect to the proof of the main theorem. We can define $\phi : S' \to \boldsymbol{P}$: $\phi^\star(X_0) = h\eta$, $\phi^\star(X_1) = x\zeta$ and $\phi^\star(X_2) = g_0 g_1 \zeta^3$. By the numerical identities of 3.4 and by 3.3 (iv) ϕ is a morphism, and by 3.3 (iii) it is birational onto the image. Since $\phi^\star(T) \equiv K_{S'} + 2F'$ then $\phi^\star(4T - 5\Pi) \equiv 4K_{S'} + 3F'$. In $H^0(S', \mathcal{O}_{S'}(4K_{S'} + 3F'))$ we consider the sublinear system $\mathcal{A}'$ given by the sections which vanish on $2F_0' + 3E'$. Since $4K_{S'} + 3F' - (2F_0' + 3E') \equiv 3L' + F' + K_{S'}$ then $\mathcal{A}' \approx H^0(S', \mathcal{O}_{S'}(3L' + E' + K_{S'}))$ and $\dim_{\mathbb{C}} \mathcal{A}' = 39$. On the other hand in $\mathcal{A}'$ there are the pulls-back of the following 40 sections: $\alpha X_0^2 X_2^2$ with $\alpha \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(3\Pi))$; $t_0 X_0 X_2 Q$ with $Q = c_0 X_0^2 + \alpha_1 X_0 X_1 + \alpha_2 X_1^2 + \alpha_3 X_1 X_2 + \alpha_4 X_2^2$ and $c_0 \in \mathbb{C}$, $\alpha_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(i\Pi))$; $c_1 X_0 X_1^3$ with $c_1 \in \mathbb{C}$; $t_0^2 \beta_1 X_1^4$ and $t_0^2 X_2 P$ with $P = \beta_2 X_1^3 + \beta_3 X_1^2 X_2 + \beta_4 X_1 X_2^2 + \beta_5 X_2^3$ where $\beta_i \in H^0(\boldsymbol{P}, \mathcal{O}_{\boldsymbol{P}}(i\Pi))$. Now it is obvious that $\phi(S') = Y$. If y is the fibre coordinate of $[2T - 5\Pi]$ on $\boldsymbol{P}$ then ϕ can be lifted to a holomorphic map $\nu : S' \to [2T - 5\Pi]$ and the image is nothing but the relative canonical model:

$$\begin{cases} t_0 y = X_0 X_2 \\ \alpha y^2 + Qy + \beta_1 X_1^4 + c_1 X_0 X_1^3 + X_2 P = 0. \end{cases}$$

This completes the proof of theorem 1. □

As in the case $p_g = 4$ we see that the locus $M_{3,4}^3$ of surfaces with $K^2 = 4$, $p_g = 3$ and $d = 3$ is irreducible and unirational. Moreover the same proof of 2.8 shows that $\mathcal{M}_{3,4}^3 = 29$.

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