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# NORMAL SUBGROUPS AND MULTIPLICITIES OF INDECOMPOSABLE MODULES

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## Introduction

Let $G$ be a finite group and $(K,\boldsymbol{o},F)$ be a $p$-modular system, where $p$ is a prime number. We assume that $K$ contains the $|G|$-th roots of unity and $F$ is algebraically closed and we put $R=\boldsymbol{o}$ or $F$. For an $R$-free finitely generated indecomposable $RG$-module $M$ and a normal subgroup $N$ of $G$, let $V$ be an indecomposable component of $M_N$, where $M_N$ is the restriction of $M$ to $N$. In this paper we give some results on the multiplicity of $V$ as a component of $M_N$ and from them we obtain properties of heights of indecomposable modules and irreducible characters. This study is inspired by Murai [8, 9].

Throughout this paper $N$ is a fixed normal subgroup of $G$ and $v$ is the $p$-adic valuation such that $v(p)=1$. All $RG$-modules are assumed to be $R$-free of finite rank. For an indecomposable $RG$-module $M$, let $\mathrm{vx}(M)$ denote a vertex of $M$. As is well known $v(\mathrm{rank}_R M)\geq v(|G:\mathrm{vx}(M|)$. We refer to Feit[1, Chap.3] and Nagao-Tsushima [10, Chap.4] for the vertex-source theory in modular representations of finite groups.

## 1. $p$-parts of multiplicities

In this section we study the $p$-parts of multiplicities of indecomposable $RN$-modules in an indecomposable decomposition of $M_N$. The following is a key result of this paper.

**Theorem 1.** *Let $V$ be a $G$-invariant indecomposable $RN$-module. Let $M$ be an indecomposable $RG$-module with vertex $Q$ and $n$ be the multiplicity of $V$ in an indecomposable decomposition of $M_N$. Then we have $v(n)\geq v(|G:QN|)$.*

Proof. Let $L$ be a subgroup of $G$ such that $L/N$ is a Sylow $p$-subgroup of $G/N$ and let

$$M_L=M_1\oplus M_2\oplus\cdots\oplus M_s,$$

where each $M_i$ is an indecomposable $RL$-module. By Mackey decomposition

$M_i$ is $(Q^{x_i} \cap L)$-projective for some $x_i \in G$. We have

$$\nu(|L:(Q^{x_i} \cap L)N|) = \nu(|G:(Q^{x_i} \cap L)N|) \geq \nu(|G:Q^{x_i}N|) = \nu(|G:QN|).$$

Let $n_i$ be the multiplicity of $V$ as an indecomposable component of $M_{i_N}$ and $Q_i$ be a vertex of $M_i$. We have $n = \sum_{i=1}^{s} n_i$. If Theorem holds for each $M_i$, then we have

$$\nu(n_i) \geq \nu(|L:Q_iN|) \geq \nu(|L:(Q^{x_i} \cap L)N|) \geq \nu(|G:QN|)$$

and hence $\nu(n) \geq \nu(|G:QN|)$. So we may assume that $G=L$. Then by a theorem of Green, there exists an indecomposable $R(QN)$-module $M_0$ such that $M$ is isomorphic to $M_0^G$. Then we have $M_N = \sum_x M_0 \otimes x$, where $x$ ranges over a set of representatives for the $QN$-cosets $(QN)x$ of $G$. Since $M_0 \otimes x$ is an $RN$-module which is $G$-conjugate to $M_{0_N}$, $n$ is divisible by $|G:QN|$. This completes the proof.

