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PERCOLATION ON THE PRE-SIERPINSKI GASKET

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1. Introduction and statements of results

In this paper, we regard percolation as a model of phase transitions. We are especially interested in problems near the *critical point*, where the phase transition occurs. We call these problems *critical behaviors*. Our purpose in this paper is to clarify the critical bahaviors of percolation on the pre-Sierpinski gasket which has self-similarity.

Until now, studies of percolation are restricted on *periodic* graphs, such as $\boldsymbol{Z}^d$. (An exact definition of periodic graph is mentioned in Kesten [1].) There are lots of conjectures and hypotheses about critical behaviors, but many of them are still unsolved rigorously (see Grimmett [2] and references therein). In high dimension lattices $\boldsymbol{Z}^d$, rigorous results for critical behaviors were obtained by Hara-Slade [3]. But in low dimensions, except a work on $\boldsymbol{Z}^2$ by Kesten [4], few rigorous results have been proved about the existence of *critical exponents* and justification of the *scaling*, *hyperscaling relations*.

For critical behaviors, *self-similarity* of the graph plays more important role than periodicity. This is a motivation to consider percolation problems on the pre-Sierpinski gasket.

We now define the pre-Sierpinski gasket. Let $\boldsymbol{O}=(0,0)$, $a_0=(1/2, \sqrt{3}/2)$, $b_0=(1,0)$. Let F_0 be the graph which consists of the vertices and edges of the triangle $\Delta \boldsymbol{O} a_0 b_0$. Let $\{F_n\}_{n=0,1,2,\dots}$ be the sequence of graphs given by

$$F_{n+1}=F_n \cup (F_n+2^n a_0) \cup (F_n+2^n b_0)$$

where $A+a=\{x+a \mid x\in A\}$ and $kA=\{kx \mid x\in A\}$. Let $F=\bigcup_{n=0}^{\infty}F_n$. We call F the *pre-Sierpinski gasket*. (Fig. 1.1) Note that $\tilde{F}=\overline{\bigcup_{n=0}^{\infty}2^{-n}F}$ become the Sierpinski gasket. Let V be the set of all vertices in F, and E the set of all edges with length 1.

We consider the Bernoulli bond percolation on the pre-Sierpiski gasket; each edges in E are *open* with probability p and *closed* with probability $1-p$ independently. Let P_p denote its distribution. We think of open bonds as permitting to go along the bond. We write $x \leftrightarrow y$ if there is an open path from x to y. Let $C(x)=\{y\in V : x\leftrightarrow y\}$. $C(x)$ is called the *open cluster* containing x. We denote by C the open cluster containing the origin.

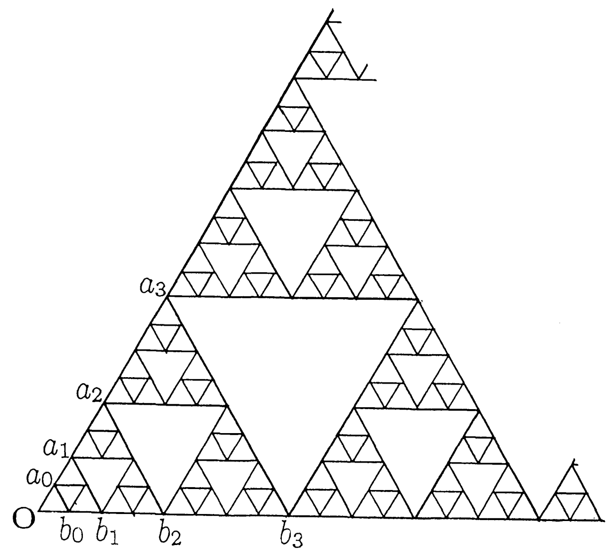

Fig. 1.1

We define two functions in a similar way as percolations on $\boldsymbol{Z}^d$.

$$\theta(p)=P_p(|C|=\infty), \quad \chi(p)=E_p(|C|\,;|C|<\infty),$$

where $|C|$ denotes the number of vertices contained in C, and E_p denotes the expectation with respect to P_p. $\theta(p)$ is called the *percolation probability*, and $\chi(p)$ is called the *mean cluster size*.

Let p_c denote the *critical point*; that is

$$p_c=\inf\{p:\theta(p)>0\}.$$

Then $p_c=1$ for the pre-Sierpinski gasket because it is finitely ramified. We note that $\chi(p)=E_p|C|$ for $p<1$.

The *correlation length* is defined by

$$\xi(p)=\lim_{n\to\infty}\left\{-\frac{1}{2^n}\log P_p(\boldsymbol{O}\leftrightarrow a_n)\right\}^{-1}. \tag{1}$$

The existence of the limit in (1) will be proved in Section 2.

We write $f(p)\approx g(p)$ *as* $p\to p_0$ if $\log f(p)/\log g(p)\to 1$ as $p\to p_0$.

We now state our main theorems:

Theorem 1.1. $\displaystyle\lim_{p\to 1}-\frac{\log\xi(p)}{\log(1-p)}=\infty$, *and* $\displaystyle\lim_{p\to 1}\frac{\log(\log\xi(p))}{\log(1-p)}=-2.$

Theorem 1.2. *Let* $D=\log 3/\log 2$. *Then*

$$E_p|C|^k \approx \{\xi(p)\}^{Dk} \quad as \quad p \to 1 \quad for\ all\ k \geq 1.$$

REMARK. Our results are quite different from the results on $\boldsymbol{Z}^d$ (see below). In physical literature, Theorem 1.1 was known by Gefen et al. [5] by using formal renormalization arguments. Our contribution is that we prove Theorem 1.1 rigorously.

