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Osaka J. Math.
33 (1996), 793–804

EQUIVARIANT LEFSCHETZ CLASSES IN ALEXANDER - SPANIER COHOMOLOGY

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(Received April 14, 1995)

1. Introduction and statement of results

Let G be a finite group, X a paracompact (and Hausdorff) G-space and $f: X \to X$ a G-map. Then the fixed point subspaces X^H of X for all subgroups $H \le G$ are also paracompact, being closed in X, and f induces maps $f^H: X^H \to X^H$, $H \le G$. To be able to define the Lefschetz number $L(f^H) \in \boldsymbol{Z}$ of f^H with respect to Alexander-Spanier cohomology with integer coefficients we require

Assumption 1.1. *The Alexander-Spanier cohomology $\bar{H}^{\cdot}(X^H; \boldsymbol{Z})$ is finite for every $H \le G$.*

Here a graded abelian group (or module) $(A^i)_{i \in \boldsymbol{N}}$ is called finite, provided that each A^i is finitely generated and $A^i = 0$ for all sufficiently large i.

Under the assumption 1.1 we can define

$$L(f^H) = \sum_{i \in \boldsymbol{N}} (-1)^i \mathrm{tr}[(f^H)^* : \bar{H}^i(X^H; \boldsymbol{Z}) \to \bar{H}^i(X^H; \boldsymbol{Z})] \in \boldsymbol{Z}, \qquad H \le G,$$

as usual. The number $L(f^H)$ depends only on the conjugacy class of H, so we get an element

$$L(f) = (L(f^H)) \in \prod_{\psi(G)} \boldsymbol{Z},$$

where $\psi(G)$ is the set of conjugacy classes of all subgroups of G.

On the other hand, we have the standard embedding

$$\chi = (\chi_H) : A(G) \to \prod_{\psi(G)} \boldsymbol{Z}$$

of the Burnside ring $A(G)$ of G; the component $\chi_H : A(G) \to \boldsymbol{Z}$ is determined by

[1] The second author passed away on August 24, 1996.

$\chi_H([S])=|S^H|$ for the class $[S]\in A(G)$ of a finite G-set S. We identify $A(G)$ with the subring $\mathrm{Im}(\chi)$ of $\Pi\boldsymbol{Z}$. Then it is well-known that a family $d=(d_H)\in\Pi\boldsymbol{Z}$ lies in $A(G)$ if and only if it satisfies the congruences

$$d_H\equiv-\sum_K\varphi(|K/H|)d_K \mod |WH|, \qquad H\leq G, \tag{1.2}$$

where φ is the Euler function and $WH=NH/H$; the summation is over $K\leq G$ such that H is a proper normal subgroup of K and K/H is cyclic.

We intend to prove that, under a suitable additional condition on X, the *equivariant Lefschetz class* $L(f)$ of f lies in $A(G)$, so that in particular the Lefschetz numbers $d_H=L(f^H)$ satisfy the congruences 1.2. It is clear that some finiteness condition on X is needed for this, as the case of the contractible free G-space $X=EG$ shows.

We formulate the finiteness condition we need in terms of the equivariant Alexander-Spanier cohomology $\bar{H}_G^{\cdot}$ of X, cf. [4].

Assumption 1.3. *There is an* $n\in\boldsymbol{N}$ *such that* $\bar{H}_G^{n+1}(X;m)=0$ *for any coefficient system* $m:\mathrm{Or}(G)^{\mathrm{op}}\to\boldsymbol{Ab}$.

REMARK 1.4. a) Assumption 1.3 holds if the covering dimension of X (and X/G) is finite. This is due to the fact that $\bar{H}_G^{\cdot}(X;m)$ can be interpreted as the ordinary cohomology of X/G with coeffecients in a certain non-constant sheaf, see Section 6 of [4].

b) In 1.3 it would suffice to consider only coefficient systems m such that, for some $K\leq G$, $m(G/K)$ is a finitely generated free abelian group and $m(G/H)=0$ for $G/H\not\geq G/K$ (see Section 3 below).

Theorem 1.5. *If G is finite and the paracompact G-space X satisfies* 1.1 *and* 1.3, *then* $L(f)\in A(G)$ *for ever G-map* $f:X\to X$.

