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THE FEFFERMAN-PHONG INEQUALITY IN THE LOCALLY TEMPERATE WEYL CALCULUS

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1. Introduction

The Fefferman-Phong inequality, which is a two derivatives Gårding type inequality, has been proved by its authors for pseudo-differential operators with symbols in the class $S^m_{1,0}$. It has been extended by HÖRMANDER [5], in his work on the Weyl calculus, to symbols in the general class $S(m,g)$ where g is a slowly varying and temperate metric satisfying the uncertainly principle. Further works on spectral theory and singularities for nonlinear hyperbolic equations showed the necessity to relax the temperacy condition on the metric and DENCKER [3], BONY-LERNER [1] introduced new classes allowing to deal with these applications. However the Fefferman-Phong inequality was not proved for these classes. In a recent work, COLOMBINI, DEL SANTO, ZUILY [2], we have also been led to consider non temperate metrics and the above mentioned inequality was required in the proof; it turned out that these metrics were locally temperate in the sense of DENCKER [3]. The purpose of this work is then to prove the Fefferman-Phong inequality for properly supported pseudo-differential operators with symbols in locally temperate classes. Unfortunately, because of the complexity of the quantification, this inequality is still not available in the general case of the Bony-Lerner classes.

2. Notations, statement of the result, examples

We first recall some definitions taken from HÖRMANDER [5] and DENCKER [3].

Let V be an n dimensional vector space and $W=V\oplus V'$ where V' is the dual of V. Elements in V will be denoted by x and those in W by w or (x,ξ).

Let G be a metric on V, assumed to be slowly varying i.e.

$$(2.1)\quad \begin{cases} \text{there exist constants } a_0>0,\ A_0\geq 1 \text{ such that for } x,y \text{ in } V: \\ G_x(x-y)\leq a_0 \Rightarrow \dfrac{1}{A_0}\leq \dfrac{G_x}{G_y}\leq A_0. \end{cases}$$

Let g be a metric on W which is also slowly varying i.e.

$$\left\{\begin{array}{l}\text{there exist constants } a_1>0,\ A_1\geq 1 \text{ such that for } w, w' \text{ in } W\text{:}\\ \qquad g_w(w-w')\leq a_1 \Rightarrow \dfrac{1}{A_1}\leq \dfrac{g_w}{g_{w'}}\leq A_1 .\end{array}\right. \tag{2.2}$$

We shall also assume that

$$\text{for every } (x,\xi) \text{ in } W, \quad G_x \leq g_{x,\xi}. \tag{2.3}$$

The space W is a symplectic space with the standard symplectic form

$$\sigma(w,w')=\langle y,\xi\rangle-\langle x,\eta\rangle \quad \text{if } w=(x,\xi),\ w'=(y,\eta)\in W. \tag{2.4}$$

The dual metric of g with respect to σ is then defined by

$$g^\sigma_{w_0}(w)=\sup_{w'\neq 0}\frac{|\sigma(w,w')|^2}{g_{w_0}(w')}. \tag{2.5}$$

The metric g will also be assumed to separate the uncertainly principle which reads

$$g_w \leq g^\sigma_w \quad \text{for every } w \text{ in } W. \tag{2.6}$$

We then define the function h on W by

$$h^2(w)=\sup_{w'\neq 0}\frac{g_w(w')}{g^\sigma_w(w')}\leq 1. \tag{2.7}$$

The metric g is said locally temperate if :

$$\left\{\begin{array}{l}\text{there exist positive constants } a_2,\ A_2 \text{ and } N\in \boldsymbol{N} \text{ such that}\\ G_x(x-y)\leq a_2 \Rightarrow g_w\leq A_2 g_{w'}(1+g^\sigma_w(w-w'))^N\\ \text{if } w=(x,\xi) \text{ and } w'=(y,\eta).\end{array}\right. \tag{2.8}$$

We introduce now the order functions. These are positive functions m on W for which one can find constants $b_j>0$, $B_j\geq 1$, $j=0,1$ and $M\in \boldsymbol{N}$ such that

$$g_w(w-w')\leq b_0 \Rightarrow \frac{1}{B_0}\leq \frac{m(w)}{m(w')}\leq B_0, \quad w,w'\in W, \tag{2.9}$$

$$\left\{\begin{array}{l}G_x(x-y)\leq b_1 \Rightarrow m(w)\leq B_1 m(w')(1+g^\sigma_w(w-w'))^M\\ \text{if } w=(x,\xi) \text{ and } w'=(y,\eta).\end{array}\right. \tag{2.10}$$

Given G, g, m as above we define the class of symbols $S(m,g)$ to be the set of C^∞ functions a on W such that

$$\text{(2.11)} \quad \begin{cases} \text{for every } k \in N \text{ one can find } C_k > 0 \text{ such that for every} \\ w, w_1, \cdots w_k \text{ in } W \text{ we have} \\ |a^{(k)}(w)(w_1, \cdots, w_k)| \le C_k m(w) \prod_{i=1}^{k} (g_w(w_i))^{1/2}. \end{cases}$$

We define a metric $\tilde{G}$ on $V \times V$ by

$$\tilde{G}_{x,y}(s,t) = G_x(s) + G_y(t) \tag{2.12}$$

and we consider the $\tilde{G}$ distance of a point $(x,y) \in V \times V$ to the diagonal

$$D(x,y) = \inf_{x_0 \in V} \tilde{G}_{x_0,x_0}(x - x_0, y - x_0). \tag{2.13}$$

If $\varepsilon > 0$, we shall set $D_\varepsilon = \{(x,y) \in V \times V,\ D(x,y) \le \varepsilon\}$.