We collect results and conjectures of the percolation on $\boldsymbol{Z}^d$. It is conjectured (see [2])

$$\xi(p) \approx |p_c - p|^{-\nu(d)} \quad \text{as} \quad p \to p_c. \tag{2}$$

The value $\nu(d)$ is called the *critical exponent*. It is proved that $\nu(d)=1/2$ for sufficiently large d (Hara-Slade [3]), and conjectured $\nu(2)=4/3$ (see [4]).

Other critical exponents considered in $\boldsymbol{Z}^d$ are as follows:

$$\chi(p) \approx |p_c - p|^{-\gamma}, \quad \frac{E_p(|C|^{k+1}; |C| < \infty)}{E_p(|C|^k; |C| < \infty)} \approx |p_c - p|^{-\Delta} \quad \text{as} \quad p \to p_c.$$

It is conjectured for $\boldsymbol{Z}^d$ that $d\nu = 2\Delta - \gamma$. This relation is one of hyperscaling relations. We note $\gamma = \Delta = \infty$ on the pre-Sierpinski gasket. So the relation $d\nu = 2\Delta - \gamma$ does not make sense on the pre-Sierpinski gasket. Accordingly we modify the hyperscaling relation as follows:

$$\{\xi(p)\}^d \approx \frac{E_p|C|^3}{\{\chi(p)\}^2} \quad \text{as} \quad p \to p_c. \tag{3}$$

If finite critical exponents ν, γ, Δ exist, then (3) is equvalent to $d\nu = 2\Delta - \gamma$.

REMARK. By Theorem 1.2, we have $E_p|C|^3 \approx \{\xi(p)\}^{3D}$ and $\chi(p) \approx \{\xi(p)\}^D$. Hence the above hyperscaling relation (3) holds when we regard D as the dimension of the pre-Sierpinski gasket. The value $D=\log 3/\log 2$ coincides with the fractal dimension of the Sierpinski gasket.

In addition, we mention site percolation on the pre-Sierpinski gasket: each vertices in V are determined to be open or closed independently. (Details will be given in Section 5). We define the correlation length $\hat{\xi}(p)$ in the same manner as (1). We have the result below;

Theorem 1.3. $\lim_{p \to 1} -\frac{\log \hat{\xi}(p)}{\log(1-p)} = \infty$, *and* $\lim_{p \to 1} \frac{\log(\log \hat{\xi}(p))}{\log(1-p)} = -1$.

The critical exponent in a usual sense is also infinite in this case. But $\log \hat{\xi}(p) \approx (1-p)^{-1}$, which is different from Theorem 1.1. We cannot see the universality of this exponent on the pre-Sierpinski gasket.

We refer to the self-avoiding walks on the Sierpinski gasket, as related works of phase transitions; Hattori-Hattori [6] and Hattori-Hattori-Kusuoka [7] construct the self-avoiding paths on two- and three-dimensional Sierpinski gasket. Before [6], Hattori-Hattori-Kusuoka [8] constructed them on the pre-Sierpinski gasket. These works also gave us a motivation to study percolation on the Sierpinski gasket.

The organization of this paper is as follows: In Section 2 we prepare for the proof of our main theorems; we construct recursion formulas of relations between events in F_n and ones in F_{n+1}. In the reminder of Section 2, we prove the existence of the correlation length. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. In Section 5 we study site percolation and prove Theorem 1.3.

2. Recursion formulas and the existence of $\xi(p)$

We introduce two connectivity functions as follows.

$$\Phi_n(p)=P_p(\boldsymbol{O}\leftrightarrow a_n \quad \text{in} \quad \Delta \boldsymbol{O} a_n b_n),$$

$$\Theta_n(p)=P_p(\boldsymbol{O}\leftrightarrow a_n \quad \text{and} \quad \boldsymbol{O}\leftrightarrow b_n \quad \text{in} \quad \Delta \boldsymbol{O} a_n b_n).$$

We write $\boldsymbol{O}\leftrightarrow a_n$ *in* $\Delta \boldsymbol{O} a_n b_n$ if there is an open path from $\boldsymbol{O}$ to a_n in $\Delta \boldsymbol{O} a_n b_n$ (contains its perimeter). We easily calculate $\Phi_0(p)=p+p^2-p^3$, $\Theta_0(p)=3p^2-2p^3$. Note that (i) $\Phi_n(p)\geq\Theta_n(p)$ by definition, (ii) if $\boldsymbol{O}\leftrightarrow a_n$ and $\boldsymbol{O}\leftrightarrow b_n$ then we have $a_n\leftrightarrow b_n$ automatically.

