Chi, Q-S. and Mo, X.
Osaka J. Math.
33 (1996), 669–696

THE MODULI SPACE OF BRANCHED SUPERMINIMAL SURFACES OF A FIXED DEGREE, GENUS AND CONFORMAL STRUCTURE IN THE FOUR-SPHERE

QUO-SHIN CHI and XIAOKANG MO

(Received March 3, 1995)

0. Introduction

Minimal immersions from a Riemann surface M into S^n were studied by Calabi ([3]) and Chern ([4]), among many authors. To each such immersion F in S^4, they found a holomorphic quartic form Q_F (to be defined in Section 1) on M. A superminimal immersion is one for which $Q_F=0$, which is always the case when $M=S^2$. In [2], Bryant studied a superminimal immersion of a higher genus into S^4 by lifting it to $\boldsymbol{CP}^3$, the twistor space of S^4. The lift of a superminimal immersion is a holomorphic curve, of the same degree as that of the immersion, which is horizontal with respect to the twistorial fibration; more precisely, it is a holomorphic curve in $\boldsymbol{CP}^3$ satisfying the differential equation $z_0dz_1-z_1dz_0+z_2dz_3-z_3dz_2=0$. Setting $z_0=1$, $z_1+z_2z_3=f$ and $z_2=g$, one can solve z_1, z_2, z_3 in terms of the meromorphic functions f and g, which serves as a kind of "Weierstrass representation". Via this representation, Bryant showed the existence of a superminimal immersion from any compact Riemann surface into S^4. However, his existence result does not specify the degree d of the immersion, which is the simplest global invariant of the surface.

In Loo ([12]) and Verdier ([17]), $f_1=z_1/z_0$ and $f_2=z_3/z_2$ were chosen in place of the aforementioned f and g. Generically, f_1 and f_2 are of degree d which satisfy $\mathrm{ram}(f_1)=\mathrm{ram}(f_2)$, where $\mathrm{ram}(f)$ denotes the ramification divisor of the meromorphic function f. This gives a scheme of constructing the moduli space of all branched superminimal surfaces in S^4 with a fixed degree d. For $M=S^2$, Loo ([12]) showed that the moduli space is connected and has dimension $2d+4$; Verdier ([17]) in addition pointed out that the moduli space has three irreducible components.

In this paper, we propose to carry the investigation over to higher genera. Let $F: M\to S^4$ be a superminimal immersion of degree d and let $\tilde{F}: M\to \boldsymbol{CP}^3$ be its horizontal lift. Let L_F be the pullback bundle via $\tilde{F}$ of the hyperplane bundle of $\boldsymbol{CP}^3$. We may regard $z_0,\cdots,z_3$ as four sections in $H^0(L_F)$ without common zeros. Now there is a natrual map $\mathcal{R}am$ which sends the 1-dimensional linear system $\langle z_0,z_1\rangle$ (called a g_d^1), i.e., the plane spanned by z_0, z_1 in the Grassmann manifold $G(2,H^0(L))$ of two-planes is $H^0(L_F)$, to the zero divisor of $z_0dz_1-z_1dz_0$

in $H^0(K\otimes L_F^2)$ with K the canonical bundle of M. Set $f_1=[z_0:z_1]$ and $f_2=[z_2:z_3]$, which are two holomorphic maps from M to $\boldsymbol{CP}^1$. One defines $\mathscr{R}am(f_1)=\mathscr{R}am(\langle z_0,z_1\rangle)$. $\mathscr{R}am(f_1)$ may be thought of as the "virtual" ramification divisor of f_1 since $\mathscr{R}am(f_1)=\mathrm{ram}(f_1)+2B$, where B is the base locus of $\langle z_0,z_1\rangle$, i.e., is the divisor of the common zeros of z_0 and z_1 counted with multiplicity. With these, one obtains $\mathscr{R}am(f_1)=\mathscr{R}am(f_2)$ as an immediate consequence of $z_0dz_1-z_1dz_0=-(z_2dz_3-z_3dz_2)$ satisfied by $\tilde{F}$. It is now clear that if we let G_d^1 be the space of all g_d^1, R_d^1 be the space of all maps $[s:t]$ from M to $\boldsymbol{CP}^1$ associated with the linear systems $\langle s,t\rangle$ in G_d^1, W_d^1 be the space of holomorphic line bundles L of degree d such that $\dim H^0(L)\geq 2$, and consider the map $\mu: R_d^1 \xrightarrow{\pi_1} G_d^1 \xrightarrow{\pi} W_d^1$ given by $[z_0:z_1]\mapsto\langle z_0,z_1\rangle\mapsto L_F$, then the moduli space of horizontal holomorphic curves of degree d of a Riemann surface M, denoted by $\mathscr{M}_d(M)$, is essentially the set of (f_1,f_2), where $\mathscr{R}am(f_1)=\mathscr{R}am(f_2)$, $\mu(f_1)=\mu(f_2)$, and $\pi_1(f_1)$ and $\pi_1(f_2)$ have disjoint base loci. (The last condition ensures that the four sections $z_0,\cdots,z_3$ have no common zeros, so that $\tilde{F}$ is of degree d.)

We can now picture $\mathscr{M}_d(M)$ as the set of such pairs (f_1,f_2) sitting over W_d^1, and thus may slice $\mathscr{M}_d(M)$ by $L\in W_d^1$. Let $\mu(f_1)=\mu(f_2)=L$, and let $x=\pi_1(f_1)$ and $y=\pi_1(f_2)\in G_d^1$. Then on the G_d^1 level, each slice is just the collection of pairs (x,y) with $x,y\in G(2,H^0(L))$ such that $\mathscr{R}am(x)=\mathscr{R}am(y)$ and x and y have disjoint base loci, where $\mathscr{R}am$ now is the restriction of a projection $\mathscr{R}$ from $\boldsymbol{P}(\wedge^2(H^0(L)))$ to $\boldsymbol{P}(H^0(K\otimes L^2))$. Notice that if $x=y$, then the branched superminimal immersion constructed out of (f_1,f_2) is totally geodesic. We assume henceforth that $x\neq y$. It follows that x and y generate a sub-Grassmann $G(2,4)$ in $G(2,H^0(L))$. By looking at the singular locus of $\mathscr{R}am$ restricted on this $G(2,4)$, one sees immediately that one can always continuously deform (x,y) to an element of the form (t,t) for some $t\in G_d^1$; consequently the connectedness of G_d^1 ([1]) enables us to assert the connectedness of $\mathscr{M}_d(M)$ when M is a Riemann surface of genus g with $d>(g+2)/2$. It should be mentioned that the connectedness of $\mathscr{M}_d(M)$ has recently been proved by Guest-Ohnita [8] via loop group analysis when the ambient sphere is of arbitrary dimension.

As to the existence of a nontotally geodesic branched surperminimal surface of degree d, one must distinguish small degrees from large ones. Notice that the existence of a nontotally geodesic branched superminimal immersion, or rather the existence of the $G(2,4)$ generated by x and y above, implies that $\dim H^0(L)\geq 4$ indeed. Employing this condition and Clifford's Theorem about special divisors on Riemann surfaces, we can show that if $\mathrm{Min}(g,6)\geq d$ the branched superminimal immersions of degree d and genus g are all totally geodesic, except in the case when $d=6$ and M is hyperelliptic, where $\mathscr{M}_6(M)$ is isomorphic to $\mathscr{M}_3(\boldsymbol{CP}^1)$. Furthermore, by analyzing all complete linear systems of degree 5 on Riemann surfaces of genus ≤ 4, we are able to conclude that all branched superminimal immersions of degree 5 and genus ≤ 4 are totally geodesic. The upshot of these results, which is the context of Theorem 1 in Section 4, is: For $g \geq 1$, all branched superminimal immersions of degree ≤ 5 from any Riemann surface into S^4 are totally geodesic.

When the degree is 6, one readily sees the existence of nontotally geodesic branched superminimal immersions if the Riemann surface is hyperelliptic: Just take a nontotally geodesic branched superminimal sphere of degree 3 and pull it back onto the Riemann surface via its branched double covering onto the sphere. In fact, that M is hyperelliptic is not fortuitous, since by looking into the interrelation between the Weierstrass points and the complete linear systems of degree 6 on nonhyperelliptic Riemann surfaces of genus 3 and 4, with the aid of Clifford's Theorem and the notion of correspondences between Riemann surfaces, we will assert in Theorem 2, Section 4, the following conclusion: For $g \geq 1$, a Riemann surface of genus g admits a nontotally geodesic branched superminimal immersion of degree 6 into S^4 if and only if the Riemann surface is hyperelliptic.

This naturally brings forward the question of classifying all nontotally geodesic superminimal immersions of degree 6 for a given hyperelliptic Riemann surface. We have succeeded in carrying out the classification for $g \neq 2$ in Theorem 3, Section 4. Namely, all the nontotally geodesic branched superminimal immersions of degree 6 from a hyperelliptic Riemann surface of genus $g \geq 3$ into S^4 are just the pullback of nontotally geodesic branched superminimal spheres of degree 3 via the branched double covering. For $g = 1$, the closure of the space of nontotally geodesic branched superminimal tori of degree 6 in the moduli space is essentially a fiber bundle over the underlying torus, where each fiber in turn is a fiber bundle over a certain cubic curve whose fiber is a principal $\mathrm{PGL}(2,\boldsymbol{C}) \times \mathrm{PGL}(2,\boldsymbol{C})$-bundle over the 4-dimensional complex Grassmann $G(2,4)$. This is to be proved in Section 6. It seems, suggested by $g = 1$, that a study of the Riemann Θ-function would lead to the classification when $g = 2$. In fact, our classification answers a question raised in [19] affirmatively for $d \leq 6$ as to whether the twisted cubic is the only curve in $\boldsymbol{CP}^3$ with a base-point-free complete g_d^3 for $\mathcal{R}am$ not to be injective.

For large degrees, exploring the "Weierstrass representation" mentioned earlier and the existence of nonspecial very ample line bundles for appropriate degrees, we prove in Theorem 4, Section 5, the existence of a nontotally geodesic branched superminimal immersion from any Riemann surface into S^4 as long as $d \geq 5g + 4$ for $g \geq 2$ ($d \geq 6$ if $g = 1$). (This lower bound is sharp for $g = 1$.) Moreover, the dimension of each irreducible component of $\mathcal{M}_d(M)$ is bounded between $2d - 4g + 4$ and $2d - g + 4$ (Theorem 5, Section 5). The upper bound is always achieved by the totally geodesic component, whereas the lower bound is realized by each nontotally geodesic component of the moduli space of branched superminimal tori of degree 6. Observe that when $g = 0$, the two dimension bounds are both equal to $2d + 4$. It is tempting to conjecture that the nontotally geodesic part of $\mathcal{M}_d(M)$ is of pure dimension $2d - 4g + 4$ for any Riemann surface (or at least for a generic) M of genus g. This would be true if the intersection of $\mathrm{Ker}\,\mathcal{R}$, the kernel of

$\mathscr{R}: \boldsymbol{P}(\wedge^2 H^0(L))) \to \boldsymbol{P}(H^0(K\otimes L^2))$, and the projective variety $\mathscr{L}=\boldsymbol{P}(\{\omega\in\wedge^2(H^0(L)): \omega\wedge\omega\wedge\omega=0\})$ were transversal.

A study of the interseciton of Ker$\mathscr{R}$ and $\mathscr{L}$ in the case $d=6$ and $g=1$ in Section 6 shows that the nontotally geodesic part of the moduli space $\mathscr{M}_6(T)$, where T is a torus, may be reducible (e.g., when T is the torus where the conformal structure is given by the Weierstrass constants $g_2=0$, $g_3=1$), so that the moduli spaces of branched superminimal tori with these conformal structures consist of more than three components (seven to be precise), although for a generic torus it is irreducible. It is likely that for a generic Riemann surface M of genus g, the nontotally geodesic part of $\mathscr{M}_d(M)$ is irreducible. Again, this would follow if the intersection Ker$\mathscr{R}$ and $\mathscr{L}$ were transversal by a result in [7].

1. Twistor theory and superminimal immersions

Since Bryant's initial work ([2]) there have been many general investigations of minimal immersions in terms of the twistorial scheme, which we will briefly present in this section; for a detailed discussion and related references see [5], [6], [9]. Given an oriented Riemannian 4-manifold N, let $O(N)$ be the orthonormal frame bundle of N. Consider the bundle of pointwise orthogonal complex structures $O(N)\times_{O(4)}O(4)/U(2)$, which has two connected components Z_+ and Z_-, called twistor spaces of N, consisting of those pointwise complex structures that are orientation-preserving and orientation-reversing, respectively. $Z_\pm$ is a 2-sphere bundle over N associated with $SO(4)$ since $SO(4)/U(2)=S^2$. The Levi-Civita connection on N induces a connection on $Z_\pm$ which splits the tangent spaces of $Z_\pm$ into vertical and horizontal spaces, $TZ_\pm=V_\pm\oplus H_\pm$. $TZ_\pm$ inherits naturally a Riemannian matric $\langle\,,\rangle$ that coincides with that of N on $H_\pm$ and that of S^2 on $V_\pm$ such that $V_\pm$ is perpendicular to $H_\pm$. One can define a Hermitian structure J on Z by setting, at $u\in Z_\pm, J$ to be the natural complex structure on V_u (the fiber of $Z_\pm$ is S^2 identified with $\boldsymbol{CP}^1$), and to be u acting on H_u (u itself is a pointwise complex structure). $(Z_-,J)((Z_+,J)$, respectively) turns out to be a complex manifold if and only if N is self-dual (anti-self-dual, respectively). Moreover, $(Z_-,J,\langle\,,\rangle)((Z_+,J,\langle\,,\rangle)$, respectively) is Kaehler-Einstein if N if Einstein with positive scalar curvature; in fact, Z_- (Z_+, respectively) is either $\boldsymbol{CP}^3$ or $F(1,2)$, where $N=S^4$ or $\boldsymbol{CP}^3$ with the standard metric, respectively.

