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SPECIAL GENERIC MAPS AND L^2-BETTI NUMBERS

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1. Introduction

A smooth map between manifolds with only definite fold singular points is called a *special generic map*. Burlet and de Rham [2] first defined a special generic map and showed that a closed 3-manifold admits a special generic map into $\boldsymbol{R}^2$ if and only if it is diffeomorphic to S^3 or the connected sum of some S^2-bundles over S^1. Furthermore, Saeki determined completely those closed manifolds which admit special generic maps into $\boldsymbol{R}^2$ in [20]. Further results are known about the topology of manifolds which admit special generic maps (see [12], [19], [20], [21], [22], [23]). In this paper we study special generic maps of closed manifolds into open manifolds and their singular sets by using L^2-Betti numbers which were introduced by Atiyah [1].

Theorem 1.1. *Let M be a closed connected n-dimensional manifold such that the q-th L^2-Betti number $b_q^{(2)}(M)$ of M is not zero for some $q \leq n/2$. Then for any open p-dimensional manifold N with $p \leq q$, M does not admit a special generic map into N.*

By using this theorem, we see that no closed hyperbolic manifold of dimension $2n$ admits a special generic map into an open p-dimensional manifold N with $p \leq n$ (see Corollary 5.2).

As is seen in [2], the topology of the singular set of a special generic map is not determined only by the topology of the source manifold. However some results are known about the relationship between the topology of the singular set of a special generic map and that of the source manifold (see [20], [22], [23]). We study the order of the fundamental group of the singular set by using the L^2-Betti numbers of the source manifold. In order to state the next theorem, we recall the definition of residually finite groups. A group G is called *residually finite* if every non-trivial element of G is mapped nontrivially in some finite quotient group of G by a homomorphism.

Theorem 1.2. *Let M be a closed connected n-dimensional manifold such that $\pi_1(M)$ has infinite order and let $f: M \rightarrow N$ be a special generic map of M into an open* *p-dimensional manifold N with $p<n$. Let $S(f)=S_1 \cup \cdots \cup S_k$ be the decomposition of the singular set of f into the connected components. If either* (i) $p \leq (n+2)/2$ *or* (ii) $\pi_1(M)$ *is residually finite, then*

$$\sum_{i=1}^{k} \frac{1}{|\pi_1(S_i)|} \leq b_{p-1}^{(2)}(M).$$

Furthermore, if $p=(n+2)/2$, then

$$\sum_{i=1}^{k} \frac{1}{|\pi_1(S_i)|} \leq \frac{1}{2} b_{p-1}^{(2)}(M).$$

Here $|\pi_1(S_i)|$ denotes the order of $\pi_1(S_i)$ and we adopt the convention that $|\pi_1(S_i)|=\infty$ if and only if $1/|\pi_1(S_i)|=0$.

For special generic maps into open orientable 3-dimensional manifolds, Saeki and Sakuma studied the case where source manifolds are 1-connected (see [20], [21], [22], [23]). When the fundamental group of the source manifold is infinite, we have the following.

Theorem 1.3. *Let $f: M \to N^3$ be a special generic map of a closed connected n-dimensional manifold ($n>3$) into an open orientable 3-dimensional manifold. If $\pi_1(M)$ has infinite order and if the first L^2-Betti number $b_1^{(2)}(M)$ of M vanishes, then each connected component of the singular set $S(f)$ of f is diffeomorphic to either the 2-sphere or the torus. Furthermore, we have*

$$|\{S \in \pi_0(S(f)); S \cong S^2\}| = \begin{cases} b_2^{(2)}(M) & \text{if } n \geq 5, \\ \frac{1}{2} b_2^{(2)}(M) & \text{if } n=4. \end{cases}$$

Here the left hand side of the equality denotes the number of connected components S of $S(f)$ such that S is diffeomorphic to S^2.

Note that $b_1^{(2)}(M)$ depends only on $\pi_1(M)$. Let $\chi(M)$ denote the Euler characteristic of M. Under the condition in Theorem 1.3, if n is even, then we have $\chi(M)=\chi(S(f))$ (see [9], [20, Proposition 3.5], [23, Corollary 2.7]). Therefore we obtain

Corollary 1.4. *Let $f: M \to N^3$ be a special generic map of a closed connected $2n$-dimensional manifold ($n \geq 2$, $n \in N$) into an open orientable 3-dimensional manifold, and let $S(f)$ be the singular set of f. If $\pi_1(M)$ has infinite order and if the first L^2-Betti number $b_1^{(2)}(M)$ vanishes, then we have $\chi(M)=2|\{S \in \pi_0(S(f)); S \cong S^2\}|$.*

Sakuma [22] has proved an analogous equality in the case where the source manifold is simply connected.

The paper is organized as follows. In Section 2, we define special generic maps and their Stein factorizations, and study their basic properties. In Section 3, we define L^2-Betti numbers and prove some propositions concerning L^2-Betti numbers of compact smooth manifolds with boundary. In Section 4, we prove our main theorems by using the results in Sections 2 and 3. In Section 5, we state some facts concerning L^2-Betti numbers and give some applications of our main results.

In this paper, we assume that all manifolds are smooth unless otherwise stated.