Proposition 2.1. *For each* $n\geq 0$ *and* $0\leq p\leq 1$,

$$\Phi_{n+1}(p)=\{\Phi_n(p)\}^2+\{\Phi_n(p)\}^3-\Phi_n(p)\{\Theta_n(p)\}^2, \tag{4}$$

$$\Theta_{n+1}(p)=3\{\Phi_n(p)\}^2\Theta_n(p)-2\{\Theta_n(p)\}^3. \tag{5}$$

Proof. Recall $\Delta \boldsymbol{O} a_n b_n=F_n$. Let $F_n'=F_n+a_n$, $F_n''=F_n+b_n$, and $c_n=(3\cdot 2^{n-1}, \sqrt{3}\cdot 2^{n-1})$. Let A_n^1 and A_n^2 be events given by

$$A_n^1=\{\boldsymbol{O}\leftrightarrow a_n \quad \text{in} \quad F_n\}\cap\{a_n\leftrightarrow a_{n+1} \quad \text{in} \quad F_n'\},$$

$$A_n^2=\{\boldsymbol{O}\leftrightarrow b_n \quad \text{in} \quad F_n\}\cap\{b_n\leftrightarrow c_n \quad \text{in} \quad F_n''\}\cap\{c_n\leftrightarrow a_{n+1} \quad \text{in} \quad F_n'\}.$$

Then we have

$$\Phi_{n+1}(p)=P_p(A_n^1)+P_p(A_n^2)-P_p(A_n^1\cap A_n^2). \tag{6}$$

Here we used the fact that a path from $\boldsymbol{O}$ to a_{n+1} goes through a_n or b_n. Since the events in F_n, F_n', F_n'' are mutually independent, $P_p(A_n^1)=\{\Phi_n(p)\}^2$, $P_p(A_n^2)=\{\Phi_n(p)\}^3$,

$P_p(A_n^1 \cap A_n^2) = \{\Theta_n(p)\}^2 \Phi_n(p)$ (Fig. 2.1). Combining these with (6) yields (4).

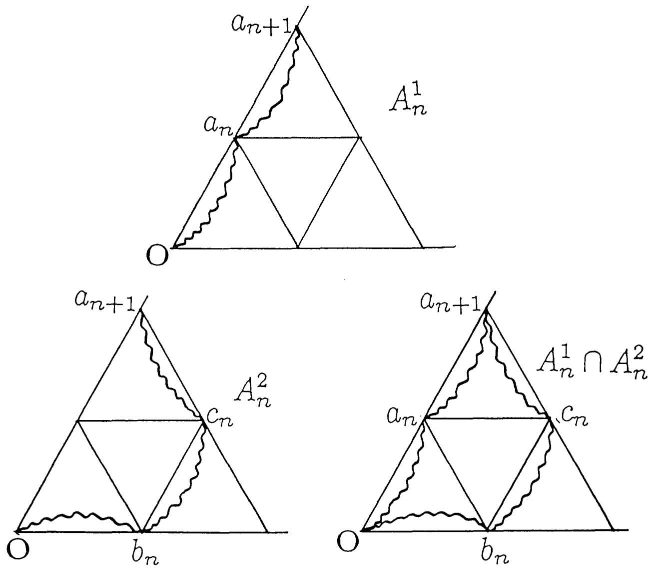

Fig. 2.1

We proceed to the proof of (5). Let B_n^1, B_n^2, B_n^3 be events given by

$$B_n^1 = \{O \leftrightarrow a_n \text{ and } O \leftrightarrow b_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ in } F_n'\} \cap \{b_n \leftrightarrow b_{n+1} \text{ in } F_n''\},$$

$$B_n^2 = \{O \leftrightarrow a_n \text{ in } F_n\} \cap \{a_n \leftrightarrow a_{n+1} \text{ and } a_n \leftrightarrow c_n \text{ in } F_n'\} \cap \{c_n \leftrightarrow b_{n+1} \text{ in } F_n''\},$$

$$B_n^3 = \{O \leftrightarrow b_n \text{ in } F_n\} \cap \{b_n \leftrightarrow b_{n+1} \text{ and } b_n \leftrightarrow c_n \text{ in } F_n''\} \cap \{c_n \leftrightarrow a_{n+1} \text{ in } F_n'\}$$

(see Fig. 2.2).
Then we have

$$\Theta_{n+1}(p) = P_p(B_n^1) + P_p(B_n^2) + P_p(B_n^3) - P_p(B_n^1 \cap B_n^2) - P_p(B_n^2 \cap B_n^3) - P_p(B_n^3 \cap B_n^1) + P_p(B_n^1 \cap B_n^2 \cap B_n^3).$$

We see easily

$$P_p(B_n^1) = P_p(B_n^2) = P_p(B_n^3) = \{\Phi_n(p)\}^2 \Theta_n(p),$$

$$P_p(B_n^1 \cap B_n^2) = P_p(B_n^2 \cap B_n^3) = P_p(B_n^3 \cap B_n^1) = P_p(B_n^1 \cap B_n^2 \cap B_n^3) = \{\Theta_n(p)\}^3.$$

(5) follows from this immediately. □

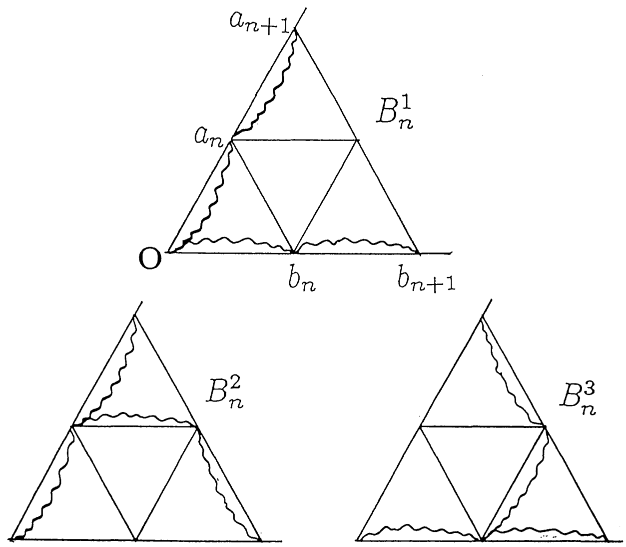

Fig. 2.2

From now on, we assume $0<p<1$. We prove the existence of the limit (1), correlation length $\xi(p)$, by using these recursions.