Let $f: M\to N$ be an immersion with the induced metric from a compact Riemann surface M into N. For each point p in M, if one assigns to f_*T_pM the natural orientation μ_p induced from M, then $(f_*T_pM)^\perp$ inherits a unique orientation τ_p such that $\mu_p\oplus\tau_p$ is the orientation of N at $f(p)$. Regarding μ_p and τ_p ($-\tau_p$, respectively) as complex structures on f_*T_pM and $(f_*T_pM)^\perp$, one can define a map $\tilde{f}_+: p\mapsto\mu_p\oplus\tau_p$ ($\tilde{f}_-: p\mapsto\mu_p\oplus-\tau_p$, respectively) from M into Z_+ (Z_-, respectively), called the twistor lifts.

Now let e_1, e_2, e_3, e_4 be an adapted orthonormal frame of M so that (e_1,e_2)

is a positively oriented frame on M, and let θ^a, ω^a_b, $1\leq a$, $b\leq 4$, be the coframe and the connection forms of N with respect to the adapted frame. Then $\omega^\alpha_i=\sum_j h^\alpha_{ij}\theta^j$, $1\leq i$, $j\leq 2$, $3\leq\alpha\leq 4$, where $\sum_{i,j} h^\alpha_{ij}\theta^i\otimes\theta^j$ is the second fundamental form. Set $H^\alpha=(h^\alpha_{11}+h^\alpha_{22})/2$, and $L^\alpha=(h^\alpha_{11}-h^\alpha_{22})/2-\sqrt{-1}h^\alpha_{12}$. Consider the (1,0)-form $\varphi=\theta^1+\sqrt{-1}\theta^2$. One observes that $Q_f=(\sum_\alpha L^\alpha L^\alpha)\varphi^4$ is a globally defined quartic form on M. Write $Q_f=S_+S_-\varphi^4$, where $S_+=L^3-\sqrt{-1}L^4$, and $S_-=L^3+\sqrt{-1}L^4$; $|S_+|$ and $|S_-|$ are globally defined smooth functions. We say that f is an isotropic isometric immersion if $Q_f\equiv 0$, and f is isotropic with positive spin (negative spin, respectovely) if $|S_+|\equiv 0$ ($|S_-|\equiv 0$, respectively).

The important fact is that the twistor lift $\tilde f$ ($\tilde f_-$, respectively) is J-holomorphic if and only if f is isotropic with positive spin (negative spin, respectively). Furthermore, $\tilde f_+$ ($\tilde f_-$, respectively) is horizontal with respect to the spliting $TZ_+=V_+\oplus H_+$ ($TZ_-=V_-\oplus H_-$, respectively) if and only if f is minimal and $\tilde f_+$ ($\tilde f_-$, respectively) is J-holomorphic; f is said to be a superminimal immersion with positive spin (negative spin, respectively) in this case. It shold be remarked that f is superminimal with both positive and negative spin if and only if f is totally geodesic; moreover, it is clear that reversing the orientation of N interchanges Z_+ and Z_-. It is for this reason that we consider only f with negative spin from now on.

2. Branched superminimal immersions in S^4

When we specialize N to S^4, the above formulation can be made explicit. To be more precise, one regards S^4 as $\boldsymbol{HP}^1$, the 1-dimensional quaternionic projective space. Let τ be the universal quaternionic line bundle over S^4 with quaternionic multiplication on the right. Then one can identify TS^4 with $\mathrm{Hom}_{\boldsymbol{H}}(\tau,\tau^\perp)$ where $\tau\oplus\tau^\perp=\boldsymbol{HP}^1\times(\boldsymbol{H}\oplus\boldsymbol{H})$. Each v in τ_p, where p is the base point of v, can be regarded as an element $\tilde v$ in $\mathrm{Hom}(T_pS^4,(\tau_p)^\perp_{\boldsymbol{R}})$ given by $\tilde v(f)=f(v)$ for $f\in\mathrm{Hom}_{\boldsymbol{H}}(\tau_p,(\tau_p)^\perp)$; $\tilde v$ is a real vector space isomorphism between T_pS^4 and $(\tau_p)^\perp_{\boldsymbol{R}}$ if $v\neq 0$. Since $(\tau_p)^\perp_{\boldsymbol{C}}=\boldsymbol{C}\oplus\boldsymbol{C}$ (regarding elements in $\boldsymbol{H}$ as z_1+jz_2 and multiplying complex numbers on the right), it is clear that $\tilde v$ then induces a complex structure on T_pS^4 which is orientation-reversing. Now since the complex structure is unaltered by changing $\tilde v$ to $\tilde v\lambda$ for any $\lambda\in\boldsymbol{C}$, if follows that Z_- is $\boldsymbol{P}(\tau_{\boldsymbol{C}})$, the complex projectivization of $\tau_{\boldsymbol{C}}$, which is $\boldsymbol{CP}^3$ with the Fubini-Study metric.

The horizontal distribution of $\boldsymbol{CP}^3=Z_-$ is easy to describe: $T\boldsymbol{CP}^3=V\oplus H$, where H is the kernel of a contact form whose pullback to $\boldsymbol{C}^4\backslash\{0\}$ is $(z_0dz_1-z_1dz_0+z_2dz_3-z_3dz_2)/\|z\|^2$, where $z_0,\cdots,z_3$ are the homogeneous coordinates of $\boldsymbol{CP}^3$. Hence a branched superminimal immersion of genus g and degree d in S^4 is the projection of a holomorphic curve $F:M\to\boldsymbol{CP}^3$ of degree d and genus

g satisfying the differential equation

$$z_0dz_1 - z_1dz_0 + z_2dz_3 - z_3dz_2 = 0. \tag{2.1}$$

We denote by $\mathscr{M}_d(M)$ the space of horizontal holomorphic curves of degree d of a fixed Riemann surface M. Let $F: M \to \boldsymbol{CP}^3$, $F(p) = [z_0(p): \cdots : z_3(p)]$, be such a horizontal curve. $z_0, \cdots, z_3$ can be interpreted as four holomorphic sections without common zeros, to be denoted by $s_0, \cdots, s_3$ from now on, on the pullback bundle $L_F = F^*\mathcal{O}(1)$, where $\mathcal{O}(1)$ is the hyperplane bundle of $\boldsymbol{CP}^3$. Define two functions f_1 and f_2 from M to $\boldsymbol{CP}^1$ by setting $f_1(p) = [s_0(p): s_1(p)]$ and $f_2(p) = [s_2(p): s_3(p)]$. Consider now

$$[s_0, s_1] = s_0ds_1 - s_1ds_0, \tag{2.2}$$

which can be viewed as a holomorphic section of $K \otimes L_F^2$. Set

$$\mathscr{R}am(f_1) = \text{ zero divisor of } [s_0, s_1] \text{ in } K \otimes L_F^2. \tag{2.3}$$

$\mathscr{R}am(f_1)$ is the ramification divisor of f_1 plus $2B$, where B is the base locus of the linear system $\langle s_0, s_1 \rangle$. With these, (2.1) merely says $[s_0, s_1] = -[s_1, s_2]$, and thus $\mathscr{R}am(f_1) = \mathscr{R}am(f_2)$.

Conversely, let L be a holomorphic line bundle of degree d over M, and let $s_0, \cdots, s_3$ be four holomorphic sections without common zeros. If $\mathscr{R}am(f_1) = \mathscr{R}am(f_2)$, then there is a constant c^2 such that $[s_0, s_1] = -c^2[s_2, s_3]$ in view of (2.3); we may assume $c = 1$ by rescaling. It follows that $[s_0 : s_1 \pm s_2 : \pm s_3]$ will define two holomorphic maps $F_\pm$ of degree d from M to $\boldsymbol{CP}^3$ which satisfy (2.1). Therefore, F can be reconstructed from the pair (f_1, f_2) up to the contact involution

$$\sigma: [z_0 : z_1 : z_2 : z_3] \mapsto [z_0 : z_1 : -z_2 : -z_3]. \tag{2.4}$$

We choose to identify [1:0] and [0:1] in $\boldsymbol{HP}^1$ with the south and north poles of s^4, respectively. Then σ induces the geodesic symmetry about the south pole on s^4. Of course, this reduces to the construction in Loo [12] when the genus is zero.

As in [1], let G_d^r be the space of all r-dimensional linear systems g_d^r of degree d on M, and let W_d^r be the space of all holomorphic line bundles L of degree d such that $\dim H^0(L) \geq r+1$. Let

$$\pi: G_d^1 \to W_d^1$$

be the natural projection. Given a $g_d^1 = \langle s, t \rangle$ with $s, t \in H^0(L)$ and $L = \pi(g_d^1)$, it is clear that g_d^1 determines the map $p: \mapsto [s(p): t(p)]$ in $\boldsymbol{CP}^1$ up to $\mathrm{Aut}(\boldsymbol{CP}^1) = \mathrm{PGL}(2, \boldsymbol{C})$. The collection of all such maps determined by G_d^1 is a principal $\mathrm{PGL}(2, \boldsymbol{C})$-bundle over G_d^1 ([14]), to be denoted by R_d^1. Let

$$\pi_1: R_d^1 \to G_d^1$$

be the natural projection. In view of (2.3) $\mathcal{R}am: R_d^1 \to S^{2g-2+2d}M$, where the target space is the $(2g-2+2d)$-fold symmetric product of M. Moreover, set

$$\mathcal{N}_d^*(M)=\{(f_1,f_2)\in R_d^1\times R_d^1:\pi\pi_1(f_1)=\pi\pi_1(f_2), \mathcal{R}am(f_1)=\mathcal{R}am(f_2), \pi_1(f_1) \text{ and } \pi_1(f_2) \text{ have disjoint base loci}\}.$$

Observe that $\mathcal{N}_d^*(M)$ recovers all the horizontal holomorphic curves of degree d in $\boldsymbol{CP}^3$ except the ones where the pair (f_1, f_2) is a constant (iff f_1 is constant then f_2 is constant by (2.1)), in which case the corresponding horizontal curves are of the form $[s:as:t:bt]$, where a, b are complex numbers and s, t are two sections of L_F; the projection of these horizontal holomorphic curves into s^4 gives totally geodesic 2-spheres passing through both the north and the south poles. To include these horizontal curves, we must enlarge $\mathcal{N}_d^*(M)$.

Recall that R_d^1 is a principal PGL(2,$\boldsymbol{C}$)-bundle over G_d^1. As in [13], if we identify $\boldsymbol{CP}^3$ with the projectivization of the space of nonzero 2×2 complex matrices, there is a natural PGL(2,$\boldsymbol{C}$)-action on $\boldsymbol{CP}^3$. Consider the associated bundle $R_d^1\times_{\mathrm{PGL}(2,\boldsymbol{C})}\boldsymbol{CP}^3=\bar{R}_d^1$ over G_d^1. Let

$$\pi_2:\bar{R}_d^1\to G_d^1$$

be the standard projection. Fix (s_1,s_2) in R_d^1. Any other (s'_1,s'_2) with $s'_i=\sum_j a_{ji}s_j$ is identified with $[(s_1,s_2),(a_{ji})]$ in $\bar{R}_d^1$. As a consequence, a horizontal curve of the form $[s:as:t:bt]$ in $\boldsymbol{CP}^3$ projects to

$$\left[(s,t),\begin{bmatrix}1 & a\\ 0 & 0\end{bmatrix}\right]\times\left[(s,t),\begin{bmatrix}0 & 0\\ 1 & b\end{bmatrix}\right]$$

in $\bar{R}_d^1\times\bar{R}_d^1$. Accordingly, we set

(2.5) $$\mathcal{N}_d(M)=\{(f_1,f_2)\in \bar{R}_d^1\times\bar{R}_d^1:\pi\pi_2(f_1)=\pi\pi_2(f_2), \mathcal{R}am(f_1)=\mathcal{R}am(f_2), \pi_2(f_1) \text{ and } \pi_2(f_2) \text{ have disjoint base loci}\}.$$

It is understood here that $\mathcal{R}am(f_1)$, for instance, is the ramification divisor of $\pi_2(f_1)\in G_d^1$. Then $\mathcal{N}_d(M)$ recovers $\mathcal{M}_d(M)$ up to the involution in (2.4). In other words, $\mathcal{N}_d(M)=\mathcal{M}_d(M)/\sigma$.

