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THE ALPERIN ARGUMENT REVISITED

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1. Introduction

It was observed by J.L. Alperin ([1]) that the Glauberman Correspondence (in the case a p-group acts on a p'-group) was a consequence of the Brauer First Main Theorem. Namely, if a p-group P acts on a p'-group G and χ is an irreducible P-invariant character of G, then χ uniquely extends to some Brauer character φ of $\Gamma = GP$. This Brauer character lies in a p-block B of defect P and, in fact, φ is the only modular character in B. If b is the p-block of $N_\Gamma(P) = \mathbf{C}_G(P) \times P$ with $b^G = B$, then b contains a unique Brauer character φ^* and $\chi^* = \varphi^*_{\mathbf{C}_G(P)}$ is irreducible. This is the Glauberman Correspondence via the Alperin Argument. (Perhaps, this is a good place to stress that, although the p-group case is certainly important, the Glauberman Correspondence is defined for general P solvable).

Later, H. Nagao extended the Glauberman Correspondence (also in the p-group case) to a noncoprime situation. If G is a normal subgroup of Γ with p-power index and χ is a Γ-invariant p-defect zero character of G, then χ is naturally associated with an irreducible p-defect zero character of $\mathbf{C}_G(P)$, where now P is some p-subgroup of Γ complementing G. Notice that Nagao's map is, again, another application of the Alperin Argument: since χ is a Brauer character and Γ/G is a p-group, then there is a unique Brauer character φ of Γ over χ (Green's Theorem); the block B of φ has a unique modular character (because B covers the block $\{\chi\}$), and the defect group P of B complements G in Γ (Fong's Theorem). Now the p-block b of $N_\Gamma(P) = \mathbf{C}_G(P) \times P$ with $b^G = B$, has a unique modular character φ^* with $\chi^* = \varphi^*_{\mathbf{C}_G(P)}$ irreducible. (The Nagao correspondence in the non p-group case was constructed in [11].)

As we see, there is an essential idea above: find blocks B with only one Brauer character and prove that Brauer First Main correspondents b satisfy the same property. If this is the case, the existence of a natural map, has been shown. In the Glauberman-Nagao conditions, this is not a problem: since $N_\Gamma(P) = P\mathbf{C}_G(P)$, it follows that every block of defect P of $N_\Gamma(P)$ has a unique modular character. To prove this fact in general, however, seems deep and it is a consequence

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of the Alperin *Weight Conjecture*. (For nilpotent blocks, this is known to be true and we suggest the reader the articles of Puig and Watanabe [14],[15],[17].)

From this point of view, we see that the existence of a general natural map from groups to local subgroups is a consequence of the Alperin Weight Conjecture together with the Alperin Argument.

We find interesting to explicitly construct this map in the cases where the Alperin conjecture is known to be true and this is, for p-solvable groups, what we do in this paper.

After some preparatory work, in Section 5 below, we will characterize the Brauer characters which lie in blocks with only one modular character. It is our belief that blocks with one Brauer character should have an strong relationship with fully ramified characters (at the very least, it is a consequence of the Alperin Weight Conjecture that the canonical character of such a block is fully ramified in its inertia group). This is proven to be true for p-solvable groups: these Brauer characters are exactly those which are induced from some character $\gamma \in \mathrm{IBr}(J)$ such that, γ_H is fully ramified with respect to N (H is a p-complement of J and N is its p'-radical). (The most relevant cases of blocks B with only one modular character will appear, in this context, quite naturally: if $H=N$ then, B is nilpotent [12]; if $J=H$ then B has defect zero.)

In Section 6, we will give an explicit construction of our map in terms of the Glauberman Correspondence. Therefore, for p-solvable groups, Theorem (6.2) below can be seen as a common generalization of the Glauberman and Nagaos maps. Some new results (which show, once again, how much Glauberman correspondents are related) will be needed.

2. Preliminaries

To obtain greater generality, we will make use of the Isaacs π-partial characters and work in π-separable groups. Recall (for those readers not interested in π-characters) that when $\pi=p'$, the set of Isaacs π-partial characters, $\mathrm{I}_\pi(G)$, is just $\mathrm{IBr}(G)$, the set of irreducible Brauer characters. We will also use π-blocks (as defined by M. Slattery [16]).

The notation is taken from [4] and [5].

We begin with some easy lemmas on partial characters that will be needed later.

If J is a subgroup of G and $\mu \in \mathrm{I}_\pi(J)$, as usual, we will denote by $\mathrm{I}_\pi(G \mid \mu)$ the set of $\varphi \in \mathrm{I}_\pi(G)$ which have μ as an irreducible constituent of φ_J.