Proposition 2.2. *There exists $\xi(p)>0$ such that*

$$\lim_{n\to\infty}\frac{\Phi_n(p)}{\exp\{-2^n/\xi(p)\}}=1.$$

REMARK. The convergence as $n\to\infty$ in Proposition 2.2 is stronger than the convergence in (1).

Proof. By (4) and $\Theta_n(p)\le\Phi_n(p)$, we have

$$\{\Phi_n(p)\}^2\le\Phi_{n+1}(p)\le\{\Phi_n(p)\}^2+\{\Phi_n(p)\}^3.$$

Hence

$$1\le\frac{\Phi_{n+1}(p)}{\{\Phi_n(p)\}^2}\le 1+\Phi_n(p).$$

Let $h_n(p)=\Phi_{n+1}(p)/\{\Phi_n(p)\}^2$. Then $1\le h_n(p)\le 2$ and $\lim_{n\to\infty}h_n(p)=1$ because $\lim_{n\to\infty}\Phi_n(p)=0$. Now

$$\frac{1}{2^n}\log\Phi_n(p)$$

$$=\frac{1}{2^n}\log\left(\{\Phi_0(p)\}^{2^n}\cdot\frac{\{\Phi_1(p)\}^{2^{n-1}}}{\{\Phi_0(p)\}^{2^n}}\cdot\frac{\{\Phi_2(p)\}^{2^{n-2}}}{\{\Phi_1(p)\}^{2^{n-1}}}\cdots\frac{\Phi_n(p)}{\{\Phi_{n-1}(p)\}^2}\right)$$

$$=\log\Phi_0(p)+\frac{1}{2}\log h_0(p)+\frac{1}{2^2}\log h_1(p)+\cdots+\frac{1}{2^n}\log h_{n-1}(p)$$

$$\leq\log\Phi_0(p)+\log 2.$$

Hence $\{\log\Phi_n(p)/2^n\}_{n=0,1,2,\cdots}$ is increasing and $\lim_{n\to\infty}\log\Phi_n(p)/2^n$ exists. Let $-\{\xi(p)\}^{-1}=\lim_{n\to\infty}\log\Phi_n(p)/2^n$. Then

$$-\frac{1}{\xi(p)}\geq\frac{1}{2^n}\log\Phi_n(p)=-\frac{1}{\xi(p)}-\left(\frac{1}{2^{n+1}}\log h_n(p)+\frac{1}{2^{n+2}}\log h_{n+1}(p)+\cdots\right)$$

$$\geq-\frac{1}{\xi(p)}-\frac{1}{2^n}\log H_n(p),$$

where $H_n(p)=\sup_{m\geq n}h_m(p)$. Therefore

$$\exp\left\{-\frac{2^n}{\xi(p)}\right\}\geq\Phi_n(p)\geq\frac{1}{H_n(p)}\exp\left\{-\frac{2^n}{\xi(p)}\right\}. \tag{7}$$

Since $\lim_{n\to\infty}H_n(p)=1$, we complete the proof. □

REMARK. Note that the function $\xi(p)$ is continuous and increasing on (0, 1) from the proof above.

Lemma 2.3. $\displaystyle\lim_{n\to\infty}\frac{P_p(O\leftrightarrow a_n)}{\exp\{-2^n/\xi(p)\}}=1.$

Proof. Recall that $\Phi_n(p)=P_p(O\leftrightarrow a_n \text{ in } F_n)$. Then

$$P_p(O\leftrightarrow a_n)-P_p(O\leftrightarrow a_n \text{ in } F_n)$$
$$\leq P_p(O\leftrightarrow b_n \text{ in } F_n, b_n\leftrightarrow c_n \text{ in } F_n'', c_n\leftrightarrow a_n \text{ in } F_n')$$
$$+P_p(O\leftrightarrow b_n \text{ in } F_n, b_n\leftrightarrow b_{n+1} \text{ in } F_n'', a_n\leftrightarrow a_{n+1} \text{ in } F_n') \text{ (Fig. 2.3)}$$
$$=2\{\Phi_n(p)\}^3.$$

So

$$1\leq\frac{P_p(O\leftrightarrow a_n)}{\Phi_n(p)}\leq 1+2\{\Phi_n(p)\}^2,$$

which implies

$$\lim_{n\to\infty}\frac{P_p(O\leftrightarrow a_n)}{\Phi_n(p)}=1.$$

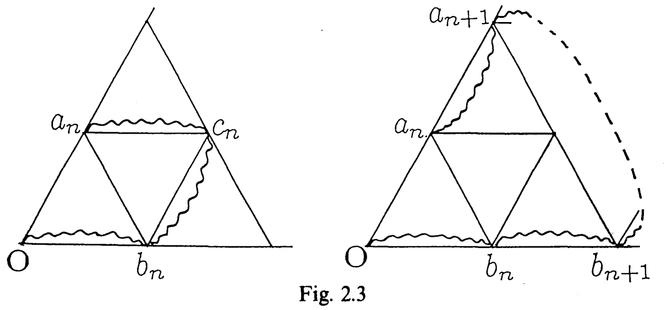

Fig. 2.3

Combining this with Proposition 2.2 completes the proof. □

3. Proof of Theorem 1.1

The next lemma is a key of the proof.