Given $0 < \varepsilon' < \varepsilon$ one can construct χ such that

$$\chi \in S(1, \tilde{G}), \quad \operatorname{supp} \chi \subset D_\varepsilon, \quad \chi = 1 \text{ on } D_{\varepsilon'}. \tag{2.14}$$

We shall call such a function properly supported.

Given $a \in \mathscr{S}(W)$ and χ properly supported we can define a pseudo-differential operator by the Weyl quantification

$$a_\chi^w u(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} \chi(x,y) a\left(\frac{x+y}{2}, \xi\right) u(y) dy\, d\xi. \tag{2.15}$$

This formula can be extended to the case where $a \in S(m,g)$, as an operator sending $C_0^\infty(V)$ into $\mathscr{E}'(V)$ and $C^\infty(V)$ to $\mathscr{D}'(V)$. The symbolic calculus for this class of operators and the L^2 continuity, when m is bounded, have been achieved by DENCKER [3]. The purpose of this work is to prove that the FEFFERMANN-PHONG inequality [4], which has been proved by HÖRMANDER [5] when the metric is temperate, still holds for locally temperate metrics and properly supported operators. Thus the main result of this work is the following.

Theorem 2.1. *Let $a \in S(h^{-2}, g)$, $a \ge 0$ on W. Let χ be properly supported. Then one can find a positive constant C such that*

$$(a_\chi^w u, u) + C\|u\|_{L^2}^2 \ge 0, \quad \textit{for } u \in C_0^\infty(V).$$

EXAMPLE 2.2. As we said before, locally temeperate metrics occur in proving Carleman estimates with singular weights (see [2]). Let $V = \boldsymbol{R}_x^n \times \boldsymbol{R}_y$, $V' = \boldsymbol{R}_\xi^n \times \boldsymbol{R}_\eta$. Let $\theta(y)$ be a C^∞ function on $\boldsymbol{R}$ such that

$$0(y)=e^y \text{ if } y\le 0,\quad 0(y)=2 \text{ if } y\ge 1,\quad 0\le\theta\le 2.$$

We set $\Phi^2(x,y;\xi,\eta)=1+0^2(y)|\xi|^2+|\eta|^2$. For G we take the flat metric on V, $G_{x,y}(s,t)=|s|^2+t^2$. Then the metric

$$g_{x,y;\xi,\eta}=dx^2+dy^2+\frac{d\xi^2+d\eta^2}{\Phi^2(x,y;\xi,\eta)}$$

is slowly varying, locally temperate, satisfies the uncertainly principle and (2.3), but is not temperate in the sense of HÖRMANDER [5].

3. Proof of Theorem 2.1

The first step is to prove an analogue of Lemma 18.6.10 in [5].

Proposition 3.1. *Let G* (resp. g) *be one positive definite quadratic form on* $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^{2n}$). *Let us assume*

$$G(t)\le g(t,\tau),\quad (t,\tau)\in\boldsymbol{R}^{2n}, \tag{3.1}$$

$$\sup_{(t,\tau)}\frac{g(t,\tau)}{g^\sigma(t,\tau)}=\lambda^2\le 1. \tag{3.2}$$

Let $a\in C^\infty(\boldsymbol{R}^{2n})$, $a\ge 0$, *be such that for each $k\in N$ one can find a constant $C_k>0$ such that for X, $T_1,\cdots,T_k$ in $\boldsymbol{R}^{2n}$*

$$|a^{(k)}(X)(T_1,\cdots,T_k)|\le C_k\lambda^{-2}\prod_{i=1}^{k}(g(T_i))^{1/2}. \tag{3.3}$$

Let $\chi\in S(1,\tilde{G})$, $\operatorname{supp}\chi\subset D_\varepsilon$, $\chi=1$ on $D_{\varepsilon'}$, $\varepsilon'<\varepsilon$. *Then there exists a constant $C>0$ such that*

$$(a_\chi^w u,u)+C\|u\|_{L^2}^2\ge 0 \quad \textit{for } u\in C_0^\infty(\boldsymbol{R}^n). \tag{3.4}$$

Here C is independent of g, G and depends on a only through a finite sum $\sum_{k=1}^{k_0} C_k$, where k_0 is independent of a.

Proof. Let us first assume $a\in\mathscr{S}(\boldsymbol{R}^{2n})$. Then we shall have $a_\chi^w=b^w$ if

$$\hat{a}\left(\frac{1}{2}(x+y),y-x\right)\chi(x,y)=\hat{b}\left(\frac{1}{2}(x+y),y-x\right)$$

where $\hat{}$ is the Fourier transform with respect to the second variable.

Taking the inverse Fourier transform we obtain

$$b(x,\xi)=\exp(-i\langle D_t,D_\eta\rangle)\left[\chi\left(x+\frac{t}{2},x-\frac{t}{2}\right)a(x,\eta)\right]\Bigg|_{\substack{t=0\\ \eta=\xi}}.$$