2. Special generic maps and their Stein factorizations

Let $f: M \to N$ be a smooth map of an n-dimensional manifold into a p-dimensional manifold with $n \geq p$. Set $S(f) = \{q \in M;\ \operatorname{rank} df_q < p\}$, which is called the *singular set* of f. A point $q \in S(f)$ is called a *fold point* if there exist local coordinates $(x_1, \cdots, x_n)$ centered at q and $(y_1, \cdots, y_p)$ centered at $f(q)$ such that f has the form:

$$y_i \circ f = x_i \ (i \leq p-1),$$
$$y_p \circ f = -x_p^2 - \cdots - x_{p+\lambda-1}^2 + x_{p+\lambda}^2 + \cdots + x_n^2$$

for some λ $(0 \leq \lambda \leq n-p+1)$. If, in addition, $\lambda = 0$ or $\lambda = n-p+1$, we call q a *definite fold point*; otherwise, we call q an *indefinite fold point*. Finally, a smooth map $f: M \to N$ of an n-dimensional manifold into a p-dimensional manifold with $n \geq p$ is a *special generic map* if all the points in $S(f)$ are definite fold points. Note that for a special generic map, $S(f)$ is a $(p-1)$-dimensional submanifold of M and that $f \mid S(f)$ is a smooth immersion.

Next we define the Stein factorization of a special generic map $f: M \to N$, where M and N are n- and p-dimensional manifolds respectively $(n > p)$ (see [2], [19], [20], [21], [22], [23]). For $q, q' \in M$, define $q \sim q'$ if $f(q) = f(q')$ and q and q' belong to the same connected component of $f^{-1}(f(q))$. Denote by W_f the quotient space of M under this equivalence relation and by $q_f: M \to W_f$ the quotient map. Furthermore, we have the unique map $f': W_f \to N$ such that $f' \circ q_f = f$. The space W_f or the commutative diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "f'"'] & \end{tikzcd}$$

is called the *Stein factorization* of f.

We state propositions which were proved in [20] for the case $N=\boldsymbol{R}^p$ (see also [2], [19]). Note that the same proof works also in our general case.

Proposition 2.1. *Let $f: M \to N$ be a special generic map of a closed n-dimensional manifold into an open p-dimensional manifold with $n>p$ and $q_f: M \to W_f$ the quotient map in its Stein factorization. Then we have the following.*

(i) *W_f is a p-dimensional manifold with boundary.*
(ii) *∂W_f is diffeomorphic to $S(f)$.*
(iii) *$(q_f)_*: \pi_1(M) \to \pi_1(W_f)$ is an isomorphism.*

REMARK 2.2. In Proposition 2.1, if N is orientable, then so is W_f.

Proposition 2.3. *Let $f: M \to N$ be a special generic map of a closed n-dimensional manifold into an open p-dimensional manifold with $n>p$ and W_f the Stein factorization of f. Then there exists a topological D^{n-p+1}-bundle E over W_f such that ∂E is homeomorphic to M.*

We have the following relationship between the Euler characteristic of the source manifold and that of the Stein factorization.

Proposition 2.4. *Let $f: M \to N$ be a special generic map of a closed $2n$-dimensional manifold ($n \in \boldsymbol{N}$) into a p-dimensional manifold with $2n>p$. Then*

$$\chi(M)=2\chi(W_f).$$

Note that in the case where p is even this proposition has been proved in [20, Proposition 3.5].

Proof of Proposition 2.4. Let E be as in Proposition 2.3. Put $X=E \cup_{\partial E} E$. Since X is an odd-dimensional closed manifold, we have $2\chi(E)-\chi(\partial E)=\chi(X)=0$. Since ∂E is homeomorphic to M, $\chi(M)=\chi(\partial E)$. Since E is homotopy equivalent to W_f, $\chi(E)=\chi(W_f)$. Therefore $\chi(M)=2\chi(W_f)$. □

EXAMPLE 2.5 (see [21]). (1) Let $S^n \subset \boldsymbol{R}^{n+1}$ be the unit sphere. The restriction of the standard projection $\boldsymbol{R}^{n+1} \to \boldsymbol{R}^p$ ($n \geq p$) to S^n is a special generic map. If $n>p$, the Stein factorization of this special generic map is diffeomorphic to D^p.

(2) Let $g: S^n \to \boldsymbol{R}^p$ be the special generic map as above and $\eta: S^m \times \boldsymbol{R}^p \to \boldsymbol{R}^{m+p}$ an embedding. Then the composition

$$f: S^m \times S^n \xrightarrow{\mathrm{id} \times g} S^m \times \boldsymbol{R}^p \xrightarrow{\eta} \boldsymbol{R}^{m+p}$$

is a special generic map. If $n>p$, then W_f is diffeomorphic to $S^m \times D^p$.

In fact, we can show that if a closed n-dimensional manifold M^n admits a special generic map $g: M^n \to \boldsymbol{R}^p$ $(n \geq p)$, then $S^m \times M^n$ admits a special generic map f into $\boldsymbol{R}^{m+p}$ by an analogous construction. If $n > p$, then the Stein factorization of f is diffeomorphic to $S^m \times W_g$.

Now we consider covering spaces of source manifolds. Let $f: M \to N$ be a special generic map of a closed connected n-dimensional manifold into an open p-dimensional manifold with $n > p$. Let $\bar{\Gamma}$ be a normal subgroup of $\pi_1(M)$ such that the index $[\pi_1(M): \bar{\Gamma}]$ is finite, and let $\pi: \bar{M} \to M$ be the covering of M associated with $\bar{\Gamma} \subset \pi_1(M)$. Set $\bar{f} = f \circ \pi$. Evidently $\bar{f}: \bar{M} \to N$ is also a special generic map. Then by Proposition 2.1 (iii), we easily see the following.

Proposition 2.6. *Under the condition above, the Stein factorization $W_{\bar{f}}$ of $\bar{f}$ is diffeomorphic to the covering space of the Stein factorization W_f of f associated with $(q_f)_*(\bar{\Gamma}) \subset (q_f)_*(\pi_1(M)) = \pi_1(W_f)$.*

3. L^2-Betti numbers and manifolds with boundary

In this section we definte L^2-Betti numbers and study their properties.

