Shibuya, T.
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SELF Δ - EQUIVALENCE OF RIBBON LINKS

TETSUO SHIBUYA

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1. Introduction

Throughout this paper, knots and links mean tame and oriented ones in an oriented 3-space R^3.

Several properties of local moves of links were studied in many papers, for example, homotopy of cobordant links in [3], [4] and boundary links in [1], [5], Δ-equivalence of links in [6], [7] and self #-equivalences of links in [8], [9] and [10].

In this paper, we investigate the self Δ-equivalence of ribbon links.

For a link ℓ, let B^3 be a 3-ball such that $\ell \cap B^3$ is a tangle illustrated in Fig. 1(a). The local move from Fig. 1(a) to (b) is called a Δ-*move*. Especially if these arcs are contained in the same component of ℓ, this move is called *a self* Δ *-move*. For two links ℓ and ℓ', if ℓ can be deformed into ℓ' by a finite se-

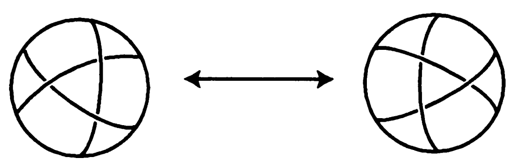

Fig. 1

quence of self Δ-moves, we say that ℓ and ℓ' are *self* Δ*-equivalent* or that ℓ is *self* Δ*-equivalent to* ℓ'.

An n-component link $\ell = k_1 \cup \cdots \cup k_n$ is called *a ribbon link* if there are n disks $\mathscr{C} = C_1 \cup \cdots \cup C_n$ in R^3 with $\partial\mathscr{C} = \ell$, $\partial C_i = k_i$, such that the singularity of $\mathscr{C}$, denoted by $\mathscr{S}(\mathscr{C})$, consists of mutually disjoint simple arcs of ribbon type, Fig. 2(b). For an arc β of $\mathscr{S}(\mathscr{C})$ in Fig. 2(b), the pre-images β^* and β'^* of β in Fig. 2(a) are called the *i-line* and the *b-line* respectively.

In this paper, we shall prove the following.

Theorem. *Any ribbon link is self* Δ*-equivalent to a trivial link.*

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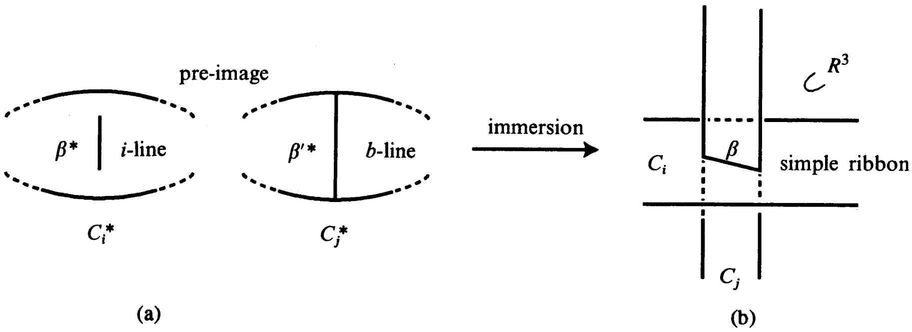

Fig. 2.

2. Deformations of links by self Δ-moves

The deformations of the following local moves from Fig. 3(a) to 3(d) accomplish by a Δ-move and an ambient isotopy. Therefore if 3 arcs of Fig. 3(a) are contained in the same component of a link, the move from Fig. 3(b) to 3(c) is a self Δ-move and so the links in Fig. 3(a) and 3(d) are self Δ-equivalent.

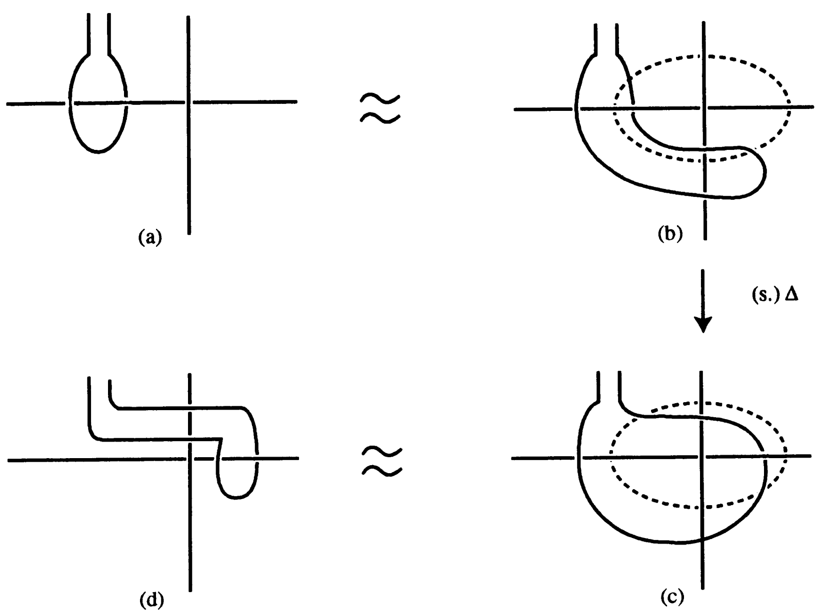

Fig. 3.