Lemma 2.1. *Suppose that J is a subgroup of a π-separable group G with π'-index and let $\mu \in \mathrm{I}_\pi(J)$ and $\varphi \in \mathrm{I}_\pi(G)$. If $\varphi_J = u\mu$ for some integer u, then $u=1$. Also, φ is the only π-partial character of G lying over μ.*

Proof. See Lemma (3.1) of [13]. ■

Lemma 2.2. *Let J be a subgroup of a π-separable group G. Let $\tau \in I_\pi(J)$ and suppose that $\tau^G \in I_\pi(G)$. If $\psi \in I_\pi(G \mid \tau)$ then $\psi = \tau^G$.*

Proof. Let H be a Hall π-subgroup of G such that $H \cap J$ is a Hall π-subgroup of J and let $\alpha \in \mathrm{Irr}(H \cap J)$ be a Fong character for τ (see (2.4) of [5]). By Lemmas 2.3 and 2.4 of [10], we know that $\alpha^H \in \mathrm{Irr}(H)$ is a Fong character for τ^G. Now, if τ is under ψ, it follows that α is an irreducible constituent of $\psi_{H \cap J}$ and hence, α^H is an irreducible constituent of ψ_H. By (2.5.b) of [5], we have that $\psi = \tau^G$, as required. ■

3. Vertices

The vertices for π-partial characters were defined in [7]. B. Huppert already noticed that every irreducible Brauer character (in a p-solvable group) is induced from a character with p'-degree. The same happens for π-partial characters: if $\varphi \in I_\pi(G)$, then $\varphi = \gamma^G$ for some $\gamma \in I_\pi(U)$ with π-degree (see (3.2) and (3.4) of [5]). It is not trivial to prove that the Hall π-complements of U only depend (up to G-conjugacy) on φ (not on U). These subgroups are called the *vertices* for φ.

If Q is a π'-subgroup of G, we will denote by $I_\pi(G \mid Q)$ the set of π-partial characters of G which have vertex Q.

Now, we want to distinguish between some of the normal irreducible constituents of $\varphi \in I_\pi(G \mid Q)$. Recall that Clifford Theory works for π-partial characters in the same way it works for Brauer characters ((3.1) and (3.2) of [5]).

DEFINITION 3.1. Let Q be a π'-subgroup of a π-separable group G and let $\varphi \in I_\pi(G \mid Q)$. If N is a normal subgroup of G and $\theta \in I_\pi(N)$ lies under φ, we say that θ is Q-good (relative to φ) if the Clifford correspondent of φ over θ has vertex Q.

Theorem 3.2. *Let $\varphi \in I_\pi(G \mid Q)$ and let $N \lhd G$, where G is π-separable. Then φ_N contains a Q-good constituent and all of them are $\mathbf{N}_G(Q)$-conjugate.*

Proof. Let $\xi \in I_\pi(N)$ be any irreducible constituent of φ_N, let I be the inertia subgroup and let $\tau \in I_\pi(I \mid \xi)$ be the Clifford correspondent of φ under ξ. Since τ induces φ, every vertex for τ is a vertex for φ. Therefore, there is $g \in G$ such that $\tau \in I_\pi(I \mid Q^g)$. Hence, if $\theta = \xi^{g^{-1}}$, $\mu = \tau^{g^{-1}}$ and $T = I^{g^{-1}}$, it follows that μ has vertex Q, μ is the Clifford correspondent of φ over θ and therefore, θ is Q-good (relative to φ). This proves the existence.

Now, suppose that η is also a Q-good irreducible constituent of φ and let $\delta \in I_\pi(U)$ be a π-degree character which induces μ and such that Q is a complement of U. By Clifford's theorem, $\eta = \theta^x$, for some $x \in G$, and hence, Q^x is a vertex for μ^x (because δ^x is a π-degree character which induces μ^x). By hypothesis, we have that Q is a vertex for μ^x, however. Hence, Q^x and Q are conjugate in T^x. Thus,

$tx \in \mathbf{N}_G(Q)$ for some $t \in T$ and we have that $\eta = \theta^x = \theta^{tx}$, as desired. ■

4. Fully ramified characters

If N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$ is G-invariant, we say that θ is *fully ramified* with respect to G if $\theta^G = e\chi$ for some $\chi \in \mathrm{Irr}(G)$ (sometimes, we will also say that χ is fully ramified with respect to N or with respect to θ). We see that a G-invariant character of N is fully ramified in G if only one irreducible character of G lies over θ. Later, we will need the following fact.

Lemma 4.1. *Let $N \lhd G$, let $\chi \in \mathrm{Irr}(G)$ and let θ be an irreducible constituent of χ_N. Then $[\chi_N, \theta]^2 \leq |G/N|$. Equality holds if θ is fully ramified with respect to G.*

Proof. Let $\psi \in \mathrm{Irr}(T)$ be the Clifford correspondent of χ over θ (where T is the inertia group of θ). We know that $[\chi_N, \theta] = [\psi_N, \theta]$. Also,

$$\theta^T = \sum_{\xi \in \mathrm{Irr}(T|\theta)} e_\xi \xi$$

and thus

$$\sum_{\xi \in \mathrm{Irr}(T|\theta)} e_\xi^2 = |T/N|.$$

Now, since $[\chi_N, \theta] = e_\psi$, everything follows. ■

In [6] (in the classical case), M. Isaacs already noticed the relationship between ordinary fully ramified characters and its π-analog. For our purposes here, it suffices to show the following.