Proposition 1. *Let $p:\mathcal{M}_d(M)\to\mathcal{N}_d(M)$ be the covering map, and let $V_1,V_2,\cdots,V_k$ be the irreducible components of $\mathcal{N}_d(M)$. Then $p^{-1}(V_1),\cdots,p^{-1}(V_k)$ are the irreducible components of $\mathcal{M}_d(M)$. Furthermore, $\mathcal{M}_d(M)$ is connected if and only if $\mathcal{N}_d(M)$ is connected.*

Proof. We claim first that σ in (2.4) is homotopic to the identity map. Indeed, each orthogonal transformation on S^4 induces naturally an automorphism on

CP^3. Consider the geodesic symmetry about the south pole on S^4, which is an orientation-preserving isometry and hence is homotopic to the identity. This homotopy induces a homotopy on CP^3 from the identity map to σ on CP^3, and thus on $\mathscr{M}_d(M)$, which interchanges the two elements in each fiber of the map p. The first statement follows from the fact that σ, being homotopic to the identity map, must leave invariant each irreducible component of $\mathscr{M}_d(M)$. The second statement is a consequence of the first. Q.E.D.

3. Basics

From now on we consider $\mathscr{N}_d(M)$ in view of Proposition 1. To understand the space $\mathscr{N}_d(M)$, we will slice by $L \in W_d^1$. Namely, fixing L we consider the space

$$\mathscr{N}_{d,L}(M) = \{(f_1, f_2) \in \mathscr{N}_d(M) : \pi\pi_2(f_1) = \pi\pi_2(f_2) = L\}.$$

Clearly, $\mathscr{N}_d(M) = \bigcup_{L \in W_d^1} \mathscr{N}_{d,L}(M)$.

Let L be a line bundle of degree d, and let $t_1, t_2, \cdots, t_m$ be a basis of $H^0(L)$. Notice that $m = d - g + 1$ by the Riemann-Roch Theorem when L is generic or when $d > 2g - 2$. The map

$$\mathscr{R} : s \wedge t \mapsto [s, t], \tag{3.1}$$

where $[s, t]$ is given in (2.2), extends to a linear map from $\wedge^2(H^0(L))$ to $H^0(K \otimes L^2)$, which can be projectivized as a rational map, still denoted by $\mathscr{R}$, from $P(\wedge^2(H^0(L)) \simeq CP^{\binom{m}{2}-1}$ to $P(H^0(K \otimes L^2)) \simeq CP^{2d+g-2}$. Note that $\mathscr{R}$ is completely determined by $\mathscr{R}(t_i \wedge t_j) = [t_i, t_j]$. Let $G(2, H^0(L))$ be the Grassmann manifold of two-planes in $H^0(L)$; $G(2, H^0(L)) \subsetneq P(\wedge^2(H^0(L)))$ via the Plücker imbedding $(s, t) \mapsto s \wedge t$, where s and t span the plane (s, t). Observe that $G(2, H^0(L))$ is characterized by the equation $x \wedge x = 0$, where $x = s \wedge t$.

Lemma 1. *The rational map $\mathscr{R}$ in* (3.1) *is regular on* $G(2, H^0(L))$.

Proof. If $\mathscr{R}(s \wedge t) = sdt - tds = 0$, then $d(t/s) = 0$ so that t is a constant multiple of s. Hence $s \wedge t = 0$, which is impossible. Q.E.D.

In accordance with this lemma we see that

$$\mathscr{N}_{d,L}(M) = \{(x, y) \in \mathscr{N}_d(M) : \pi_2(x),\ \pi_2(y) \in G(2, H^0(L)),\ \mathscr{R}(\pi_2(x)) = \mathscr{R}(\pi_2(y)),\ \pi_2(x) \text{ and } \pi_2(y) \text{ have disjoint base loci}\}.$$

Lemma 2. *$\mathscr{R}$ restricted to $G(2, H^0(L))$ is a finite map.*

Proof. $\mathscr{R}$ is induced by a linear map, and can therefore be regarded as a projection whose center does not intersect $G(2,H^0(L))$ by Lemma 1. Hence $\mathscr{R}$ is a finite map on $G(2,H^0(L))$ [16]. Q.E.D.

Lemma 3. *If* $(x,y)\in\mathscr{N}_{d,L}(M)$, $x\neq y$, *is represented by* $\pi_2(x)=[e_1\wedge e_2]$ *and* $\pi_2(y)=[e_3\wedge e_4]$ *in* $G(2,H^0(L))$, *then* e_1, e_2, e_3, e_4 *are linearly independent in* $H^0(L)$. *Here,* [ ] *denotes projectivization.*

Proof. Since $\mathscr{R}(\pi_2(x))=\mathscr{R}(\pi_2(y))$, we have $\mathscr{R}(e_1\wedge e_2-\lambda e_3\wedge e_4)=0$ for some $\lambda\in \boldsymbol{C}$ on the Euclidean level; we may assume $\lambda=1$ by rescaling. So $[v]=[e_1\wedge e_2-e_3\wedge e_4]\notin G(2,H^0(L))$ by Lemma 1. Hence $v\wedge v\neq 0$. This implies e_1, e_2, e_3, e_4 are independent. Q.E.D.

In light of Lemma 3, we now restrict our consideration from $H^0(L)$ to a 4-dimensional linear subsystem $V_4\subset H^0(L)$. Let $G(2,V_4)\subset G(2,H^0(L))$ be the Grassmann manifold of 2-planes in V_4.

Lemma 4. $\mathscr{R}$ *restricted to* $G(2,V_4)$ *is either a one-to-one map or a branched double covering onto its image.*

Proof. It is wellknown that $G(2,V_4)\subset \boldsymbol{P}(\wedge^2 V_4)\simeq \boldsymbol{CP}^5$ is a smooth hyperquadric. If $\mathscr{R}$ is not one-to-one on $G(2,V_4)$, $\mathscr{R}$ restricted to $\boldsymbol{P}(\wedge^2 V_4)$ must have a center, which cannot intersect $G(2,V_4)$ by Lemma 1, and therefore must be a single point. This shows that $\mathscr{R}(\boldsymbol{P}(\wedge^2 V_4))\simeq \boldsymbol{CP}^4$. Since $\dim G(2,V_4)=4$ and $\mathscr{R}$ is a finite map, the image of $G(2,V_4)$ must have dimension 4 and is therefore the entire $\boldsymbol{CP}^4$. The fact that $G(2,V_4)$ is a quadric implies that $\mathscr{R}$ is a branched double covering. Q.E.D.

The connectedness of $\mathscr{N}_d(M)$ is now immediate from Lemma 4 since one can always deform (x,y) in $\mathscr{N}_d(M)$ to some (t,t) on the singular locus of $G(2,V_4)$ of the map $\mathscr{R}$. In [8], the connectedness of $\mathscr{M}_d(M)$ is proved for any S^n.

4. Moduli spaces of small degree

One consequence of Lemma 3 is that to construct branched superminimal immersions which are not of the form $(f,Af)\in\mathscr{N}_d^*(M)$, where f is of degree d and $A\in \mathrm{PGL}(2,\boldsymbol{C})$, i.e., which are not totally geodesic, it is necessary that one start with a line bundle $L\in W_d^1$ such that $\dim H^0(L)\geq 4$, i.e., $L\in W_d^3$. However, there is no W_d^r when M is generic and the Brill-Noether number $(r+1)(d-r)-rg<0$. So, we have the following.

Proposition 2. *Let M be a generic Riemann surface of genus* $g\geq 1$. $\mathscr{N}_d(M)$ *is empty if* $d<(g+2)/2$. $\mathscr{N}_d^*(M)$ *consisits of* (f,Af), *where f is of degree d and*

$A\in \mathrm{PGL}(2,\boldsymbol{C})$, if $(g+2)/2\leq d<(3g+12)/4$; so all the corresponding branched superminimal immersions in S^4 are totally geodesic. Here, any Riemann surface of $g=1$, 2 or 3 is considered generic.

Proof. Take $r=1$ in the Brill-Noether number, which is <0 if $d<(g+2)/2$, in which case there are no W_d^1 for a generic Riemann surface. Similarly, take $r=3$ in the Brill-Noether number, which is <0 when $d<(3g+12)/4$; hence there are no W_d^3 for a generic Riemann surface. Finally, for any Riemann surface of $g=1,2$ or 3 with a line bundle L of a given degree within the bounds, one checks by the Riemann-Roch Theorem that $\dim H^0(L)\leq 3$. Q.E.D.

On the other hand Clifford's Theorem enables us to look into the case of a small degree d for any Rimann surface. Recall first that Clifford's Theorem ([10]) states that if $L\in W_d^r-W_d^{r+1}$ with $d\leq 2g-2$, then $d\geq 2r$; furthmore if $d=2r$, then either L is trivial, or $L=K$, the canonical bundle, or the Riemann surface M is hyperelliptic with the branched double covering $\phi:M\to \boldsymbol{CP}^1$ and $L=(\phi^*\mathcal{O}(1))^r$.

Proposition 3. *If $\mathrm{Min}(g,6)\geq d$, then $\mathcal{N}_d^*(M)=\{(f,Af): f$ is of degree d and $A\in \mathrm{PGL}(2,\boldsymbol{C})\}$. Hence the branched superminimal immersions from M into S^4 are all totally geodesic, except in the case when M is hyperelliptic and $d=6$, in which case $\mathcal{M}_6(M)$ is isomorphic to $\mathcal{M}_3(\boldsymbol{CP}^1)$, the moduli space of horizontal rational curves of degree 3.*

Proof. If there is an $(x_0,y_0)\in \mathcal{N}_{d,L}(M)$, $x_0\neq y_0$, then $L\in W_d^r$ with $r\geq 3$ as mentioned earlier. By Clifford's Theorem $6\geq d\geq 2r\geq 6$. However this is possible only when $d=6=2r$. Now $L\neq K$, the canonical bundle, since $6=d=2g-2$ implies $g=4$ while we assume that $g\geq 6$. Hence Clifford's Theorem infers that the Riemann surface is hyperelliptic, $L=(\phi^*\mathcal{O}(1))^3$ and $H^0(L)$ is generated by $(z)^i\circ\phi$, $0\leq i\leq 3$, where $\phi:M\to \boldsymbol{CP}^1$ is the branched double covering and $z\in \boldsymbol{C}$ (one regards $\boldsymbol{CP}^1$ as $\boldsymbol{C}\cup\{\infty\}$). Therefore $G(2,H^0(L))$ is comprised of $f\circ\phi$, where $f: M\to \boldsymbol{CP}^1$ and $\deg(f)\leq 3$. Now since $d(f\circ\phi)=df\circ d\phi$, we see that

$$\mathcal{R}am(f\circ\phi)=\mathcal{R}am(\phi)+\phi^{-1}(\mathcal{R}am(f)).$$

It follows that $\mathcal{R}am(f\circ\phi)=\mathcal{R}am(g\circ\phi)$ if and only if $\mathcal{R}am(f)=\mathcal{R}am(g)$. Consequently, the proposition will be true if we can verify that all the maps from M to $\boldsymbol{CP}^1$ of degree 6 come from $G(2,H^0(L))$. But this is the case since all the complete g_d^r with $d\leq g$ (in our case $d=6$ and $r\leq 3$) on a hyperelliptic curve is of the form $rg_2^1+p_1+p_2+\cdots+p_{d-2r}$, where no two of the points p_i are invariant under the involution of M induced by ϕ and g_2^1 is the linear system corresponding to ϕ [10]; hence the g_d^r, $r\leq 2$, will be ruled out since they have base locus $p_1,\cdots,p_{d-2r}$. Q.E.D.

REMARK. In the same vein as in the proof of Proposition 3, let M be a hyperelliptic curve and $d \leq g$. Then the moduli space of branched superminimal immersions from M into S^4 is isomorphic to that of branched superminimal spheres of degree $d/2$.

Before proceeding with further examples of small degree, we first consider a general situation. Let $t_1, t_2, \cdots, t_m$ span $H^0(L)$. Consider the curve $\psi: M \to \boldsymbol{CP}^{m-1}$ given by $\psi(p) = [t_1(p): \cdots : t_m(p)]$. The first associated curve ψ_1 of ψ, i.e., the set of the tangents of ψ, lies in $G(2,m)$ indentified with $G(2, H^0(L))$. Via the Plücker imbedding, $\psi_1 \in \boldsymbol{P}(\wedge^2(H^0(L))$.

Lemma 5. *Let k be the dimension of the smallest linear subspace containing ψ_1 in $\boldsymbol{P}(\wedge^2(H^0(L))$. Then the dimension of the center of the projection $\mathcal{R}$ in* (3.1) *is equal to* $\binom{m}{2} - k - 2$.