**Proposition 1.** *Let $M$ be an indecomposable RG-module such that* $\nu(\mathrm{rank}_R M) = \nu(|G:\mathrm{vx}(M)|)$, *then there exists an indecomposable component $V$ of $M_N$ which satisfies the following.*

(i) $\nu(\mathrm{rank}_R V) = \nu(|N:\mathrm{vx}(V)|)$,

(ii) *Let $n$ be the multiplicity of $V$ in an indecomposable decomposition of $M_N$. There exists a vertex $P$ of $M$ such that $P \cap N$ is a vertex of $V$, $T(V) \supset P$ and $\nu(n) = \nu(|T(V):PN|)$, where $T(V)$ is the inertial group of $V$ in $G$.*

Proof. Let $\{V_1, V_2, \cdots, V_t\}$ be a set of representatives (up to isomorphism) for the $G$-conjugacy classes of indecomposable components of $M_N$ and $\tilde{V}_i$ be the direct sum of all $RN$-modules which is $G$-conjugate to $V_i$. We can set

$$M_N \cong \sum_{i=1}^{s} \oplus n_i \tilde{V}_i,$$

where $n_i$ is the multiplicity of $V_i$. Here we fix some i for a while and let $T_i$ be the inertial group of $V_i$. We put

$$M_{T_i} = M_1 \oplus M_2 \oplus \cdots \oplus M_t \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_u,$$

where $M_j$ is an indecomposable $RT_i$-module such that $V_i$ is a component of $M_{j_N}$, and $L_j$ is an indecomposable $RT_i$-module such that $V_i$ is not a component of $L_{j_N}$. Let $Q$ be a vertex of $M$. By Mackey decomposition, $M_j$ is $(Q^{y_j} \cap T_i)$-projective for some $y_j \in G$. By Theorem 1 we have

$$\nu(n_i) \geq \min_{1 \leq j \leq t} \{\nu(|T_i:\mathrm{vx}(M_j)N|)\} \geq \min_{1 \leq j \leq t} \{\nu(|T_i:(Q^{y_j} \cap T_i)N|)\}.$$

Hence we have

$$(1)\qquad \begin{aligned} \nu(n_i \mathrm{rank}_R \tilde{V}_i) &= \nu(n_i) + \nu(|G: T_i|) + \nu(\mathrm{rank}_R V_i) \\ &\geq \min_{1 \leq j \leq t} \{\nu(|G:(Q^{y_j} \cap T_i)N|)\} + \nu(\mathrm{rank}_R V_i) \\ &\geq \nu(|G: QN|) + \nu(|N: \mathrm{vx}(V_i)|). \end{aligned}$$

On the other hand $V_i$ is $(Q^{x_i} \cap N)$-projective for some $x_i \in G$. Therefore we have

$$\begin{aligned} \nu(n_i \mathrm{rank}_R \tilde{V}_i) &\geq \nu(|G: QN|) + \nu(|N:(Q^{x_i} \cap N)|) \\ &= \nu(|G: QN|) + \nu(|Q^{x_i}N: Q^{x_i}|) = \nu(|G: Q|). \end{aligned}$$

By the assumption we may assume that $\nu(n_i \mathrm{rank}_R \tilde{V}_i) = \nu(|G:Q|)$. Then $\nu(\mathrm{rank}_R V_i) = \nu(|N:\mathrm{vx}(V_i)|)$ and $|\mathrm{vx}(V_i)| = |Q \cap N|$. On the other hand for some $M_j$ we have $\nu(n_i \mathrm{rank}_R \tilde{V}_i) - \nu(\mathrm{rank}_R V_i) = \nu(|G:\mathrm{vx}(M_j)N|) = \nu(|G:(Q^{y_j} \cap T_i)N|) = \nu(|G:Q^{y_j}N|)$. This implies $Q^{y_j} \subset T_i$, $\nu(n_i) = \nu(|T_i: Q^{y_j}N|)$ and $PN = Q^{y_j}N$, where $P$ is a vertex of $M_j$ which is contained in $Q^{y_j}$. Since $V_i$ is a component of $M_{j_N}$, we have $|Q \cap N| = |\mathrm{vx}(V_i)| \leq |P \cap N|$. Hence we have $Q^{y_j} \cap N = P \cap N$, so $Q^{y_j} = P$ and $P \cap N$ is a vertex of $V_i$. This completes the proof.