Let Γ be a countable group and $l^2(\Gamma)$ the space of formal sums $\Sigma_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma$ with complex coefficients λ_γ satisfying $\Sigma_{\gamma \in \Gamma} |\lambda_\gamma|^2 < \infty$. $l^2(\Gamma)$ is a Hilbert space with the inner product given by

$$\langle \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma, \sum_{\gamma \in \Gamma} \mu_\gamma \cdot \gamma \rangle = \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \overline{\mu_\gamma}$$

for $\Sigma_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma, \Sigma_{\gamma \in \Gamma} \mu_\gamma \cdot \gamma \in l^2(\Gamma)$. The *von Neumann algebra* $\mathcal{N}(\Gamma)$ of Γ is the algebra of bounded operators from $l^2(\Gamma)$ to $l^2(\Gamma)$ which commute with the right Γ-action on $l^2(\Gamma)$. A *finitely generated Hilbert $\mathcal{N}(\Gamma)$-module P* is a Hilbert space together with a continuous left $\mathcal{N}(\Gamma)$-module structure such that there exists an isometric $\mathcal{N}(\Gamma)$-module embedding into $\oplus_{i=1}^r l^2(\Gamma)$ for some $r \in \boldsymbol{N}$. A map $f: U \to V$ between finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules always means a bounded operator from U to V which commutes with multiplication by $\mathcal{N}(\Gamma)$.

Let $\mathcal{N}'(\Gamma)$ denote the algebra of $\mathcal{N}(\Gamma)$-module maps from $l^2(\Gamma)$ to $l^2(\Gamma)$. The *von Neumann trace* $\mathrm{tr}(f)$ of an element $f \in \mathcal{N}'(\Gamma)$ is the complex number $\langle f(e), e \rangle$, where $e \in \Gamma$ is the unit element. For an $\mathcal{N}(\Gamma)$-module map $f: \oplus_{i=1}^n l^2(\Gamma) \to \oplus_{i=1}^n l^2(\Gamma)$, we consider f to be an $(n \times n)$-matrix $(f_{i,j})$ over $\mathcal{N}'(\Gamma)$ and define $\mathrm{tr}(f) = \Sigma_{i=1}^n \mathrm{tr}(f_{i,i})$. For a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module P, Let pr $: \oplus_{i=1}^n l^2(\Gamma) \to \oplus_{i=1}^n l^2(\Gamma)$ be a projection whose image is isometrically $\mathcal{N}(\Gamma)$-isomorphic to P. The *von Neumann dimension* of P is defined by $\dim_{\mathcal{N}(\Gamma)}(P) = \mathrm{tr}(\mathrm{pr})$. Note that $\dim_{\mathcal{N}(\Gamma)}(P)$ is a well-defined nonnegative real number (see [3], [7]). A sequence $0 \to U \xrightarrow{j} V \xrightarrow{q} W \to 0$ of finitely generated Hilbert

$\mathcal{N}(\Gamma)$-modules is *weakly exact* if j is injective, $\overline{\mathrm{im}(j)} = \ker(q)$ and $\overline{\mathrm{im}(q)} = W$, where the bar means the closure.

Lemma 3.1 ([5], [13]). *For finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules U, V and W, we have the following.*

(i) $\dim_{\mathcal{N}(\Gamma)}(U) = 0$ *if and only if* $U = 0$.

(ii) *If* $U \subset V$, *then* $\dim_{\mathcal{N}(\Gamma)}(U) \leq \dim_{\mathcal{N}(\Gamma)}(V)$.

(iii) *If* $0 \to U \xrightarrow{j} V \xrightarrow{q} W \to 0$ *is weakly exact, then* $\dim_{\mathcal{N}(\Gamma)}(V) = \dim_{\mathcal{N}(\Gamma)}(U) + \dim_{\mathcal{N}(\Gamma)}(W)$.

For more information about finitely generated Hilbert $\mathcal{N}(\Gamma)$-modules, we refer to [7], [13], [14] and [18].

Let X be a finite connected CW-complex and $A \subset X$ a CW-subcomplex. Let $\pi: \tilde{X} \to X$ be the universal covering and put $\tilde{A} = \pi^{-1}(A)$. We adopt the convention that $\pi_1(X)$ acts from the left on the universal covering and on its cellular chain complex. Consider a group homomorphism $\phi: \pi_1(X) \to \Gamma$. Let $C_*(X, A; \phi^* l^2(\Gamma))$ denote the finitely generated Hilbert $\mathcal{N}(\Gamma)$-chain complex $l^2(\Gamma) \otimes_{\mathbf{Z}\pi_1(X)} C_*(\tilde{X}, \tilde{A})$, where the right $\pi_1(X)$-action on $l^2(\Gamma)$ is induced by $\phi: \pi_1(X) \to \Gamma$. Let $c_*^{(2)}$ denote the differentials. The *p-th L^2-homology* of X with coefficients in $\phi^* l^2(\Gamma)$ is defined by

$$H_p(X, A; \phi^* l^2(\Gamma)) = \ker(c_p^{(2)}) \big/ \overline{\mathrm{im}(c_{p+1}^{(2)})}.$$

Since we take the quotient by the closure of the image, this is again a finitely generated Hilbert $\mathcal{N}(\Gamma)$-module. Define the *p-th L^2-Betti number* of X with coefficients in $\phi^* l^2(\Gamma)$ by

$$b_p(X, A; \phi^* l^2(\Gamma)) = \dim_{\mathcal{N}(\Gamma)} H_p(X, A; \phi^* l^2(\Gamma)).$$

Note that if Γ is the trivial group, then the L^2-Betti number is equal to the ordinary Betti number $\dim_{\mathbf{Q}} H_p(X, A; \mathbf{Q})$. In the case where $\Gamma = \pi_1(X)$ and $\phi = \mathrm{id}$, we write $l^2(\Gamma)$ instead of $\mathrm{id}^* l^2(\Gamma)$ and put

$$b_p^{(2)}(X, A) = b_p(X, A; l^2(\Gamma)).$$

It is known that L^2-Betti numbers are homotopy invariants. The following proposition is proved in [13], [15] (see also [3], [4], [5], [7], [14]).