Proof. Observe first that in homogeneous coordinates (see (2.3) for notation) $\psi_1 = [\psi \wedge \psi'] = [\cdots : [t_i, t_j] : \cdots]$, where we use ψ and ψ' to also denote the Euclidean lift and derivative of ψ. Hence any linear relation $\Sigma a_{ij}[t_i, t_j] = 0$ gives rise to the element $\Sigma a_{ij} t_i \wedge t_j$ which lies in the center of the projection $\mathcal{R}$ in view of (3.1), and vice versa. Q.E.D.

Lemma 6. *Let $[1 : z^{1+\alpha_1} : z^{2+\alpha_1+\alpha_2} : z^{3+\alpha_1+\alpha_2+\alpha_3}]$ be the canonical form of a linearly full curve ψ in $\boldsymbol{CP}^3$ around $z=0$. Here we only display the first term in each Taylor series. Then ψ_1, the first associated curve of ψ, is linearly full in $\boldsymbol{CP}^5 \supset G(2,4)$ if $\alpha_1 \neq \alpha_3$.*

Proof. Assume $\alpha_3 < \alpha_1$. A straightforward computation shows that ψ_1 assumes the form $[1 : z^a : z^b : z^c : z^d : z^e]$, where $a = 1 + \alpha_2$, $b = 2 + \alpha_2 + \alpha_3$, $c = 2 + \alpha_1 + \alpha_2$, $d = 3 + \alpha_1 + \alpha_2 + \alpha_3$ and $e = 4 + \alpha_1 + 2\alpha_2 + \alpha_3$. It follows that $a < b < c < d < e$. So the curve is linearly full in $\boldsymbol{CP}^5$. Q.E.D.

We now study the case when $g=2$ and $d=5$ so that $d=2g+1$. Let L be a line bundle of degree 5 over M of genus 2; $\dim H^0(L) = 4$ by the Riemann-Roch Theorem. As mentioned before Lemma 5, any basis $t_1, \cdots, t_4$ of $H^0(L)$ generates a curve $\psi: M \to \boldsymbol{CP}^3$ of degree 5 which is an imbedding in our case (any L of degree $d \geq 2g+1$ is very ample). Conversely, the plane cut of any imbedded space curve of $g=2$ and $d=5$ in $\boldsymbol{CP}^3$ gives a line bundle of degree 5. From now on we identify M with $C = \psi(M)$ in $\boldsymbol{CP}^3$. Pick a point p on C and consider the projection π_p in $\boldsymbol{CP}^3$ whose center is p. $\pi_p(C) = C'$ is a curve of degree 2 or 4 in $\boldsymbol{CP}^2$ because π_p has mapping degree 4. If $\deg(C') = 2$, then C' is a conic. π_p

may be regarded as a branched double covering from C onto the Riemann sphere; hence the canonical bundle $K=p_1+p_2$, where $\{p_1,p_2\}=\pi_p^{-1}(x)$ for any $x\in C'$. (For simplicity in notation, we regard "=" in $K=p_1+p_2$, etc., as the divisor p_1+p_2 defining K.) Now pick a line joining x and some y on C' with $\pi_p^{-1}(x)$ as given above and $\pi_p^{-1}(y)=\{p_3,p_4\}$. Then $D=p+p_1+p_2+p_3+p_4$ is a plane cut of C defining L. It follows that $L=p+2K$. Otherwise, $\deg(C')=4$ and C' has a unique ordinary double point x by the genus formula ([15]). Let $\{q_1,q_2\}=\pi^{-1}(x)$; p, q_1, q_2 are collinear. The projection whose center is the line $\overline{pq_1q_2}$ is a meromorphic function of order 2 whose poles are, say, p_1 and p_2; so $K=p_1+p_2$. Now $D=p+q_1+q_2+p_1+p_2$ is a plane cut of C defining L. We obtain $L=p+q_1+q_2+K$, where $q_1+q_2\neq K$.

Proposition 4. *Let M be a Riemann surface of genus 2. Then all the branched superminimal immersions of degree 5 from M into S^4 are totally geodesic.*

Proof. Consider $\dim H^0(L-i\cdot q)$, $0\leq i\leq 3$, for an arbitrary point q; it is equal to $(4-i)+\dim H^0(K-L+i\cdot q)$ by the Riemann-Roch Theorem, which is $4-i$ if $i\leq 2$ since the degree of $K-D+i\cdot q$ is negative. Now let $i=3$.

Case 1. $L=p+2K$. $H^0(K-L+3q)$ is equal to $H^0(3q-p-2\#)$ with $K=2\#$ for some fixed Weierstrass point # chosen once and for all (recall that M is hyperelliptic).

If $H^0(3q-p-2\#)\neq 0$, there will be a meromorphic function f assuming the only pole of order at most 3 at q and zeros of order at least 1 and 2 at p and #, respectively. f cannot be of order 3; for otherwise, q and 2# are all the zeros of f, so that if we let ω be a holomorphic form whose zero is 2#, then $f^{-1}\omega$ will be a meromorphic form with a single pole q of order 1, which is absurd. Thus f can only be of order 2. However, this implies that q will eliminate either p or #.

If $q=p$, then $2p=2\#$ and so $L=5p$ with p a Weierstrass point. Now $\dim H^0(i\cdot p)=1,1,2,2,3,4$ for $i=0,1,2,3,4,5$ since p is a Weierstrass point. We have $\dim H^0(L-i\cdot p)=\dim H^0((5-i)p)=4,3,2,2,1,1,0$ for $i=0,1,2,3,4,5,6$. It follows that near p, ψ assumes the parametric form $[1:z:z^3:z^5]$ so that $(\alpha_1,\alpha_2,\alpha_3)$ given in Lemma 6 is $(0,1,1)$; in particular $\alpha_1\neq\alpha_3$ at p. Lemma 6 then implies that the first associated curve of ψ is nondegenerate in $\boldsymbol{CP}^5$. Lemma 5 in turn asserts that $\mathscr{R}$ has no center, i.e., $\mathscr{R}$ is injective. In other words, the branched superminimal immersion constructed is totally geodesic, which is what we intend to conclude.

Hence we may now assume $q\neq p$ and so $q=\#$. But then f will be a meromorphic function of order 1 with pole # and zero p, which is impossible unless $p=\#=q$, so that one more time we obtain $L=5p$ with p a Weierstrass point.

Therefore we may now assume that $\dim H^0(3q-p-2\#)=0$, i.e., $\dim H^0(L-3q)=1$ for all q. In summary, we have $\dim H^0(L-i\cdot q)=4-i$, $0\leq i\leq 3$, for all q. This is equivalent to saying that near any q, ψ is of the form $[1:z:z^2:z^m]$ with $m\geq 3$; in particular, $\alpha_1=\alpha_2=0$ for all points. However, there must be a point at which

$\alpha_3 \neq 0$ by the Plücker formula ([10]), and thus at this point $\alpha_1 \neq \alpha_3$. Hence again $\mathscr{R}$ is injective, and the branched superminimal immersion is totally geodesic.

Case 2. $L=p+q_1+q_2+K$ with $q_1+q_2 \neq K$. As explained above, C' must be a curve of degree 4 in $\boldsymbol{CP}^2$ having only an ordinary double point. Since ψ is imbedded, at a ramified point p the curve ψ is of the form $[1:z:z^{2+\alpha_2}:z^{3+\alpha_2+\alpha_3}]$, where $\alpha_1=0$. If $\alpha_3 \neq 0$, then $\alpha_1 \neq \alpha_3$ and we are done by Lemmas 5 and 6. Hence we may assume $\alpha_1=\alpha_3=0$ at all ramified points p. We claim that this case cannot occur. To this end, observe that the projection π_p in $\boldsymbol{CP}^3$ with center p maps C to C' whose only singularity, being the image of p, is a cusp of the form $(z^{1+\alpha_2}, z^{2+\alpha_2})$ in affine coordinates, so that $\alpha_2=1$ since the singularity must be an ordinary simple cusp. Thus $(\alpha_1,\alpha_2,\alpha_3)=(0,1,0)$ at all ramified points p. Now, for $1 \leq k \leq 3$ the Plücker formula $\Sigma_k(4-i)\alpha_k=32$ ([10]) implies that there are 16 ramified points for ψ. On the other hand, since the tangent line to ψ at a ramified point is of contact order 3, we must have $p=q_1=q_2$ in $L=p+q_1+q_2+K$. Hence $L=3p+K$ with $2p \neq K$ for all ramified points p; in particular p is not a Weierstrass point. Fixing one ramified point p_0, for any ramified point $p \neq p_0$ we have $L-K=3p_0=3p$, so that there is a meromorphic function assuming the single pole and zero of order 3 at p_0 and p, respectively. However, $\dim H^0(3p_0)=2$, i.e., $3p_0$ defines a single g_3^1, we therefore see that all the ramified points belong to this g_3^1, each of ramification index 2. In particular, the total ramification index of the g_3^1 is ≥ 32, which is absurd, since the total ramification is 8 by the Riemann-Hurwitz formula. Q.E.D.

We are now ready to characterize all branched superminimal immersions of degree ≤ 5.

Theorem 1. *Let M be a Riemann surface of genus $g \geq 1$. Then all the branched superminimal immersions of degree $d \leq 5$ from M into S^4 are totally geodesic.*

Proof. Proposition 9 below in Section 6 solves the case $g=1$. Proposition 2 takes care of $g=2$ when $d \leq 4$, while $d=5$ is handled by Proposition 4. The case $g=3$ follows from Proposition 2. For $g=4$, Proposition 3 gives the result as long as $d \leq 4$. However, when $g=4$ and $d=5$, we have $d<2g-2$; hence Clifford's Theorem implies that $d \geq 2r$, i.e., $\dim H^0(L) \leq 3$ for any bundle L of degree 5. Finally, Proposition 3 settles $g \geq 5$. Q.E.D.

We now move on to the case $d=6$. We first study the case $g=3$ and $d=6$ so that $d=2g$. Let M be a nonhyperelliptic Riemann surface of genus 3 and let L be a line bundle over M of degree 6. As before let ψ be the curve in $\boldsymbol{CP}^3$ associated with L. Since $\deg(L-p-q)=\deg(K)$, we see that $L=K+p+q$ if and only if $\dim H^0(L-p-q)=3$, if and only if ψ is not imbedded (recall that ψ is imbedded if and only if $\dim H^0(L-p-q)=\dim H^0(L)-2$ for all p and q

[10]). Notice that when $p=q$, $L=K+2p$ means that the curve ψ is not an immersion at p, whereas when $p\neq q$, $L=K+p+q$ signifies that ψ is an immersion but is not one-to-one.

Assume now that ψ is an imbedded curve so that $L\neq K+x+y$ for any x and y. Identify M with $C=\psi(M)$. Pick a point $p\in C$. Let $C'=\pi_p(C)$ be the projection of C where π_p has center p. Since $\deg(C')=5$, for a generic point p, C' has three ordinary double points x, y and z by the genus formula; in general these singularities may collapse so that higher singularity may result. Let $\{x_1,x_2\}$, $\{y_1,y_2\}$ and $\{z_1,z_2\}$ be the preimages of x, y and z, respectively, via π_p. The pair in each set is collinear with p; denote these three lines by l_1, l_2, l_3. The projections π_1, π_2 and π_3, whose centers are l_1, l_2 and l_3 respectively, are meromorphic functions of order 3 whose poles are, say $\{p_1,p_2,p_3\}$, $\{q_1,q_2,q_3\}$ and $\{r_1,r_2,r_3\}$, respectively, so that $\dim H^0(p_1+p_2+p_3)=2$ by nonhyperellipcy, i.e., $\dim H^0(K-p_1-p_2-p_3)=1$. In other words, there is a point p_0 such that p_0, p_1, p_2, p_3 are collinear on the canonical curve ϕ_K imbedded in $\boldsymbol{CP}^2$ so that $K=p_0+p_1+p_2+p_3$. Similarly there are q_0 and r_0 collinear with q_i and r_i, $1\leq i\leq 3$, respectively, on ϕ_K. Since $D=p+x_1+x_2+p_1+p_2+p_3$ is a hyperplane cut defining L, we see that ([15])

$$L=p+x_1+x_2+K-p_0. \tag{4.1}$$

Similar identities hold when x_i are replaced by y_i and z_i and p_0 by q_0 and r_0, repectively, $1\leq i\leq 2$. In particular, $x_1+x_2-p_0=y_1+y_2-q_0$ by (4.1), i.e., $x_1+x_2+q_0=y_1+y_2+p_0$, or x_1, x_2 and q_0 are collinear on ϕ_K; similarly x_1, x_2 and r_0 are collinear on ϕ_K. We see then that x_1, x_2, q_0 and r_0 are collinear on ϕ_K so that $K=x_1+x_2+q_0+r_0$. Substituting this into (4.1) gives

$$L=2K+p-p_0-q_0-r_0. \tag{4.2}$$

Sublemma 1. *Notation is as above. Let ψ be immersed in $\boldsymbol{CP}^3$, and let p be a point at which the first associated curve is singular (i.e., $\dim H^0(L-3p)=3$). Then there is a meromorphic function of order 3 whose only pole is p. In particular, p is a Weierstrass point. Moreover, $L=s_1+s_2+s_3+3p$ where s_1, s_2 and s_3 are collinear on ϕ_K.*

Proof. Retaining the assumption that ψ is imbedded, we consider the correspondence $T(p)=p_0+q_0+r_0$. T is of valence -1, i.e., $T(p)-p$ is independent of p, which is clear since $T(p)-p=2K-L$ by (4.2). Moreover, T has no united points, i.e., there are no points p for which $p\in T(p)$, which follows because $p\in T(p)$ would force, say $p=p_0$, and thus by (4.1) $L=K+x_1+x_2$ so that ψ would be singular. Therefore, the Cayley-Brill formula ([10]) asserts that $\deg(T^{-1})=3$, i.e., for each point p_0, there are three points p, p' and p'' such that $p_0\in T(p)$, $T(p')$ and $T(p'')$. By the definition of p_0, this means that the points p_1, p_2 and p_3 introduced above belong to three plane cuts through p, p' and p'', respectively; in particular, p_1, p_2 and p_3 are collinear. The projection whose center is the line $\overline{p_1p_2p_3}$ is a meromorphic function H of order 3, whose poles may be chosen to be p, x_1 and x_2 since these six points are coplanar.