Lemma 4.2. *Let $\varphi \in \mathrm{I}_\pi(G)$, where G is π-separable, let N be a normal π-subgroup of G and let H be any Hall π-subgroup of G. Suppose that $\varphi_N = e\theta$, where $\theta \in \mathrm{Irr}(N)$. Then $\mathrm{I}_\pi(G \mid \theta) = \{\varphi\}$ if and only if θ is fully ramified with respect to H. Also, in this case, φ_H is irreducible.*

Proof. Suppose that θ is fully ramified with respect to H and write $\mathrm{Irr}(H \mid \theta) = \{\xi\}$. Since θ is G-invariant, notice that $\varphi_H = e\xi$ for some integer e. Now, since θ is fully ramified and Lemma (2.1), we have that $\varphi_H = \xi$ and $\mathrm{I}_\pi(G \mid \xi) = \varphi \in \mathrm{I}_\pi(G \mid \theta) = \{\varphi\}$, as required.

Conversely, if $\mathrm{I}_\pi(G \mid \theta) = \{\varphi\}$, by Lemma (2.7) of [16], φ has π-degree and therefore $\varphi_H \in \mathrm{Irr}(H)$ (by (2.6) of [5]). Now let $\beta \in \mathrm{Irr}(H \mid \theta)$. Since β is a constituent of $(\beta^G)_H$, there is some $\mu \in \mathrm{I}_\pi(G)$ lying over β (and hence, over θ). Therefore, $\mu = \varphi$ and hence $\beta = \varphi_H$. This finishes the proof of the Lemma. ■

5. Blocks with one modular character

The goal of this section is to show that a modular character φ is the only modular character of its block if and only if φ is induced from some $\gamma \in \mathrm{I}_\pi(U)$ such that, if K is a Hall π-subgroup of U, then γ_K is an irreducible character fully ramified with respect to some normal π-subgroup of U. Arguing by induction, and using Fong Theory, one direction is easily obtained. To prove that such a φ is unique in its block is more complicated, we believe.

Next we show that there is no essential difference (concerning φ) about whether the character γ_K is fully ramified with respect to some normal π-subgroup of U or with respect to $\mathbf{O}_\pi(U)$. If we wish to control the irreducible ordinary characters of the block, this precision will have some importance, however.

To state more clearly our results, it seems convenient to make a definition. If U is a π-separable group we say that (U,γ,θ) is a *uniqueness triple* if the following conditions are satisfied:

(a) $\gamma \in \mathrm{I}_\pi(U)$ and $\theta \in \mathrm{Irr}(M)$, where $M = \mathbf{O}_\pi(U)$, and

(b) if K is a Hall π-subgroup of U, then γ_K is irreducible and fully ramified with respect to θ.

We say in this case that γ is a *uniqueness character.*

If (U,γ,θ) is a uniqueness triple, notice that $\gamma(1)$ is a π-number since γ_K is irreducible. Also, θ is U-invariant and $\mathrm{I}_\pi(U \mid \theta) = \{\gamma\}$ by Lemma 4.2.

Lemma 5.1. *Let G be a π-separable group. let $\varphi \in \mathrm{I}_\pi(G)$ and let H be a Hall π-subgroup of G. Suppose that φ_N is a multiple of $\theta \in \mathrm{Irr}(N)$, where N is a normal π-subgroup of G and θ is fully ramified with respect to H. Then there exists a uniqueness triple (V,γ,θ) such that $\varphi = \gamma^G$ and $N \subseteq V$. Also, induction defines a bijection from* $\mathrm{Irr}(V \mid \gamma)$ *onto* $\mathrm{Irr}(G \mid \theta)$.

Proof. By Lemma 4.2, we know that φ_H is irreducible and fully ramified with respect to N. In particular, φ has π-degree. Let $O = \mathbf{O}_\pi(G)$ and let $\eta \in \mathrm{Irr}(O)$ be an irreducible constituent of φ_O. Then, Lemma (4.2) gives us $\mathrm{I}_\pi(G \mid \eta) = \{\varphi\}$. If η is G-invariant, again by Lemma (4.2), η is fully ramified with respect to H, (G,φ,η) is a uniqueness triple and the result follows in this case. So we may assume that T, the inertia group of η, is proper in G. Since φ has π-degree notice that T has π-index.