Corollary 1.6. *In the situation of* 1.5,

$$L(f^H)\equiv-\sum_K\varphi(|K/H|)L(f^K) \mod |WH|, \qquad H\leq G,$$

where the summation is as in 1.2. □

The proof of 1.5 is carried out in Sections 2 and 3.

Let now G be a compact Lie group. We write $H\leq G$ to indicate that H is a closed subgroup of G. Let X be a paracompact G-space which satisfies 1.1, and $f:X\to X$ a G-map. Then we can again define an element

$$L(f)=(L(f^H))\in\prod_{\psi(G)}\boldsymbol{Z};$$

here $\psi(G)$ is the set of conjugacy classes of all closed subgroups of G. In Section 4 we shall use 1.5 to prove

Theorem 1.7. *Suppose G is a compact Lie group and the paracompact G-space X has finite covering dimension, finite orbit type and satisfies* 1.1. *Then $L(f)\in A(G)$ for every G-map $f\colon X\to X$.*

We also prove a Lefschetz fixed point formula in Section 4.

Theorem 1.8. *In the situation of* 1.7, *the Lefschetz number*

$$L(g,X)=\sum_{i\geq 0}(-1)^i\mathrm{tr}(g:\bar{H}^i(X^H;\boldsymbol{Z})\to\bar{H}^i(X^H;\boldsymbol{Z}))$$

of the left translation by $g\in G$ is equal to the Euler characteristic $\chi(X^C)$, where C is the closure or the subgroup generated by g.

The final Section 5 gives an application to fixed point sets of finite group actions on contractible spaces.

To end this introduction, we discuss some earlier work on equivariant Lefschetz classes. The equivariant Euler characteristic of a finite group acting on a locally compact space of finite covering dimension was treated by Verdier in [14] by using sheaf cohomology and derived category techniques. Oliver proved Theorem 1.8 for finite groups in his thesis [9], but the result was never published. In [2] tom Dieck discusses the equivariant Euler characteristic of a compact G-ENR, where G is a compact Lie group, using multiplicativity of the Euler characteristic in relative fibrations. He also poses the question of finding the most general assumptions on the space X for which the Lefschetz numbers of a self-G-map decompose as in Corollary 1.6.

In [6] it is shown that an equivariant self-map of a finite G-CW-complex X, where G is a finite group, has an equivariant Lefschetz class in $A(G)$. The construction of [6] uses cellular homology of the fixed point sets of X. In [8] it is proved that, for a compact Lie group G, any equvariant self-map of a finite-dimensional G-CW-complex X of finite orbit type has an equivariant Lefschetz class in $A(G)$, provided that X satisfies 1.1. The second author has also shown (unpublished) that the construction of an equivariant Lefschetz class is possible for more general G-spaces, if one uses singular homology; the needed finiteness assumption analogous to 1.3 is then formulated in terms of equivariant singular cohomology of X.

All these previous constructions are based on being able to calculate the traces appearing in the definition of the Lefschetz class on the chain level. In this paper, the lack of a naturally defined homology dual to Alexander-Spanier cohomology forces us to use the dualization process of Section 3.

This work was completed while the second author was visiting the Mittag-Leffler Institute, Djursholm, Sweden. He wishes to thank for the hospitality.

2. Reduction to the case of a single orbit type

In this section, and in section 3 below, G is finite, X is a paracompact G-space and $f: X \to X$ is a G-map. Let $(H_1), (H_2), \cdots, (H_r)$ be the distinct conjugacy classes of subgroups of G, ordered in such a way that

$$(H_i) < (H_j) \Rightarrow i > j$$

(in particular, $H_1 = G$ and $H_r = \{e\}$, the trivial subgroup). For $j \in \{1, 2, \cdots, r\}$ we set

$$X^{(H_j)} = GX^{H_j} = \{x \in X | (H_j) \leq (G_x)\},$$
$$X_j = X^{(H_1)} \cup X^{(H_2)} \cup \cdots \cup X^{(H_j)}.$$

Then

$$\varnothing = X_0 \subset X_1 \subset \cdots \subset X_{r-1} \subset X_r = X$$

are closed G-subsets of X, and $x \in X_j \backslash X_{j-1}$ if and only if $(G_x) = (H_j)$. The G-map $f: X \to X$ restricts to G-maps $f_j: X_j \to X_j$ $(0 \leq j \leq r)$.

