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SEIBERG-WITTEN INVARIANTS ON NON-SYMPLECTIC 4-MANIFOLDS

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Let X be an oriented, closed Riemannian 4-manifold. There is an integral cohomology class which reduces mod (2) to the second Stieffel-Whiney class $w_2(X)$. This integral cohomology class induces a $Spin^c$-structure on X. Seiberg and Witten in [10] introduced a new invariant on X which is a differential-topological invariant. Taubes in [9] proved that every closed symplectic 4-manifold has a non-trivial Seiberg-Witten invariant. The Seiberg-Witten invariants of connected sums of 4-manifolds with $b_2^+>0$ identically vanish. Kotschick, Morgan and Taubes in [8] showed that there are closed oriented 4-manifolds with nontrivial Seiberg-Witten invariants which do not admit symplectic structures. They considered the case which is the first Betti number $b_1(N)=0$. We would like to generalize their theorem by giving a certain condition instead of $b_1(N)=0$, of course our case will cover their case. We introduce their theorem:

Theorem ([8]). *Let X be a manifold with a nontrivial Seiberg-Witten invariant with $b_2^+(X)>1$, and let N be a manifold with $b_1(N)=b_2^+(N)=0$ whose fundamental group has a nontrivial finite quotient. Then $M=X\sharp N$ has a non-trivial Seiberg-Witten invariant but does not admit any symplectic structure.*

Let M be a closed symplectic 4-manifold and let $M=X\sharp N$ be a smooth connected sum decomposition. By the vanishing theorem of Seiberg-Witten invariants and non-trivial Seiberg-Witten invariants for symplectic manifolds, one of the summands, say it N, has a negative definite intersection form. By Donaldson's Theorem [5] there is a basis $\{e_1,\cdots,e_n\}$ of the free part of $H^2(N,\mathbf{Z})$ such that in this basis the intersection form of N is diagonal, where n is the rank of $H^2(N,\mathbf{Z})$. An element $\alpha\in H^2(N,\mathbf{Z})$ is said to be characteristic if the intersection number $\alpha\cdot x=x\cdot x$ mod (2) for any $x\in H^2(N,\mathbf{Z})$. If α is characteristic, then $\alpha\equiv w_2(N)$ modulo 2.

Lemma 1. *Let N be a closed oriented Riemannian 4-manifold with $b_2^+(N)=0$ and let $\{e_1,\cdots,e_n\}$ is a basis for the free part of $H^2(N,\mathbf{Z})$ such that $e_i\cdot e_j=-\delta_{ij}$.*

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Then

1. $e = e_1 + \cdots + e_n$ *is characteristic.*
2. $\alpha = (1+\lambda_1)e_1 + \cdots + (1+\lambda_n)e_n$ *is characteristic if and only if the* λ_i *are even.*

Proof. It is sufficient to consider the free elements in the proof because the intersection numbers with torsion elements are zero. Let $x = x_1e_1 + \cdots x_ne_n \in H^2(N,\mathbf{Z})$ where the x_i are integers $i = 1,\cdots,n$.

Then

$$\alpha \cdot x = -(1+\lambda_1)x_1 - \cdots - (1+\lambda_n)x_n \quad \text{and}$$
$$x \cdot x = -x_1^2 - \cdots - x_n^2.$$

$\alpha \cdot x = x \cdot x \mod (2)$ for all $x \in H^2(N,\mathbf{Z})$.

$\Leftrightarrow -(1+\lambda_1)x_1 - \cdots - (1+\lambda_n)x_n = -x_1^2 - \cdots - x_n^2 \mod (2)$ for all $x_1,\cdots,x_n$.

$\Leftrightarrow \lambda_1x_1 + \cdots + \lambda_nx_n = 0 \mod (2)$ for all $x_1,\cdots,x_n$.

$\Leftrightarrow \lambda_1,\cdots,\lambda_n$ are even.

If the fundamental group $\pi_1(N)$ of N has a non-trivial finite quotient, then there is a connected covering of N with the cardinality of fiber >1 and so is a connected sum with N.

Lemma 2 ([8]). *Let $M = X\sharp N$ be a closed symplectic 4-manifold which decomposes as a connected sum. If N has a negative definite intersection form then its fundamental group does not admit nontrivial finite quotient.*

We recall briefly the Seiberg-Witten invariants for a compact, oriented Riemannian 4-manifold X with $b_2^+(X) > 1$.