If μ is the Clifford correspondent of φ over η, we have that $(\mu_{T\cap H})^H = \varphi_H$ (observe that $TH = G$, because T has π-index in G). Also, if $\beta \in \mathrm{Irr}(T \cap H \mid \eta)$, then β^H is irreducible and lies over θ. Therefore, $\beta^H = \varphi_H$ and thus $\beta = \mu_{T\cap H}$ by the uniqueness in the Clifford Correspondence. So we see that the hypotheses of the lemma are satisfied with the character μ and the normal π-subgroup O of T. Therefore, by induction, there exists a uniqueness triple (V,γ,τ) with $\gamma^T = \mu$ and

$O \subseteq V$ and such that induction defines a bijection $\mathrm{Irr}(V|\tau) \to \mathrm{Irr}(T|\eta)$. We see that (V,γ,τ) is a uniqueness triple inducing φ such that $N \subseteq V$. To finish the proof of the lemma, by the Clifford correspondence, it suffices to show that $\mathrm{Irr}(G|\theta) = \mathrm{Irr}(G|\eta)$. If $\chi \in \mathrm{Irr}(G)$ lies over θ, it follows that each irreducible constituent of χ_H lies over θ. Therefore, χ_H is a multiple of φ_H. Since we chose η lying under φ, everything follows. ■

Lemma 5.2. *Let G be a π-separable group, let $\varphi \in \mathrm{I}_\pi(G)$ and suppose that $\gamma^G = \varphi$, where (U,γ,θ) is a uniqueness triple. Assume further that $\mathbf{O}_\pi(G) \subseteq U$. If $\varphi_{\mathbf{O}_\pi(G)}$ is homogeneous, then $U = G$.*

Proof. First of all, let Q and K be a π-complement and a Hall π-subgroup of U, respectively. Since γ has π-degree, observe that Q is a vertex for φ. Now, let $N = \mathbf{O}_\pi(G) \subseteq O = \mathbf{O}_\pi(U)$, and write $\varphi_N = e\eta$, where $\eta \in \mathrm{Irr}(N)$. Let $M/N = \mathbf{O}_{\pi'}(G/N)$. If $\chi \in \mathbf{B}_\pi(G)$ lifts φ (see (2.3) of [5]), it follows that η is an irreducible constituent of χ_N. Since η is G-invariant, we have that η extends to M and by (6.3) of [4], there is a unique extension $\hat{\eta}$ of η to M lying in $B_\pi(M)$. By Lemma (5.4) of [4], we have that $\mathbf{B}_\pi$-characters with π-degree are π-special (we will use this fact several times). Hence, observe that $\hat{\eta}$ is π-special. Since normal irreducible constituents of $\mathbf{B}_\pi$-characters are again $\mathbf{B}_\pi$-characters ((7.5) of [4]), by (6.2.b) of [4], it follows that χ_M is a multiple of $\hat{\eta}$, and therefore $(M,\hat{\eta})$ is a subnormal factorable pair in the sense of Section 3 of [4]. By Definition (5.1) of [4], it follows that there exists a subgroup W containing M together with a π-degree character ψ such that $\psi^G = \chi$. Since χ lifts φ, it follows that ψ lifts some $\mu \in \mathrm{I}_\pi(W)$ with $\mu^G = \varphi$. Since, μ has π-degree, we have that the π-complements of W are vertices for φ. Hence, some G-conjugate of Q, say Q^g, is a π-complement of W. Then, $M = N(Q^g \cap M) = N(Q \cap M)$ and therefore, $M \subseteq U$.

Now, since O/N is a normal π-subgroup of U/N, we have that O/N centralizes M/N. But $\mathbf{C}_{G/N}(M/N) \subseteq M/N$ and therefore, $O = N$. Hence, $\theta = \eta$.

By hypothesis, we know that γ_K is irreducible and fully ramified with respect to N. If $\mu \in \mathrm{I}_\pi(G|\eta)$, μ_N is a multiple of η, and therefore μ_K is a multiple of γ_K. By Lemma (2.1), μ_U contains γ. Now, by Lemma (2.2), $\mu = \gamma^G = \varphi$. Hence, $\mathrm{I}_\pi(G|\eta) = \{\varphi\}$. By Lemma (4.2), we have that φ_H is fully ramified with respect to η, where we choose $K \subseteq H$ a Hall π-subgroup of G. Notice that, since $(\gamma^G)_H$ is irreducible, by Mackey, we have that $UH = G$ and hence $H \cap U = K$. Now, we have that $\theta^H = e\varphi_H$ with $e^2 = |H/N|$ and, also, we have that $\theta^K = d\gamma_K$ with $d^2 = |K/N|$. Now, $\theta^H = (\theta^K)^H = d(\gamma^G)_H = d\varphi_H$ and thus $d = e$ and $H = K$. Hence, $U = G$, as required. ■

To control the ordinary characters of the block, the following lemma will be needed.

Lemma 5.3. *Suppose that $G=UV$, where U,V are subgroups of G. Assume that α is a character of U such that $(\alpha_{U\cap V})^V$ is irreducible. If $\xi_{U\cap V}$ is a multiple of $\alpha_{U\cap V}$ for every $\xi\in \mathrm{Irr}(U\mid \alpha_{U\cap V})$, then induction defines a bijection from $\mathrm{Irr}(U\mid \alpha_{U\cap V})$ onto $\mathrm{Irr}(G\mid (\alpha_{U\cap V})^V)$.*