Lemma 3.1. *There exists* $\varepsilon>0$ *such that*

$$2\leq\frac{\xi(p+3(1-p)^3)}{\xi(p)}\leq 4 \quad \textit{for} \quad 1-\varepsilon<p<1.$$

Proof. We introduce

$$\begin{aligned}(8)\qquad \Psi_n(p)&=1-P_p(\boldsymbol{O}\not\leftrightarrow a_n,\ \boldsymbol{O}\not\leftrightarrow b_n,\ a_n\not\leftrightarrow b_n \text{ in } F_n)\\ &=3\Phi_n(p)-2\Theta_n(p).\end{aligned}$$

Here $\boldsymbol{O}\not\leftrightarrow a_n$ *in* F_n means that there exists no open path from $\boldsymbol{O}$ to a_n in F_n. By (4) and (5),

$$\Theta_{n+1}(p)=S(\Theta_n(p),\Psi_n(p)),$$
$$\Psi_{n+1}(p)=T(\Theta_n(p),\Psi_n(p)),$$

where $S,T:\boldsymbol{R}^2\to\boldsymbol{R}$ are functions defined by

$$S(x,y)=-\frac{2}{3}x^3+\frac{4}{3}x^2y+\frac{1}{3}xy^2,$$

$$T(x,y)=\frac{2}{9}x^3+\frac{4}{3}x^2-\frac{7}{3}x^2y+\frac{4}{3}xy+\frac{1}{9}y^3+\frac{1}{3}y^2.$$

Let D be a subset of $\boldsymbol{R}^2$ defined by $D=\{(x,y):0<x\leq y<1\}$. We see $\partial S/\partial x$,

$\partial S/\partial y$, $\partial T/\partial x$, $\partial T/\partial y > 0$ for $(x,y)\in D$. Indeed,

$$\frac{\partial S}{\partial x} = -2x^2 + \frac{8}{3}xy + \frac{1}{3}y^2 = 2x(y-x) + \frac{2}{3}xy + \frac{1}{3}y^2 > 0,$$

$$\frac{\partial S}{\partial y} = \frac{4}{3}x^2 + \frac{2}{3}xy > 0,$$

$$\frac{\partial T}{\partial x} = \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{8}xy + \frac{4}{3}y \geq \frac{2}{3}x^2 + \frac{8}{3}x - \frac{14}{3}xy + \frac{2}{3}y^2 + \frac{2}{3}y$$

$$= \frac{2}{3}(y-x)^2 + \frac{8}{3}x(1-y) + \frac{2}{3}y(1-x) > 0,$$

$$\frac{\partial T}{\partial y} = -\frac{7}{3}x^2 + \frac{4}{3}x + \frac{1}{3}y^2 + \frac{2}{3}y$$

$$= \frac{4}{3}x(1-x) + \frac{1}{3}(y^2 - x^2) + \frac{2}{3}(y - x^2) > 0.$$

Therefore if (x_1,y_1), $(x_2,y_2)\in D$ and $x_1 < x_2$ and $y_1 < y_2$, then

$$S(x_1,y_1) < S(x_2,y_2), \qquad T(x_1,y_1) < T(x_2,y_2). \tag{9}$$

Note that $\Psi_n(p) = \Theta_n(p) + 3\{\Phi_n(p) - \Theta_n(p)\} \geq \Theta_n(p)$ for all n by (8). Hence $(\Theta_n(p), \Psi_n(p)) \in D$. Calculating $\Theta_n(p)$ and $\Psi_n(p)$ directly from the recursions, we have

$$\Theta_n(p) = 1 - 3(1-p)^2 - (12n-6)(1-p)^4 + 6(1-p)^5 + (-48n^2 + 120n - 15)(1-p)^6 + \cdots, \tag{10}$$

$$\Psi_n(p) = 1 - 3(1-p)^4 - 24n(1-p)^6 + \cdots \tag{11}$$

for $n \geq 2$. For $1 - 1/\sqrt{3} < p < 1$, let $\tilde{p} = p + 3(1-p)^3$. Then we have

$$\Theta_3(\tilde{p}) - \Theta_2(p) = 6(1-p)^4 + 213(1-p)^6 + \cdots,$$

$$\Psi_3(\tilde{p}) - \Psi_2(p) = 12(1-p)^6 + \cdots.$$

Note that $\Theta_2(p)$, $\Psi_2(p)$, $\Theta_3(\tilde{p})$, and $\Psi_3(\tilde{p})$ are polynomials of finite degree. Hence we can take $\varepsilon_1 > 0$ in such a way that $\Theta_2(p) < \Theta_3(\tilde{p})$ and $\Psi_2(p) < \Psi_3(\tilde{p})$ for $1 - \varepsilon_1 < p < 1$. By (9), We have

$$\Theta_3(p) = S(\Theta_2(p), \Psi_2(p)) < S(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Theta_4(\tilde{p}),$$

$$\Psi_3(p) = T(\Theta_2(p), \Psi_2(p)) < T(\Theta_3(\tilde{p}), \Psi_3(\tilde{p})) = \Psi_4(\tilde{p}).$$