To prove Theorem, we consider some deformations of ribbon links by self Δ-moves in this section.

Suppose that $\ell = k_1 \cup \cdots \cup k_n$ is a ribbon link. Then ℓ can be obtained by a fusion of a trivial link, [2]. Hence there are mutully disjoint disks $\mathscr{D}=\bigcup_{i=1}^{n}\mathscr{D}_i$, $\mathscr{D}_i=\bigcup_{j=0}^{p_i}D_{ij}$ and bands $\mathscr{B}=\bigcup_{i=1}^{n}\mathscr{B}_i$, $\mathscr{B}_i=\bigcup_{j=1}^{p_i}B_{ij}$ such that $\partial(\mathscr{D}\cup\mathscr{B})=\ell$, $\partial(\mathscr{D}_i\cup\mathscr{B}_i)=k_i$ and $\mathscr{S}(\mathscr{D}\cap\mathscr{B})$ consists of arcs of ribbon type. By deforming $\mathscr{D}\cup\mathscr{B}$ suitably, we may choose that $\mathscr{D}_i\subset R^2[i]$ and $\partial B_{ij}\cap\partial\mathscr{D}_i=\partial B_{ij}\cap\partial(D_{i0}\cup D_{ij})$ for $i=1,\cdots,n$ and $j=1,\cdots,p_i$, see Fig. 4.

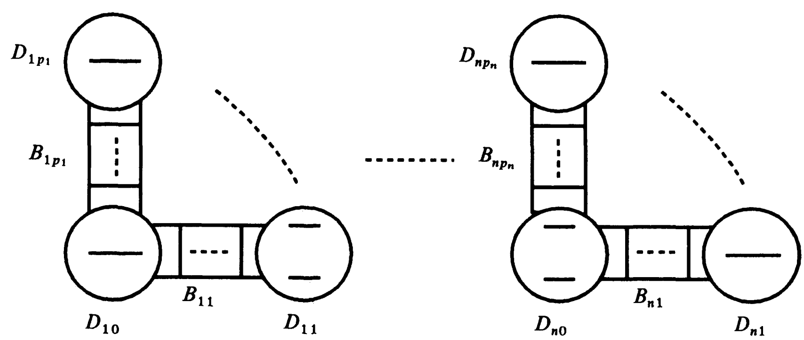

Fig. 4.

Lemma 1. *If there is a band B_{ij} such that $\mathscr{S}(B_{ij}\cap D_{i0})$ (or $\mathscr{S}(B_{ij}\cap D_{ij})$ is empty, then $\mathscr{S}(B_{rs}\cap D_{i0})$* (resp. *$\mathscr{S}(B_{rs}\cap D_{ij})$*) *for $r\neq i$, if not empty, can be deformed into D_{ij}* (resp. *D_{i0}*) *by an ambient isotopy without increasing $\sharp\mathscr{S}(\mathscr{B}_i\cap\mathscr{D}_i)$, where $\sharp\mathscr{S}(x)$ means the number of arcs of $\mathscr{S}(x)$.*

Proof. If $\mathscr{S}(B_{ij}\cap D_{i0})$ (or $\mathscr{S}(B_{ij}\cap D_{ij})$) is empty, $B_{ij}\cup D_{i0}$ (resp. $B_{ij}\cup D_{ij}$) is a non-singular disk. Hence we can perform the following deformation which does not increase $\sharp\mathscr{S}(\mathscr{B}_i\cap\mathscr{D}_i)$, Fig. 5.

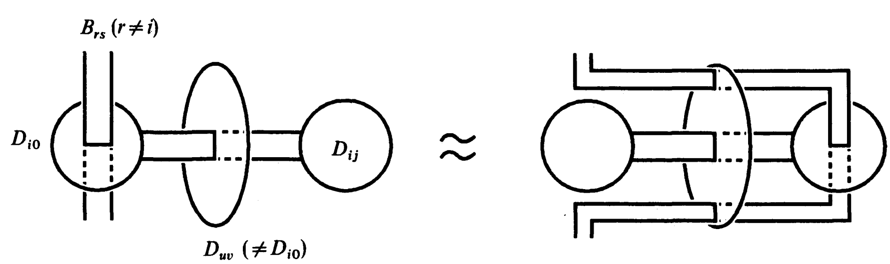

Fig. 5.