Now if ψ is imbedded, then $\dim H^0(L-3p)=3$ means that the tangent line to ψ at p has contact order at least 3 and so $p=x_1=x_2$; hence the function H defined above is a function with a single pole of order 3 at p. By (4.1), we have $L=K+3p-p_0=p_1+p_2+p_3+3p$.

If ψ is immersed but not imbedded, then $L=K+x+y$ with $x\neq y$. $\dim H^0(L-3p)=3$ is equivalent to $\dim H^0(3p-x-y)=2$ by the Riemann-Roch Theorem, which gives the existence of such a function H of order 3 whose only pole is p. In particular, let s_1, x and y be the zeros of H. Then $3p=x+y+s_1$ and so there is point s_2 such that $3p+s_2=x+y+s_1+s_2=K$. Substituting this into $L=K+x+y$ yields $L=x+y+s_2+3p$. Q.E.D.

Sublemma 2. *Let p be an immersed point of a nondegenerate nonhyperelliptic curve ψ of degree 6 in CP^3. Suppose the first associated curve of ψ is singular at p. Then the tangent line to ψ at p is of contact order 3. In particular, $\alpha_1=0$ and $\alpha_2=1$ at p.*

Proof. The contact order must be at least 3. If the contact order is 4, then the projection in CP^3 whose center is the tangent line at p will be a meromorphic function of order at most 2, which is impossible since the curve is nonhyperelliptic. Q.E.D.

Proposition 5. *Let M be a nonhyperelliptic Riemann surface of genus 3. Then all the branched superminimal immersions from M into S^4 of degree 6 are totally geodesic.*

Proof. Suppose there is a nontotally geodesic branched superminimal immersion generated by a line bundle of degree 6. As usual let ψ be the curve in CP^3 associated with L.

Case 1. ψ is immersed. By Lemma 6, $\alpha_1=\alpha_3=0$ for all points. Take a ramified point q of ψ. $(\alpha_1,\alpha_2,\alpha_3)=(0,1,0)$ by Sublemma 2. Now the formula $\Sigma_k(4-k)\alpha_k=48$, $1\leq k\leq 3$, asserts that there are 24 ramified points on ψ, while Sublemma 1 says that these 24 points are all Weierstrass points. On the other hand, the Plücker formula applied to the canonical curve ϕ_K, which is imbedded in CP^2, gives $\Sigma(2\beta_1+\beta_2)=(g-1)g(g+1)=24$ summed over all Weierstrass points, which were just proved to be ≥ 24 in number, where ϕ_K assumes the parametric form $[1:z^{1+\beta_1}:z^{2+\beta_2}]$. Since $\beta_1=0$ for all p, we see that there are exactly 24 Weierstrass points with $\beta_2=1$ for all of them. In other words, all the Weierstrass points are ordinary flexes.

Now given two Weierstrass points p and p', by Sublemma 1 we have $L=K+3p-s_p=K+3p'-s_{p'}$ for some points s_p and $s_{p'}$. Let $\tilde{p}+3p=K=\tilde{p}'+3p'$ (the tangent line to ϕ_K at p intersects ϕ_K at $\tilde{p}$). Then we see that $\tilde{p}+s_p=\tilde{p}'+s_{p'}$. Therefore either $\tilde{p}'=\tilde{p}$, or $\tilde{p}'=s_p$ since M is nonhyperelliptic. Thus fixing p, the 24 Weierstrass points p' are divided into the set S_1 where $\tilde{p}'=\tilde{p}$, in which case the tangent lines to ϕ_K at p' are all through $\tilde{p}$, and the set S_2, where $\tilde{p}'=s_p$, in which case the tangent lines to ϕ_K at p' are all through s_p. We may thus assume that S_1 contains at least 12 Weierstrass points without loss of generality. Howerer, the projection in $\boldsymbol{CP}^2$ whose center is p on ϕ_K gives a meromorphic function π_p of order 3 for which p' in S_1 have ramification 2 except when $p'=p$, where the ramification index is 1. Now the Riemann-Hurwitz formula says that the total ramification for π_p is 10, while the sub-total ramification over these Weierstrass points is at least $2\cdot 11+1=23$, which is a contradiciton.

Case 2. ψ is not immersed. Then $L=K+2p$ for some p. By the Riemann-Roch Theorem $\dim H^0(L-i\cdot p)=4-i+\dim H^0((i-2)p)=4,3,3,2,1$ for $i=0,1,2,3,4$, respectively, so that with the fact that $\alpha_1=\alpha_3$ for all points we have, near p, that ψ is of the parametric form $[1:z^2:z^3:z^5]$ with $(\alpha_1,\alpha_2,\alpha_3)=(1,0,1)$ at p. The existence of z^5 implies $1=\dim H^0(L-5p)$ so that $\dim H^0(3p)=2$ by the Riemann-Roch Theorem; in particular, p is a Weierstrass point.

Let $q\neq p$ be a ramified point. $\dim H^0(L-i\cdot q)=4-i+\dim H^0(i\cdot q-2p)=4,3,2$ for $i=0,1,2$, respectively. Hence q is an immersed point. Sublemma 2 then infers that $(\alpha_1,\alpha_2,\alpha_3)=(0,1,0)$ so that ψ is of the form $[1:z:z^3:z^4]$ near q. As a consequence of the nonexistence of z^2, we have $H^0(L-3q)=2$, or equivalently $\dim H^0(3q-2p)=1$; in particular q is also a Weierstrass point and there is a point s_0 such that $3q=2p+s_0$. Let s_1 and s_2 be such that $s_1+3q=K=s_2+3p$ (p and q are Weierstrass points which have contact of order at least 3 to ϕ_K). Then $p+s_2=s_0+s_1$. Hence either $3q=3p$, in which case the tangent lines to ϕ_K at p and q pass through $s_1=s_2$, or $p=s_1$, in which case the tangent line to ϕ_K at q passes through p; we divide such points q into two sets U_1 and U_2, respectively. We are now in a familiar situation that we saw in Case 1. The number of ramified points of ψ is 23 (total ramification at p is 4 and is 2 at $q\neq p$), so that we may assume U_1 contains at least 12 of them for instance. However the projection with center p_2 in $\boldsymbol{CP}^2$ is a meromorphic function of order 3 which has $q\in U_1$ as ramified points and whose total ramification is 10, which is absurd. Q.E.D.

The upper limit of the degree d of a special line bundle L (L is special if $H^0(K\otimes L^{-1})\neq 0$) is $2g-2$ by the Riemann-Roch Theorem. Let $g=4$ and $d=2g-2=6$. Let M be a nonhyperelliptic Riemann surface of genus 4. Recall that M is the intersection of a quadric surface Q and a cubic surface C in $\boldsymbol{CP}^3$ ([10]). If Q is nonsingular, Q has two one-parameter families of independent rulings L_1 and L_2 ([10]), where the one-parameter t for L_1 (s for L_2, respectively) runs along a fixed line in L_2 (a fixed line in L_1, respectively), such that any two different lines in the same ruling are not coplanar whereas any two lines from the two different rulings are coplanar. On the other hand, Q degenerates to a cone if it is singular and the two rulings coincide. Each line $l_t \in L_1$ ($l'_s \in L_2$, respectively) intersects the cubic surface C in three points p_1, p_2, p_3. The two projections from $\boldsymbol{CP}^3$ to $\boldsymbol{CP}^1$ whose centers are the lines l_t and l'_s give rise to two meromorphic functions of order 3 on M; hence there are at least two g^1_3. To see that there are exactly two g^1_3 for a nonsingular Q, let $q_1+q_2+q_3$ be a divisor defining a g^1_3. Since $\dim H^0(q_1+q_2+q_3)=\dim H^0(K-q_1-q_2-q_3)=$ the number of independent planes containing q_1, q_2, q_3 in $\boldsymbol{CP}^3$ by the Riemann-Roch Theorem, it follows that $\dim H^0(q_1+q_2+q_3)=2$ and q_1, q_2 and q_3 are collinear. The line through q_1, q_2 and q_3 must belong to one of the rulings; therefore, there are exactly two g^1_3 for a nonsingular Q. In particular, if Q degenerates to a cone, then $l_t=l'_s$ and so there is a unique g^1_3.

Proposition 6. *Let M be a nonhyperelliptic Riemann surface of genus* 4. *Then all the branched superminimal immersions of degree* 6 *from M into S^4 are totally geodesic.*

Proof. Since $d=2g-2$, L must be the canonical bundle so that the corresponding curve ψ is nothing but the canonical curve in $\boldsymbol{CP}^3$, which is imbedded. We identify M with $C=\psi(M)$. We recall that on a canonical curve, p is an unramified point for all of the associated curves of ψ if and only if p is a non-Weierstrass point. Accordingly, we assume that q is a Weierstrass point in what follows. By Lemma 6 and Sublemma 2, once more we have $(\alpha_1, \alpha_2, \alpha_3)=(0,1,0)$ for all q. We claim that this case cannot occur. For, first note that the Plücker formula gives that the number of Weierstrass points is $(g-1)g(g+1)/2=30$. However, since 3 is not a Weierstrass gap value at q we see that all $3q$ belong to the two g^1_3 (if the quadric surface Q mentioned above is nondegenerate) ; one of these g^1_3 therefore contains at least 15 Weierstrass points q at which the ramification index of this g^1_3 is 2. By the Riemann-Hurwitz formula, the total ramification of the g^1_3, which is 12, must be greater than or equal to the subtotal ramification index evaluated at these Weierstrass points, which is at least $2\cdot 15=30$. This is a contradiction. Q.E.D.

REMARK. Equivalently put, Propositions 4 through 6 say that all nonhyperelliptic space curves of degree 5 and genus 2, degree 6 and genus 3, and degree 6 and genus 4 have nondegenerate first associated curves in $\boldsymbol{CP}^5$.

We are ready to characterize the Riemann surfaces of genus ≥ 1 for which there exist nontotally geodesic branched superminimal immersions into S^4.

Theorem 2. *Let M be a Riemann surface of genus $g \geq 1$. M admits a nontotally* *geodesic branched superminimal immersion of degree* 6 *into* S^4 *if and only if* M *is hyperelliptic.*

Proof. If M is hyperelliptic of any genus, then M admits a nontotally geodesic branched superminimal immersion. More precisely, let $\phi: M \to \boldsymbol{CP}^1$ be the branched double covering and let (f_1, f_2) be a pair of meromorphic functions on $\boldsymbol{CP}^1$ which gives rise to a nontotally geodesic branched superminimal sphere. Then $(f_1 \circ \phi, f_2 \circ \phi)$ is a pair which generates a nontotally geodesic branched superminimal immersion on M. Conversely, Proposition 3 takes care of $g \geq 6$. For $g=4$ and 5, we have $2g-2 \geq 6=d$. Hence $\dim H^0(L)=4$ by Clifford's Theorem if one can construct a nontotally geodesic branched superminimal immersion on L. L is not the canonical bundle for $g=5$ since $2g-2 \neq d$; Clifford's Theorem then concludes that M is hyperelliptic. Finally, Propositions 5 and 6 finish the cases $g=3$, 4. Q.E.D.

Theorem 2 brings forward the question of classifying $\mathcal{M}_6(M)$ for a hyperelliptic surface M of genus g. We will do it for $g \geq 3$ in this section.