Estimating repeatedly as above, we have $\Theta_n(p) < \Theta_{n+1}(\tilde{p})$, $\Psi_n(p) < \Psi_{n+1}(\tilde{p})$ for $n \geq 2$. Combining this with (8) yields $\Phi_n(p) < \Phi_{n+1}(\tilde{p})$. So

$$\frac{\log \Phi_n(p)}{2^n} < 2 \cdot \frac{\log \Phi_{n+1}(\tilde{p})}{2^{n+1}}.$$

This implies $\xi(p)^{-1} \geq 2 \cdot \xi(\tilde{p})^{-1}$, that is $\xi(\tilde{p}) / \xi(p) \geq 2$ for $1 - \varepsilon_1 < p < 1$.

We now proceed to the estimate from the opposite side. By using (10) and (11) again, we see

$$\Theta_4(\tilde{p}) - \Theta_2(p) = -6(1-p)^4 + 141(1-p)^6 + \cdots$$
$$\Psi_4(\tilde{p}) - \Psi_2(p) = -12(1-p)^6 + \cdots.$$

Hence we can take $\varepsilon_2 > 0$ such that $\Theta_4(\tilde{p}) < \Theta_2(p)$ and $\Psi_4(\tilde{p}) < \Psi_2(p)$ for $1 - \varepsilon_2 < 1$. So we have $\Theta_{n+2}(\tilde{p}) < \Theta_n(p)$ and $\Psi_{n+2}(\tilde{p}) < \Psi_n(p)$. Therefore $\xi(\tilde{p}) / \xi(p) \leq 4$ for $1 - \varepsilon_2 < p < 1$, which completes the proof. □

Proof of Theorem 1.1. Let $g(p) = \log \xi(p)$. Since $\xi(p)$ is an increasing function, $g(p)$ is also increasing. Suppose that p is sufficiently large to satisfy $g(p) > 0$. Let

$$m = \liminf_{p \to 1} -\frac{\log g(p)}{\log(1-p)} \geq 0, \qquad M = \limsup_{p \to 1} -\frac{\log g(p)}{\log(1-p)}.$$

First, we prove $m \geq 2$. Suppose $m < 2$, and pick $\delta > 0$ with $m + \delta < 2$. Let

$$h(x) = \frac{1}{(x - 3x^3)^{m+\delta}} - \frac{1}{x^{m+\delta}}.$$

Applying the L'Hospital theorem, we see $\lim_{x \to 0} h(x) = 0$. So we take p_0 such that

$$h(1-p) < \frac{1}{2} \log 2 \quad \text{for} \quad 0 < 1 - p < 1 - p_0 \tag{12}$$

and $1 - p_0 < \varepsilon$. (ε is given in Lemma 3.1.)

Let

$$f(p) = p + 3(1-p)^3. \tag{13}$$

We define $\{p_n\}_{n=1,2,\cdots}$ by $f(p_0) = p_1$, $f(p_n) = p_{n+1}$ inductively. Then $p_0 < p_1 < \cdots < p_n < 1$, and $\lim_{n \to \infty} p_n = 1$. By (13) and Lemma 3.1, we have

$$\log 2 \leq g(p_{n+1}) - g(p_n),$$

and hence

$$g(p_0) + n \log 2 \leq g(p_n). \tag{14}$$

Take $N=N(p_0)\in \boldsymbol{N}$. By assumpion, there exists t such that $p_N<t<1$ and

$$(15)\qquad -\frac{\log g(t)}{\log(1-t)}<m+\delta.$$

For this t, there exists unique $N'=N'(t)$ such that $p_{N'}\le t<p_{N'+1}$. By (15) and $1-p_{N'+1}<1-t$, we have

$$\begin{aligned}(16)\qquad g(t)&<\frac{1}{(1-p_{N'+1})^{m+\delta}}\\ &=\left\{\frac{1}{(1-p_{N'+1})^{m+\delta}}-\frac{1}{(1-p_{N'})^{m+\delta}}\right\}\\ &\qquad+\left\{\frac{1}{(1-p_{N'})^{m+\delta}}-\frac{1}{(1-p_{N'+1})^{m+\delta}}\right\}+\cdots+\frac{1}{(1-p_0)^{m+\delta}}\\ &=h(1-p_{N'})+h(1-p_{N'-1})+\cdots+h(1-p_0)+\frac{1}{(1-p_0)^{m+\delta}}\\ &<\frac{1}{2}(N'+1)\log 2+\frac{1}{(1-p_0)^{m+\delta}}.\end{aligned}$$

The last inequality follows from (12). On the other hand, $g(p_0)+N'\log 2\le g(p_{N'})$ $\le g(t)$ by (14). Combining this with (16) yields

$$(17)\qquad \frac{1}{2}(N-1)\log 2<\frac{1}{2}(N'-1)\log 2<\frac{1}{(1-p_0)^{m+\delta}}-g(p_0).$$

Here we used $N<N'$ for the first inequality. We can pick $N(p_0)$ so large that (17) does not hold. This yields a contradiction. Hence we have $m\ge 2$.