Let b_1, b_2 be two arcs of $\mathscr{S}(B_{ij}\cap\mathscr{D}_i)$ and B the connected component of $B_{ij}-b_1-b_2$ such that $\partial B\supset b_1\cup b_2$. Then b_1 and b_2 are said to be *adjacent* on

B_{ij} with respect to $\mathscr{D}_i$ if $B \cap \mathscr{D}_i = \phi$.

Lemma 2. *The order of two arcs b_1, b_2 of $\mathscr{S}(B_{ij} \cap \mathscr{D}_i)$ which are adjacent on B_{ij} with respect to $\mathscr{D}_i$ can be exchanged without increasing $\sharp\mathscr{S}(\mathscr{B}_i \cap \mathscr{D}_i)$ by a finite sequence of self Δ-moves.*

Proof. Let D_{ip} be the disk of $\mathscr{D}_i$ which contains b_1 and α an arc on D_{ip} which connects a point of ∂b_1 and one of $\partial D_{ip} - \partial\mathscr{B}$ with $\alpha \cap \mathscr{B} = \alpha \cap \partial b_1$, Fig. 6(a). We deform $N(\alpha \cup b_1 : D_{ip})$, the regular neighborhood of $\alpha \cup b_1$ on D_{ip}, towards b_2 along B, Fig. 6(a), (b) and perform twice self Δ-moves, Fig. 6(c) and obtain Lemma 2.

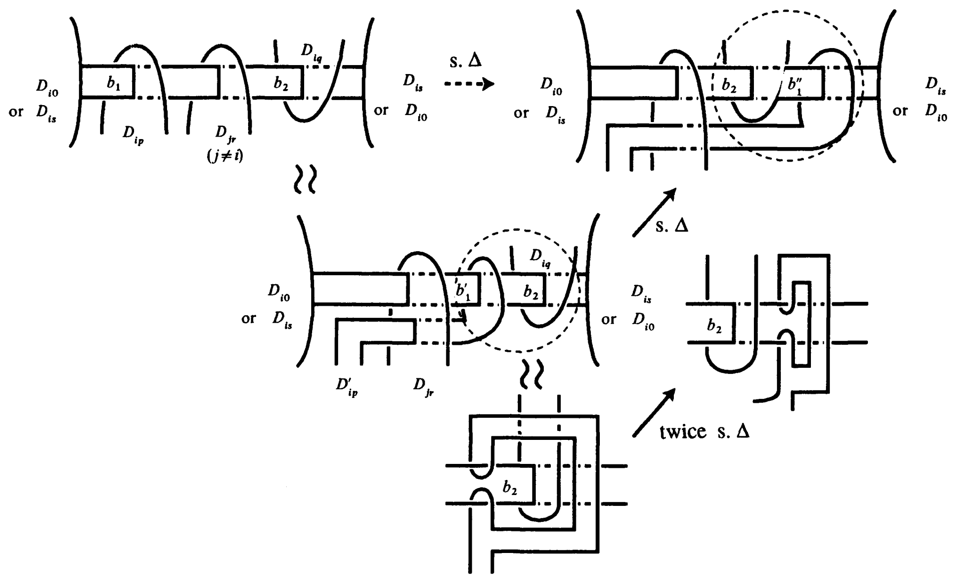

Fig. 6.

Let D''_{ip} be the disk obtained from D_{ip} by the deformations of Lemma 2, Fig. 6(c). Then we easily see that the followings by the above deformations: D''_{ip} is non-singular and $D''_{ip} \cap (\mathscr{D}_i - D_{ip}) = \phi$, namely $\mathscr{D}'_i = (\mathscr{D}_i - D_{ip}) \cup D''_{ip}$ is a union of mutually disjoint non-singular disks and, if $B \cap D_{jr} \neq \phi$ for $j \neq i$, Fig. 6(a), $D''_{ip} \cap D_{jr}$ contains arcs of ribbon type whose b-lines and i-lines of pre-images of $D''_{ip} \cap D_{jr}$ are contained in those of D''_{ip} and D_{jr} respectively.

Let b be an arc of $\mathscr{S}(B_{ij} \cap \mathscr{D}_i)$ which is nearest to $B_{ij} \cap \partial D_{ij}$.

Lemma 3. *If $b \subset \mathscr{D}_i - D_{ij}$, b can be removed without increasing $\sharp\mathscr{S}(\mathscr{B}_i \cap \mathscr{D}_i)$ by a finite sequence of self Δ-moves.*

Proof. First we consider the case that $(\mathscr{B} - \mathscr{B}_i) \cap D_{ij} = \phi$.