By Proposition 2.4 in [3] and Theorem 18.4.11 in [5], the mapping $a\mapsto b$ has a continuous extension to a weakly continuous linear mapping from $S(\lambda^{-2},g)$ to $S(\lambda^{-2},g)$. Moreover the remainder term

$$b(x,\xi)-\sum_{j=0}^{N}(-i\langle D_t,D_\eta\rangle)^j\left[\chi\left(x+\frac{t}{2},x-\frac{t}{2}\right)a(x,\eta)\right]\Bigg|_{\substack{t=0\\ \eta=\xi}}\cdot\frac{1}{j!}$$

is weakly continuous with value in $S(\lambda^{N-1},g)$ and its semi-norms in this space depend on N and on the semi-norms of a in $S(\lambda^{-2},g)$ of χ in $S(1,\tilde{G})$. Since $\chi=1$ near the diagonal we get, if $N=1: b=a+r$ with $r\in S(1,g)$. Therefore $a_\chi^w=a^w+r^w$. Now, by Lemma 18.6.10 in [5] we have $(a^w u,u)\geq -C_0\|u\|_{L^2}^2$ with C_0 independent of a and g. On the other hand, since $r\in S(1,g)$ we have by Theorem 18.6.3 in [5]: $|(r^w u,u)|\leq C_1\|u\|_{L^2}^2$, where C_1 depends on a fixed number of semi-norms of r in $S(1,g)$, therefore on a fixed number of semi-norms of a in $S(\lambda^{-2},g)$ and of χ in $S(1,\tilde{G})$. It follows that $(a_\chi^w u,u)\geq C\|u\|_{L^2}^2$ with C as claimed. $\diamond$

Now we return to general a, g, G and we localize the problem in the balls constructed in Lemma 18.4.4 of [5].

Let G, g be metrics on V and W, satisfying (2.1) to (2.8).

Let $\rho>0$ be so small that, with a_j, A_j defined in (2.1) to (2.8),

$$\rho<a_1,\quad 32A_0^3\rho\leq a_2,\quad 8A_0^4\rho\leq a_0. \tag{3.5}$$

Let us set with $\rho_0<\rho$,

$$B_\nu=\{w:g_{w_\nu}(w-w_\nu)\leq\rho_0\}\subset U_\nu=\{w:g_{w_\nu}(w-w_\nu)\leq\rho\}\subset U_\nu'$$
$$U_\nu'=\{w:g_{w_\nu}(w-w_\nu)\leq a_1\}.$$

Let $\varphi_\nu\in C_0^\infty(B_\nu)$ be real such that $\sum_\nu\varphi_\nu^2=1$ and $\theta_\nu\in C_0^\infty(U_\nu)$, $\theta_\nu\geq 0$, $\theta_\nu=1$ on B_ν, φ_ν and θ_ν uniformly bounded in $S(1,g)$. If we set we get $a_\nu=\theta_\nu a$ we get $a=\sum_\nu\varphi_\nu a_\nu\varphi_\nu$ and the semi-norms of a_ν in $S(h^{-2},g)$ are uniformly bounded by those of a in $S(h^{-2},g)$.

We also fix $\varepsilon>0$ so small that

$$16A_0^6\varepsilon\leq a_2,\quad 8A_0^4\varepsilon\leq a_0 \tag{3.6}$$

and we take a properly supported χ with $\operatorname{supp}\chi\subset D_\varepsilon$, $\chi=1$ near the diagonal. We also assume that χ is real and $\chi(y,x)=\chi(x,y)$. Since $a_\nu\geq 0$, it follows from Proposition 3.1, that

(3.7) $$(a^w_{\nu\chi}u,u)+C\|u\|^2_{L^2}\geq 0, \quad u\in C_0^\infty,$$

where C is independent of G, g and ν. Applying (3.7) to $\varphi^w_{\nu\chi}u$, instead of u, we get

$$(\varphi^w_{\nu\chi}a^w_{\nu\chi}\varphi^w_{\nu\chi}u,u)+C(\varphi^w_{\nu\chi}\varphi^w_{\nu\chi}u,u)\geq 0$$

since $(\varphi^w_{\nu\chi})^*=\varphi^w_{\nu\chi}$, because χ is real and $\chi(x,y)=\chi(y,x)$.

Taking the sum with respect to ν we get

(3.8) $$\Big(\underbrace{\sum_\nu \varphi^w_{\nu\chi}a^w_{\nu\chi}\varphi^w_{\nu\chi}}_{(1)},u,u\Big)+C\Big(\underbrace{\sum_\nu \varphi^w_{\nu\chi}\varphi^w_{\nu\chi}}_{(2)},u,u\Big)\geq 0.$$

Lemma 3.2. *The operator* $\sum_\nu \varphi^w_{\nu\chi}\varphi^w_{\nu\chi}$ *is bounded on* L^2.

Proof. As in the proof of the L^2 continuity in [5] we shall use the Cotlar-Knapp-Stein Lemma. Let us set $A_\nu=\varphi^w_{\nu\chi}\cdot\varphi^w_{\nu\chi}$. We have to show

(3.9) $$\sup_\nu\sum_\mu\|A_\nu A^*_\mu\|^{1/2}\leq C, \quad \sup_\nu\sum_\mu\|A^*_\nu A_\mu\|^{1/2}\leq C.$$

The proof of course is completly symmetric for the two terms. Let us consider the firstt. We have (here the norm is the L^2-L^2 operator norm), $\|A_\nu A^*_\mu\| \leq\|\varphi^w_{\nu\chi}\|\cdot\|\varphi^w_{\nu\chi}\varphi^w_{\mu\chi}\|\cdot\|\varphi^w_{\mu\chi}\|$.

Since $\varphi_\nu\in S(1,g)$ has uniformly bounded semi-norms in this space, $\varphi^w_{\nu\chi}$ is bounded in L^2 with uniformly bounded operator norm (see the Remark after Theorem 4.3 in [3]). Therefore the first part of (3.9) will be a consequence of the following estimate.

