Ballico, E. and Keem, C.
Osaka J. Math.
33 (1996), 663–667

FAMILIES OF SMOOTH *k*-GONAL CURVES WITH ANOTHER FIXED PENCIL

EDOARDO BALLICO and CHANGHO KEEM

(Received September 4, 1995)

0. Introduction

The actors of this paper are the same as the ones in [3], but the problems and methods are completely different (empty intersection). The actors are smooth curves, C, with 2 fixed pencils, say a g_k^1 and a g_b^1, which do not exist on curves with general moduli and that induce a birational morphism from C to a curve Y on a quadric surface $Q := \boldsymbol{P}^1 \times \boldsymbol{P}^1$, Y of bidegree (k,b). Indeed, while in [3] we studied a fixed such C, here we will study suitable families of such curves C.

In this paper we will work always in characteristic 0. In the first section we will prove (using very strongly [10]) the following result.

Theorem 0.1. *For all integers k, b, n with $0 \leq n \leq bk-b-k+1$ and either $k \geq 4$ and $b \geq 10$ or $k \geq 5$ and $b \geq 8$, the smooth scheme $W((k,b),n)$ parametrizing the set of all nodal irreducible curves in Q of bidegree (k,b) and with geometric genus $g := bk-b-k+1-n$ is irreducible.*

This theorem shows the power of the method introduced in [10] and refined very much in [11].

In the second section we will give a first step toward the Brill-Noether theory of special divisors on the general such curve C with as image $Y \subset Q$ a nodal curve, i.e. a curve $Y \in W((k,b),n)$. Remember that such a Brill-Noether theory is still in its infancy for curves not with general moduli. For interesting results for the case of general k-gonal curves, see [6] and [2]. In section 2 we will prove the following Brill-Noether type result.

Theorem 0.2. *Fix integers g, k, b, r, d with $r \geq 2$, $4 \leq k \leq b$, $2k-2 \leq g \leq bk-b-k+1$, $(r+1)d < r(2k+r-1)$. Let $S(g;k,b)$ be the constructible subset of the moduli space M_g of smooth curves of genus g parametrizing the curves, C, with a fixed pair of pencils, the first of degree k and the second of degree b, inducing a birational morphism from C onto a curve $Y \subset Q := \boldsymbol{P}^1 \times \boldsymbol{P}^1$. Then $S(g;k,b)$ is irreducible and a general $C \in S(g;k,b)$ has no g_d^r, only finitely many g_k^1 and no g_m^1 with $m < k$. Furthermore, C has Clifford index $k-2$.*

The proof of Theorem 0.2 was inspired by [7], §4.

This paper dues its existence to the stimulating atmosphere of Max-Planck-Institut für Mathematik (Bonn), where the two authors met and worked. The first named author was partially supported by MURST and GNSAGA of CNR (Italy). The second named author was partially supported by GARC-KOSEF and Ministry of Education (South Korea).

1. Irreducibility of family of nodal curves on Q.

Recall that we always work in characteristic 0. We will introduce now some notations. We will be interested in smooth genus g currves with two base point free pencils, a g_k^1 and a g_b^1, inducing a birational morphism onto a smooth quadric $Q \subset \boldsymbol{P}^3$. Hence we fix integers g, k, b with $2 \le k \le b$ and $0 \le g \le kb-k-b+1$; set $n := bk-b-k+1$. Let $W((k,b),n)$ be the subscheme of the Hilbert scheme Hilb(Q) of Q parametrizing the integral curves of bidegree (k,b) and with n nodes as only singularities. Let $V((k,b),n)$ be the subscheme of the Hilbert scheme Hilb(Q) of Q parametrizing the union for all $x>0$ of the nodal curves with x irreducible components and at least $n+x-1$ nodes. It is classical (see e.g. [12], lemma 2.2, or [1], §3 and §4, for modern complete proofs) that $W((k,b),n)$ is smooth and equidimensional of dimension $bk+b+k-n$ and that $V((k,b),n)$ is its closere in the part of Hilb(Q) parametrizing reduced nodal curves.

Proof of Theorem 0.1. The main step in the proof follows with only very minor modifications from the proof of the correspoding result for $\boldsymbol{P}^2$ given in [10] using (and introducing) a very poweful method of degeneration of the surface Q. Hence in the first step of the proof of Theorem 0.1 we will only give the very minor modifications assuming that the reader simultaneously reads [10]. The second step of the proof of Theorem 0.1 contains only classical material and we chosed [8] as reference. In the third (and last) step we conclude with a monodromy argument.