Proof. Write $\beta=(\alpha_{U\cap V})^V$ and let $\xi\in\mathrm{Irr}(U\mid\alpha_{U\cap V})$. By hypothesis, $\xi_{U\cap V}=e\alpha_{U\cap V}$ and then, $(\xi^G)_V=e\beta$. Therefore, we may write

$$\xi^G=\sum_{\chi\in\mathrm{Irr}(G\mid\beta)}[\xi^G,\chi]\chi.$$

Let χ be an irreducible constituent of ξ^G. Then $\chi_V=d\beta$, where, clearly, $d\leq e$. Now,

$$[\chi_{U\cap V},\alpha_{U\cap V}]=d[\beta_{U\cap V},\alpha_{U\cap V}]=d[\beta,\alpha_{U\cap V}]=d[\beta,\beta]=d.$$

Now, we may write $\chi_U=\xi+\Xi$, for some character Ξ. Hence, $\chi_{U\cap V}=\xi_{U\cap V}+\Xi_{U\cap V}=e\alpha_{U\cap V}+\Xi_{U\cap V}$ and therefore, $e\leq d$. Hence, we conclude that $e=d$ and that $\Xi_{U\cap V}$ does not contain $\alpha_{U\cap V}$. It follows that $\chi=\xi^G$ is irreducible and that induction defines a one to one map. Clealy, the map is surjective. ■

Theorem 5.4. *Let G be a separable group, let $\varphi\in \mathrm{I}_\pi(G)$ and let B be the block of φ. Then $\mathrm{I}_\pi(B)=\{\varphi\}$ if and only if $\varphi=\gamma^G$, where (U,γ,θ) is a uniqueness triple. In this case, induction defines a bijection from $\mathrm{Irr}(U\mid\theta)$ onto $\mathrm{Irr}(B)$.*

Proof. We argue by induction on $|G|$. Write $N=\mathbf{O}_\pi(G)$ and suppose first that $\mathrm{I}_\pi(B)=\{\varphi\}$. We want to show that φ is induced from a uniqueness character. Let μ be an irreducble constituent of φ_N. If δ is the Clifford correspondent of φ over μ, by (2.10) of [16], we know that δ is also unique in its block. Since δ induces φ, by the inductive hypothesis, we may assume that the character μ is G-invariant. In this case, by page 73 of [16], we have that $\mathrm{I}_\pi(B)=\mathrm{I}_\pi(G\mid\mu)$. Now, by Lemma (4.2), we have that φ is itself a uniqueness character and in this case, we are done.

Assume now that $\varphi=\gamma^G$, where (U,γ,θ) is a uniqueness triple. We want to prove that φ is unique in its block an that induction defines a bijection from $\mathrm{Irr}(U\mid\theta)$ onto $\mathrm{Irr}(B)$.

Write $O=\mathbf{O}_\pi(U)$ and let K be a Hall π-subgroup of U. Since $N\cap U\subseteq O$, observe that $\mathbf{O}_\pi(UN)=ON$ and that KN is a Hall π-subgroup of UN. Write $\alpha=\gamma_K$.

Now, notice that γ^{UN} is an irreducible character with π-degree (since γ also has π-degree and $|UN:U|$ is a π-number). Therefore, by (2.6) of [6], $\beta=\alpha^{KN}=(\gamma^{UN})_{KN}$ is irreducible.

By hypothesis, write $\alpha_N=e\theta$, where $e^2=|K:O|$ and notice that we may write $\beta_{ON}=(\alpha^{KN})_{ON}=e\theta^{ON}$. Hence, if η is an irreducible constituent of θ^{ON}, we have that $[\beta_{ON},\eta]\geq e$. Since $|KN:ON|=|K:O|=e^2$, by Lemma (4.1), we conclude that

$\beta_{ON}=e\eta$, and necessarily, that $\theta^{ON}=\eta$. Therefore, we have found a uniqueness triple (UN,γ^{UN},η) inducing φ and containing N. We wish to replace (V,γ,θ) by (UN,γ^{UN},η) and this will be possible if we are able to show that induction defines a bijection from $\mathrm{Irr}(U|\theta)$ onto $\mathrm{Irr}(UN|\eta)$.

If $\chi\in\mathrm{Irr}(U)$ lies over θ, notice that χ_K is multiple of α, because no other irreducible character of K than α lies over θ. Hence, we are in the hypotheses of Lemma (5.3) with the group UN. This lemma tells us that induction defines a bijection from $\mathrm{Irr}(U|\alpha)=\mathrm{Irr}(U|\theta)$ onto $\mathrm{Irr}(UN|\beta)=\mathrm{Irr}(UN|\eta)$. So, as we see, there is no loss of generality if we assume $N\subseteq O$.

Now, let ν be an irreducible constituent of θ_N and let T be the inertia subgroup of ν in G. We have that $T\cap U$, $T\cap K$ and $T\cap O$ are the inertia groups of ν in U, K, and O (respectively), and we write γ_0, α_0 and θ_0 for the Clifford correspondents of γ, α and θ (respectively).

First of all, note that $\gamma_0^G=\gamma^G=\varphi$, and thus γ_0^T is the Clifford correspondent of φ over ν. If $T=G$, notice that Lemma (5.2) implies that $U=G$. In this case, by Lemma (4.2), we have that $\mathrm{I}_\pi(G|\theta)=\{\varphi\}$ and by the results of [16] mentioned above, we have that $\mathrm{I}_\pi(B)=\{\varphi\}$ and $\mathrm{Irr}(B)=\mathrm{Irr}(G|\theta)$. So we may assume that T is proper in G.