(3.10) $$\begin{cases} \text{For every } l\in \boldsymbol{N} \text{ one can find } C_l>0 \text{ such that for every } \nu \\ \qquad \|\varphi^w_{\nu\chi}\varphi^w_{\mu\chi}\|\leq C_l(1+d_{\nu\mu})^{-l}. \\ \text{One can find } l_0\in\boldsymbol{N} \text{ such that} \\ \qquad \sup\limits_\nu\sum\limits_\mu(1+d_{\nu\mu})^{-l_0}<+\infty. \end{cases}$$

The proof of 3.10 is close to that of Theorem 4.3 in [3] and we give it for sake of completess and the convenience of the later use in the proof of Lemma 3.3. Let $\psi\in S(1,\tilde{G})$, $\psi=1$ on $\{(x,y):\exists z:\chi(x,z)\chi(z,y)\neq 0\}$.

Then by formula (4.22) in [3] we can write

(3.11) $$\varphi^w_{\nu\chi}\varphi^w_{\mu\chi}=\theta^w_{\nu\mu\psi}$$

$$
(3.12) \qquad \begin{cases} \theta_{\nu\mu}(x,\xi)=\exp\left(\dfrac{i}{2}\sigma(D_z,D_\zeta;D_t,D_\tau)\right)[F_{x,\xi}(z,t,\zeta,\tau)]|_{t=\tau=0}^{\zeta=z=0} \\ F_{x,\xi}(z,t,\zeta,\tau)=\chi(x+z+t,x+z-t)\chi(x+z+t,x-z+t) \\ \qquad\qquad \varphi_\nu(x+z,\xi+\zeta)\varphi_\mu(x+t,\xi+\tau). \end{cases}
$$

We claim that, denoting by πU_ν the projection of U_ν on V:

$$
(3.13) \qquad \inf_{t\in\pi U_\nu} G_t(x-t)>A_0^2\varepsilon \Rightarrow \operatorname{supp} F_{x,\xi}=\phi.
$$

Indeed let $(z,t,\zeta,\tau)\in\operatorname{supp}F_{x,\xi}$. Then $x+z\in\pi U_\nu$, $x+t\in\pi U_\nu$, $(X,Y)\in\operatorname{supp}\chi$, $(X,Z)\in\operatorname{supp}\chi$ where $X=x+z+t$, $Y=x+z-t$, $Z=x-z+t$. It follows that $D(X,Y)\le\varepsilon$, $D(X,Z)\le\varepsilon$. By Lemma 2.2 in [3], since $\varepsilon\le a_0$, we get $G_X(X-Y)\le 4A_0\varepsilon$, $G_X(X-Z)\le 4A_0\varepsilon$. It follows that $G_X(X-\frac{X+Y}{2})=\frac{1}{4}G_X(X-Y)\le A_0\varepsilon\le a_0$ so by (2.1),

$$
G_{x+z}(z)=\frac{1}{4}G_{\frac{X+Y}{2}}(X-Z)\le\frac{A_0}{4}G_X(X-Z)\le A_0^2\varepsilon.
$$

Since $t=x+z\in\pi U_\nu$ this contradicts (3.13).

It follows that, on the support of $\theta_{\nu\mu}$ we have

$$
(3.14) \qquad \begin{cases} \inf\limits_{t\in\pi U_\nu} G_t(x-t)\le A_0^2\varepsilon \\ \inf\limits_{z\in\pi U_\nu} G_z(x-z)\le A_0^2\varepsilon. \end{cases}
$$

We shall show that this implies

$$
(3.15) \qquad \begin{cases} \forall t\in\pi U_\nu \quad G_t(x-t)\le 2A_0^4\varepsilon+4A_0\rho \qquad (\le\min(a_0,a_2)) \\ \forall z\in\pi U_\mu \quad G_z(x-z)\le 2A_0^4\varepsilon+4A_0\rho \qquad (\le\min(a_0,a_2)). \end{cases}
$$

Indeed let $t_0\in\pi U_\nu$ be such that $G_{t_0}(x-t_0)\le A_0^2\varepsilon$. Since t_0, t are in πU_ν we have $G_{t_\nu}(t-t_\nu)\le g_{w_\nu}(w-w_\nu)\le\rho$, $G_{t_\nu}(t_0-t_\nu)\le\rho$. Since $\rho\le a_0$ we obtain $G_t\le A_0G_{t_\nu}\le A_0^2G_{t_0}$. Therefore

$$
G_t(x-t)\le 2[G_t(x-t_0)+G_t(t_0-t)]\le 2[A_0^4\varepsilon+2A_0\rho].
$$

◇

It follows from [5], formula (18.4.12) that, for every $k\in N$ one can find $C_k>0$, independent of ν such that

$$
(3.16) \qquad |\theta_{\nu\mu}(w)|\le C_k\left(1+\inf_{\substack{w'\in U_\nu\\ w''\in U_\mu}}[g^\sigma_{w'}(w-w'')+[g^\sigma_{w''}(w-w')]\right)^{-k}.
$$

We would like to replace the right hand side of (3.16) by

$$\left(1+\inf_{w'\in U_\upsilon} g^\sigma_w(w-w')+\inf_{w''\in U_\mu} g^\sigma_w(w-w'')\right)^{-k}.$$

This will follow from

$$g^\sigma_w(w-w')+g^\sigma_w(w-w'')\leq C(1+g^\sigma_{w'}(w-w'')+g^\sigma_{w''}(w-w'))^{n_0} \tag{3.17}$$

for some n_0 independent of w, w', w''.