Let B be the connected component of $B_{ij} - b$ which contains $\partial B_{ij} \cap \partial D_{ij}$ and

D the disk of $\mathscr{D}_i - D_{ij}$ which contains b and α an arc on D which connects a point of ∂b and one of ∂D such that $\alpha \cap \mathscr{B} = \partial\alpha \cap \partial b$.

Since b is contained in $D(\subset \mathscr{D}_i - D_{ij})$, $B \cup D_{ij}$ is non-singular. Therefore we can deform $N(\alpha \cup b : D)$ along $B \cup D_{ij}$ from Fig. 7(a) to 7(f) whose deformations

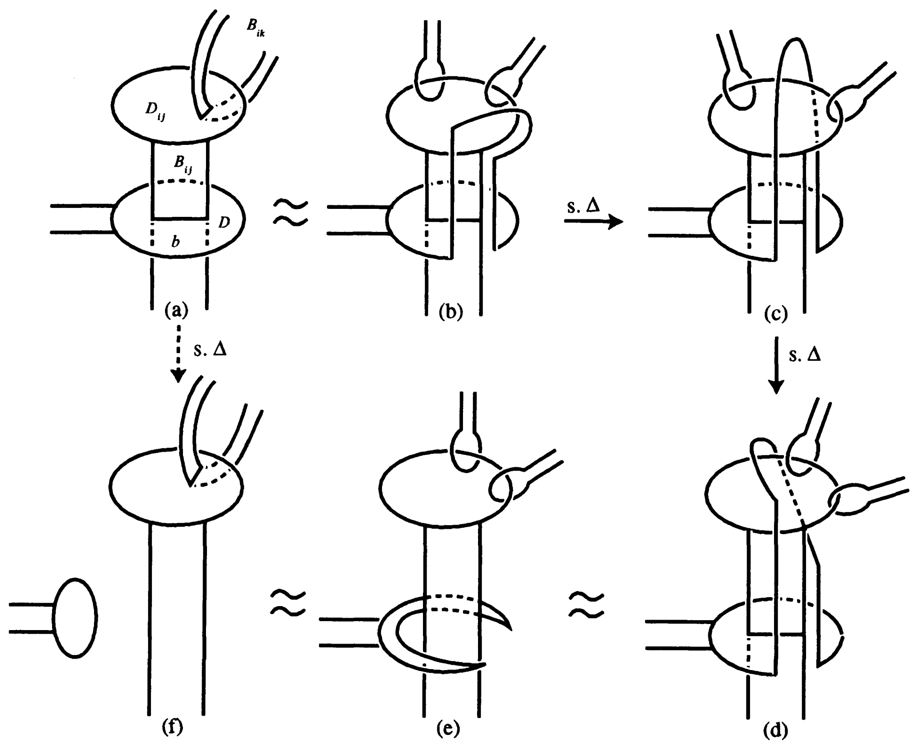

Fig. 7.

can be accomplished by a finite sequence of self Δ-moves, for $(\mathscr{B} - \mathscr{B}_i) \cap D_{ij} = \phi$, and an ambient isotopy of R^3. Although Fig. 7(a) is the figure such that $\mathscr{S}(B \cap (\mathscr{D} - \mathscr{D}_i)) = \phi$, we can perform the above deformations even if $\mathscr{S}(B \cap (\mathscr{D} - \mathscr{D}_i)) \neq \phi$, because $B \cup D_{ij}$ is non-singular.

Next we consider the case that $(\mathscr{B} - \mathscr{B}_i) \cap D_{ij} \neq \phi$, namely there is a band B_{uv} $(u \neq i)$ such that $B_{uv} \cap D_{ij} \neq \phi$. Let γ be an arc on $B \cup D_{ij}$ which connects a point of $\partial(B_{uv} \cap D_{ij})$ and one of the interior of b, Fig. 8(a), such that $\gamma \cap \mathscr{S}(\mathscr{B} \cap D_{ij}) = \partial\gamma \cap \partial B_{uv}$. We deform B_{uv} along γ from Fig. 8(a) to 8(b) and obtain B'_{uv}. Now we perform the above deformations for each such a band B_{uv}, $u \neq i$, and apply the deformations of Fig. 7 to Fig. 8(b) and obtain 8(c) and deform B'_{uv} from Fig. 8(c) to 8(d) and obtain a band B''_{uv}.