Now, since γ_0 induces γ and γ has π-degree, observe that $(T\cap U)K=U$ and hence, that $T\cap K$ is a Hall π-subgroup of $T\cap U$. Also, since $\alpha=(\gamma_0^U)_K=(\gamma_{0T\cap K})^K$, we have that $\alpha_0=\gamma_{0T\cap K}$, by uniqueness of the Clifford correspondents.

Now, we claim that θ_0 is fully ramified with respect to $T\cap K$. First of all, since θ is K-invariant (by hypothesis) and θ_0 is the Clifford correspondent of θ over ν, it follows that θ_0 is $T\cap K$-invariant. If ε is any irreducible character of $T\cap K$ lying over θ_0, then ε lies over ν and by the Clifford Correspondence, we know that ε^K is irreducible. Now, $(\varepsilon^{(T\cap K)O})_O=(\varepsilon_{T\cap O})^O$ contains $(\theta_0)^O$ which contains θ. Therefore, ε^K lies over θ and hence, $\varepsilon^K=\alpha$. Now, by the uniqueness in the Clifford Correspondence, since ε lies over ν and induces α, necessarily $\varepsilon=\alpha_0$ and the claim is proved.

Now, by Mackey, we have that $((\gamma_0)_{T\cap K})^K=(\gamma_0^U)_K$ is irreducible. Moreover, if $\xi\in\mathrm{Irr}(T\cap U|\gamma_{0T\cap K})$ then ξ lies over θ_0. Since θ_0 is fully ramified with respect to $T\cap K$ and it is $T\cap U$-invariant, it follows that $\xi_{T\cap K}$ is a multiple of $\gamma_{0T\cap K}$. So we are in the hypothesis of Lemma 5.3. By applying Lemma 5.3, we have that induction defines a bijection $\mathrm{Irr}(T\cap U|\theta_0)\to\mathrm{Irr}(U|\theta)$.

We wish to apply the inductive hypothesis to the character γ_0^T. However, since we do not know whether or not $T\cap O$ is $\mathbf{O}_\pi(T)$, we apply Lemma 5.1 to conclude that there exists a uniqueness triple (V,ξ_0,τ_0) with $T\cap O\subseteq V$ such that $\xi_0^{T\cap U}=\gamma^0$. Also, we know that induction defines a bijection $\mathrm{Irr}(V|\tau_0)\to\mathrm{Irr}(T\cap U|\theta_0)$.

Now, we see that (V,ξ_0,τ_0) is a uniqueness triple inducing γ_0^T. Since T is proper in G, by the inductive hypothesis, we have that the block b of $(\gamma_0)^T$ has a unique modular character and that induction is a bijection $\mathrm{Irr}(U|\tau_0)\to\mathrm{Irr}(b)$. By

Theorem 2.10 of [16], we conclude that B has a unique modular character.

Finally, since induction defines a bijection in each of the following cases $\mathrm{Irr}(V|\tau_0) \to \mathrm{Irr}(b)$, $\mathrm{Irr}(b) \to \mathrm{Irr}(B)$, $\mathrm{Irr}(V|\tau_0) \to \mathrm{Irr}(T\cap U|\theta_0)$, and $\mathrm{Irr}(T\cap U|\theta_0) \to \mathrm{Irr}(U|\theta)$, we have that induction is also a bijection from $\mathrm{Irr}(U|\theta) \to \mathrm{Irr}(B)$, as required. ■

6. A correspondence of characters

First of all, we need the following fact.

Lemma 6.1. *Suppose that $G=UN$, where G is a π-separable group, U is a subgroup of G and N is normal π-subgroup of G. If Q is a π-complement of U, then $\mathbf{N}_G(Q)=\mathbf{N}_U(Q)\mathbf{C}_N(Q)$.*

Proof. We argue by induction on $|G|$. Let $M=N\mathbf{N}_G(Q)$ and since Q is a π-complement of $M\cap U$, if $M<G$, the result follows by induction. So we may assume that QN is a normal subgroup of G. Hence, $Q(U\cap N)$ is normal in U and the Frattini argument gives us $U=(U\cap N)\mathbf{N}_U(Q)$. Now, $G=UN=\mathbf{N}_U(Q)N$ and therefore, $\mathbf{N}_G(Q)=\mathbf{N}_U(Q)\mathbf{C}_N(Q)$. ■

Now we are ready to prove our main result. The solvability assumption we make on the Hall π-complements comes from the results in [8].