Assume now that X satisfies 1.1. Denote $\bar{H}^{\cdot}(\cdot) = \bar{H}^{\cdot}(\cdot\,; \boldsymbol{Z})$.

Lemma 2.1. *$\bar{H}^{\cdot}(X_j^H)$ is finite for every $H \leq G$ and $j \in \{1, 2, \cdots, r\}$.*

Proof. This is done by induction, the case $j = 0$ being trivial. Let $j \in \{1, 2, \cdots, r\}$ and denote $A = X_j$, $B = X_{j-1}$, $K = H_j$. By induction, $\bar{H}^{\cdot}(B^H)$ is finite for every $H \leq G$, and we must show that $\bar{H}^{\cdot}(A^H)$ is finite for every $H \leq G$.

In Lemma 4.2 of [5] it was shown that the G-map

$$G/K \times_{WK} (A^K, B^K) \to (A, B), \qquad [gK, a] \mapsto ga,$$

induces an isomorphism

$$\bar{H}^{\cdot}((A, B)^H) \xrightarrow{\simeq} \bar{H}^{\cdot}((G/K \times_{WK} (A^K, B^K))^H)$$

for every $H \leq G$ (here $WK = NK/K$); this fact is based on the strong excision property of Alexander-Spanier cohomology on paracompact spaces. Similarly, for $H \leq G$, the natural map

$$(G/K)^H \times_{WK}(A^K, B^K) \to (G/K \times_{WK}(A^K, B^K))^H$$

induces an isomorphism in Alexander-Spainer cohomology (cf. 6.4 of [5]). Taking 6.6 of [5] into account, too, we see that

$$\bar{H}^{\cdot}(A^H, B^H) \cong \prod_S \bar{H}^{\cdot}(A^K, B^K), \qquad S=(G/K)^H/WK, \qquad H \leq G. \tag{2.2}$$

The exact cohomology sequence of the pair $(A^K, B^K)=(X^K, B^K)$ implies that $\bar{H}^{\cdot}(A^K, B^K)$ is finite, and so, by 2.2, $\bar{H}^{\cdot}(A^H, B^H)$ is finite for $H \leq G$. Finally, by the cohomology sequence of the pair (A^H, B^H), $\bar{H}^{\cdot}(A^H)$ is finite. □

Let us denote $h_j=(f_j, f_{j-1}):(X_j, X_{j-1}) \to (X_j, X_{j-1})$, $L(h_j)=(L(h_j^H)) \in \Pi \boldsymbol{Z}$ and

$$L(h_j^H)=\sum_{i \in \boldsymbol{N}}(-1)^i \operatorname{tr}[(f_j^H, f_{j-1}^H)^*: \bar{H}^i(X_j^H, X_{j-1}^H) \to \bar{H}^i(X_j^H, X_{j-1}^H)]$$

$(H \leq G,\ 1 \leq j \leq r)$. From the cohomology sequences of the pairs (X_j^H, X_{j-1}^H), $H \leq G$, we see that

$$L(f_j)=L(f_{j-1})+L(h_j), \qquad 1 \leq j \leq r.$$

Hence, to prove that $L(f)=L(f_r) \in A(G)$, it is enough to show that $L(h_j) \in A(G)$ for every $j \in \{1, 2, \cdots, r\}$.

Pick $j \in \{1, 2, \cdots, r\}$ and denote $A=X_j$, $B=X_{j-1}$ $K=H_j$ as in the proof of 2.1, and $h=h_j:(A, B) \to (A, B)$. It follows from 2.2 that, for $H \leq G$,

$$L(h^H)=\frac{|(G/K)^H|}{|WK|} \cdot L(h^K). \tag{2.3}$$

In Section 3 below we shall prove

Lemma 2.4. *If X satisfies* 1.1 *and* 1.3, *then $L(h^K) \in \boldsymbol{Z}$ is divisible by $|WK|$. Moreover, if G is a cyclic group generated by g, then $L(g^K)=\chi(X^G)$ if $K=G$ and* 0 *otherwise.*

The proof of Theorem 1.5 is now clear. Namely, it follows from 2.3 that

$$L(h)=\frac{L(h^K)}{|WK|} \cdot [G/K] \in A(G);$$

thus $L(h_j) \in A(G)$ for every $j \in \{1,2,\cdots,r\}$, and, by the above remarks, this implies that $(f) \in A(G)$.