Consider a hyperelliptic Riemann surface M of genus 3. Let L be a line bundle of degree 6 over M and let ψ be the curve of degree 6 associated with L in $\boldsymbol{CP}^3$. Assume ψ is imbedded and identify M with $C=\psi(M)$. For a point $p \in M$ consider the projection π_p whose center is p. $C'=\pi_p(C)$ is a curve of degree 5 in $\boldsymbol{CP}^2$ which has a unique triple point as singularity by the genus formula and the fact that a hyperelliptic Riemann surface of genus ≥ 3 has no meromorphic functions of order 3 (so that the singularity cannot be a double point). Let this singular point be x and let $\pi^{-1}(x)=\{p_1, p_2, p_3\}$. As before, p, p_1, p_2 and p_3 are collinear, and the projection whose center is this line is a meromorphic function of order 2; let the pole of this function be a Weierstrass point $\sharp$ chosen once and for all. We have

$$L=2\sharp+p+p_1+p_2+p_3. \tag{4.3}$$

Note that $K=4\sharp$. Consider the correspondence $T(p)=p_1+p_2+p_3$. T has valence 1 since $T(p)+p=L-2\sharp$. Furthermore $\deg(T^{-1})=3$; for otherwise, if $p_1 \in T(q)$ for $q \neq p, p_2, p_3$, then p, q, $T(p)$ and $T(q)$ would be coplanar so that $\deg(C) \geq 7$. It follows from the Cayley-Brill formula that T has 12 united points. Now since the tangent line to ψ at a ramified point p is of contact order ≥ 3, we see that we may assume $p_2=p_3=p$ in (4.3) so that on the one hand

$$L=2\sharp+3p+p_1, \tag{4.4}$$

and on the other hand p is a united point. Hence the number of ramified points ≤ 12.

Proposition 7. *Let* M *be a hyperelliptic Riemann surface of genus* 3. *A* *nontotally geodesic branched superminimal immersion of degree* 6 *from* M *into* S^4 *is the pullback of a branched superminimal sphere of degree* 3 *via the branched double covering* $\phi: M \to \boldsymbol{CP}^1$.

Proof. Let L be the line bundle of degree 6 generating the nontotally geodesic branched supeminimal immersion. As before let ψ be the holomorphic curve in $\boldsymbol{CP}^3$ associated with L.

In what follows, we will assume that $L \neq 6\#$; otherwise it is just the conclusion of this proposition, because $\psi = [1:\phi:\phi^2:\phi^3]$ then.

Case 1. ψ is nonsingular. $\alpha_1 = \alpha_3 = 0$ for all $q \in M$ by Lemma 6. Let p be a ramified point of ψ. By (4.4) $\dim H^0(L - i \cdot p) = (4-i) + \dim H^0(2\# + (i-3)p - p_1) = 4,3,2,2$ if $i = 0,1,2,3$ since ψ is an imbedding and its first associated curve is singular at p. Now $\dim H^0(L-4p) = \dim H^0(2\# + p - p_1)$, and moreover $H^0(2\# + p) = H^0(2\#)$ because M has no meromorphic functions of order 3. We see that $\dim H^0(L-4p) = \dim H^0(2\# + p - p_1) = 2$ or 1 if $p = p_1$ or $p \neq p_1$, respectively. (If $\dim H^0(2\# + p - p_1) = 2$ when $p \neq p_1$, then $p_1 = \#$ and $\# + p = 2\#$ since $\dim H^0(\# + p) = 2$; hence $p = p_1 = \#$ and $L = 6\#$ by (4.4), which is excluded.) It follows that either $p = p_1$ where $(\alpha_1, \alpha_2, \alpha_3) = (0,2,0)$ and $L = 4p + 2\#$, or $p \neq p_1$ and $(\alpha_1, \alpha_2, \alpha_3) = (0,1,0)$.

We now estimate the number of ramified points p for which $L = 4p + 2\#$. Pick one such point p_0. Any other such p satisfies $4p = 4p_0 = L - 2\#$. On the other hand, $\dim H^0(4p_0) = 2$ by the Riemann-Roch Theorem since $K \neq 4p_0$ (or else $L = 6\#$). We assert then that all these ramified points belong to the g^1_4 generated by $4p_0$, each of ramification index 3. Since the Riemann-Hurwitz formula says that the total ramification index of this g^1_4 is 12, it follows that there are at most 4 ramified points p such that $L = 4p + 2\#$. The formula $\Sigma_k (4-k)\alpha_k = 48$, $1 \leq k \leq 3$, for ψ then implies that there are at least 16 ramified points such that $L = 3p + p_1 + 2\#$ with $p \neq p_1$. This is a contradiction since we mentioned preceding this proposition that there are at most 12 ramified points.

Case 2. ψ is singular. $L = K + x + y$ for some x and y; since we assume that $K \neq 6\#$, we have $x + y \neq 2\#$. Now $\dim H^0(L - i \cdot \#) = 4 - i + \dim H^0(i \cdot \# - x - y) = 4,3,2$ for $i = 0,1,2$ clearly. If $\dim H^0(L - 3\#) = 1$, then since $\dim H^0(L - 4\#) = \dim H^0(x+y) = 1$ (recall $x + y \neq 2\#$) we have that near $\#$, ψ assumes the parametric form $[1:z:z^2:z^m]$ with $m \geq 4$ so that $\alpha_1 \neq \alpha_3$ at $\#$, which is ruled out by Lemma 6. Thus $\dim H^0(L - 3\#) = 2$, i.e., $\dim H^0(3\# - x - y) = 1$. Hence there is a point z such that $3\# = x + y + z$. However, this forces $x = \#$ or $y = \#$; for on the one hand one of x, y and z must equal $\#$ since there are no meromorphic functions of order 3, and on the other hand $z \neq \#$, or else $x + y = 2\#$. Assume $x = \#$, so that $L = 5\# + y$, with $y \neq \#$. Now $\dim H^0(L - 5\#) = \dim H^0(y) = 1$, and $\dim H^0(L - 6\#) = \dim H^0(y - \#) = 0$. We conclude that near $\#$, ψ is of the form $[1:z:z^3:z^5]$, so that $\alpha_1 \neq \alpha_3$ at $\#$, which is impossible by Lemma 6. Q.E.D.

Theorem 3. *Let* M *be a hyperelliptic surface of genus* $g \geq 3$. *Then*

$\mathcal{M}_6(M)=V_1\cup V_2$, where V_1 is the totally geodesic part $\simeq\bar{R}^1_6$ (see Section 2 for notation), and V_2 is isomorphic to the nontotally geodesic part of $\mathcal{M}_3(\boldsymbol{CP}^1)$. V_1 and V_2 are identified along the singular locus of $\mathcal{M}_3(\boldsymbol{CP}^1)$. In particular, nontotally geodesic branched superminimal immersions of degree 6 from M into S^4 are the pullback of nontotally geodesic branched superminimal spheres of degree 3 via the branched double covering of M onto $\boldsymbol{CP}^1$. Furthermore, $\mathcal{M}_6(M)\simeq\mathcal{M}_3(\boldsymbol{CP}^1)$ only when $g\geq 6$.

Proof. For the first statement, Proposition 3 takes care of $g\geq 6$. For $g=4$ or 5, since $6\leq 2g-2$, Clifford's Theorem suffices for the conclusion. $g=3$ is finished by Proposition 7. We are left with showing that $\mathcal{M}_3(\boldsymbol{CP}^1)$ is not isomorphic to $\mathcal{M}_6(M)$ for $3\leq g\leq 5$. It is enough to exihbit a g^1_6 which does not come from $G(2,H^0(L))$, where $L=(\phi^*(\mathcal{O}(1)))^3$ with ϕ the branched double covering onto $\boldsymbol{CP}^1$. To this end, observe first of all that a $g^1_6\in G(2,H^0(L))$ gives rise to a meromorphic g of degree 6 on M of the form $f\circ\phi$, where f is meromorphic of degree 3 on $\boldsymbol{CP}^1$, so that the polar divisor $(g)_\infty$ of g is invariant under the involution τ of M. Now pick a non-Weierstrass point p such that $p\neq\tau(p)$ and consider the divisor 6p. The Weierstrass gap values at p are (1,2,3), (1,2,3,4) and (1,2,3,4,5) for $g=3,4$ and 5, respectively, since p is a non-Weierstrass point. It follows that there are meromorphic functions of order 6 whose only pole is p; take such a function g of order 6. Then $(g)_\infty=6p$, which is not invariant under τ; hence $g\neq f\circ\phi$ for any f that is a rational function of degree 3 over $\boldsymbol{CP}^1$. Q.E.D.

We will classify the case $g=1$ and $d=6$ in Section 6. Contrary to $g\geq 3$, lots of nontotally geodesic branched superminimal tori exist.

5. Moduli spaces of large degree

In contrast with small degrees, we will next show that when the degree d is sufficiently large nontotally geodesic branched superminimal immersions of genus $g\geq 1$ are abundant.

Recall that given a Riemann surface M of genus $g\geq 2$ ($g\geq 1$, respectively), let $d\geq g+3$ (≥ 3, respectively). Then M is rationally equivalent to a curve of degree d with at most ordinary nodes as singularities. This follows from the wellknown fact that a line bundle L of degree d is very ample if $d\geq 2g+1$. Furthermore, for $g\geq 2$, there exists a nonspecial very ample line bundle of degree d if $d\geq g+3$ ([11]).

Before proving the existence of a branched superminimal immersion of a sufficiently large degree d into S^4, we recall that a branched superminimal immersion assumes the parametric form

$$[1:y-2^{-1}x\,dy/dx:x:2^{-1}dy/dx], \tag{5.1}$$

where x and y are arbitrary meromorphic functions on the Riemann surface ([2]). Notice that one can interpret $[1:x:y]$ as an algebraic curve in $\boldsymbol{CP}^2$.

Lemma 7. *Let $F=[1:x:y]$ be a plane curve with dual curve F^*. Let $\alpha(p)$ and $\beta(p)$ be the pole order of x and y at p, respectively. Then the branched superminimal immersion G given in* (5.1) *is of degree equal to* $\deg(x)+\deg(F^*)-\sum_{p\in M}(\varepsilon(p)+\eta(p)+\theta(p)+\zeta(p))$, *where* $\varepsilon(p)=\mathrm{Max}$ *(pole order of* $y-x\,dy/dx$, 0), *if* $\alpha(p)=\beta(p)$. $\eta(p)=\beta(p)-\mathrm{Max}$ *((pole order of* $y-2^{-1}x\,dy/dx)-\alpha(p)$, 0), *if* $\beta(p)=2\alpha(p)$. $\theta(p)=\alpha(p)$, *if* $\alpha(p)<\beta(p)$ *and* $\beta(p)\neq 2\alpha(p)$. $\zeta(p)=\beta(p)$, *if* $\alpha(p)>\beta(p)$. ($\varepsilon(p), \eta(p), \theta(p), \zeta(p)=0$ *elsewhere.*)

Proof. We know

$$F^*=[1:x\,dy/dx-y:dy/dx].$$

If dy/dx is identically zero, the lemma is trivially true. Assume therefore that $dy/dx\neq 0$. We will count the number of points of intersection of G (F^*, respectively) and the plane $P_1=\{[s:t:u:0]\}$ (the plane $P_2=\{[s:t:0]\}$, respectively). Let $\sigma(p)$ be the difference between the intersection multiplicities of $G\cap P_1$ and $F^*\cap P_2$ at p.

Case 1. $x=a_0+a_1z^{\alpha}+\cdots$ and $y=b_0+b_1z^{\beta}+\cdots$ around $z=0$ identified with $p\in M$. Then dy/dx is a zero of order β-α at p (if $\beta>\alpha$ of course). All the other coordinate functions for F^* and G are holomorphic around $z=0$. Hence $\sigma(p)=0$.

Case 2. $x=a_0+a_1z^{\alpha}+\cdots$ and $y=z^{-\beta}+b_1z^{-\beta+1}+\cdots$ Then p is a pole of order $\alpha+\beta$ for dy/dx, and all other coordinate functions for F^* and G have poles of order $\leq\alpha+\beta$. In other words $P_1\cap G$ and $P_2\cap F^*$ are empty at p, and so $\sigma(p)=0$.

Case 3. $x=z^{-\alpha}+a_1z^{-\alpha+1}+\cdots$ and $y=b_0+b_1z^{\beta}+\cdots$ Then p is a zero of order $\alpha+\beta$ for dy/dx. The second coordinate functions for both F^* and G are holomorphic around $z=0$, *whereas* x, having a pole of order α at p, contributes α to the intersection multiplicity of $G\cap P_1$. Hence $\sigma(p)=\alpha(p)$.

Case 4. $x=z^{-\alpha}+a_1z^{-\alpha+1}+\cdots$ and $y=z^{-\beta}+b_1z^{-\beta+1}+\cdots$ Then G is of the form $[(1:(1-\beta/2\alpha)z^{-\beta}:z^{-\alpha}:(\beta/\alpha)z^{\alpha-\beta}]$ and F^* of the form $[1:(1-\beta/\alpha)z^{-\beta}:(\beta/\alpha)z^{\alpha-\beta}]$. (We only exhibit the leading term of each Taylor series.) (a): If $\alpha=\beta$, then $G\cap P_1$ is of intersection multiplicity α while $F^*\cap P_2$ is of intersection multiplicity equal to the pole order of $y-x\,dy/dx$ at p, which is $\leq\alpha$. Hence $\sigma(p)=\alpha(p)-\varepsilon(p)$. (b): If $\beta=2\alpha$, then the intersection multiplicity of $F^*\cap P_2$ is α while the intersection multiplicity of $G\cap P_1$ is Max ((pole order of $y-2^{-1}x\,dy/dx)-\alpha$, 0). Hence $\sigma(p)=\alpha(p)-\eta(p)$. (c): If $\alpha(p)<\beta(p)$ and $\beta(p)\neq 2\alpha(p)$, then both $G\cap P_1$ and $F^*\cap P_2$ have intersection multiplicity equal to α. Hence $\sigma(p)=\alpha(p)-\theta(p)$. (d): If $\alpha(p)>\beta(p)$, then $G\cap P_1$ is of intersection multiplicity $2\alpha-\beta$ while $F^*\cap P_2$ is of intersection multiplicity α. Hence $\sigma(p)=\alpha(p)-\zeta(p)$. *Adding* $\sigma(p)$ in the four cases gives the result. Q.E.D.