Theorem 6.2. *Let G be a π-separable group with a solvable Hall π-complement. Suppose that $\varphi=\gamma^G\in \mathrm{I}_\pi(G)$, where $\gamma\in \mathrm{I}_\pi(U)$ and (U,γ,θ) is a uniqueness triple. Let K be a Hall π-subgroup of U, so that that $\gamma_K\in\mathrm{Irr}(K)$ is fully ramified with respect to $\mathbf{O}_\pi(U)$. Write where $\gamma_{\mathbf{O}_\pi(U)}=e\theta$, where $\mathrm{Irr}\mathbf{O}_\pi(U))$ and let Q be a π-complement of U. If $\theta^*\in\mathrm{Irr}(\mathbf{C}_{\mathbf{O}_\pi(U)}(Q))$ is the Glauberman correspondent of θ, then:*
(a) *The character θ^* is fully ramified with respect to $\mathbf{N}_K(Q)$. If $\theta^{*\mathbf{N}_K(Q)}=e^*\delta^*$, for some $\delta^*\in\mathrm{Irr}(\mathbf{N}_K(Q))$, then δ^* extends to a unique $\gamma^*\in\mathrm{I}_\pi(\mathbf{N}_U(Q))$ and $\gamma^{*\mathbf{N}_G(Q)}\in\mathrm{I}_\pi(\mathbf{N}_G(Q))$.*
(b) *Assume that $\varphi=\mu^G$, where $\mu\in\mathrm{I}_\pi(V)$. Suppose that H is a Hall π-subgroup of V and that μ_H is fully ramified with respect to $\mathbf{O}_\pi(V)$. If Q is a π-complement of V, then $\gamma^{*\mathbf{N}_G(Q)}=\mu^{*\mathbf{N}_G(Q)}$.*

Proof. We simultaneously porve parts (a) and (b) by induction on the order of G.

Write $\gamma_K=\delta$, $O=\mathbf{O}_\pi(U)$ and notice, since Q is a π-complement of U, that $U=K\mathbf{N}_U(Q)$. By Theorem (6.3) of [8], we have that the number of I_π-characters of $O\mathbf{N}_U(Q)$ lying over θ and with vertex Q is 1 (because $\mathrm{I}_\pi(U|\theta)=\{\gamma\}$, γ has π-degree and, since Q is a π-complement of U, then γ has vertex Q). Now, by Proposition (6.4) of [8], we have that the number of I_π-characters of $\mathbf{N}_U(Q)$ over θ^* with vertex Q equals 1. Since Q is a π-complement of U, notice that (inside U) to have

vertex Q is equivalent to have π-degree. Since all I_π-characters of $\mathbf{N}_U(Q)$ have π-degree (just consider a $\mathbf{B}_\pi$-lifting and observe that the normal π'-subgroup Q must lie in its kernel), it follows that $|\mathrm{I}_\pi(\mathbf{N}_U(Q)\,|\,\theta^*)|=1$. Let γ^* be the unique character in this set. Notice that since θ is $\mathbf{N}_U(Q)$-invariant, by the uniqueness in the construction of the Glauberman map, θ^* is $\mathbf{N}_U(Q)$-invariant. Now, by Lemma (4.2), we have that $(\gamma^*)_{\mathbf{N}_K(Q)}=\delta^*\in\mathrm{Irr}(\mathbf{N}_K(Q))$ and θ^* is fully ramified with respect to $\mathbf{N}_K(Q)$.

To complete part (a), we need to show that $\gamma^{*\mathbf{N}_G(Q)}\in\mathrm{I}_\pi(\mathbf{N}_G(Q))$. This will require some work, however.

Let $N=\mathbf{O}_\pi(G)$. First of all observe that, by the first step in the proof of Theorem (5.4), we have that θ^{ON} is irreducible and fully ramified with respect to $(\gamma^{UN})_{KN}$. Now, by Lemma (6.1), notice that $\mathbf{N}_{UN}(Q)=\mathbf{N}_U(Q)\mathbf{C}_N(Q)$. Also, by Lemma (3.5) of [10], notice that $\mathbf{C}_O(Q)\mathbf{C}_N(Q)=\mathbf{C}_{ON}(Q)$.

By the first part of the proof, we know that $(\theta^{ON})^*\in\mathrm{Irr}(\mathbf{C}_{ON}(Q))$ is fully ramified with respect to an irreducible character $(\delta^{KN})^*\in\mathrm{Irr}(\mathbf{N}_K(Q)\mathbf{C}_N(Q))$ which has a unique extension $(\gamma^{UN})^*\in\mathrm{I}_\pi(\mathbf{N}_{UN}(Q))$.

Now, by Theorem A of [7], it follows that $(\theta^*)^{\mathbf{C}_{ON}(Q)}=(\theta^{ON})^*$. Therefore,

$$(\delta^{*\mathbf{N}_K(Q)\mathbf{C}_N(Q)})_{\mathbf{C}_{ON}(Q)}=d^*\theta^{*\mathbf{C}_{ON}(Q)}$$

where $d^{*2}=|\mathbf{N}_K(Q):\mathbf{C}_O(Q)|$. Since $\theta^{*\mathbf{C}_{ON}(Q)}$ is fully ramified with respect to the group $\mathbf{N}_K(Q)\mathbf{C}_N(Q)$ it follows that $(\delta^*)^{\mathbf{N}_K(Q)\mathbf{C}_N(Q)}=(\delta^{KN})^*$ is irreducible and consequently, $(\gamma^*)^{\mathbf{N}_{UN}(Q)}=(\gamma^{UN})^*$ is also irreducible. Hence, to prove that γ^* induces irreducibly to $\mathbf{N}_G(Q)$, we may assume that $N\subseteq U$. The same happens to prove part (b): we may assume that N is also contained in V.