Let $e \in H^2(X,\mathbf{Z})$, with $e \equiv w_2(X) \mod (2)$.

The cohomology class e defines a *Spin*^{c}-structure on X. Let $W^+(W^-) \to X$ be the positive (negative respectively) spinor bundle on X and $L = \det(W^+)$ the determinant line bundle of W^+. Let $\tau : \mathrm{End}(W^+) \to \Lambda^+(T^*X) \otimes \mathbf{C}$ be the adjoint of Clifford multiplication. A connection A on L with the Levi-Civita connection on T^*X defines a covariant derivative $\nabla_A : \Gamma(W^+) \to \Gamma(W^+ \otimes T^*X)$. The composition of ∇_A and Clifford multiplication define a Dirac operator

$$D_A : \Gamma(W^+) \to \Gamma(W^-).$$

For each connection on L $A \in \mathscr{A}(L)$ and $\phi \in \Gamma(W^+)$, the equations

$$\begin{cases} D_A\phi = 0 \\ F_A^+ = \dfrac{1}{4}\tau(\phi \otimes \phi^*) \end{cases}$$

are called the Seiberg-Witten monopole equations. The gauge group $C^{\infty}(X, U(1))$ of the complex line bundle L acts on the space of solutions of the monopole equations. The moduli space $\mathfrak{M}(X,e)$ is the quotient of the space of solutions by the gauge group. Then the moduli space is generically a compact smooth manifold with its dimension $-(1/4)(2\chi(X)+3\sigma(X))+(1/4)c_1(L)^2$ and defines canonically an invariant which is so called the Seiberg-Witten invariants. For details see [5].

Let X and N be compact oriented 4-manifolds. Let $\alpha \in H^2(X,\boldsymbol{Z})$ and $\beta \in H^2(N,\boldsymbol{Z})$ such that $\alpha \equiv w_2(X)$ mod (2), $\beta \equiv w_2(N)$ mod (2). Let $M = X \# N$, then $\alpha+\beta \equiv w_2(M)$ mod (2). Let the complex line bundles $L_\alpha \to X$, $L_\beta \to N$, $L_{\alpha+\beta} \to M$ with their Chern classes $c_1(L_\alpha)=\alpha$, $c_1(L_\beta)=\beta$ and $c_1(L_{\alpha+\beta})=\alpha+\beta$ respectively. We can easily calculate the virtual dimensions of the moduli spaces.

Lemma 3. $\dim \mathfrak{M}(M, \alpha+\beta) = \dim \mathfrak{M}(X,\alpha) + \dim \mathfrak{M}(N,\beta) + 1.$

Proof. The Euler characteristic is $\chi(M)=\chi(X)+\chi(N)-2$. The signature is $\sigma(M)=\sigma(X)+\sigma(N)$. The first Chern classes are $c_1(L_{\alpha+\beta})=c_1(L_\alpha)+c_1(L_\beta)$ and $\alpha \cdot \beta = 0$. Thus

$$\begin{aligned}\dim \mathfrak{M}(M, \alpha+\beta) &= -\frac{1}{4}(2\chi(M)+3\sigma(M))+\frac{1}{4}c_1(L_{\alpha+\beta})^2 \\ &= \left[-\frac{1}{4}(2\chi(X)+3\sigma(X))+\frac{1}{4}c_1(L_\alpha)^2\right] \\ &\quad + \left[-\frac{1}{4}(2\chi(N)+3\sigma(N))+\frac{1}{4}c_1(L_\beta)^2\right]+1 \\ &= \dim \mathfrak{M}(X,\alpha)+\dim \mathfrak{M}(N,\beta)+1.\end{aligned}$$

Let N have a negative definite intersection form.

As in Lemma 1, let $\{e_1,\cdots,e_n\}$ be a basis of the free part of $H^2(N,\boldsymbol{Z})$. If $\alpha=(1+\lambda_1)e_1+\cdots+(1+\lambda_n)e_n$ and the λ_i are even, the α is characteristic.

Lemma 4. *If* $4b_1(N)=2\lambda_1+\cdots+2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2$, *then* $\dim \mathfrak{M}(N,\alpha)=-1$.

Corollary 5. *If X is a symplectic manifold and K is the canonical line bundle on X, and $M=X\#N$, then* $\dim \mathfrak{M}(M, c_1(K)+\alpha)=\dim \mathfrak{M}(X, c_1(K))=0$.