If we set $w=(x,\xi)$, $w'=(y,\eta)$, $w''=(z,\zeta)$, it follows from (3.15) that $G_y(x-y)$ and $G_z(x-z)$ are bounded by a_2. It follows from (2.8) that

$$g_{w'}\leq A_2 g_w(1+g^\sigma_{w'}(w-w'))^N$$
$$g_{w''}\leq A_2 g_w(1+g^\sigma_{w''}(w-w''))^N$$

which implies

$$g^\sigma_w\leq A_2 g^\sigma_{w'}(1+g^\sigma_{w'}(w-w'))^N$$
$$g^\sigma_w\leq A_2 g^\sigma_{w''}(1+g^\sigma_{w''}(w-w''))^N$$

so

$$g^\sigma_w(w-w')+g^\sigma_w(w-w'')\leq A_2(2+g^\sigma_{w'}(w-w')+g^\sigma_{w''}(w-w''))^{N+1}.$$

It follows that (3.17) will be a consequence of

$$g^\sigma_{w'}(w-w')+g^\sigma_{w''}(w-w'')\leq C(1+g^\sigma_{w'}(w-w'')+g^\sigma_{w''}(w-w'))^q \tag{3.18}$$

ffor some $q\in \boldsymbol{N}$.

Let us set $w_1=w'+w''-w$. Since $w'=(y,\eta)\in U_\nu$, $w''=(z,\zeta)\in U_\mu$ (3.13) implies that $G_y(x-y)\leq a_0$, $G_z(x-z)\leq a_0$. We deduce from (2.1) that $G_y(x-z)\leq A_0^2 G_z(x-z)$ $\leq 2A_0^6\varepsilon+4A_0^3\rho\leq a_2$. Since $w_1-w'=w''-w=(z-x,\zeta-\xi)$ and $w'=(y,\eta)$, (2.8) implies

$$g_{w'}\leq A_2 g_{w_1}(1+g^\sigma_{w'}(w''-w_1))^N$$
$$g_{w''}\leq A_2 g_{w_1}(1+g^\sigma_{w''}(w''-w_1))^N.$$

This implies

$$\begin{cases} g^\sigma_{w_1}(w-w'')\leq A_2(1+g^\sigma_{w'}(w-w''))^{N+1} \\ g^\sigma_{w_1}(w-w')\leq A_2(1+g^\sigma_{w''}(w-w'))^{N+1}. \end{cases} \tag{3.19}$$

By symmetry one gets

$$(3.20)\qquad \begin{cases} g^\sigma_{w'} \le A_2 g^\sigma_{w_1}(1+g^\sigma_{w_1}(w'-w_1))^N \\ g^\sigma_{w''} \le A_2 g^\sigma_{w_1}(1+g^\sigma_{w_1}(w''-w_1))^N. \end{cases}$$

It follows from (3.19) and (3.20) that

$$g^\sigma_{w'}(w-w')+g^\sigma_{w''}(w-w'')\le C(A_2)(1+g^\sigma_{w'}(w-w'')+g^\sigma_{w''}(w-w'))^q$$

for some q. This is (3.18). Therefore (3.17) is proved and by (3.16) we get

$$(3.21)\qquad |\theta_{\nu\mu}(x,\xi)|\le C_l\left(1+\inf_{w'\in U_\nu} g^\sigma_w(w-w')+\inf_{w''\in U_\mu} g^\sigma_w(w-w'')\right)^{-l}.$$

Let $w'_0\in U_\nu$ and $w''_0\in U_\mu$ be such that

$$(3.22)\qquad g_w(w-w'_0)=\inf_{w'\in U_\nu} g_w(w-w'),\quad g_w(w-w''_0)=\inf_{w''\in U_\mu} g_w(w-w'').$$

Since $w'_0\in U_\nu$ it follows from (3.15) that $G_{x'_0}(x-x'_0)\le a_2$. Then (2.8) implies $g^\sigma_{w'_0}\le A_2 g^\sigma_w(1+g^\sigma_w(w-w'_0))^N$. By the slow variation of g in U_ν we get

$$\begin{aligned} g^\sigma_{w_\nu}(w'_0-w''_0) &\le A_1 g^\sigma_{w'_0}(w'_0-w''_0)\le A_1A_2 g^\sigma_w(w'_0-w''_0)(1+g^\sigma_w(w-w'_0))^N \\ &\le 2A_1A_2[g^\sigma_w(w'_0-w)+g^\sigma_w(w-w''_0)](1+g^\sigma_w(w-w'_0))^N \\ &\le 2A_1A_2(1+g^\sigma_w(w-w'_0)+g^\sigma_w(w-w''_0))^{N+1}. \end{aligned}$$

It follows from (3.21) and (3.22) that if we set

$$(3.23)\qquad d_{\nu\mu}=\inf_{\substack{w'\in U_\nu\\ w''\in U_\mu}} g_{w_\nu}(w'-w'')$$

we have for every $l\in N$,

$$(3.24)\qquad |\theta_{\nu\mu}(x,\xi)|\le C_l(1+d_{\nu\mu})^{-l}.$$

The same estimate is true for every semi-norm of $\theta_{\nu\mu}$ in $S(1,g)$. It follows from [3] (Remark after Theorem 4.3) that first part of (3.10) is valid. Let us now show the second part. We fix so we may assume that $g_{w_\nu}(t)=|t|^2$. We have

$$\sum_{\mu=1}^{\mu_0}(1+d_{\nu\mu})^{-l_0}=\sum_{k=1}^{k_0}\sum_{k-1\le d_{\nu\mu}<k}(1+d_{\nu\mu})^{-l_0}\le\sum_{k=1}^{k_0}k^{-l_0}\operatorname{card}M_k$$

where $M_k=\{\mu: d_{\nu\mu}<k\}$. We shall show that card $M_k\le Ck^q$ for some q, which will prove (3.10).