Proposition 3.2. (i) *$b_0^{(2)}(X)$ and $b_1^{(2)}(X)$ depend only on the fundamental group $\pi_1(X)$.*

(ii) *Let $\bar{p}: \bar{X} \to X$ be an n-sheeted finite covering and $\bar{A} = \bar{p}^{-1}(A)$. Then*

$$b_p^{(2)}(\bar{X}, \bar{A}) = n \cdot b_p^{(2)}(X, A).$$

(iii) *If the image of* $\phi:\pi_1(X)\to\Gamma$ *is finite, then*

$$b_0(X;\phi^*l^2(\Gamma))=\frac{1}{|\mathrm{im}(\phi)|}.$$

Otherwise

$$b_0(X;\phi^*l^2(\Gamma))=0.$$

(iv) *Denoting the Euler characteristic of* X *by* $\chi(X)$, *we have*

$$\chi(X)=\sum_{p\geq 0}(-1)^p\cdot b_p(X;\phi^*l^2(\Gamma)).$$

(v) *Let* M *be a compact connected orientable triangulable* (*topological*) *manifold of dimension* n *and* $\phi:\pi_1(M)\to\Gamma$ *a group homomorphism. Then*

$$b_p(M,\partial M;\phi^*l^2(\Gamma))=b_{n-p}(M;\phi^*l^2(\Gamma)),$$

where ∂M *is the boundary of* M.

REMARK 3.3. For a compact connected non-orientable triangulable (topological) manifold M of dimension n, we have

$$b_p^{(2)}(M,\partial M)=b_{n-p}^{(2)}(M)$$

by Proposition 3.2 (ii) and (v). However this equality is not true in general coefficients. For example, when $M=\boldsymbol{R}P^2$ and Γ is the trivial group, $b_0(M;\phi^*l^2(\Gamma))=1$ and $b_2(M;\phi^*l^2(\Gamma))=0$.

Now we consider applications of L^2-Betti numbers to manifolds with boundary.

Proposition 3.4. *Let* M *be a compact connected p-dimensional manifold with nonempty boundary* ∂M *and* $\partial M=S_1\cup\cdots\cup S_k$ *the decomposition of* ∂M *into the connected components. If* $\pi_1(M)$ *has infinite order, then*

$$\sum_{i=1}^{k}\frac{1}{|\pi_1(S_i)|}\leq b_{p-1}^{(2)}(M).$$

Proof. Let π denote the fundamental group of M and $\tilde{p}:\tilde{M}\to M$ the universal covering. Set $\tilde{S}_i=\tilde{p}^{-1}(S_i)$ $(i=1,2,\cdots,k)$ and $\partial\tilde{M}=\tilde{p}^{-1}(\partial M)$. Under the condition in Proposition 3.4, we have the exact sequence of chain complex

$$0 \to l^2(\pi)\otimes_{Z\pi}\left(\bigoplus_{i=1}^{k} C_*(\tilde{S}_i;Z)\right) \to l^2(\pi)\otimes_{Z\pi}C_*(\tilde{M};Z) \to l^2(\pi)\otimes_{Z\pi}C_*(\tilde{M},\partial\tilde{M};Z) \to 0.$$

Let $(j_i)_*:\pi_1(S_i)\to\pi$ be the homomorphism induced by the inclusion $j_i:S_i\to M$. We see easily that $l^2(\pi)\otimes_{Z\pi}(\oplus_{i=1}^{k}C_*(\tilde{S}_i;Z))$ is naturally isomorphic to $\oplus_{i=1}^{k}C_*(S_i;((j_i)_*)^*l^2(\pi))$. Therefore we have the exact sequence of chain complex

$$0\to \bigoplus_{i=1}^{k} C_*(S_i;((j_i)_*)^*l^2(\pi)) \to C_*(M;l^2(\pi)) \to C_*(M,\partial M;l^2(\pi)) \to 0.$$

Therefore we have the following.

Lemma 3.5 ([4], [5], [14]). *Under the condition above, we have the weakly exact sequence*

$$\begin{aligned}\cdots &\to \bigoplus_{i=1}^{k} H_q(S_i;((j_i)_*)^*l^2(\pi)) \to H_q(M;l^2(\pi)) \to H_q(M,\partial M;l^2(\pi))\\ &\to \bigoplus_{i=1}^{k} H_{q-1}(S_i;((j_i)_*)^*l^2(\pi)) \to H_{q-1}(M;l^2(\pi)) \to H_{q-1}(M,\partial M;l^2(\pi))\\ &\to \cdots\\ &\to \bigoplus_{i=1}^{k} H_0(S_i;((j_i)_*)^*l^2(\pi)) \to H_0(M;l^2(\pi)) \to H_0(M,\partial M;l^2(\pi)) \to 0.\end{aligned}$$

By Lemmas 3.5 and 3.1, we have

$$\sum_{i=1}^{k} b_0(S_i;((j_i)_*)^*l^2(\pi)) \le b_1^{(2)}(M,\partial M)+b_0^{(2)}(M).$$

By our assumption that $|\pi_1(M)|=\infty$ together with Proposition 3.2 (iii), we have $b_0^{(2)}(M)=0$. By Proposition 3.2 (v) (see also Remark 3.3), we have $b_1^{(2)}(M,\partial M)=b_{p-1}^{(2)}(M)$. Since

$$b_0(S_i;((j_i)_*)^*l^2(\pi_1(M)))=\frac{1}{|(j_i)_*(\pi_1(S_i))|}\ge\frac{1}{|\pi_1(S_i)|}$$

by Proposition 3.2 (iii), we have $\Sigma_{i=1}^{k}(1/|\pi_1(S_i)|)\le b_{p-1}^{(2)}(M)$. □

The following proposition is essentially due to [13].