As a corollary of the proposition we have the following for $N$-projective indecomposable modules (see Karpilovsky [3, Chapter 12]).

**Corollary 1.** *Let $M$ be an $N$-projective indecomposable $RG$-module and $V$ be an indecomposable component of $M_N$ with multiplicity $n$. Then* $\mathrm{vx}(V)$ *is a vertex of $M$ and* $\nu(\mathrm{rank}_R M) \geq \nu(|G:N|) + \nu(\mathrm{rank}_R V)$. *Moreover if* $\nu(\mathrm{rank}_R M) = \nu(|G:\mathrm{vx}(M)|)$ *then* $\nu(n) = \nu(|T(V):N|)$ *and* $\nu(\mathrm{rank}_R V) = \nu(|N:\mathrm{vx}(V)|)$.

Proof. Let $\tilde{V}$ be the direct sum of the $G$-conjugates of $V$. By the assumption $\mathrm{vx}(V)$ is a vertex of $M$ and $M_N \cong \oplus n\tilde{V}$. Hence from the arguments in the proof of the above proposition we have $\nu(\mathrm{rank}_R M) \geq \nu(|G:\mathrm{vx}(M)N|) + \nu(\mathrm{rank}_R V) = \nu(|G:N|) + \nu(\mathrm{rank}_R V)$. The latter also follows from it.

From the above corollary we have the following, which is shown implicitly in Knörr [6].

**Corollary 2.** *Let $M$ be an indecomposable $RG$-module with source $S$. If* $\nu(\mathrm{rank}_R M) = \nu(|G:\mathrm{vx}(M)|)$, *then* $p \nmid \mathrm{rank}_R S$.

Proof. Let $Q$ be a vertex of $M$. By Green correspondence we may assume that $Q$ is normal and $S$ is an $RQ$-module. Here we can put $N = Q$ and $V = S$ in

Corollary 1. By the assumption and since $Q=\mathrm{vx}(V)$ we have $p\nmid \mathrm{rank}_R S$.

Y. Tsushima and M. Murai pointed out independently that if $G$ is $p$-solvable the converse of Corollary 2 is true. This follows from Green correspondence and the fact that if $G$ is $p$-solvable then the equality $v(n)=v(|T(V):N|)$ holds in Corollary 1.

Moreover if $G$ is $p$-solvable then for an irreducible $FG$-module $M$, $v(\dim_F M)=v(|G:\mathrm{vx}(M)|)$ by Hemernik-Michler [2, Theorem 2.1]. Hence Proposition 1 and Corollary 2 combined with Clifford's theorem imply the following.

**Corollary 3.** *Suppose that $G$ is $p$-solvable. Let $M$ be an irreducible $FG$-module and $V$ be an irreducible constituent of $M_N$ with multiplicity $n$. Then $V$ has $P\cap N$ as a vertex and $v(n)=v(|T(V):PN|)$, where $P$ is a vertex of $M$. Moreover if $S$ is a source of $M$, then $p\mid \dim_F S$.*

Let $B$ be a $p$-block of $G$ with defect group $D$. In [8] Murai extends the heights of characters to $RG$-modules. For an $RG$-module $U$ in $B$ the height $\mathrm{ht}(U)$ is defined by $\mathrm{ht}(U)=v(\mathrm{rank}_R U)-v(|G:D|)$. In particular when $U$ is indecomposable, $U$ is of height 0 if and only if $v(\mathrm{rank}_R U)=v(|G:\mathrm{vx}(U)|)$ and $\mathrm{vx}(U)$ is $G$-conjugate to $D$. Let $b$ be a $p$-block of $N$ covered by $B$. Since by Knörr [5, Prop.4.2], a defect group of $b$ is $G$-conjugate to $D\cap N$, we see by Proposition 1 that if an indecomposable $RG$-module $U$ lying in $B$ is of height 0 then an indecomposable component $V$ of $U_N$ lying in $b$ is of height 0 (see [8, Theorem 4.11]). We can also get this fact from the following, which Murai proved by using the arguments of the proof of Proposition 1.