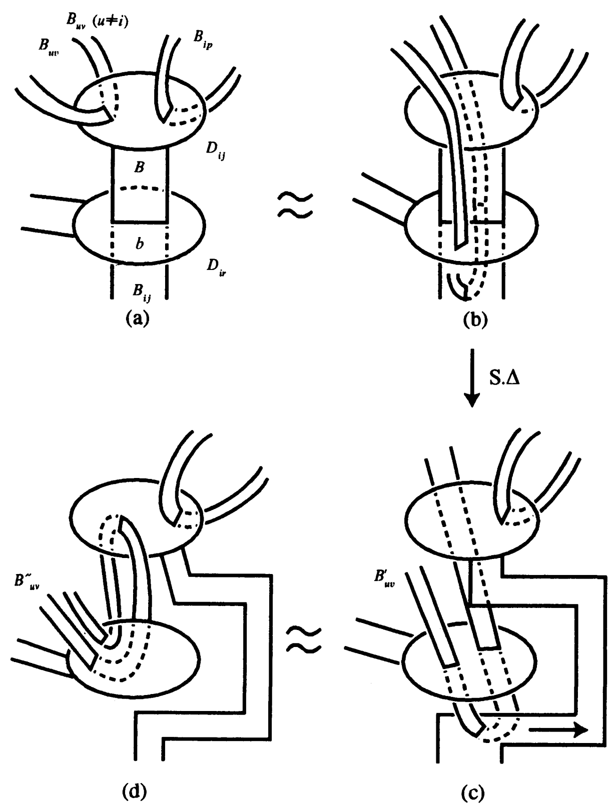

Fig. 8.

By the deformations of Fig. 7 and 8, we can eliminate b without increasing $\#\mathscr{S}(\mathscr{B}_i \cap \mathscr{D}_i)$.

Lemma 4. *If there are two arcs b_1, b_2 of $\mathscr{S}(B_{ij} \cap \mathscr{D}_i)$ which are adjacent on B_{ij} with respect to $\mathscr{D}_i$ such that $b_1 \subset D_{ij}$, $b_2 \subset \mathscr{D}_i - D_{ij}$ and b_2, $\partial B_{ij} \cap \partial D_{ij}$ are contained in the different connected components of $B_{ij} - b_1$, see Fig.* 9(a). *Then b_2 can be removed without increasing $\#\mathscr{S}(\mathscr{B}_i \cap \mathscr{D}_i)$ by a finite sequence of self Δ-moves.*

Proof. Let b_1, b_2 be the pair nearest to β $(=\partial B_{ij} \cap \partial D_{ij})$ on B_{ij} satisfying the conditions of Lemma 4 and B, B_1 the connected components of $B_{ij} - b_1 - b_2$, $B_{ij} - b_2$ with $\partial B \supset b_1 \cup b_2$, $\partial B_1 \supset \beta$ respectively.

First we consider the case that $\mathscr{S}(B_1 \cap D_{ij}) = b_1$. Let α be an arc on D_{ij} such that α connects a point of ∂b_1 and one of $\partial D_{ij} - \partial B_{ij}$ with $\alpha \cap \mathscr{B} = \partial\alpha \cap \partial b_1$. Deform $N(\alpha \cup b_1 : D_{ij})$ along B towards b_2, Fig. 9 (a), (b), and perform twice self Δ-moves and obtain a disk D'_{ij} from D_{ij}, Fig. 9(c). Hence we can remove b_2

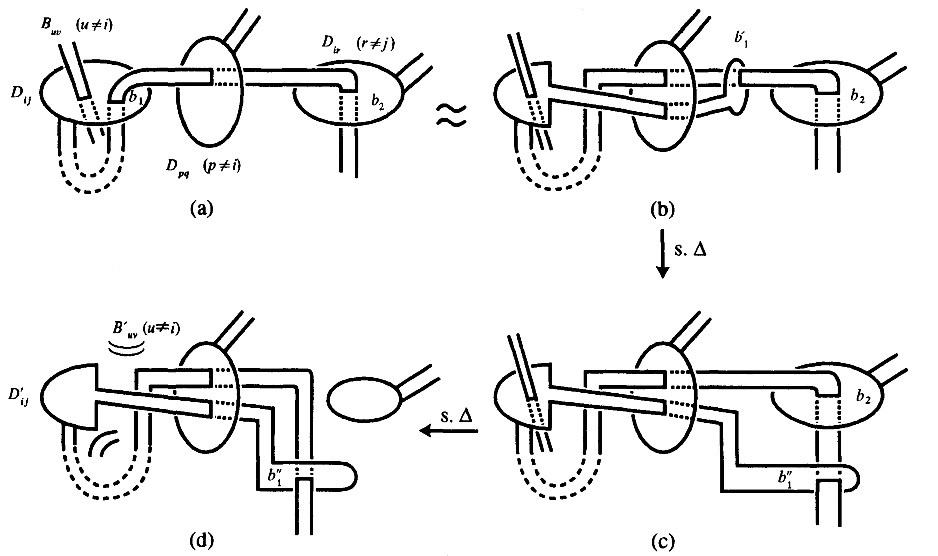

Fig. 9.