Proposition 3.6. *Let M be a compact connected orientable 3-dimensional manifold with nonempty boundary. Suppose that $\pi_1(M)$ has infinite order and that the first L^2-Betti number $b_1^{(2)}(M)$ vanishes. Then each connected component of ∂M is diffeomorphic to either the 2-sphere or the torus. Furthermore,*

$$|\{S \in \pi_0(\partial M); S \cong S^2\}| = b_2^{(2)}(M).$$

Proof. Let $M = M_1 \# \cdots \# M_r$ be the prime decomposition. As is seen in the proof of [13, Proposition 6.5], if $|\pi_1(M)| = \infty$ and $b_1^{(2)}(M) = 0$, then the prime decomposition of M must consist of homotopy 3-spheres, 3-disks and either

A. A prime manifold M' with infinite fundamental group and vanishing $b_1^{(2)}(M')$

or

B. Two prime manifolds M^1 and M^2 with fundamental groups isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z}$.

In case A, M' is $S^1 \times S^2$ or is irreducible. If M' is irreducible and has nonempty boundary, then [13, Lemma 6.4] implies that its boundary components are tori. In case B, we easily see $\partial M_i = \emptyset$ $(i = 1,2)$. Therefore each component of ∂M is diffeomorphic to either the 2-sphere or the torus. By Proposition 3.2 (iv), we have $\chi(M) = \Sigma_{p \geq 0}(-1)^p \cdot b_p^{(2)}(M)$. By our assumption, $b_q^{(2)}(M) = 0$ for $q \neq 2$. Therefore

$$b_2^{(2)}(M) = \chi(M) = \frac{1}{2}\chi(\partial M).$$

Since each connected component of ∂M is diffeomorphic to either the 2-sphere or the torus,

$$\chi(\partial M) = 2|\{S \in \pi_0(\partial M); S \cong S^2\}|.$$

This completes the proof. □

REMARK 3.7. If the fundamental group of a compact connected orientable 3-dimensional manifold M with nonempty boundary is finite, then each connected component of the boundary is diffeomorphic to S^2. However the number of connected components is not equal to the second L^2-Betti number in general. For example, the second L^2-Betti number of the 3-disk is zero.

For the case where M is non-orientable, we have the following.

Proposition 3.8. *Let M be a compact connected non-orientable 3-dimensional manifold with nonempty boundary. Suppose that $\pi_1(M)$ has infinite order and that the first L^2-Betti number $b_1^{(2)}(M)$ vanishes. Then each connected component of ∂M is diffeomorphic to the 2-sphere, the torus, the real projective plane $\boldsymbol{RP}^2$ or the Klein bottle. Furthermore,*

$$|\{S \in \pi_0(\partial M); S \cong S^2\}| + \frac{1}{2}|\{S \in \pi_0(\partial M); S \cong \boldsymbol{RP}^2\}| = b_2^{(2)}(M).$$

Proof. Let $p: \tilde{M} \to M$ be the orientable double covering of M. By Proposition

3.2 (ii), the first L^2-Betti number of $\bar{M}$ is also zero. Therefore each connected component of $\partial\bar{M}$ is diffeomorphic to either the 2-sphere or the torus by Proposition 3.6. Hence each connected component of ∂M is diffeomorphic to S^2, T^2, $\boldsymbol{RP}^2$ or the Klein bottle.

If S is a connected component of ∂M which is diffeomorphic to S^2, then $p^{-1}(S)$ is diffeomorphic to the disjoint union of two 2-spheres. If S' is a connected component of ∂M which is diffeomorphic to $\boldsymbol{RP}^2$, then $p^{-1}(S')$ is diffeomorphic to S^2. If S'' is a connected component of ∂M which is diffeomorphic to neither S^2 nor $\boldsymbol{RP}^2$, then each connected component of $p^{-1}(S'')$ is diffeomorphic to the torus.

By Proposition 3.6, $|\{S\in\pi_0(\partial\bar{M});S\cong S^2\}|=b_2^{(2)}(\bar{M})$. Hence

$$2|\{S\in\pi_0(\partial M);S\cong S^2\}|+|\{S\in\pi_0(\partial M);S\cong \boldsymbol{RP}^2\}|=b_2^{(2)}(\bar{M}).$$

By Proposition 3.2 (ii), we have $b_2^{(2)}(\bar{M})=2b_2^{(2)}(M)$, and hence the required equality follows. □

4. Proof of main theorems

Proposition 4.1. *Let $f:M\to N$ be a special generic map of a closed connected n-dimensional manifold into an open p-dimensional manifold with $n>p$. Denote by W_f the Stein factorization of f. Then for $q\le n-p$, we have*

$$b_q^{(2)}(M)=b_q^{(2)}(W_f).$$

Proof. By Proposition 2.3, there exists a D^{n-p+1}-bundle E over W_f such that ∂E is homeomorphic to M. The composition of the inclusion $i:\partial E\to E$ and the projection $\pi:E\to W_f$ is homotopic to the composition of the homeomorphism from ∂E to M and $q_f:M\to W_f$. Since both $(q_f)_*:\pi_1(M)\to\pi_1(W_f)$ and $\pi_*:\pi_1(E)\to\pi_1(W_f)$ are isomorphisms, $i_*:\pi_1(\partial E)\to\pi_1(E)$ is also an isomorphism. Therefore by the exact sequence

$$0\to C_*(\partial E;l^2(\pi_1(M)))\to C_*(E;l^2(\pi_1(M)))\to C_*(E,\partial E;l^2(\pi_1(M)))\to 0,$$

we have the weakly exact sequence

$$\begin{aligned}&\cdots\to H_{q+1}(E,\partial E;l^2(\pi_1(M)))\\&\to H_q(\partial E;l^2(\pi_1(M)))\to H_q(E;l^2(\pi_1(M)))\to H_q(E,\partial E;l^2(\pi_1(M)))\to\cdots.\end{aligned}$$