**Proposition 2.** *Let $B$ and $b$ be as in the above, and $M$ be an indecomposable $RG$-module lying in $B$. Let*

$$M_N \cong \sum_{i=1}^{t} \oplus n_i V_i$$

*be a decomposition of $M_N$ to the sum of indecomposable $RN$-submodules. Then we have* $\mathrm{ht}(M)\geq \min\{\mathrm{ht}(V_i)\mid 1\leq i\leq t\}$

Proof. We may assume that $\{V_1, V_2, \cdots, V_s\}$ $(s\leq t)$ is a set of representatives for the $G$-conjugacy classes of indecomposable components of $M_N$ and that $V_i$ $(1\leq i\leq s)$ belongs to $b$. Let $D$ be a defect group of $B$ such that $D\cap N$ is a defect group of $b$. Using the notations in the proof of Proposition 1, from (1) we have

$$\begin{aligned} v(n_i \mathrm{rank}_R \tilde{V}_i) &\geq v(|G:QN|)+v(\mathrm{rank}_R V_i) \\ &\geq v(|G:DN|)+v(|N:D\cap N|)+\mathrm{ht}(V_i) \end{aligned}$$

$$=\nu(|G:D|)+\mathrm{ht}(V_i)$$

where $1\leq i\leq s$. This implies the inequality in the proposition.

## 2. Heights of irreducible characters

Let $\chi$ be an irreducible character in $B$ and $\zeta$ be an irreducible constituent of $\chi_N$ in $b$, where $B$ and $b$ are as in §1. Let $X$ be an indecomposable $oG$-lattice affording $\chi$ and let $Z$ be an indecomposable component of $X_N$ which lies in $b$ and $\mathrm{ht}(X)\geq\mathrm{ht}(Z)$ ($Z$ exists by Proposition 2). Then $\mathrm{rank}_o Z$ is a multiple of $\zeta(1)$, and hence $\mathrm{ht}(Z)\geq\mathrm{ht}(\zeta)$. Since $\mathrm{ht}(\chi)=\mathrm{ht}(X)$, we have $\mathrm{ht}(\chi)\geq\mathrm{ht}(\zeta)$ as in [9, Lemma 2.2]. On the other hand, by Proposition 2, for an irreducible $FG$-module $M$ in $B$ and an irreducible constituent $V$ of $M_N$ in $b$, we have $\mathrm{ht}(M)\geq\mathrm{ht}(V)$ ([9, Lemma 3.2]). We shall show that $\mathrm{ht}(\chi)=\mathrm{ht}(\zeta)$ and $\mathrm{ht}(M)=\mathrm{ht}(V)$ when a defect group $D$ of $B$ is contained in $N$.

The following is shown from the results of Külshammer-Robinson [7], and the converse is proved in Robinson [11, Lemma 4.4].

**Lemma 1** (Külshammer-Robinson). *Let $\chi$ be an irreducible character of $G$ and $\zeta$ be an irreducible constituent of $\chi_N$ with multiplicity $n$. If $\chi$ is afforded by an $N$-projective $oG$-lattice $M$ then we have $\nu(n)=\nu(|T(\zeta):N|)$.*

Proof. Suppose that $\chi$ is afforded by an $N$-projective indecomposable $oG$-lattice $M$ and let $V$ be an indecomposable component of $M_N$. Then from the argument in the proof Corollary 1 we have $\mathrm{rank}_o M=m\,\mathrm{rank}_o V$, where $m$ is a natural number with $\nu(m)\geq\nu(|G:N|)$. Since $\nu(\chi(1)/\zeta(1))\geq\nu(m)\geq\nu(|G:N|)$ because $m$ divides $\chi(1)/\zeta(1)$, we see $\nu(n)\geq\nu(|T(\zeta):N|)$. On the other hand, as is well known $n$ divides $|T(\zeta):N|$. Therefore we have $\nu(n)=\nu(|T(\zeta):N|)$.