Now if $\mu\in M_k$ there exists $w'_\nu\in U_\nu$, $w''_\mu\in U_\mu$ such that $g^\sigma_{w_\nu}(w'_\nu-w''_\mu)<k$. It follows from (2.3) that

(3.25) $$g^{\sigma}_{w'_v}(w'_v - w''_\mu) \le A_1 k.$$

Now, by (3.15), if $w \in \operatorname{supp} \theta_{\nu\mu}$, $w=(x,\xi)$ we have $G^{\sigma}_{x'_v}(x-x'_v) \le a_0$ and $G_{x''_\mu}(x-x''_\mu) \le a_0$. It follows that $G_x(x-x'_v)+G_x(x-x''_\mu) \le 4A_0^5\varepsilon+8A_0^2\rho$ so $G_x(x''_\mu-x'_v) \le 8A_0^5\varepsilon+16A_0^2\rho$. Therefore $G_{x'_v}(x'_v-x''_\mu) \le 8A_0^6\varepsilon+16A_0^3\rho \le a_2$ by (3.5), (3.6). We deduce from (2.8) that $g^{\sigma}_{w''_\mu}(t) \le C g^{\sigma}_{w'_v}(t)(1+g^{\sigma}_{w'_v}(w'_v-w''_\mu))^N$. It follows from (2.6) and (3.25) that $g^{\sigma}_{w''_\mu}(t) \le C(A_1)|t|^2 k^N$. Therefore $|t| \le \delta k^{-N/2}$ imlies $g_{w''_\mu}(t) \le C(A_1)\delta^2$ so:

$$|z-w''_\mu| \le \delta k^{-N/2} \Rightarrow g_{w_\mu}(z-w''_\mu) \le C'(A_1)\delta^2.$$

Now $g^{1/2}_{w_\mu}(z-w_\mu) \le g^{1/2}_{w_\mu}(z-w''_\mu)+g^{1/2}_{w_\mu}(w_\mu-w''_\mu) \le C'(A_1)\delta+\rho \le a_1$ if δ is small enough. Therefore

(3.26) $$|z-w''_\mu| \le \delta k^{-N/2} \Rightarrow z \in U'_\mu.$$

Now, $|z-w'_v| \le |z-w''_\mu|+|w''_\mu-w'_v| \le \delta k^{-N/2}+g^{1/2}_{w_v}(w''_\mu-w'_v) \le \delta k^{-N/2}+k^{1/2}$, by (2.6). Then,

(3.27) $$|z-w''_\mu| \le \delta k^{-N/2} \Rightarrow |z-w'_v| \le 2k^{1/2}.$$

Let us set $V_\mu=\{z:|z-w''_\mu| \le \delta k^{-N/2}\}$. We deduce from (3.27)

(3.28) $$\bigcup_{\mu \in M_k} V_\mu \subset \{z:|z-w'_v| \le 2k^{1/2}\}.$$

Now there is a bound for the number of V_μ which can intersect (since it is true for U'_μ and $V_\mu \subset U'_\mu$. Therefore (3.28) implies

$$C_1 \sum_{\mu \in M_k} m(V_\mu) \le m\left(\bigcup_{\mu \in M_k} V_\mu\right) \le C_0 k^{n/2}.$$

It follows that card M_k. $(\delta k^{-N/2})^n \le C k^{n/2}$. This completes the proof of Lemma 3.2. $\diamond$

We consider now the term (1) in (3.8). Using Theorem 3.3 in [3] we get

(3.29) $$\varphi^w_{v\chi} a^w_{v\chi} \varphi^w_{v\chi} = (\varphi_v^2 a_v)^w_\psi + r^w_{v\psi},$$

where $\psi=1$ on the support of χ, $r_v \in S(1,g)$ and

(3.30) $$|r_v^{(k)}(w)(w_1,\cdots,w_k)| \le C\prod_{i=1}^{k} g^{1/2}_w(w_i)\|a\|_{S(h^{-2},g)}.$$

Since $\left(\sum_v \varphi_v^2 a_v\right) = a^w_\psi$ we get

$$(a_\psi^w u,u)=\Big(\sum_\nu \varphi_{\nu\chi}^w a_{\nu\chi}^w \varphi_{\nu\chi}^w u,\, u\Big)-\Big(\Big(\sum_\nu r_\nu\Big)_\psi^w u,u\Big).$$

Lemma 3.3. *The operator $(\Sigma r_\nu)_\psi^w$ is bounded on L^2.*

If we prove this Lemma, it will follow from (3.8) and Lemma 3.2, that $(a_\psi^w u,u)\geq -C\|u\|_{L^2}^2$. Since, by Corollary 2.5 in [3], we have $a_\psi^w - a_\chi^w = r_\chi^w$ with $r\in S(1,g)$, we shall have $(a_\chi^w u,u)\geq -C\|u\|_{L^2}^2$ for every properly supported χ, which proves Theorem 2.1.

Proof of Lemma 3.3. We shall estimate the semi-norms in $S(1,g)$ of the symbol of $r_{\nu\psi}^w$ more precisely than (3.30). We split the proof into two cases. Let $w=(x,\xi)\in W$ be fixed.