We proceed to prove $M\le 2$. Suppose $M>2$. Pick $\delta>0$ such that $M-\delta>2$. Let

$$h(x)=\frac{1}{(x-3x^3)^{M-\delta}}-\frac{1}{x^{M-\delta}}.$$

Note that $\lim_{x\to 0}h(x)=\infty$. Then by a similar argument as above, we lead a contradiction. Hence $M\le 2$, which concludes $m=M=2$. □

4. Proof of Theorem 1.2

First, we estimate the probability $P_p((1/9)\cdot 3^n\le |C|\le (9/2)\cdot 3^n)$. Let $M=\sup\{m: \boldsymbol{O}\leftrightarrow a_m \text{ or } b_m\}$. We define two conditional probabilities

$$U_n(p)=P_p(\boldsymbol{O}\leftrightarrow a_n,\ \boldsymbol{O}\not\leftrightarrow b_n \text{ in } F_n\mid M=n),$$

$$V_n(p)=P_p(O\leftrightarrow a_n,\ O\leftrightarrow b_n \text{ in } F_n \mid M=n).$$

Clearly

(18) $$2U_n(p)+V_n(p)=1,$$

and

(19) $$V_n(p)=\frac{P_p(O\leftrightarrow a_n,\ O\leftrightarrow b_n \text{ in } F_n,\ O\not\leftrightarrow a_{n+1},\ O\not\leftrightarrow b_{n+1})}{P_p(M=n)}.$$

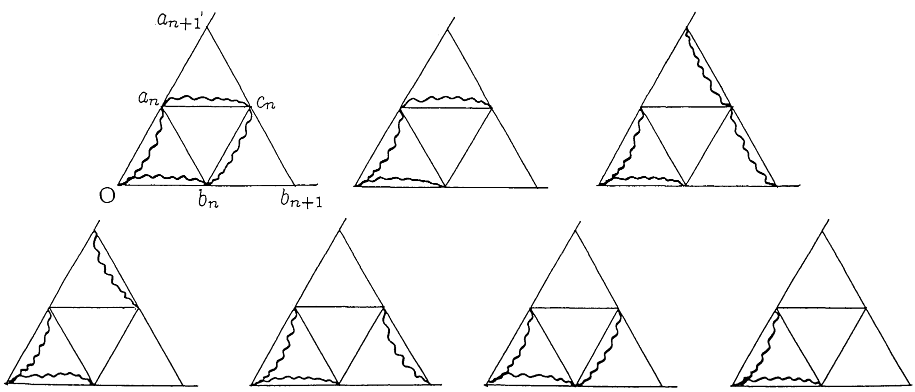

Fig. 4.1

We consider the event of the numerator of (19), $\{O\leftrightarrow a_n,\ O\leftrightarrow b_n \text{ in } F_n,\ O\not\leftrightarrow a_{n+1},\ O\not\leftrightarrow b_{n+1}\}$. We divide the case into seven parts as Fig. 4.1. Since the events in F_n, F_n', F_n'' are independent, we have

(20) $$V_n(p)=\frac{\Theta_n(1-2\Phi_n-\Phi_n^2+4\Phi_n\Theta_n-2\Theta_n^2)}{P_p(M=n)}.$$

Here we denoted $\Phi_n=\Phi_n(p)$, $\Theta_n=\Theta_n(p)$ briefly. Note that

(21)
$$\begin{aligned}
&P_p(M=n)\\
&=P_p(M\geq n)-P_p(M\geq n+1)\\
&=2\Phi_n-\Theta_n-(2\Phi_{n+1}-\Theta_{n+1})\\
&=2\Phi_n-\Theta_n-2\Phi_n^2-2\Phi_n^3+2\Phi_n\Theta_n^2+3\Phi_n^2\Theta_n-2\Theta_n^3
\end{aligned}$$

by (4). Hence by (18),

(22) $$U_n(p)=\frac{1}{2}\{1-V_n(p)\}$$

$$=\frac{(\Phi_n-\Theta_n)(1-\Phi_n-\Phi_n^2+\Phi_n\Theta_n)}{P_p(M=n)}.$$

Let

(23) $$n_0=n_0(p)=\sup\{n:\Theta_n(p)\geq\frac{2}{3}\}.$$

Lemma 4.1. *$V_n(p)\geq\frac{2}{9}$ if $n<n_0$.*

Proof. From (18), it is enough to show

(24) $$\frac{V_n(p)}{2U_n(p)}\geq\frac{2}{7}.$$

Let

$$\kappa(x,y)=\frac{y(1-2x-x^2+4xy-2y^2)}{2(x-y)(1-x-x^2+xy)}.$$

By (20) and (22), (24) follows from the following:

(25) $$\kappa(x,y)\geq\frac{2}{7}\quad\text{for}\quad\frac{2}{3}\leq x<1,\ \frac{1}{2}(3x-1)<y<x.$$

The second condition in (25) comes from the fact that

(26) $$3\Phi_n(p)-2\Theta_n(p)=\Psi_n(p)<1.$$

Let $y/x=t$. Then the domain of (25) is $2/3\leq x<1/(3-2t)$, $2/3\leq t<y<1$. And

$$\kappa(x,tx)=\frac{t}{2(1-t)}\{1-\frac{x+(-3t+2t^2)x^2}{1-x-(1-t)x^2}\}.$$

Now let

$$\lambda(x)=\frac{x+(-3t+2t^2)x^2}{1-x-(1-t)x^2}.$$

From a direct calculation,

$$\lambda'(x)=\frac{(1+2t-2t^2)x^2+2(-3t+2t^2)x+1}{\{1-x-(1-t)x^2\}^2}.$$

We see that if $2/3 \leq t < 1$, $\lambda'(x) > 0$ for $2/3 \leq x < 1/(3-2t)$. Therefore

$$\kappa(x,tx) > \kappa\left(\frac{1}{3-2t}, \frac{t}{3-2t}\right) = \frac{t}{5-4t} \geq \frac{2}{7}. \qquad \square$$

Next, we estimate the expectation of $|C|$ on condition that $M=n$ $(n<n_0)$.