by applying Lemma 3, because $B_1 \cup D'_{ij}$ is non-singular. Moreover we easily see that $D'_{ij} \cap (\mathscr{D}_i - D_{ij}) = \phi$ and that, if $\mathscr{S}(D'_{ij} \cap D_{pq}) \neq \phi$ for $p \neq i$, Fig. 9(c), it consists of arcs of ribbon type whose b-lines and i-lines of pre-images of $\mathscr{S}(D'_{ij} \cap D_{pq})$ are contained in those of D'_{ij} and D_{pq} respectively.

Next we consider the case that $\mathscr{S}(B_1 \cap D_{ij}) - b_1 \neq \phi$. In this case, we apply the above deformations to each of $\mathscr{S}(B_1 \cap D_{ij})$ in turn such that we obtain a disk D''_{ij} from D_{ij} which satisfies that $B_1 \cup D''_{ij}$ is non-singular and so we can remove b_2 by applying Lemma 3.

Hence we obtain Lemma 4.

3. Proof of Theorem

Now we are ready to prove Theorem.

Proof of Theorem. Suppose that ℓ is a ribbon link and let $\mathscr{B}_i$, $\mathscr{D}_i$ $(\subset R^2[i])$ be mutually disjoint bands, disks respectively such that $\mathscr{B}_i \cup \mathscr{D}_i$ are situated in Fig. 4.

First we consider the case that $\mathscr{S}(\mathscr{B}_1 \cap \mathscr{D}_1) = \phi$, namely $\mathscr{B}_1 \cup \mathscr{D}_1$ is non-singular. Hence if there is a band B_{rs} $(r \neq 1)$ such that $B_{rs} \cap D_{1j} \neq \phi$ for $j \neq 0$, we can deform $B_{rs} \cap D_{1j}$ into D_{10} by Lemma 1. Therefore we may assume that $B_{rs} \cap D_{1j} = \phi$ for each $r \neq 1$, $j \neq 0$ and so we may deform $\mathscr{B}_1 \cup \mathscr{D}_1$ into $R^2[1]$ by an ambient isotopy of R^3 with $D_{10} \cup (\mathscr{D} - \mathscr{D}_1)$ fixed for $\mathscr{D} = \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_n$.

Next we consider the case that $\mathscr{S}(\mathscr{B}_1 \cap \mathscr{D}_1) \neq \phi$. If there is a band B_{1i} such that $\mathscr{S}(B_{1i} \cap \mathscr{D}_1) = \phi$, we may deform $B_{1i} \cup D_{1i}$ into R^2 [1] by the similar deformation as above with $\mathscr{D} - D_{1i}$ fixed. Hence we may assume that $\mathscr{S}(B_{1i} \cap \mathscr{D}_1) \neq \phi$ for each $i = 1, \cdots, n$. In this case, let us apply Lemmas 2, 3 and 4 to $\mathscr{B}_1 \cup \mathscr{D}_1$ in the following way.

Suppose that $B_{1u}\cap\mathscr{D}_{1v}$ contains an arc b of ribbon type for some $u\neq v$. Let b be nearest to β $(=\partial B_{1u}\cap\partial D_{1u})$ among $B_{1u}\cap(\mathscr{D}_1-D_{1u})$ and B a connected component of $B_{1u}-b$ which contains β. If $\mathscr{S}(B\cap D_{1u})=\phi$, b can be removed by Lemma 3 and if $\mathscr{S}(B\cap D_{1u})\neq\phi$, b can be removed by Lemma 4 because of the choice of b.

We perform the above discussion in turn as far as possible. By the deformations of Lemmas 2, 3 and 4, $\mathscr{D}_j$ are fixed for $j\geqq 2$, hence $\mathscr{D}_j\subset R^2[j]$, on the other hand, $\mathscr{D}_1=D_{10}\cup\cdots\cup D_{1p_1}$ and $\mathscr{B}(=\mathscr{B}_1\cup\cdots\cup\mathscr{B}_n)$ may be deformed. Now we write these deformed $\mathscr{D}_1$ and $\mathscr{B}$ by $\mathscr{D}'_1(=D'_{10}\cup\cdots\cup D'_{1p_1})$ and $\mathscr{B}'(=\mathscr{B}'_1\cup\cdots\cup\mathscr{B}'_n)$ respectively. Although $\mathscr{D}'_1$ may not be contained in $R^2[1]$, each disk D'_{1i} of $\mathscr{D}'_1$ is non-singular and any two of $\mathscr{D}'_1$ are mutually disjoint and $\mathscr{S}(\mathscr{D}'_1\cap\mathscr{D}_j)$, if not empty for $j\geqq 2$, consists of arcs of ribbon type whose b-lines and i-lines of pre-images of $\mathscr{S}(\mathscr{D}'_1\cap\mathscr{D}_j)$ are contained in pre-images of $\mathscr{D}'_1$ and $\mathscr{D}_j$ respectively.