As is seen in the proof of [20, Proposition 3.1], E is the associated D^{n-p+1}-bundle of a smooth S^{n-p}-bundle over W_f. Hence we easily see that E is triangulable. By Proposition 3.2 (v) and Remark 3.3, $b_{q+1}^{(2)}(E,\partial E)=b_{n-q}^{(2)}(E)$

$=b_{n-q}^{(2)}(W_f)=0$ for $q\leq n-p$, since E is homotopy equivalent to W_f. Similarly, we have $b_q^{(2)}(E,\partial E)=0$. Therefore by the exact sequence above, we have $b_q^{(2)}(\partial E)=b_q^{(2)}(E)=b_q^{(2)}(W_f)$ for $q\leq n-p$. Since ∂E is homeomorphic to M, we have $b_q^{(2)}(\partial E)=b_q^{(2)}(M)$. Hence $b_q^{(2)}(M)=b_q^{(2)}(W_f)$ for $q\leq n-p$. □

Proof of Theorem 1.1. Let M be a closed connected n-dimensional manifold such that $b_q^{(2)}(M)\neq 0$ for some $q\leq n/2$. If M admits a special generic map into an open p-dimensional manifold ($p\leq q$), then $b_r^{(2)}(W_f)=0$ for all $r\geq p$. By Proposition 4.1, we have $b_r^{(2)}(M)=0$ for all r with $p\leq r\leq n-p$. Since $p\leq q\leq n-q\leq n-p$, this is a contradiction. □

Let $f:M\to N$ be a special generic map of a closed connected n-dimensional manifold into an open p-dimensional manifold with $n>p$. Let W_f be the Stein factorization of f and $q_f:M\to W_f$ the quotient map.

Lemma 4.2. *If $\pi_1(M)$ is residually finite, then there exists a nested sequence of normal subgroups of $\pi_1(M)$, $\cdots\subset\Gamma_{m+1}\subset\Gamma_m\subset\cdots\subset\Gamma_1\subset\Gamma_0=\pi_1(M)$, such that the following conditions are satisfied:*

(i) *The index $[\pi_1(M):\Gamma_m]$ is finite for all $m\geq 0$.*

(ii) *The intersection $\bigcap_{m\geq 0}\Gamma_m$ is the trivial group.*

(iii) *The covering space M_1 of M associated with Γ_1 and the covering space W_1 of W_f associated with $(q_f)_*(\Gamma_1)$ are orientable.*

Proof. Since $(q_f)_*$ is an isomorphism, there exists a normal subgroup Γ_1 of finite index satisfying the condition (iii).

Since $\pi_1(M)-\{e\}$ (e is the unit element) is countable, there exists a bijection $g:N\to\pi_1(M)-\{e\}$. By the residual finiteness of $\pi_1(M)$, there exist a finite group G_i and a homomorphism $\phi_i:\pi_1(M)\to G_i$ such that $\phi_i(g(i))$ is not the unit element for each i.

Put $\Gamma_m=\Gamma_1\cap(\bigcap_{i=1}^{m-1}\ker\phi_i)$ ($m\geq 2$). Then the sequence $\cdots\subset\Gamma_{m+1}\subset\Gamma_m\subset\cdots\subset\Gamma_1\subset\Gamma_0=\pi_1(M)$ satisfies the conditions (i), (ii) and (iii). □

Consider any sequence $(\Gamma_m)_{m\geq 0}$ satisfying the conditions in Lemma 4.2. Let $p_m:M_m\to M$ be the covering of M associated with $\Gamma_m\subset\pi_1(M)$. Suppose that $f:M\to N$ is a special generic map into an open p-dimensional manifold, and put $f_m=f\circ p_m$. Then we have the following.

Lemma 4.3. *Suppose the conditions above and that $|\pi_1(M)|=\infty$. Denote by $S(f_m)$ the singular set of f_m and by $\sharp S(f_m)$ the number of connected components of $S(f_m)$. Then we have*

$$\lim_{m\to\infty}\frac{\#S(f_m)}{[\pi_1(M):\Gamma_m]}\leq b^{(2)}_{p-1}(M).$$

Note that the left hand side of the inequality is always convergent, since the sequence $(\#S(f_m)/[\pi_1(M):\Gamma_m])_{m\geq 0}$ is monotone decreasing and each term is positive.

Proof of Lemma 4.3. By Proposition 2.6, the Stein factorization of f_m is diffeomorphic to the covering space W_m of W_f associated with $(q_f)_*(\Gamma_m)$, and hence it is orientable for $m\geq 1$. Since M_m and W_m are orientable for $m\geq 1$, $\#S(f_m)\leq b_{p-1}(M_m)+1$ for $m\geq 1$ by [20, Proposition 3.15], where $b_{p-1}(M_m)=\dim_Q H_{p-1}(M_m;Q)$. Hence $\#S(f_m)/[\pi_1(M):\Gamma_m]\leq(b_{p-1}(M_m)+1)/[\pi_1(M):\Gamma_m]$.