Similarly, Theorem 1.8 follows in the case where G is finite.

3. Proof of Lemma 2.4

Assume that the paracompact G-space X satisfies 1.1 and 1.3. Let A, B, K and $h:(A,B)\to(A,B)$ be as in 2.4. We now begin the proof of the first assertion of Lemma 2.4, that is, the divisibility of $L(h^K)\in \boldsymbol{Z}$ by by $|WK|$.

We consider the cochain complex $C^{\cdot}=\bar{C}^{\cdot}(A^K,B^K;\boldsymbol{Z})$, the Alexander-Spanier cochain complex of (A^K,B^K) with integer coefficients. Let $D.=D(C^{\cdot})=\mathrm{Hom}_{\boldsymbol{Z}}(C^{\cdot},\boldsymbol{Z}^{\cdot})$ be the dual chain complex of $C^{\cdot}$ in the sense of [7, Section 1]; here $\boldsymbol{Z}^{\cdot}$ is the canonical injective resolution

$$\text{(3.1)}\qquad \boldsymbol{Q}\to\boldsymbol{Q}/\boldsymbol{Z}$$

of $\boldsymbol{Z}$, regarded as a cochain complex with $\boldsymbol{Q}$ in degree zero. The action of WK on (A^K,B^K) makes $C^{\cdot}$ and $D.$ into complexes of $\boldsymbol{Z}WK$-modules. Since $H^i(C^{\cdot})$ is finitely generated over $\boldsymbol{Z}$ for every i (by 1.1), the natural morphism $C^{\cdot}\to D(D.)=\mathrm{Hom}_{\boldsymbol{Z}}(D.,\boldsymbol{Z}.)$ is a quasi-isomorphism by 1.3 of [7]; here $\boldsymbol{Z}.$ is 3.1 regarded as a chain complex with again $\boldsymbol{Q}$ in degree zero.

Let $\alpha:C^{\cdot}\to C^{\cdot}$ be the cochain map induced by $h^K:(A^K,B^K)\to(A^K,B^K)$, and $\beta:D.\to D.$ the dual chain map. Let

$$L(\alpha)=\sum_{i\in\boldsymbol{N}}(-1)^i\mathrm{tr}[H^i(\alpha):H^i(C^{\cdot})\to H^i(C^{\cdot})],$$

$$L(\beta)=\sum_{i\in\boldsymbol{N}}(-1)^i\mathrm{tr}[H^i(\beta):H^i(D.)\to H^i(D.)].$$

We have $L(h^K)=L(\alpha)=L(\beta)$, because $H.(D.)$ and $\mathrm{Hom}_{\boldsymbol{Z}}(H^{\cdot}(C^{\cdot}),\boldsymbol{Z})$ are isomorphic modulo torsion by 1.2 of [7]. We intend to compute $L(\beta)$ on the chain level by choosing a suitable projective resolution of $D.$.

Because each $H_i(D.)$ is finitely generated, there is, by standard homological algebra, a $\boldsymbol{Z}WK$-projective resolution $E.\to D.$ of $D.$ such that each E_i is a finitely generated $\boldsymbol{Z}WK$-module. In particular, each E_i is a finitely generated free abelian group. We claim that $E.$ can be replaced with a resolution $F.\to D.$ with the additional property that $F_i=0$ whenever $i>n$; here $n\in\boldsymbol{N}$ is as in 1.3. The obstruction to finding such a resolution $F.$ lies in $\mathrm{Ext}^{n+1}_{\boldsymbol{Z}WK}(D.,B_n)$, where $B_n\subset E_n$ is the submodule of n-boundaries (see Brown [1, VIII(2.1)]). Therefore it is enough to prove the