Case 1: $g_{w_\upsilon}(w-U_\nu)=\inf_{w'\in U_\upsilon} g_{w_\upsilon}(w-w')\leq\rho.$

We know by the construction of the balls U_ν (see [5] Lemma 18.4.4) that there is at most N_0 (independent of w) such ν. By (3.30) we get

$$\sum_{\nu\geq 1}|r_\nu^{(k)}(w)(w_1,\cdots,w_k)|\leq N_0\cdot C\prod_{i=1}^k g_{w_\upsilon}^{1/2}(w_i)\|a\|_{S(h^{-2},g)}.$$

Case 2: $g_{w_\upsilon}(w-U_\nu)>\rho.$

Then $w\notin U_\nu$ and $\varphi_\nu^2 a_\nu(w)=0$. Let us set $\varphi_{\nu\chi}^w a_{\nu\chi}^w\varphi_{\nu\chi}^w=d_{\nu\psi}^w$; then by (3.18) in [3] and (3.29) it is easy to see that $r_\nu(w)=d_\nu(w)$. We want to show that for every k, l in $\boldsymbol{N}$,

$$|d_\nu^{(k)}(w)(w_1,\cdots,w_k)|\leq C_{kl}h^{-2}(w_\nu)\prod_{i=1}^k g_{w_\upsilon}^{1/2}(w_i)(1+g_{w_\upsilon}^\sigma(w-U_\nu))^{-l/2}, \tag{3.31}$$

where C_{kl} are independent of w, w_i and ν.

This will follow essentially from the proof of Theorem 2.2.1 in [1]. Indeed, according to Bony-Lerner, a symbol c is said to be confined in U_ν if for every integers k, l one can find constants C_{kl} such that for every X in U_ν and every w_i in W we have

$$|c^{(k)}(X)(w_1,\cdots,w_k)|\leq C_{kl}m(X)\prod_{i=1}^k g_{w_\upsilon}^{1/2}(w_i)(1+g_{w_\upsilon}^\sigma(X-U_\nu))^{-l/2}.$$

We shall show that since a_ν is supported in U_ν and φ_ν is confined in U_ν then $a_{\nu\chi}^w\varphi_{\nu\chi}^w=b_\nu^w$, with b_ν confined in U_ν. Then since φ_ν is supported in U_ν, the same argument will show that d_ν is confined in U_ν, which proves (3.31).

Accorting to (3.8) in [3] we have

$$b_\nu(X)=\pi^{-2n}\iint e^{2i\sigma(T-X,Z-X)}a_\nu(Z)\varphi_\nu(T)\chi(x+z-t,t+z-x)\cdot\chi(t+z-x,x+t-z)dZ\,dT, \tag{3.32}$$

where $X=(x,\xi)$, $T=(t,\tau)$, $Z=(z,\zeta)$.

Let $X_0=(x_0,\xi_0)$ be such that $g_{w_\upsilon}(X_0)=1$, $(g^\sigma_{w_\upsilon}(X))^{1/2}=\sigma(X,X_0)$. Then

$$\frac{1}{2}\langle X_0,D_Z\rangle(2i\sigma(T-X,Z-X))=\sigma(T-X,X_0)=[g^\sigma_{w_\upsilon}(T-X)]^{1/2}.$$

It follows that

$$\left(1+\frac{1}{2}\langle X_0,D_Z\rangle\right)^k e^{2i\sigma(T-X,Z-X)}=(1+g^\sigma_{w_\upsilon}(T-X)^{1/2})^k e^{2i\sigma(T-X,Z-X)}.$$

Therefore

$$b_\nu(X)=\pi^{-2n}\iint e^{2i\sigma(T-X,Z-X)}(1+g^\sigma_{w_\upsilon}(T-X)^{1/2})^{-k}\varphi_\nu(T)\left(1+\frac{1}{2}\langle X_0,D_Z\rangle\right)^k [a_\nu(Z)\chi(x+z-t,t+z-x)\chi(t+z-x,x+t-z)]dZ\,dT.$$

Now since φ_ν and a_ν are confined in U_ν and $\chi\in S(1,\tilde{G})$, the Leibniz formula will give

$$\begin{aligned}(3.33)\quad &|b_\nu(X)|\\ &\le C\|\varphi_\nu\|_{0,l}\|a_\nu\|_{k,N}\|\chi\|^2_{S(1,\tilde{G})}\iint(1+g^\sigma_{w_\upsilon}(T-X))^{-k/2}(1+g^\sigma_{w_\upsilon}(T-U_\nu))^{-l/2}\\ &\cdot(1+g^\sigma_{w_\upsilon}(T-U_\nu))^{-N}(G^{1/2}_{x+z-t}(x_0)+G^{1/2}_{t+z-x}(x_0)+G^{1/2}_{x+t-z}(x_0))^k dZ\,dT.\end{aligned}$$

Now in the integral defining b_ν, since a_ν is supported in U_ν, we get exactly as in (3.15)

$$\forall s\in\pi U_\nu\quad G_s(x-s)\le 2A_0^4\varepsilon+4A_0\rho\quad(\le\min(a_0,a_2)). \tag{3.34}$$

It follows that on the support of the function inside the integral in (3.32) we have

$$\begin{cases} G_{x+z-t}\le A_0G_z\\ G_{t+z-x}\le A_0G_z\\ G_{x+t-z}\le A_0G_z.\end{cases}$$

Since $z\in\pi U_\nu$ (because $Z\in\operatorname{supp}a_\nu\subset U_\nu$) we get $G_z\le A_0^2G_{x_\upsilon}$; by (2.3), $G_{x_\upsilon}\le g_{w_\upsilon}$, therefore

$$G^{1/2}_{x+z-t}(x_0)+G^{1/2}_{t+z-x}(x_0)+G^{1/2}_{x+t-z}(x_0)\le C(A_0)g_{w_\upsilon}(X_0)=C(A_0).$$