REMARK. It is important to understand the geometric contents of this lemma. In CP^2, pick any three independent points A, B, C and set up the projective coordinate system such that $A=[1:0:0]$, $B=[0:1:0]$, $C=[0:0:1]$. Given a Riemann surface and a holomorphic map $f: M \to CP^2$, the projection with center C (B, respectively) onto the line AB (line AC, respectively) gives the meromorphic function x (y, respectively). The cases in Lemma 7 can be rephrased as follows: Case 1 holds if $f(p) \in CP^2 \setminus$line BC, the affine part of CP^2. Case 2 holds if $f(p)=C$ and $f(M)$ is transversal to line BC. Case 3 holds if $f(p)=B$ and $f(M)$ is transversal to line BC. Case 4.a. holds if $f(p) \in$ line BC and $f(p) \neq B, C$. Case 4.b. and 4.c. hold if $f(p)=C$ and $f(M)$ is tangent to line BC. Case 4.d. holds if $f(p)=B$ and $f(M)$ is tangent to line BC.

Corollary 1. *Notation as in Lemma 7 and the above remark, let M be rationally equivalent to $f(M)$.*
i) *If $f(M)$ does not pass through the points B and C, and if line BC intersects $f(M)$ transversally, then* $\deg(G)=\deg(F)+\deg(F^*)$.
(ii) *$f(M)$ does not pass through the points B, C, and line BC intersects $f(M)$ transversally with the exception of one generic point to which line BC is tangent, then* $\deg(G)=\deg(F)+\deg(F^*)-1$.
(iii) *$f(M)$ is through C but not through B, and if line BC intersects $f(M)$ transversally except for one generic point different from C to which line BC is tangent, then* $\deg(G)=\deg(F)+\deg(F^*)-2$.
(iv) *If line BC is tangent to $C \in f(M)$ as a generic tangent line, and if line BC is transversal to $f(M)$ otherwise, then* $\deg(G)=\deg(F)+\deg(F^*)-3$.

Proof. (i) is true since it is Case 4.a. in Lemma 7 with $(\alpha,\beta)=(1,1)$ for any point of intersection of line BC and $f(M)$. Hence $\varepsilon(p)=\eta(p)=\theta(p)=\zeta(p)=0$, and $\deg(x)=\deg(F)$.

(ii) holds since it is Case 4.a. with $(\alpha,\beta)=(1,1)$ for $\deg(F)-1$ points of intersection at which line BC intersects $f(M)$ transversally, where $\varepsilon(p)=\eta(p)=\theta(p)=\zeta(p)=0$. Moreover, it is Case 4.a. for the point of tangency at which $(\alpha,\beta)=(2,2)$, where $\varepsilon(p)=1$, $\eta(p)=\theta(p)=\zeta(p)=0$. $\deg(x)=\deg(F)$ in this case.

(iii) holds since it is (ii) above at all points of intersection of line BC and $f(M)$ other than C. At C, it is Case 2 in Lemma 7 with $\varepsilon(p)=\eta(p)=\theta(p)=\zeta(p)=0$. Furthermore, $\deg(x)=\deg(F)-1$ since C is the projection center of x and $C \in f(M)$.

(iv) holds since it is (i) above for all points of intersection of line BC and $f(M)$ other than C. At C, it is Case 4.b. with $(\alpha,\beta)=(1,2)$, where $\eta(p)=2$, $\varepsilon(p)=\theta(p)=\zeta(p)=0$. $\deg(x)=\deg(F)-1$ in this case for the same reason as in (iii). Q.E.D.

Theorem 4. *Let M be a Riemann surface of genus $g \geq 2$ ($g=1$, respectively). If $d \geq 5g+4$, (≥ 6, respectively), then there is a nontotally geodesic branched superminimal* *immersion of degree d from M to* S^4. *The immersion is generically one-to-one.*

Proof. Pick a plane curve F of degree $d_1 \geq g+3$ (≥ 3 if $g=1$) with only δ nodes as singularities. By the Plücker formula, $g=(d_1-1)(d_1-2)/2-\delta$. Let d_2 be the degree of the dual curve of $F; d_2=d(d\text{-}1)-2\delta$. We have $d_1+d_2=2g+3d_1-2 \geq 5g+7$ (≥ 9 if $g=1$) and any two consecutive d_1+d_2 differ by 3. Now Corollary 1 implies that any such d_1+d_2 and the two numbers between two consecutive d_1+d_2 are achieved as the degree of a nontotally geodesic branched superminimal immersion; consequently $5g+7-3=5g+4$ ($=6$ if $g=1$) is the first degree that occurs as the degree of a nontotally geodesic branched superminimal immersion in this procedure. That the immersion is generically one-to-one follows from inspecting (5.1). Q.E.D.

REMARK. The lower bound for the degree d in Theorem 4 is sharp when $g=1$, as we will show in Proposition 9 that all the branched superminimal immersions of degree ≤ 5 are totally geodesic if $g=1$. However, it is not sharp for $g \geq 2$. For example, the above lower bound is 14 when $g=2$. Now take a plane quartic curve F of genus 2 with a simple cusp of multiplicity 2 ([15]). The Plücker formula shows that $\deg(F^*)=9$ so that $\deg(G)=13$. (Notation is as in Corollary 1.) Hence Corollary 1 infers that 10 is a better lower bound. On the other hand, one can easily construct examples of degree 6 and 8 when $g=2$ (degree ≤ 5 is excluded by Theorem 1); given the branched double cover $\phi: M \to \boldsymbol{CP}^1$, $x \circ \phi$, where $x \in \mathcal{M}_3(\boldsymbol{CP}^1)$ or $\mathcal{M}_4(\boldsymbol{CP}^1)$, will be examples. It is not clear if there are nontotally geodesic branced superminimal immersions of degree 7 and 9 for $g=2$.

With the existence result in Theorem 4, we now estimate the dimension of $\mathcal{M}_d(M)$.

Lemma 8. *Notation is as in* (2.7). *For each* $x \in G_d^1$, *there are only finitely many* $y \in G_d^1$ *for which* $\mathcal{R}am(x)=\mathcal{R}am(y)$.

Proof. Let $L_1=\pi(x)$ and $L_2=\pi(y)$. $\mathcal{R}am(x)=\mathcal{R}am(y)$ implies $K \otimes (L_1)^2 = K \otimes (L_2)^2$, and hence $(L_1)^2=(L_2)^2$. So there are only finitely many such L_2. Now apply Lemma 2. Q.E.D.

In the following theorem we refer to a Riemann surface of genus g as being "generic" if G_d^1 is an irreducile variety of dimension equal to the Brill-Noether number $2d-g-2$. For example, all Riemann surfaces are generic in this sense if $d \geq 2g-1$, or $d \geq 2g-2$ since G_d^1 is the canonical blowup of $W_d^1 \simeq J(M)$ at the canonical bundle K regarded as a point in $J(M)$ ([1]), or when M is sufficiently general in the moduli space of Riemann surfaces of genus g so that the Brill-Noether Theory applies.

Theorem 5. *Let M be a generic Riemann surface of genus g in the above sense. Then the dimension of each irreducible component of $\mathcal{M}_d(M)$ is between $2d-4g+4$ and $2d-g+4$, where the upper bound is achieved by the totally geodesic component.*

Proof. A glance at (2.7) shows that we need to impose the condition $\mathcal{R}am(x)=\mathcal{R}am(y)$ for $(x,y)\in G_d^1\times G_d^1$ to find the dimension of $\mathcal{N}_d(M)$. Now since $\mathcal{R}am$ maps G_d^1 to $S^{2g-2+2d}M$, the $(2g-2+2d)$-fold symmetric product of M, $\mathcal{R}am(x)=\mathcal{R}am(y)$ imposes at most $2g-2+2d$ conditions to carve out a subvariety of $G_d^1\times G_d^1$ of dimension $4d-2g-4$. Hence the set $S=\{(x,y)\in G_d^1\times G_d^1\colon \mathrm{Ram}(x)=\mathrm{Ram}(y)\}$ is of dimension $\geq(4d-2g-4)-(2g-2+2d)=2d-4g-2$. Notice that both x and y give rise to 3-dimensional meromorphic functions, respectively. So $\dim\mathcal{N}_d(M)\geq(2d-4g-2)+6$, which is the lower bound. Here, we do not need to worry about the other two conditions, namely, $\pi(x)=\pi(y)$ and x and y have disjoint base loci as given in (2.7), since once we are given a $(x_0,y_0)\in S$ satisfying the two extra conditions, then any element (x,y) in the irreducible component of S containing (x_0,y_0) will satisfy $\pi(x)=\pi(y)$, by continuity, due to Lemma 8; moreover, for $(x\ y)$ near (x_0,y_0), x and y will have disjoint base loci, by continuity again. To obtain the upper bound, observe that for each $x\in\pi^{-1}(L)$ with $L\in W_d^1$, there are only finitely many (x,y) in S by Lemma 8. Hence $\dim S\leq 2d-g-2$, and so $\dim\mathcal{N}_d(M)\leq(2d-g-2)+6$, which is the upper bound. Q.E.D.

When $g=0$, the upper and the lower bounds in Theorem 5 are identical. Hence $\mathcal{M}_d(M)$ is of pure dimension $2d+4$, which is obtained in [12], [17]. When $g=1$, we will show in section 6 that the lower bound is achieved for $d=6$.

We now look at Theorem 5 from a different point of view, which will facilitate the calculations to follow in the next section. Recall the map $\mathcal{R}\colon G(2,H^0(L))\to \boldsymbol{P}(H^0(K\otimes L^2))$ defined in (3.1). Let x and y, $x=[e_1\wedge e_2]$, and $y=[e_3\wedge e_4]$, in $G(2,H^0(L))$ satisfy $\mathcal{R}(x)=\mathcal{R}(y)$. Then $[e_1\wedge e_2-e_3\wedge e_4]$ is the projection center of $\mathcal{R}$ restricted to $G(2,V_4)$, *where* V_4 is spanned by $e_1,\cdots,e_4$. Observe that $\omega=e_1\wedge e_2-e_3\wedge e_4$ satisfies $\omega\wedge\omega\wedge\omega=0$. Conversely, a skew-symmetric form ω satisfying $\omega\wedge\omega\wedge\omega=0$ is either of rank 2 of the form $e_1\wedge e_2$, or of rank 4 of the form $e_1\wedge e_2-e_3\wedge e_4$. It is now clear that each point ω in the intersection $\mathcal{T}$ of $\mathrm{Ker}\,\mathcal{R}$ and the projective variety $\mathcal{L}=\boldsymbol{P}(\{\omega\in\wedge^2(H^0(L))\colon\omega\wedge\omega\wedge\omega=0\})$ in $\boldsymbol{P}(\wedge^2(H^0(L))$ is the center of the restriction of $\mathcal{R}$ to $G(2,V_4)$ for some 4-dimensional linear subsystem V_4 spanned by some e_0,e_1,e_2,e_3. (By Lemma 1, this intersection cannot contain a form ω of rank 2.) Then $f_1=[e_0:e_1]$ and $f_2=[e_2:e_3]$ give rise to a superminimal immersion. Now since $\dim\mathcal{L}=4k-11$ if $\dim H^0(L)=k$, a simple dimension count says that $\dim\mathcal{L}\geq 2d-5g-6$. In particular $\mathcal{L}$ is nonempty for every $d\geq(5g+6)/2$. (See [19] for a better bound for a general Riemann surface.) Varying $L\in J(M)$, we must add $g=\dim J(M)$ to the lower bound, which again gives the lower bound in Theorem 5.

REMARK. Note, however, that the above consruction does not supersede Theorem 4, because elements f_1 and f_2 which come from $\mathscr{T}$ might have common base loci so that the degree would be lower than d. What Theorem 4 implies is that for a sufficiently large d, there is always a line bundle L of degree d for which $\operatorname{Ker}\mathscr{R}\cap\mathscr{L}$ contains an element $e_1\wedge e_2-e_3\wedge e_4$ where $e_1,\cdots,e_4$ have disjoint base loci. In any event, the above construction does show the existence of nontotally geodesic branched superminimal immersions of degree $\leq(5g+7)/2$.

6. The case $g=1$

Let M be a Riemann surface of genus 1, and let L be a line bundle of degree d. Then L is the bundle associated with the divisor $d\cdot p$ for some point p. By applying the translation $p\mapsto 0$ on the torus, we may assume without loss of generality that p is 0, so that $H^0(L)$ is generated by the d sections 1, $\wp,\wp',\wp'',\cdots,\wp^{(d-2)}$.