By [16, Theorem 0.1], we have

$$\lim_{m\to\infty}\frac{b_{p-1}(M_m)}{[\pi_1(M):\Gamma_m]}=b^{(2)}_{p-1}(M).$$

Since $|\pi_1(M)|=\infty$,

$$\lim_{m\to\infty}\frac{\#S(f_m)}{[\pi_1(M):\Gamma_m]}\leq\lim_{m\to\infty}\frac{b_{p-1}(M_m)+1}{[\pi_1(M):\Gamma_m]}=b^{(2)}_{p-1}(M). \qquad \square$$

REMARK 4.4. Hempel has shown that the fundamental group of a compact 3-dimensional manifold whose prime decomposition consists of non-exceptional manifolds (i.e. manifolds finitely covered by a manifold which is homotopy equivalent to a Haken, Seifert or hyperbolic manifold) is residually finite (see [11]). As another example of residually finite groups, it is known that every finitely generated group possessing a faithful representation into $GL(n,F)$ for a field F is residually finite. For more information about residually finite groups, we refer to [10, Chapter 15] and [17].

Proof of Theorem 1.2. Let $f:M\to N$ be a special generic map of a closed connected n-dimensional manifold into an open p-dimensional manifold with $p<n$ and $S(f)=S_1\cup\cdots\cup S_k$ the decomposition of $S(f)$ into the connected components.

First we consider the case where $\pi_1(M)$ is residually finite with infinite order. Let $(\Gamma_m)_{m\in N}$, $(p_m)_{m\in N}$ and $(f_m)_{m\in N}$ be as in Lemmas 4.2 and 4.3. Let $|\pi_0(p_m^{-1}(S_i))|$ denote the number of elements of $\pi_0(p_m^{-1}(S_i))$. Clearly,

$$\#S(f_m)=\sum_{i=1}^{k}|\pi_0(p_m^{-1}(S_i))|\geq\sum_{i=1}^{k}\frac{[\pi_1(M):\Gamma_m]}{|\pi_1(S_i)|}.$$

Hence $\#S(f_m)/[\pi_1(M):\Gamma_m]\geq\Sigma_{i=1}^{k}1/|\pi_1(S_i)|$ for $m\geq 0$. Therefore, by Lemma 4.3, we have $b^{(2)}_{p-1}(M)\geq\Sigma_{i=1}^{k}1/|\pi_1(S_i)|$.

If $p\leq(n+1)/2$ and $\pi_1(M)$ has infinite order, then the result follows from

Propositions 2.1 (ii), 3.4 and 4.1.

If $p=(n+2)/2$ and $\pi_1(M)$ has infinite order, then by Propositions 2.4 and 3.2 (v) and Remark 3.3, we have

$$2\left(\sum_{q=1}^{n/2-1}(-1)^q\cdot b_q^{(2)}(W_f)+(-1)^{n/2}\cdot b_{n/2}^{(2)}(W_f)\right)=2\sum_{q=1}^{n/2-1}(-1)^q\cdot b_q^{(2)}(M)+(-1)^{n/2}\cdot b_{n/2}^{(2)}(M).$$

By Proposition 4.1, $b_q^{(2)}(W_f)=b_q^{(2)}(M)$ for $q\leq n/2-1$. Therefore $2b_{n/2}^{(2)}(W_f)=b_{n/2}^{(2)}(M)$. Using this equality and Propositions 2.1 (ii) and 3.4, we obtain the required result. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Using Proposition 4.1, we obtain $b_q^{(2)}(M)=b_q^{(2)}(W_f)$ for $q=0,1$. Therefore by our assumption, $b_q^{(2)}(W_f)=0$ for $q=0,1$. By Proposition 3.6, each connected component of ∂W_f is diffeomorphic to either the 2-sphere or the torus, and $|\{S\in\pi_0(\partial W_f);S\cong S^2\}|=b_2^{(2)}(W_f)$. By Proposition 2.1 (ii), $S(f)$ is diffeomorphic to ∂W_f, and thus each connected component of $S(f)$ is diffeomorphic to either the 2-sphere or the torus.

Furthermore, if $n\geq 5, b_2^{(2)}(M)=b_2^{(2)}(W_f)$ by Proposition 4.1. Hence $|\{S\in\pi_0(S(f));S\cong S^2\}|=b_2^{(2)}(M)$.

If $n=4$, then $\chi(M)=2\chi(W_f)$ by Proposition 2.4. By our assumption and Poincaré duality, $b_q^{(2)}(M)=0$ for $q\neq 2$. Therefore by using Proposition 3.2 (iv), we obtain $b_2^{(2)}(M) = \chi(M) = 2\chi(W_f) = 2b_2^{(2)}(W_f)$. Hence $|\{S\in\pi_0(\partial W_f);S\cong S^2\}|=b_2^{(2)}(W_f)=(1/2)b_2^{(2)}(M)$. This completes the proof of Theorem 1.3. □

Analogously, we have the following by Propositions 3.8 and 4.1.

Proposition 4.5. *Let $f:M\to N^3$ be a special generic map of a closed connected n-manifold ($n>3$) into an open non-orientable 3-manifold. If $\pi_1(M)$ has infinite order and if the first L^2-Betti number $b_1^{(2)}(M)$ of M vanishes, then each connected component of the singular set $S(f)$ of f is diffeomorphic to the 2-sphere, the torus, the real projective plane $\boldsymbol{RP}^2$ or the Klein bottle. Furthermore, we have*

$$|\{S\in\pi_0(S(f));S\cong S^2\}|+\frac{1}{2}|\{S\in\pi_0(S(f));S\cong \boldsymbol{RP}^2\}|=\begin{cases} b_2^{(2)}(M) & \text{if } n\geq 5,\\[2ex] \dfrac{1}{2}b_2^{(2)}(M) & \text{if } n=4.\end{cases}$$

5. Applications

In this section, we give some applications of the theorems which have been proved in §4. The following theorem is an immediate consequence of [8, p. 176].

Theorem 5.1 (Donnelly and Xavier). *Let M be a closed connected Riemannian* anifold of dimension 2n($n \in N$) *with sectional curvatures K pinched by*

$$-1 \leq K \leq -c^2 < -\left(\frac{2n-2}{2n-1}\right)^2.$$

Then the L^2-Betti number $b_p^{(2)}(M)$ of M vanishes for $p \neq n$, and $b_n^{(2)}(M) \neq 0$.