By the above discussions, we may assume that $\mathscr{S}(B'_{1i}\cap\mathscr{D}'_1)=\mathscr{S}(B'_{1i}\cap D'_{1i})$ for $i=1,\cdots,p_1$ and that each of $\mathscr{S}(B'_{1i}\cap D'_{1i})\neq\phi$.

By sliding $\partial B'_{1i}\cap\partial D'_{10}$ along an arc of $\partial B'_{1i-1}$ such that B'_{1i} connects an arc of $\partial D'_{1i-1}$ and one of $\partial D'_{1i}$ for $i=1,\cdots,p_1-1$ and $\mathscr{B}'_1=\bigcup_i B'_{1i}$, Fig. 10 (b). We denote the bands obtained by the above deformations of $\mathscr{B}'_1$ by $\bar{\mathscr{B}}_1=\bigcup_i\bar{B}_{1i}$, where B'_{1p_1} does not be deformed and so $\bar{B}_{1p_1}=B'_{1p_1}$.

After the above deformations, $\sharp\mathscr{S}(\bar{\mathscr{B}}_1\cap\mathscr{D}'_1)$ may increase, see Fig. 10(b).

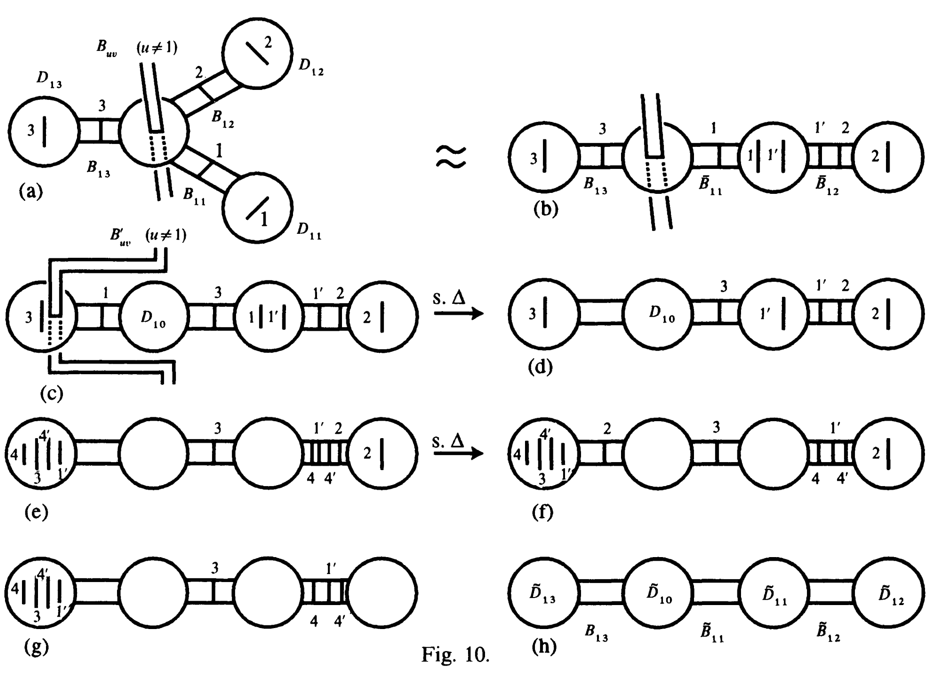

Fig. 10.

Sublemma. *We can deform* $\mathscr{S}(\bar{\mathscr{B}}_1 \cap (\mathscr{D}'_1 - D'_{1p_1}))$ *into* D'_{1p_1} *by a finite sequence of self* Δ*-moves.*