Assertion 3.2. *Under the assumptions* 1.1 *and* 1.3, $\mathrm{Ext}^{n+1}_{\boldsymbol{Z}WK}(D.,M)=0$ *for any finitely generated* $\boldsymbol{Z}WK$*-module* M, *which is free as an abelian group.*

Proof. Take a $\boldsymbol{Z}WK$-projective resolution $P.\to\boldsymbol{Z}$ of $\boldsymbol{Z}$. By the Künneth formula, $P.\otimes_{\boldsymbol{Z}}E.\to\boldsymbol{Z}\otimes_{\boldsymbol{Z}}E.\cong E.\to D.$ is a resolution of $D.$, and the $\boldsymbol{Z}WK$-modules $(P.\otimes_{\boldsymbol{Z}}E.)_i$ are projective. Thus we have

$$\begin{aligned}\operatorname{Ext}^{n+1}_{ZWK}(D_\cdot, M) &= H^{n+1}(\operatorname{Hom}_{ZWK}(P_\cdot \otimes_Z E_\cdot, M))\\ &\cong H^{n+1}(\operatorname{Hom}_{ZWK}(P_\cdot, \operatorname{Hom}_Z(E_\cdot, M)))\\ &= \operatorname{Ext}^{n+1}_{ZWK}(Z, \operatorname{Hom}_Z(E_\cdot, M)).\end{aligned}$$

Because M is a finitely generated abelian group, $\operatorname{Hom}_Z(E_\cdot, M) \cong \operatorname{Hom}_Z(E_\cdot, Z) \otimes_Z M$. By comparing the universal coeffcient sequence of [7, 1.2], for $D(D_\cdot)$ and the ordinary universal coefficient sequence for $\operatorname{Hom}_Z(E_\cdot, Z)$, we see that the natural map $D(D_\cdot) \to \operatorname{Hom}_Z(E_\cdot, Z)$ is a quasi-isomorphism. Thus $\operatorname{Hom}_Z(E_\cdot, Z) \otimes_Z M$ is quasi-isomorphic to $C^\cdot \otimes_Z M$. Because M is finitely generated free abelian, this complex is in turn isomorphic to $\bar{C}^\cdot(A^K, B^K; M)$. Therefore

$$\begin{aligned}\operatorname{Ext}^{n+1}_{ZWK}(D_\cdot, M) &\cong \operatorname{Ext}^{n+1}_{ZWK}(Z, \bar{C}^\cdot(A^K, B^K; M))\\ &= H^{n+1}(WK; \bar{C}^\cdot(A^K, B^K; M)).\end{aligned}$$

Let $m: \operatorname{Or}(G)^{\mathrm{op}} \to \boldsymbol{Ab}$ be a coefficient system such that $m(G/K) = M$ and $m(G/H) = 0$ for $G/H \not\cong G/K$. By 4.5 and 5.1 of [5] we have

$$H^{n+1}(WK; \bar{C}^\cdot(A^K, B^K; M)) \cong \bar{H}^{n+1}_G(A, B; m).$$

Next we prove the following

Subassertion 3.3. *$\bar{H}^\cdot_G(A, B; m) \cong \bar{H}^\cdot_G(X; m)$ for the coefficient system m chosen above.*

Proof. In fact we show that the equivariant Alexander-Spanier cochain complexes $\bar{C}^\cdot_G(A, B; m)$ and $\bar{C}^\cdot_G(X; m)$ are isomorphic.

First of all, it is clear that $\bar{C}^\cdot_G(X; m) \cong \bar{C}^\cdot_G(A; m)$, because $X^H = A^H$ whenever $m(G/H) \neq 0$ (i.e. $(H) = (K)$). To show that $\bar{C}^\cdot_G(A; m) \cong \bar{C}^\cdot_G(A, B; m)$, we must prove that $\bar{C}^\cdot_G(B; m) = 0$. By Section 3 of [5], an element of $\bar{C}^i_G(B; m)$ can be identified with a family $\gamma = (\gamma_H)_{H<G}$ of ordinary cochains $\gamma_H \in \bar{C}^i(B^H; m(G/H))$ such that for any G-map $u: G/H \to G/H'$, the compatibility condition