Step 1. We fix $P \in Q$. For each integer $m>1$ Q has a degeneration to the following reducible, reduced and connected surface $S(m)$. $S(m)$ is the union (with only double normal crossing as singularities) of the blowing up, $S(1)$, of Q at P and $m-1$ surfaces isomorphic to F_1. To build $S(m)$ the glueing curve on the surfaces isomorphic to F_1 are exactly the same as in [10]. The glueing curve on the surfaces isomorphic to F are exactly the same as in [10]. The glueing curve on $S(1)$ is the strict transform of the line of type (1,0) (i.e. in the system of lines of Q inducing the degree b pencil) through P. Set $d := k+b$ and use the same notations (V, V_D^{***}, and so on) to denote the corresponding varieties introduced in [10] (with "(k,b)" instead of "d" as superscript). The references now will be to the intermediate lemmas and propositions in [9]. In our situation Proposition 1.1 is OK with $\dim(V) = d-e+g-1$ instead of $2d-e+g-1$, since K_Q has bidegree

$(-2,-2)$. The same small modification must be made in Remarks 1.1.1 and 1.1.2, in the computation of obstruction spaces in Proposition 1.4 (which gives Corollary 1.5) and in the dimensional estimates of [10], §2. These dimensional estimates carry over because we never find a negative integer as lower bound for the dimension of a fiber of the suitable morphisms involved. The theory behind [10], §2, works with only very trivial notational changes and we conclude that every component, W, of $W((k,b),n)$ has in its closure in Hilb(Q) a curve T union of k lines of type (1,0) and b lines of type (0,1) (hence in $V((k,b),n)$).

Step 2. Let M be the set of reduced curves of types (k,b) on Q which are union of $k+b$ lines. Note that M is irredudible and contained in $V((k,b),n+g)$. Hence M is in the closure of the set $W((k,b),n+g)$ of integral nodal curves of bidegree (k,b) and with geometric genus 0. Note that $W((k,b),n+g)$ is irreducible, since it is an open subset of the pairs of a degree k and a degree b pencil on $\boldsymbol{P}^1$. It was explained in [8], first part of §4 before Proposition 4.1, that this imppllies that W contains $W((k,b),n+g)$. To check that $W=W((k,b),n)$, i.e. to prove Theorem 0.1, it is sufficient to show that $W((k,b),n+g)$ is contained in a unique component of $V((k,b),n)$. As explained in [8], §1, as in the case of plane nodal curves, it is sufficient to show that the monodromy action induced moving the rational nodal curve in $W((k,b),n+g)$ on the set of $n+g$ nodes contains the alternating group A_{n+g}. Since $k\geq 4$ and $b\geq 10$ or $k\geq 5$ and $b\geq 8$, we have $n+g>24$. Hence by the classification of 4-transitive finite permutation groups (see [4], §5) it is sufficient to check that the monodromy group G is least 4-transitive.

Step 3. We assume to have checked the 3-transivity of G and we will prove that it is 4-transitive. Ther proofs that G is 1-transitive, and then that it is 2-transitive and then that it is 3-transitive, are exactly the same, but work with weaker restrictions on k and b. We fix 5 points $P(1),\cdots,P(5)$ on Q and we consider the set $A(1)$ of all rational curves $A\in W((k-2,b-2),\ kb-k-b+3)$ with $P(j)\in A$ for $j=1,2,3$. Let $B(1)$ be the corresponding set for the rational curves $B\in W((2,2),1)$. Let $(AB(1))$ be the sets of pairs (A,B) with $A\in A(1)$, $B\in B(1)$ and A intersecting transversally B. By the assumptions on k and b we have card$(A\cap B)\geq 6$ for every such pair and we consider one of the points of $A\cap B$ different from $P(1),\cdots,P(5)$ as a non assigned node (in the sense of [12]) of the curve $A\cup B\in V((k,b),n+g)$. Note that given any 2 points $Q(1)$, $Q(2)\in Q$ we may find a family $\{A\{t\}\}$ (resp. $\{B\{t\}\}$) of curves of $A(1)$ (resp. $A(2)$) with $\{Q(1),Q(2)\}\subset A\{t\}\, W B\{t\}$ for all values of the parameter t. Since $Q\times Q$ is integral, we may find an integral such family with for a value t' (resp. t'') of the parameter $P(4)$ (resp. $P(5)$) as non assigned node of $A\{t'\}\cup A\{t'\}\in V((k,b),n+g)$ (resp. $A\{t''\}\cup A\{t''\}\in V((k,b),n+g)$. Hence we conclude the proof of Theorem 0.1.