We deduce from (3.33),

(3.35) $|b_\nu(X)|$

$$\leq C\|\varphi_\nu\|_{0,l}\|a_\nu\|_{k,N}\|\chi\|^2_{S(1,\tilde{G})}\iint(1+g^\sigma_{w_\upsilon}(T-X))^{-k/2}(1+g^\sigma_{w_\upsilon}(T-U_\nu))^{-l/2} \cdot(1+g^\sigma_{w_\upsilon}(Z-U_\nu))^{-N}dZ\,dT.$$

Now

$$1+g^\sigma_{w_\upsilon}(X-U_\nu)\leq C(1+g^\sigma_{w_\upsilon}(X-T)+g^\sigma_{w_\upsilon}(T-U_\nu)).$$

$$1+g^\sigma_{w_\upsilon}(X-U_\nu)\leq C(1+g^\sigma_{w_\upsilon}(X-T))(1+g^\sigma_{w_\upsilon}(T-U_\nu)). \tag{3.36}$$

On the other hand for every w' in U_ν

$$1+g^\sigma_{w_\upsilon}(Z-w_\nu)\leq 1+2g_{w_\upsilon}(Z-w')+2g_{w_\upsilon}(w'-w_\nu)\leq 3+2g^\sigma_{w_\upsilon}(Z-w')$$

because $g\leq g^\sigma$ and the radius of U_ν is smaller than one. It follows that

$$(1+g^\sigma_{w_\upsilon}(Z-U_\nu))^{-1}\leq C(1+g_{w_\upsilon}(Z-w_\nu))^{-1}. \tag{3.37}$$

Taking in (3.35) $N=2n+1$, $k=l+2n+1$, we deduce from (3.36) and (3.37)

$$|b_\nu(X)|\leq C\|\varphi_\nu\|_{0,l}\|a_\nu\|_{2n+1+l,2n+1}(1+g^\sigma_{w_\upsilon}(X-U_\nu))^{-l/2} \cdot\iint(1+g^\sigma_{w_\upsilon}(X-T))^{-(n+1/2)}(1+g_{w_\upsilon}(Z-w_\nu))^{-(n+1/2)}dZ\,dT.$$

Since the product of the determinant of g_{w_υ} and $g^\sigma_{w_\upsilon}$ is equal to one and

$$\|\varphi_\nu\|_{0,l}\leq M,\quad \|a_\nu\|_{2n+1+l,2n+1}\leq C_{l,n}\sup_{X\in U_\upsilon}h^{-2}(X),$$

we get our claim. We estimate the derivatives by the same method. This proves (3.31).

Now, from the definition (2.7) of h we have for every w' in U_ν,

$$g^\sigma_{w_\upsilon}(w-w')\geq h^{-2}(w_\nu)g_{w_\upsilon}(w-w').$$

Since we are in case 2 it follows that

$$g^\sigma_{w_\upsilon}(w-U_\nu)\geq \rho h^{-2}(w_\nu). \tag{3.38}$$

Now, let $w'_0=(x'_0,\xi'_0)\in U_\nu$ be such that

$$g^\sigma_{w_\upsilon}(w-U_\nu)=g^\sigma_{w_\upsilon}(w-w'_0). \tag{3.39}$$

It follows from (3.34) that $G_{x'_0}(x-x'_0)\leq a_2$ and from (2.8)

$$g_{w_0'} \leq A_2 g_w (1 + g^{\sigma}_{w_0'}(w - w_0'))^N.$$

By the slow variation of of g in U_v we deduce

$$g_{w_v} \leq A_1 A_2 g_w (1 + A_1 g^{\sigma}_{w_v}(w - w_0'))^N.$$

From (3.31), (3.38) and (3.39) we get

$$|r_v^{(k)}(w)(w_1, \cdots, w_k)| \leq C_{klN} \prod_{i=1}^{k} g_w^{1/2}(w_i)(1 + g^{\sigma}_{w_v}(w - U_v))^{-\frac{l}{2} + \frac{N}{2} + 1}. \tag{3.40}$$

Assume $w \in U_\lambda$. Using the slow variation of g in U_λ and (3.34) we get

$$g^{\sigma}_{w\lambda} \leq A_1 g^{\sigma}_w \leq A_1 A_2 g^{\sigma}_{w_0'}(1 + g^{\sigma}_{w_0'}(w - w_0'))^N$$

and therefore

$$g^{\sigma}_{w\lambda}(w - w_0') \leq A_1 A_2 (1 + A_1 g^{\sigma}_{w_v}(w - w_0'))^{N+1}.$$

$$1 + \inf_{\substack{w' \in U_v \\ w'' \in U_\lambda}} g^{\sigma}_{w_\lambda}(w' - w'') \leq C(A_1, A_2)(1 + g^{\sigma}_{w_v}(w - U_v))^{N+1}. \tag{3.41}$$

Recall that we defined in (3.23) $d_{\lambda v}$ by

$$d_{\lambda v} = \inf_{\substack{w' \in U_v \\ w'' \in U_\lambda}} g^{\sigma}_{w''}(w' - w'').$$

By (3.40), (3.41) and the slow variation of g on U_λ we get for every $k, q \in N$

$$|r_v^{(k)}(w)(w_1, \cdots, w_k)| \leq C_{kq} \prod_{i=1}^{k} g_w^{1/2}(w_i)(1 + d_{\lambda v})^{-q}. \tag{3.42}$$

Using the second part of (3.10) we obtain that the semi-norms of Σr_v are uniformly bounded in $S(1, g)$; it follows that the operator $(\Sigma r_v)^w_\psi$ is bounded on L^2, by Theorem 4.3 in [3]. The proof is complete. $\diamond$

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