$$B(u)^*(\gamma_H) = m(u)_*(\gamma_{H'}) \in \bar{C}^i(B^{H'}; m(G/H)) \tag{3.4}$$

holds; here $B(u): B^{H'} \to B^H$ is the map induced by u. By the choice of m, $\gamma_H = 0$ automatically, if $(H) \neq (K)$. To show that $\gamma = 0$, it is enough to see that $\gamma_K = 0$. But by (3.4), γ_K restricts to zero in $\bar{C}^i(B^{H'}; m(G/K))$ for any $H' > K$; because

$$B^K = \bigcup_{H'>K} B^{H'},$$

the claim $\gamma_K = 0$ follows. This finishes the proof of the Subassertion. □

We have now shown that

$$\operatorname{Ext}^{n+1}_{ZWK}(D_\cdot, M) \cong \bar{H}^{n+1}_G(X; m).$$

This latter group vanishes by 1.3, so the Assertion is proved. □

Choose now a resolution $F. \to D.$ such that each F_i is a finitely generated projective ZWK-module and $F_i=0$ for $i>n$. The chain map $\beta: D. \to D.$ can be lifted to a chain map $\beta': F. \to F.$. Because each F_i is a finitely generated abelian group, we can compute the Lefschetz number $L(\beta)$ we are interested in as follows:

$$L(\beta)=L(\beta')=\sum_{i\in N}(-1)^i \mathrm{tr}[H_i(\beta'): H_i(F.) \to H_i(F.)]$$

$$=\sum_{i\in N}(-1)^i \mathrm{tr}[\beta'_i: F_i \to F_i].$$

Hence, to prove that $L(\beta)$ is divisible by $|WK|$, it is enough to show that $\mathrm{tr}(\beta'_i)$ has this property for every $i\in N$. Since F_i is ZWK-projective, there is a finitely generated ZWK-module F'_i such that $F_i \oplus F'_i$ is ZWK-free. It follows that

$$\mathrm{tr}(\beta'_i)=\mathrm{tr}[\beta'_i \oplus 0: F_i \oplus F'_i \to F_i \oplus F'_i]$$

is divisible by $|WK|$. This concludes the proof of the first claim in Lemma 2.4, finishing at the same time the proof of Theorem 1.5.

Let finally G be cyclic generated by g and let β correspond to g^K. By equivariance, β_i and β'_i act through multiplication by g as an element of the base ring ZWK of the modules D_i and F_i. By a theorem of Swan ([13, Theorem 8.1]), the module $Q\otimes F_i$ is free over QWK for each i. Hence β'_i permutes freely a Q-base of $Q\otimes F_i$ and therefore $\mathrm{tr}(\beta'_i)=0$ if $WK=G/K$ is nontrivial. It follows that $L(\beta)=L(\beta')=\chi(X^G)$ if $K=G$ and 0 otherwise. Using (2.3) and adding up gives Theorem 1.8 in case G is finite.

4. The case of a compact lie grop G

Let G be a compact Lie group. We use the standard embedding

$$A(G) \to \prod_{\psi(G)} Z$$

(see tom Dieck [3, Section IV.5]) to identify the Burnside ring $A(G)$ of G with a subring of ΠZ. Elements of ΠZ are regarded as functions $\psi(G) \to Z$, with $\psi(G)$ topologized via the Hausdorff metric ([3, Section IV.3]). It is well-known that a family $d=(d_H)\in \Pi Z$ lies in $A(G)$ if and only if it satisfies the following conditions:

i) d is continuous;

(4.1) ii) $d_H=d_K$ if $H \trianglelefteq K \leq G$ and K/H is a torus;

iii) $d_H \equiv -\sum_K \varphi(|K/H|)d_K \bmod |L/H|$ if $H \trianglelefteq L \leq G$ and L/H is finite;

in iii) the summation is over $K \leq L$ such that H is a proper normal subgroup of K and K/H is cyclic.

Assume now that the hypotheses of Theorem 1.7 hold and $f: X \to X$ is a G-map. To prove 1.7, we must verify that $L(f) \in \Pi \boldsymbol{Z}$ satisfies conditions 4.1.i), ii) and iii).