Proof. For the proof use $c_1(K)^2=2\chi+3\sigma$ and Lemma 3, 4.

Proof of Lemma 4. The virtual dimension of the moduli space is

$$\dim \mathfrak{M}(N,\alpha)=-\frac{1}{4}(2\chi(N)+3\sigma(N))+\frac{1}{4}\alpha^2$$

$$= -\frac{1}{4}\{2(2-2b_1(N)+b_2(N))+3(-b_2(N))\}$$

$$+\frac{1}{4}[(1+\lambda_1)e_1+\cdots+(1+\lambda_n)e_n]^2$$

$$= -\frac{1}{4}[4-4b_1(N)-b_2(N)]+\frac{1}{4}[-(1+\lambda_1)^2-\cdots-(1+\lambda_n)^2]$$

$$= -\frac{1}{4}[4-4b_1(N)+2\lambda_1+\lambda_1^2+\cdots+2\lambda_n+\lambda_n^2]$$

$$= -1, \quad \text{since } 4b_1(N)=2\lambda_1+\cdots 2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2.$$

REMARK 1. For the equation $4b_1(N)=2\lambda_1+\cdots+2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2$,

1. If $\lambda_2=\cdots=\lambda_n=0$, $b_1(N)=6$ and $\lambda_1=4$ or -6, then the equation holds.
2. If $\lambda_1=\cdots=\lambda_n=0=b_1(N)$, then the equation also holds.

Theorem 6. *Let X have a nontrivial Seiberg-Witten invariant and let N have a negative definite intersection form. If there are even integers λ_i, $i=1\cdots n$ such that $4b_1(N)=2\lambda_1+\cdots+2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2$, then the connected sum $M=X\#N$ has a nontrivial Seiberg-Witten invariant.*

Proof. Suppose N has a negative definite intersection form. As in Lemma 4, choose $\alpha=(1+\lambda_1)e_1+\cdots+(1+\lambda_n)e_n$ such that $4b_1(N)=2\lambda_1+\cdots+2\lambda_n+\lambda_1^2+\cdots+\lambda_n^2$ and the λ_i are even. Then α is characteristic by Lemma 1 and there is a *Spin*^{c}-structure on N with first Chern class α. The Seiberg-Witten monopole equation is

$$\begin{cases} D_A\psi=0 \\ F_A^+ =\frac{1}{4}\tau(\psi\otimes\psi^*). \end{cases}$$

For a generic metric on N there is no non-abelian solution of the equations since $\dim\mathfrak{M}(N,\alpha)=-1$. We have a unique abelian solution $(A_\alpha,0)$ given by the zero section of the positive spinor bundle and a connection A_α whose curvature is the harmonic form representing $\alpha=(i/2\pi)F_{A_\alpha}\in H^2(N,\boldsymbol{R})$. The given *Spin*^{c}-structure $e\in H^2(X,\boldsymbol{Z})$ on X and α induce a *Spin*^{c}-structure on M. By choosing generic metrics on $[X\backslash D^4]\cup[0,\infty)\times S^3$ and $[N\backslash D^4]\cup[0,\infty)\times S^3$, and product metric on the cylinder $S^3\times\boldsymbol{R}$ and connecting them, we have a Riemannian metric on $M=X\#N$. The solutions of the Seiberg-Witten equations in $\mathfrak{M}(M,e+\alpha)$ are given by gluing the solutions in $\mathfrak{M}(X,e)$ on X to the unique solution $(A_\alpha,0)$ in $\mathfrak{M}(N,\alpha)$ on N.

In particular, $\dim \mathfrak{M}(M, e+\alpha) = \dim \mathfrak{M}(X, e)$.

By combining Lemma 1 to Theorem 6 we have the following Theorem.

Theorem 7. *Let X be a manifold with a nontrivial Seiberg-Witten invariant defined by $e \in H^2(X, Z)(b_2^+(X) > 1)$, and let N be a manifold with negative definite intersection form. If there are even integers λ_i, $i = 1 \cdots n$ such that $4b_1(N) = 2\lambda_1 + \cdots + 2\lambda_n + \lambda_1^2 + \cdots + \lambda_n^2$ and that the fundamental group of N has a nontrivial finite quotient, then the connected sum $X \# N$ has a nontrivial Seiberg-Witten invariant but does not admit any symplectic structure.*

According to the Remark 1, Theorem 7 covers the Theorem [8] and there are many examples which are not included in Theorem [8].

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