Since θ is Q-invariant, by (13.27) of [3], let ν be a Q-invariant irreducible constituent of θ_N. Let T be the inertia subgroup of ν in G and let $\gamma_0\in\mathrm{I}_\pi(T\cap U)$, $\delta_0\in\mathrm{Irr}(T\cap K)$ and $\theta_0\in\mathrm{Irr}(T\cap O)$ be the Clifford correspondents of γ, δ and θ over ν, respectively. By the second step in the proof of Theorem (5.4), we know that θ is fully ramified with respect to δ_0 and that $(\gamma_0)_{T\cap K}=\delta_0$.

Since γ_0 induces γ, it has π-degree, and since Q is a π-complement of $T\cap U$, it follows that $\varphi_0=(\gamma_0)^T$ (which is the Clifford correspondent of φ overν) has vertex Q. Hence ν is a Q-good constituent of φ. To prove part (b), notice that we may change V by some $\mathbf{N}_G(Q)$-conjugate. Since Q-good irreducible constituents of φ are $\mathbf{N}_G(Q)$-conjugate, it follows that we may assume that τ lies over ν (where $\mu_{\mathbf{O}_\pi(V)}=f\tau$). Therefore $(\mu_0)^T=\varphi_0$.

If $T=G$, by Lemma (5.2), $U=V=G$ and there is nothing to prove. So we may assume that T is proper in G.

By Lemma (5.1), there exists a subgroup $U_0\subseteq U\cap T$ together with a character $\alpha_0\in\mathrm{I}_\pi(U_0)$ such that $\alpha_0^{U_0}=\gamma_0$, and $\alpha_{0\mathbf{O}_\pi(U_0)}=d_0\eta_0$ with $\eta_0\in\mathrm{Irr}(\mathbf{O}_\pi(U_0))$ is fully ramified with respect to $U_0\cap K$. Notice that since γ_0 induces γ and γ has π-degree, $U\cap T$ has π-index in U and hence $K\cap T$ is Hall π-subgroup of $U\cap T$. By the same

argument, $K \cap U_0$ is a Hall π-subgroup of U_0. Also, we may certainly U_0 such that Q is a Hall π-complement of U_0 and such that $O \cap T \subseteq \mathbf{O}_\pi(U_0)$.

By induction, we have that $(\alpha_0)^{*\mathbf{N}_T(Q)}$ is irreducible. By the definition, we know that α_0^* lies over η_0^*. Also, by (13.29) of [3], it follows that η_0^* lies over θ_0^* and that θ_0^* lies over ν^*. Since $\mathbf{N}_T(Q)$ is the inertia group of ν^* in $\mathbf{N}_G(Q)$ (by the uniqueness in the Glauberman Correspondence), by the Clifford Correspondence, we have that $\alpha_0^{*\mathbf{N}_G(Q)}$ is irreducible. In particular, $\alpha_0^{*\mathbf{N}_U(Q)}$ is also irreducible. Since $\theta_0^{*\mathbf{C}_O(Q)} = \theta^*$, we have that $\alpha_0^{*\mathbf{N}_U(Q)}$ lies over θ^*. Therefore, $\alpha_0^{*\mathbf{N}_U(Q)} = \gamma^*$ and therefore, $\gamma^{*\mathbf{N}_G(Q)}$ is irreducible, as requered.

Finally, if $(V_0, \beta_0, \varepsilon_0)$ is a uniqueness triple with $V_0 \subseteq V \cap T$, $\beta_0 \in \mathbf{I}_\pi(V_0)$ such that $\beta_0^{V_0} = \mu_0$ and $\beta_{0\mathbf{O}_\pi(V_0)} = d_0\varepsilon_0$, where $\varepsilon_0 \in \mathrm{Irr}(\mathbf{O}_\pi(V_0))$, chosen in the same way that we chose (U_0, α_0, η_0) for γ_0, since $\gamma_0^T = \varphi_0 = \mu_o^T$, by induction we have that

$$(\alpha_0^*)^{\mathbf{N}_T(Q)} = (\beta_0^*)^{\mathbf{N}_T(Q)}.$$

Hence,

$$(\alpha_0^*)^{\mathbf{N}_G(Q)} = (\beta_0^*)^{\mathbf{N}_G(Q)}$$

and since $(\alpha_0^*)^{\mathbf{N}_U(Q)} = \gamma^*$ and $(\beta_0^*)^{\mathbf{N}_U(Q)} = \mu^*$, then

$$(\gamma^*)^{\mathbf{N}_G(Q)} = (\mu^*)^{\mathbf{N}_G(Q)}$$

and the proof is finished. ■

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