For i), let $H_i \to H$ with respect to the Hausdorff metric in $\psi(G)$. Because X has finite orbit type, it follows from [3, IV (3.3) and (3.4)], that $X^{H_i} = X^H$ for all sufficiently large i. Thus $L(f^{H_i}) = L(f^H)$ for all sufficiently large i, proving 4.1.i) for $L(f)$.

To prove 4.1.iii) for $L(f)$, let $H \trianglelefteq L \leq G$ such that L/H is finite. Then $L \leq NH$, so X^H can be regarded as an L/H-space. As such X^H satisfies 1.1, and also 1.3 (because X^H has finite covering dimension). Applying Theorem 1.5 to the L/H-map $f^H: X^H \to X^H$, we obtain the congruence 4.1.iii).

Finally, to prove 4.1.ii) for $L(f)$, let $H \trianglelefteq K \leq G$ and assume that K/H is a torus. In fact it is enough to consider the case $K/H \cong S^1$. Let p be a prime number. We can find a sequence of subgroups $K_i \in \psi(G)$ such that $H \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \leq K$, $K_i/H \cong \boldsymbol{Z}/p^i\boldsymbol{Z}$ for every i and $K_i \to K$ in $\psi(G)$. As in the proof of 4.1.i) above, we have $L(f^K) = L(f^{K_i})$ for sufficiently large i. On the other hand, taking $L = K_1$ in 4.1.iii), we see that $L(f^H) \equiv L(f^{K_1}) \bmod p$, and similarly $L(f^{K_1}) \equiv L(f^{K_2}) \bmod p$, etc. Thus we see that $L(f^H) \equiv L(f^K) \bmod p$. This holds for every prime p, whence the assertion $L(f^H) = L(f^K)$ follows.

We shall conclude by showing how Theorem 1.8 reduces to the finite group case treated in Section 3. We may assume that G is topologically cyclic generated by an element g acting on X. As the extension $G_0 \to G \to \pi_0(G)$ is split we can choose a sequence of elements g_i of finite order in the component of g in $\pi_0(G)$ such that the subgroups H_i generated by the g_i converge to G in the Hausdorff metric. Then the left translations by g_i and g have the same Lefschetz numbers by homotopy invariance and $\chi(X^{H_i}) = \chi(X^G)$ for sufficiently large i by continuity. Thus the Lefschetz number of g equals the Euler characteristic $\chi(X^G)$ by the finite group case. This proves Theorem 1.8.

5. Actions on contractible spaces

In this section we study the fixed point sets of finite group actions on finite dimensional contractible spaces. We require that all fixed point sets have finite cohomology in the sense of Assumption 1.1. The result is a variation of Oliver's theory [10], which classifies the fixed point sets of actions on finite contractible complexes.

Let p and q be primes.

Lemma 5.1. *Let a finite group G have normal subgroups $P \trianglelefteq H \trianglelefteq G$ such that P is of p-power order, G/H is of q-power order and H/P is cyclic. If G acts* *on an acyclic finite dimensional paracompact space X so that the fixed point sets X^K have finite cohomology for every $K \leq G$, then*

$$\chi(X^H)=1 \quad \textit{and} \quad \chi(X^G) \equiv 1 \pmod{q}.$$

REMARK. Cohomology refers to integral Alexander-Spanier (alias Čech) cohomology. In the examples below the fixed point sets are either CW-complexes or manifolds, so that singular cohomology can be used as well.

Proof. Since P is a p-group and X is acyclic, finite dimensional and paracompact, X^P is Z_p-acyclic by Smith theory. As the integral cohomology of X^P is finitely generated, it consists of torsion. The Lefschetz fixed point formula 1.8 for the cyclic group H/P acting on X^P gives $\chi(X^H)=1$. Finally, either by Smith theory again, or by the congruences 1.6 applied to the G/H-space X^H, we have $\chi(X^G) \equiv 1 \pmod{q}$. □