**Lemma 2.** *Let $M$ be an $N$-projective irreducible $FG$-module and $V$ be an irreducible component of $M_N$ with multiplicity $n$. Then we have $\nu(n)=\nu(|T(V):N|)$. Moreover $n$ is equal to the multiplicity $m$ of $M$ as an indecomposable component of $V^G$.*

Proof. We may assume that $V$ is $G$-invariant. We put $E=\mathrm{End}_{FG}(V^G)$ and let $e$ be a primitive idempotent of $E$ corresponding to $M$, *i.e.*, $M=eV^G=(eE)V$. Then as is well known $E$ is isomorphic to a twisted group algebra $F(\bar{G},\varphi)$ over $F$ with factor set $\varphi$, where $\bar{G}=G/N$. Moreover $\dim_F M=(\dim_F(eE))(\dim_F V)$ and hence $n=\dim_F(eE)$. By Humphreys [3], there exists a central $p'$-extension $\hat{G}$ of $\bar{G}$ such that $F(\bar{G},\varphi)$ is isomorphic to a direct sum of some block ideals of $F\hat{G}$. Now as $M$ is irreducible, $eE$ is irreducible. Hence $n$ is equal to the dimension of an irreducible and projective $F\hat{G}$-module, so we have $\nu(n)=\nu(|\hat{G}|)=\nu(|G:N|)$.

By the way $m$ is equal to the dimension of the irreducible $E$-module corresponding to $eE$. But $eE$ is irreducible, hence $m$ is equal to $\dim_F(eE)$. This completes the proof.

**Proposition 3.** *Let $B$ be a $p$-block of $G$ with defect group $D$ and $b$ be a $p$-block of $N$ covered by $B$. Assume that $D$ is contained in $N$. Then for an irreducible character $\chi$ in $B$ and for an irreducible constituent $\zeta$ of $\chi_N$ in $b$, we have* $\mathrm{ht}(\chi)=\mathrm{ht}(\zeta)$. *We also have* $\mathrm{ht}(M)=\mathrm{ht}(V)$ *for an irreducible $FG$-module $M$ in $B$ and an irreducible constituent $V$ of $M_N$ in $b$.*

Proof. We may assume $D$ is a defect group of $b$. By the assumption and Lemma 1 we have $v(\chi(1))=v(|G:N|)+v(\zeta(1))=v(|G:D|)+\mathrm{ht}(\zeta)$. This implies $\mathrm{ht}(\chi)=\mathrm{ht}(\xi)$. By the former of Lemma 2 we also have $v(\dim_F M)=v(|G:N|)+v(\dim_F V)=v(|G:D|)+\mathrm{ht}(V)$. Hence $\mathrm{ht}(M)=\mathrm{ht}(V)$. This completes the proof.

For a $p$-block $B$ of $G$ let $\mathrm{Ker}(B)$ be the kernel of $B$ and let $\text{mod-Ker}(B)=\cap\,\mathrm{Ker}\,M$, where $M$ runs over the irreducible $FG$-modules in $B$. After [8], let $\mathrm{Irr}^0(B)$ be the set of irreducible characters of height 0 in $B$ and let $\mathrm{Ker}^0(B)=\cap\,\mathrm{Ker}\,\chi$, where $\chi$ runs through $\mathrm{Irr}^0(B)$. As is well known, $\mathrm{Ker}(B)$ is a $p'$-group and $\text{mod-ker}(B)$ is $p$-nilpotent. By [8, Lemma 5.1], we have $\mathrm{Ker}(B)\subset\mathrm{Ker}^0(B)\subset\text{mod-Ker}(B)$.