Proposition 8. *Let M be a Riemann surface of genus* 1 *and let L be a line bundle over M of degree $d\leq 5$. Then $\mathscr{R}:\wedge^2(H^0(L))\to H^0(K\otimes L^2)=H^0(L^2)$ is* injective. Hence the moduli space $\mathscr{M}_d(M)$ consists only of totally geodesic branched superminimal immersions.

Proof. Recall the notations in (2.3) and (3.1). Observe that each of $[1,\wp^{(i)}]$ and $[\wp^{(i)},\wp^{(j)}]$, $0\leq i,j\leq d-2$, consists only of all even or all odd order terms in the polar part of its Laurent expansion. If $d\leq 4$, then an easy computation shows that the orders of the leading terms in the Laurent expansions of $[1,\wp^{(i)}]$ and $[\wp^{(i)},\wp^{(j)}]$, $0\leq i,j\leq d-2$, are all different; thus these bracketed quantities are independent in $H^0(L^2)$. So $\mathscr{R}$ is injective. For the case $d=5$, one checks similarly that those brackets with odd order terms are independent. The only possibility that $\mathscr{R}$ might have a kernel would be resulted from nontrivial linear relations among $[1,\wp']$, $[1,\wp^{(3)}]$, $[\wp,\wp']$, $[\wp,\wp'']$, and $[\wp,\wp^{(3)}]$. Differentiating the wellknown differential equation $(\wp')^2=4\wp^3-g_2\wp-g_3$ sufficiently many times, we obtain

$$\begin{bmatrix} \wp'' \\ \wp'''' \\ \wp\wp''-(\wp')^2 \\ \wp\wp''''-\wp'\wp''' \\ \wp'\wp'''-(\wp'')^2 \end{bmatrix} = \begin{bmatrix} -g_2/2, & 0, & 6, & 0, & 0 \\ -12g_3, & -18g_2, & 0, & 120, & 0 \\ g_3, & g_2/2, & 0, & 2, & 0 \\ 0, & 0, & -6g_2, & 0, & 72 \\ -(g_2)^2/4, & -12g_3, & -6g_2, & 0, & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \wp \\ (\wp)^2 \\ (\wp)^3 \\ (\wp)^4 \end{bmatrix}.$$

It is clear that 1, $\wp$, $(\wp)^2$, $(\wp)^3$, $(\wp)^4$ are linearly independent, and a straightforward calculation shows that the determinant of the above 5×5 matrix is $-27\times 2^{10}((g_2)^3-27(g_3)^2)\neq 0$ for a torus. Therefore $\mathscr{R}$ is injective. Q.E.D.

Corollary 2. *For $d\geq 5$, $\mathscr{R}$ is surjective.*

Proof. Since $\dim \wedge^2(H^0(L)) = \dim H^0(L^2) = 10$ if L is of degree 5, Proposition 8 shows that $\mathscr{R}: \wedge^2(H^0(L)) \to H^0(L^2)$ is bijective. If L is of degree 6, then among the brackets $[1, \mathfrak{p}^{(i)}]$ and $[\mathfrak{p}^{(i)}, \mathfrak{p}^{(j)}]$, $0 \le i$, $j \le 4$, $[\mathfrak{p}^{(2)}, \mathfrak{p}^{(4)}]$, and $[\mathfrak{p}^{(3)}, \mathfrak{p}^{(4)}]$ are independent of each other and of all the other brackets since the leading terms in their Laurent expansions are of order 11 and 12, respectively, while others have order ≤ 10. Hence the dimension of the linear subspace generated by all these brackets in $H^0(L^2)$ is of dimension at least 12, i.e., $\dim \mathscr{R}(\wedge^2(H^0(L)) \ge 12$. Therefore $\dim \mathscr{R}(\wedge^2(H^0(L)) = 12$ since $\dim \mathscr{R}(\wedge^2(H^0(L)) \le \dim H^0(L^2) = 12$. In other words, $\mathscr{R}$ is surjective. Exactly the same reasoning takes care of all $d \ge 6$. Q.E.D.

REMARK. Corollary 2 had been proved in [18]. However, our proof is elementary.

Now since the projective codimension of $\operatorname{Ker} \mathscr{R}$ is precisely $2d + g - 1 = 2d$ if L is of degree d by Corollary 2, a glance at the construction of $\mathscr{T}$ suggests that $\dim \mathscr{T} = 2d - 5g - 6 = 2d - 11$. We will show in the next proposition that this is true if $d = 6$.

Proposition 9. *Let M be a torus and L be a line bundle of degree* 6. *Then* $\dim \mathscr{T} = 1$. *Hence the nontotally geodesic irreducible components of $\mathscr{M}_6(M)$ all have dimension equal to* 12; *in particular the lower bound in Theorem* 5 *is achieved by these components.*

Proof. Let $e_1, e_2, \cdots, e_6$ be a basis of $H^0(L)$. The Euclidean dimension of the kernel of $\mathscr{R}$ is 3. Let E_1, E_2, E_3 be a basis of $\operatorname{Ker} \mathscr{R}$; E_1, E_2, E_3 are linear combinations of $e_i \wedge e_j$, $1 \le i$, $j \le 6$. Let $\omega = xE_1 + yE_2 + zE_3$, $x, y, z \in \boldsymbol{C}$, be any element in $\mathscr{T}$. Rewriting $\omega \wedge \omega \wedge \omega$ as a multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6$ and incorporating the fact that ω satisfies $\omega \wedge \omega \wedge \omega = 0$, we see that ω is defined by a nonvoid homogeneous polynomial of degree 3 in x, y, z. In other words, $\mathscr{T}$ is defined by a plane cubic curve. Hence $\dim \mathscr{T} = 1$. Q.E.D.

When $g = 0$, the nontotally geodesic part of $\mathscr{M}_d(\boldsymbol{CP}^1)$ is irreducible, and hence $\mathscr{M}_d(\boldsymbol{CP}^1)$ consists of two irreducible components ([13], [17]). This is not the case in general when $g \ge 1$ as the following proposition shows.

Proposition 10. *Let M be a torus. The nontotally geodesic part of $\mathscr{M}_6(M)$ can be reducible, although for a generic torus it is irreducible.*

Proof. It suffiecs to find a torus for which $\mathscr{T}$ is reducible and one for which $\mathscr{T}$ is irreducible. Consider the torus where $g_2 = 0$ and $g_3 = 1$. Set $e_1 = 1$, and $e_i = \mathfrak{p}^{(i-2)}$, $2 \le i \le 6$. Recall that a linear relation $\Sigma_{i,j} x_{ij}[e_i, e_j] = 0$ gives $\Sigma_{i,j} x_{ij} e_i \wedge e_j$ in $\operatorname{Ker} \mathscr{R}$, and vice versa. Comparing the coefficients in the Laurent expansions of all $[e_i, e_j]$ via the identity

$$\wp = 1/z^2 + \sum_{i=1}^{\infty} a_{2i} z^{2i}, \quad \text{where } a_2 = g_2/20,\ a_4 = g_3/28, \quad \text{and}$$

$$a_{2(n+1)} = 3(n-1)^{-1}(2n+5)^{-1} \sum_{i=1}^{n-1} a_{2i} a_{2(n-i)}$$

with $n \geq 2$, one ends up with three generators $E_1 = 108e_1 \wedge e_3 + e_3 \wedge e_6 - 5e_4 \wedge e_5$, $E_2 = 72e_1 \wedge e_2 - e_2 \wedge e_6 + e_3 \wedge e_5$, $E_3 = -e_1 \wedge e_6 + 60e_2 \wedge e_4$ for $\mathrm{Ker}\,\mathscr{R}$. (We leave out the details of calculation.) Let $\omega = xE_1 + yE_2 + zE_3$. Then $\omega \wedge \omega \wedge \omega = 0$ results in the equation $yz^2 - 3x^2y = 0$, which is the union of three lines defining $\mathscr{T}$, so $\mathscr{T}$ is reducible. On the other hand, setting $g_2 = 1$ and $g_3 = 0$ yields $E_1 = 5e_1 \wedge e_4 - e_2 \wedge e_6 + 5e_3 \wedge e_5$, $E_2 = -48e_1 \wedge e_2 - e_1 \wedge e_6 + 60e_2 \wedge e_4$, $E_3 = -72e_2 \wedge e_3 - e_3 \wedge e_6 + 5e_4 \wedge e_5$. Hence $\omega \wedge \omega \wedge \omega = 0$ with $\omega = xE_1 + yE_2 + zE_3$ asserts that $-x^3 + 12xy^2 + 24yz^2 = 0$, which is the torus with $g_2 = 1$ and $g_3 = 0$ defining $\mathscr{T}$; thus $\mathscr{T}$ is irreducible. Q.E.D.

7. Concluding remarks

Propositions 9 and 10 point to the challenging question whether the nontotally geodesic part of $\mathscr{M}_d(M)$ is of pure dimension $2d - g + 4$ for any Riemann surface of genus g, and whether it is irreducible, so that the moduli space of branched superminimal surfaces of degree d consists of three irreducible components, for a generic Riemann surface of genus g.

As for the compactification of $\mathscr{N}_d(M)$, and so for that of $\mathscr{M}_d(M)$, a glance at (2.5) suggests that the space

$$\bar{\mathscr{N}}_d(M) = \{(f_1, f_2) \in \bar{R}_d^1 \times \bar{R}_d^1 : \pi\pi_2(f_1) = \pi\pi_2(f_2),\ \mathscr{R}am(f_1) = \mathscr{R}am(f_2)\}$$

is the natural candidate, which Loo adopted in [13] when the genus $g = 0$. Whether this is true in the higher genus case depends on whether $\pi_2(f_1)$ and $\pi_2(f_2)$ having disjoint base loci encountered in (2.5) is a generic condition; for if it is not a generic condition, we will have an irreducible component of $\mathscr{M}_d(M)$ which is comprised entirely of branched superminimal immersions of degree lower than d, $\bar{\mathscr{N}}_d(M)$ will then be too large to be the compactification.

We suspect that the answers to these questions are all affirmative for a generic Riemann surface of genus g, which would follow if the intersection of $\mathrm{Ker}\,\mathscr{R}$ and $\mathscr{L}$ were transversal for all L in $J(M)$.

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and H. Harris: Geometry of Algebraic Curves,

Volume I, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.

[2] R. Bryant: *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Differ. Geom. **17** (1981), 455–473.

[3] E. Calabi: *Minimal immersions of surfaces in Euclidean spheres*, J. Differ. Geom. **1** (1967), 111–125.

[4] S.S. Chern: *On the minimal immersions of the two-sphere in a space of constant curvature*, Problems in Analysis, Princeton University Press, Princeton, 1970, 27–40.

[5] J. Eells and S. Salamon: *Twistorial constructions of harmonic maps of surfaces into four-manifolds*, Ann. Scuola Norm. Sup. Pisa **12** (1985), 589–640.

[6] Th. Friedrich: *On surfaces in four-spaces*, Ann. Global Analysis and Geometry **2** (1984), 257–287.

[7] W. Fulton and J. Hansen: *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. Math. **110** (1979), 159–166.

[8] M. Guest and Y. Ohnita: *Group actions and deformations for harmonic maps*, J. Math. Soc. Japan **45** (1993), 671–704.

[9] G. Jensen and M. Rigoli: *Twistor and Gauss lifts of surfaces in four-manifolds*, Contemporary Mathematics **101** (1989), 197–232.

[10] P. Griffiths and J. Harris: Principles of Algebraic Geometry, Weily-Interscience, New York, Chishester, Brisbane, Toronto, 1978.

[11] R. Hartshorne: Algebraic Geometry, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[12] B. Loo: *The space of harmonic maps of S^2 into S^4*, Trans. Amer. Math. Soc. **313** (1989), 81–102.

[13] B. Loo: *On the compactification of the moduli space of branched minimal immersions of S^2 into S^4*, preprint.

[14] M. Namba: Families of Meromorphic functions on Compact Riemann Surfaces, Lecture Notes in Math. **767**, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[15] M. Namba: Geometry of Projective Algebraic Curves, Pure and Applied Math. **88**, Marcel Decker, New York, 1984.

[16] I.R. Shafarevish: Basic Algebraic Geometry, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[17] J.L. Verdier: *Applications hormoniques de S^2 dans S^4*, Geometry of Today (E. Arbarello, C. Procesi, E. Strickland, eds.), Gionate di Geometria, Roma 1984 (Progress in Mathematics 60), Birkhaüser, Boston, 1985, 267–282.

[18] J. Wahl: *Gauss maps on algebraic curves*, J. Differ. Geom **32** (1990), 77–98.

[19] J. Wahl: *Introduction to Gaussian maps on an algebraic curve*, London Math. Soc. Lecture Notes Ser. **179** (1992), 304–323, Cambridge Univ. Press, Cambridge.

Quo-Shin Chi
Department of Mathematics
Washington University
St. Louis, MO 63130
USA
e-mail: chi@math.wustl.edu

Xiaokang Mo
Department of Mathematics
University of Kansas
Lawrence, KS 66045
USA
e-mail: mo@math.ukans.edu