Let $\mathcal{G}_p^q$ be the class of finite groups G having normal subgroups $P \trianglelefteq H \trianglelefteq G$ as in Lemma 5.1. Let $\mathcal{G}_p^1$ and $\mathcal{G}_1^q$ be the classes of such G where $H=G$ and $P=\{e\}$, respectively, and denote $\mathcal{G}^q = \bigcup_{p \geq 1} \mathcal{G}_p^q$, $\mathcal{G} = \bigcup_{q \geq 1} \mathcal{G}^q$. For a group G not of prime power order, we define an integer $m(G)$ as follows:

DEFINITION 5.2. If $G \in \mathcal{G}^1$, then $m(G)=0$. If $G \notin \mathcal{G}^1$, then $m(G)$ is a product of distinct primes, and $q \mid m(G)$ if and only if $G \in \mathcal{G}^q$. In particular, $m(G)=1$ if and only if $G \notin \mathcal{G}$.

This coincides with Oliver's number $m(G)$ by Theorem 5 of [10]. Notice that the assertion of Lemma 5.1 can now be reformulated as $\chi(X^G) \equiv 1 \pmod{m(G)}$. The following theorem gives a converse :

Theorem 5.3. *Let G be a finite group not of prime power order and let F be a finite demensional complex with finite cohomology. Then F is the fixed point set X^G of an action of G on some finite demensional contractible complex X with $H^{\cdot}(X^H; Z)$ finite for every $H \leq G$ if and only if $\chi(F) \equiv 1 \pmod{m(G)}$.*

Proof. The necessity of the condition follows from Lemma 5.1. Conversely, if $\chi(F) \equiv 1 \pmod{m(G)}$, then Oliver shows that F can be embedded equivariantly into a G-resolution X of F by attaching finitely many G-cells to F (Theorem 2 of [10]). This means that $X^G=F$, X is n-dimensional, $(n-1)$-connected (for some $n \geq 2$), and $H_n(X)$ is a projective ZG-module. Adding countably many free n-cells, $H_n(X)$ can be made into a free ZG-module, and then free $(n+1)$-cells may be added to produce a contractible complex with fixed point set F. By construction, all fixed point sets have finite cohomology. □

In the case where F is finite, Oliver shows further that there is a projective obstruction to realizing X as a finite complex. This leads to an integer n_G, defined for each group G not of prime power order, such that a finite complex F is the fixed point set of an action of G on some finite contractible complex if and only if $\chi(F)\equiv 1 \pmod{n_G}$ ([10, Cor. p. 167]). The integer $m(G)$ divides n_G and they agree for certain 2-hyperelementary metacyclic groups G having $m(G)=2$ and $n_G=4$ ([10, Cor. p. 171],[11, Th. 7]).

Given a smooth action of a finite group G on a euclidean space V, it is natural to ask for a geometric condition on the G-space V to guarantee the finiteness of the cohomology of the fixed point sets V^H. One possibility is to require that V admits a smooth compactification V^*, i.e. a compact smooth G-manifold containing V as an open invariant dense set such that $V^*\setminus V$ is a smooth G-submanifold. Such a V^* could be e.g. a disk, a projective space or a sphere. If V has a smooth compactification V^*, then V is actually equivariantly homotopy equivalent to a finite G-complex : the complement of an open regular neighbourhood of $V^*\setminus V$ in V^* with respect to some G-triangulation is a G-deformation retract of V. Thus the non-finite homotopy types of Theorem 5.3 cannot occur in this case. Quinn has realized them as locally linear actions on a disk. The following example is taken from [12, pp. 363–365].

EXAMPLE 5.4. Let G be the finite group of order 60 generated by a and b with the relations $a^{15}=b^4=1$ and $bab^{-1}=a^2$. By Oliver [11, p.263], $n_G=4$ but $m(G)=2$. Hence every finite contractible G-complex X has $\chi(X^G) \pmod 4$. Quinn constructs a locally linear action of G on a disk D, smooth outside a fixed boundary point $x\in(\partial D)^G$, such that $\chi(D^G)\equiv 3 \pmod 4$. Notice that the fixed point sets D^H are locally flat submanifolds of D, in particular compact topological manifolds, and hence have finite cohomology. However, D is not equivariantly homotopy equivalent to a finite G-complex.

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