**Theorem 2.** *Let $B$ be a $p$-block of $G$ with defect group $D$. Then we have* $\mathrm{Ker}^0(B)=\bigcap_{x\in G}(\mathrm{Ker}(B)D')^x$, *where $D'$ is the commutator subgroup of $D$. In particular if $D$ is abelian then* $\mathrm{Ker}^0(B)=\mathrm{Ker}(B)$.

Proof. Let $Q$ be a Sylow $p$-subgroup of $\mathrm{Ker}^0(B)$. Then $\mathrm{Ker}^0(B)=\mathrm{Ker}(B)Q$ by [8, Lemma 5.1]. Since $Q$ is contained in a vertex of an $\boldsymbol{o}G$-lattice affording $\chi\in\mathrm{Irr}^0(B)$, we may assume that $Q\subset D$. Let $B_0$ be the Brauer correspondent of $B$ in $N_G(D)$. For any $\zeta\in\mathrm{Irr}^0(B_0)$ there is $\chi\in\mathrm{Irr}^0(B)$ such that $\zeta$ is a constituent of $\chi_{N_G(D)}$ (cf. [8, Prop.1.8]). Hence $\mathrm{Ker}^0(B)\cap N_G(D)\subset\mathrm{Ker}^0(B_0)$. In particular $Q\subset\mathrm{Ker}^0(B_0)\cap D$. By Proposition 3 for any $\zeta\in\mathrm{Irr}(\mathrm{B}_0)$, $\zeta$ belongs to $\mathrm{Irr}^0(B_0)$ if and only if an irreducible constituent of $\zeta_D$ is linear. Therefore we see $\mathrm{Ker}^0(B_0)\cap D=D'$. So we have $Q\subset D'$.

Put $H=\bigcap_{x\in G}(\mathrm{Ker}(B)D')^x$ and let $\chi$ be any element of $\mathrm{Irr}^0(B)$. Then there exists $\zeta\in\mathrm{Irr}^0(B_0)$ such that $\zeta$ is a constituent of $\chi_{N_G(D)}$. By the above argument $\mathrm{Ker}\,\zeta\supset D'$. Hence $\chi_{\mathrm{Ker}(B)D'}$ has the trivial character of $\mathrm{Ker}(B)D'$ as an irreducible constituent, so $\chi_H$ has the trivial character of $H$ as an irreducible constituent. Therefore $\mathrm{Ker}\,\chi\supset H$ and hence we have $\mathrm{Ker}^0(B)\supset H$. Since $\mathrm{Ker}^0(B)\subset H$, we have $\mathrm{Ker}^0(B)=H$.

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## References


[1] W. Feit: The representation theory of finite groups, North-Holland, Amsterdam, 1982.
[2] W. Hamernik-G. Michler: *On vertices of simple modules in p-solvable groups*, Math. Sem. Giessen, **121** (1976), 147–162.
[3] J.F. Humphreys: *Projective modular representations of finite groups*, J. London Math. Soc. (2), **16** (1977), 51–66.
[4] G. Karpilovsky: Group Representations Volume 3, North-Holland, 1994.
[5] R. Knörr: *Blocks, vertices and normal subgroups*, Math. Z., **148** (1976), 53–60.
[6] R. Knörr: *On the vertices of irreducible modules*, Ann. Math., **110** (1979), 487–499.
[7] B. Külshammer-G.R. Robinson: *Characters of relative projective modules*, II, J. London Math. Soc. (2), **36** (1987), 59–67.
[8] M. Murai: *Block induction, normal subgroups and characters of height zero*, Osaka J. Math., **31** (1994), 9–25.
[9] M. Murai: *Normal subgroups and heights of characters.*
[10] H. Nagao-Y. Tsushima: Representations of finite groups, Academic Press, Boston, Tokyo, 1988.
[11] G.R. Robinson: *Local structure, vertices & Alperin's conjecture.*


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