2. Brill-Noether theory

In this section we will prove Theorem 0.2 and then give a related remark.

Proof of Theorem 0.2: The irreducibility statement follows from Theorem 0.1 Without using it, the proof below would show the existence of a component W of $S(g;k,b)$ whose general member C satisfies the thesis of Theorem 0.2. The assertion on Cliff(C) follows from the other assertions because if Cliff(C) is not computed by a pencil, then C has infinitely many pencils of degree Cliff$(C)+3$ ([5], th. 2.3).

Let M be an integral curve and $L\in \mathrm{Pic}(M)$; recall that the degree $\deg(L)$ of L is the leading coefficient of the Hilbert polynomial p_L with $p_L(x):=\chi(L^{\otimes x})$; if $t: U\to M$ is the normalization map, we have $\deg(L)=\deg(t^*(L))$. A g^r_d on M is just given (as in the smooth case) by a degree d line bundle, L, and a vector space $V\subseteq H^0(M,L)$ with $\dim(V)=r+1$. Since $\mathcal{O}_U/t^*(\mathcal{O}_M)$ has finite support, it is easy to check that a g^r_d on M induces a g^r_d on U with associated line bundle $t^*(L)$.

The proof of Theorem 0.2 is by induction on g, starting from the case $g+2=2k$, i.e. when we may take as C a curve with general moduli. Indeed it is easy to check that for general (C,g^1_k,g^1_b) this will define an element of $S(g;k,b)$ and that the two linear systems will map C onto a nodal curve of Q. Note that $\rho(2k-2,r,d)$ $=(r+1)d-r(r+2k-1)$. Assume $g\leq bk-b-k$ and the result true for (g,k,b). Assume that the theorem fails for $g+1$ and a certain r, d. Take (Y,g^1_k,g^1_b) general for (g,k,b); let Y' be the corresponding image of Y in Q (hence with exactly n nodes as singularities). Consider a general one dimensional flat family $\{Y'_t\}_{t\in T\setminus\{o\}}$ of curves in $W((k,b),n+1)$ with limit Y' in Hilb(Q) at $\{o\}$. Let $\{Y_t\}_{t\in T\setminus\{o\}}$ be the corresponding flat family of normalizations. We may find $\{Y'_t\}_{t\in T}$ such that $\{Y_t\}_{t\in T\setminus\{o\}}$ has as flat limit at $\{o\}$ the partial normalization of Y' at all nodes except one; call P this node. Let S be the total space of this deformation. We may assume the existence of a line bundle A' on $S\setminus Y''$ inducing on each Y_t with $t\neq\{o\}$ a g^r_d (i.e. restricting T around $\{o\}$ having $r+1$ sections on $S\setminus Y''$). Since $S\setminus\{P\}$ is smooth, A extends to a line bundle on $S\setminus\{P\}$. The key point is the following claim.

CLAIM. S is smooth at P.

The claim (i.e. with the terminology of [11] that P is a strongly smoothable point of Y' and Y'') is true because the assumptions of [11], Proposition 1.1, are satisfied by [12], lemma 2.2, or [1], §3 and §4. The claim implies that the line bundle A' extends to a line bundle A on all S. Hence by semicontinuity $A\mid Y''$ induces a g^r_d on Y''. Hence, by the first part of the proof we obtain a g^r_d on Y, contradiciton.

REMARK 2.1. In the induction used in the proof of Theorem 0.2 we started from the Brill-Noether range, i.e. from a genus g' curve with $\rho(g',l,k)\geq 0$. If b is much bigger than k, for many g, r, d we obtain better results using the known facts (see in particular [6] and [2]) on general k-gonal curves. In the same way we may obtain bounds for the dimensions of the schemes of all g^r_d's for a general curve $C \in S(g; k, b)$.

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Edoardo Ballico
Department of Mathematics
University of Trento
38050 Povo (TN), Italy
e-mail: ballico@science.unitn.it

Changho Keem
Department of Mathematics
College of Natural Science
Seoul National University
Seoul, 151–742, Korea
e-mail: ckeem@math.snu.ac.kr