Proof. Suppose that there is a band B'_{uv} $(u \neq 1)$ of $\mathscr{B}'_u$ such that $B'_{uv} \cap D'_{10} \neq \phi$. As $\mathscr{S}(B'_{1i} \cap \mathscr{D}'_1) = \mathscr{S}(B'_{1i} \cap D'_{1i})$, $B'_{1p_1} \cup D'_{10}$ is non-singular and so we can deform $B'_{uv} \cap D'_{10}$ along B'_{1p_1} into D'_{1p_1}, Fig. 10(b), (c) by the deformation of Lemma 1. As a result, $\mathscr{S}(\bar{\mathscr{B}} \cap D'_{10}) = \phi$ for $\bar{\mathscr{B}} = (\mathscr{B}' - \mathscr{B}'_1) \cup \bar{\mathscr{B}}_1$ and $\bar{B}_{11} \cup B'_{1p_1} \cup D'_{10}$ $(=E_{11})$ is non-singular and so we may exchange the order of $\mathscr{S}(B'_{1p_1} \cap \mathscr{D}'_1)$ and $\mathscr{S}(\bar{B}_{11} \cap \mathscr{D}'_1)$ on E_{11} by Lemma 2, see Fig. 10(b), (c). Then we can remove the arcs "1" of ribbon type by Lemma 3, Fig. 10(d). By repeating the above deformation if necessary, we obtain the disks, denoted by D'_{11}, E_{11} again, such that $D'_{11} \cup E_{11}$ is non-singular and so the arcs "1′" of $\mathscr{S}(B_{11} \cap D'_{11})$ on D'_{11} can be deformed into D'_{1p_1} by an ambient isotopy, Fig. 10(d), (e). (By this deformation, $\sharp\mathscr{S}(\bar{\mathscr{B}}_1 \cap \mathscr{D}'_1)$ may increase.) If $D'_{11} \cap (\mathscr{B}' - \mathscr{B}'_1) \neq \phi$ in Fig. 10 (e), we perform the same deformation as Fig. 10(b), (c) and transfer it into D'_{1p_1}. After the above, $E_{11} \cup \bar{B}_{12} \cup D'_{11}$ $(=E_{12})$ is non-singular and so we can exchange the arcs of "2" and "1′", "3", "4", "4′" by Lemma 2, Fig. 10(f), and we may remove "2" by Lemma 3, Fig. 10(g).

By repeating the above process successively, each arc of $\mathscr{S}(\bar{\mathscr{B}}_1 \cap \mathscr{D}'_1)$ on $\mathscr{D}'_1$ can be deformed into D'_{1p_1}. Now we obtain Sublemma.

Therefore E_{1p_1} $(=\bar{B}_{11} \cup \cdots \cup \bar{B}_{1p_1-1} \cup B'_{1p_1} \cup D'_{10} \cup \cdots \cup D'_{1p_1-1})$ is non-singular and so we can deform $E_{1p_1} \cup D'_{1p_1}$ $(=C_1)$ into $R^2[1]$ by an ambient isotopy with $\mathscr{D}_i$ fixed for $i \geqq 2$ because each b-line of pre-image of $C_1 \cap \mathscr{D}_i$ $(i \geqq 2)$ is contained in pre-image of C_1.

By the above deformations, we obtain $\tilde{\mathscr{B}}_1 \cup \tilde{\mathscr{D}}_1$ $(\subset R^2[1])$ from $\mathscr{B}'_1 \cup \mathscr{D}'_1$. We can perform the above deformations with $\mathscr{D} - \mathscr{D}_1$ fixed. Hence $\mathscr{D}_i$ are contained in $R^2[i]$ respectively and so $(\tilde{\mathscr{B}}_1 \cup \tilde{\mathscr{D}}_1) \cap \mathscr{D}_i = \phi$ for $i \geqq 2$.

Next we perform the same deformations to $\mathscr{B}'_2 \cup \mathscr{D}_2$. As $\tilde{\mathscr{B}}_1 \cap \mathscr{D}_2$ is empty, we can perform the deformations of Lemmas 1, 2, 3 and 4 with $\tilde{\mathscr{B}}_1 \cup \tilde{\mathscr{D}}_1$ fixed. As a result, we obtain mutually disjoint non-singular disks $\tilde{\mathscr{B}}_2$, $\tilde{\mathscr{D}}_2$ from $\mathscr{B}_2$, $\mathscr{D}_2$ such that $\tilde{\mathscr{B}}_2 \cup \tilde{\mathscr{D}}_2 \subset R^2[2]$ and $\tilde{\mathscr{B}}_1 \cup \tilde{\mathscr{D}}_1 \subset R^2[1]$.

By repeating the above discussion successively, we see that ℓ can be deformed into a trivial link by a finite sequence of self Δ-moves, namely ℓ is self Δ-equivalent to a trivial link.

Now the proof is complete.

CONJECTURE. *Suppose that two links* ℓ *and* ℓ' *are cobordant,* [3], [4], *in* R^3. *Then they are self* Δ*-equivalent.*

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Department of Mathematics
Osaka Institute of Technology
Asahi, Osaka 535
Japan