The following corollary is an immediate consequence of the theorem above and Theorem 1.1.

Corollary 5.2. *Suppose that M is a closed connected Riemannian manifold of dimension $2n$ with sectional curvatures K pinched by*

$$-1 \leq K \leq -c^2 < -\left(\frac{2n-2}{2n-1}\right)^2.$$

Then for any open p-dimensional manifold N with $p \leq n$, M does not admit a special generic map into N.

By this corollary, we see that no closed hyperbolic manifold of dimension $2n$ admits a special generic map into an open p-dimensional manifold N with $p \leq n$.

Lück showed the following theorem.

Theorem 5.3 (Lück [15]). *If $F \to E \to S^1$ is a fibration of connected finite CW-complexes, then all the L^2-Betti numbers of E vanish.*

By Theorems 1.2 and 5.3, we have the following.

Corollary 5.4. *Let M be the connected sum $M_1 \# \cdots \# M_r$ of closed connected n-dimensional manifolds such that M_i fibers over S^1 for all i. Suppose that M admits a special generic map into an open p-dimensional manifold with $p < n$. If either* (i) *$p \leq (n+2)/2$ or* (ii) *$\pi_1(M)$ is residually finite, then the fundamental group of each connected component of the singular set has infinite order.*

Proof. In the case where $p=2$, each connected component of the singular set is diffeomorphic to S^1 and hence we have the consequence.

In the case where $p \geq 3$, by Theorem 5.3 and [13, Proposition 3.7], $b_i^{(2)}(M)=0$ for all i with $2 \leq i \leq n-2$. Therefore Theorem 1.2 implies the consequence. □

REMARK 5.5. (i) For a fibration $F \to E \to S^1$, $\pi_1(E)$ is residually finite if and only if $\pi_1(F)$ is residually finite (see [10, p. 180]).

(ii) It is known that the free product of residually finite groups is residually finite (see [6, p. 27]). Therefore in Corollary 5.4, if $n \geq 3$ and if $\pi_1(M_i)$ is residually finite for all i, then $\pi_1(M)$ is residually finite.

EXAMPLE 5.6. Set $M = S^1 \times S^5 \# S^1 \times S^2 \times S^3$. By Example 2.5 and [20, Lemma 5.4] M admits a special generic map $f: M \to \boldsymbol{R}^4$. The ordinary third Betti number $b_3(M)$ of M is equal to 2. By [20, Proposition 3.15], $\# S(f) \leq (1/2) b_3(M) + 1 = 2$, where $\# S(f)$ is the number of connected components of $S(f)$. By Corollary 5.4, the fundamental group of each connected component of $S(f)$ has infinite order.

In the case of special generic maps into open orientable 3-dimensional manifolds, by Theorems 1.3 and 5.3, we have the following.

Corollary 5.7. *Let M be a closed connected n-dimensional manifold ($n > 3$) which fibers over S^1. If M admits a special generic map into an open orientable 3-dimensional manifold, then the singular set is diffeomorphic to a union of tori. Furthermore if the ordinary first Betti number of the fiber is zero, then the singular set is diffeomorphic to the torus.*

In the case where $n = 4$, if the ordinary first Betti number of the fiber is zero and if M is orientable, this corollary follows from [23, Corollary 4.4].

Proof of Corollary 5.7. If M admits a special generic map f into an open orientable 3-dimensional manifold, the singular set is diffeomorphic to a union of tori by Theorems 1.3 and 5.3.

If the ordinary first Betti number of the fiber is zero, then the ordinary first Betti number $b_1(M)$ of M is equal to 1. Let W_f be the Stein factorization of f. Since ∂W_f is diffeomorphic to $S(f)$, we have $\chi(S(f)) = \chi(\partial W_f) = 2\chi(W_f)$. By Proposition 2.1 (iii), we have $b_1(W_f) = b_1(M) = 1$. Hence $\chi(W_f) = b_0(W_f) - b_1(W_f) + b_2(W_f) = 1 - 1 + b_2(W_f) = b_2(W_f)$, and consequently $b_2(W_f) = (1/2)\chi(S(f))$. Since $S(f)$ is a union of tori, $b_2(W_f) = (1/2)\chi(S(f)) = 0$. By using the homology exact sequence for $(W_f, \partial W_f)$ and Poincaré duality, we see easily that $|\pi_0(\partial W_f)| = b_0(\partial W_f) \leq b_2(W_f) + 1$. Therefore $\# S(f) \leq b_2(W_f) + 1 = 1$. Thus $S(f)$ is diffeomorphic to the torus. $\square$

In the case of special generic maps into open non-orientable 3-dimensional manifolds, we have the following.

Corollary 5.8. *Let M be a closed connected n-dimensional manifold ($n > 3$) which fibers over S^1. If M admits a special generic map into an open non-orientable 3-dimensional manifold, then the singular set is diffeomorphic to a union of tori and Klein bottles. Furthermore if the first homology group $H_1(F; \boldsymbol{Z}/2\boldsymbol{Z})$ of the fiber F*

with coefficient in $Z/2Z$ is trivial, then the singular set is diffeomorphic to either the torus or the Klein bottle.

Proof. If M admits a special generic map f into an open non-orientable 3-dimensional manifold, the singular set is diffeomorphic to a union of tori and Klein bottles by Theorem 5.3 and Proposition 4.5.

In the case where $H_1(F;Z/2Z)=0$, we can obtain the result in a way simillar to that in the proof of Corollary 5.7 by using $\dim_{Z/2Z}H_i(W_f;Z/2Z)$ instead of the ordinary Betti numbers $b_i(W_f)$. □

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