Lemma 4.2. $E_p(|C||M=n) \geq \frac{2}{9} \cdot 3^n$ *if* $n < n_0$.

To prove the above Lemma, we use the following inequality:

Lemma 4.3. *For all* $a \in F_n$,

$$P_p(\boldsymbol{O} \leftrightarrow a \quad in \quad F_n) \geq \Phi_n(p). \tag{27}$$

Proof. Besides (27), we introduce a similar inequality:

$$P_p(a \leftrightarrow a_n \text{ or } a \leftrightarrow b_n) \geq P_p(\boldsymbol{O} \leftrightarrow a_n \text{ or } \boldsymbol{O} \leftrightarrow b_n) \text{ for all } a \in F_n. \tag{28}$$

We prove (27) and (28) by induction at the same time. If $n=0$, clearly both of them hold. Suppose (27) and (28) for $n=k$.

We prove (27) for $n=k+1$ at first. By symmetry, it is sufficient to prove the cases (i) $a \in F_k$ and (ii) $a \in F'_k$.

(i) Suppose $a \in F_k$. By using (4), we see $\Phi_k(p) \geq \Phi_{k+1}(p)$. Indeed, suppose $\Phi_k(p) \geq 1/3$, then

$$\begin{aligned} \frac{\Phi_{k+1}}{\Phi_k} &= \Phi_k + \{\Phi_k\}^2 - \{\Theta_k\}^2 \\ &\leq \Phi_k + \{\Phi_k\}^2 - \left(\frac{3\Phi_k - 1}{2}\right)^2 \\ &\leq -\frac{5}{4}(1-\Phi_k)^2 + 1 \\ &\leq 1. \end{aligned} \tag{29}$$

Here we used (26). Combining this with assumption, we see (27) for $n=k+1$ in this case.

(ii) Suppose $a \in F'_k$. Let C_n^1, C_n^2, C_n^3 be events given by

$$C_n^1 = \{\boldsymbol{O} \leftrightarrow a_n \text{ and } \boldsymbol{O} \not\leftrightarrow c_n \text{ in } F_n \cup F''_n\},$$

$$C_n^2 = \{\boldsymbol{O} \not\leftrightarrow a_n \text{ and } \boldsymbol{O} \leftrightarrow c_n \text{ in } F_n \cup F''_n\},$$

$$C_n^3=\{O\leftrightarrow a_n \text{ and } O\leftrightarrow c_n \text{ in } F_n\cup F_n''\}.$$

We see

$$\begin{aligned}
&P_p(O\leftrightarrow a \text{ in } F_{k+1})\\
&=P_p(C_k^1)P_p(a_k\leftrightarrow a \text{ in } F_k')+P_p(C_k^2)P_p(c_k\leftrightarrow a \text{ in } F_k')\\
&\qquad+P_p(C_k^3)P_p(a_k\leftrightarrow a \text{ or } c_k\leftrightarrow a \text{ in } F_k')\\
&\geq(\Phi_k-\Phi_k\Theta_k)\cdot\Phi_k+(\Phi_k-\Theta_k)\Phi_k\cdot\Phi_k+\Phi_k\Theta_k\cdot 2(\Phi_k-\Theta_k)\\
&=\Phi_k^2+\Phi_k^3-\Phi_k\Theta_k^2=\Phi_{k+1}.
\end{aligned}$$

Here we used assumption for the inequality. We thus obtain (27) for $n=k+1$. We proceed to prove (28) for $n=k+1$.

(i) Suppose $a\in F_k$. Let D_n^1, $D_n^2,\cdots,D_n^5$ be events given by

$$D_n^1=\{a_n\leftrightarrow a_{n+1} \text{ or } a_n\leftrightarrow b_{n+1} \text{ in } F_k'\cup F_k''\},$$

$$D_n^2=\{b_n\leftrightarrow a_{n+1} \text{ or } b_n\leftrightarrow b_{n+1} \text{ in } F_k'\cup F_k''\},$$

$D_n^3=D_n^1\cap(D_n^2)^c$, $D_n^4=(D_n^1)^c\cap D_n^2$, $D_n^5=D_n^1\cap D_n^2$. We see

$$\begin{aligned}
&P_p(a\leftrightarrow a_{k+1} \text{ or } a\leftrightarrow b_{k+1})\\
&=P_p(D_k^3)P_p(a\leftrightarrow a_k \text{ in } F_k)+P_p(D_k^4)P_p(a\leftrightarrow b_k \text{ in } F_k)\\
&\qquad+P_p(D_k^5)P_p(a\leftrightarrow a_k \text{ or } a\leftrightarrow b_k \text{ in } F_k)\\
&\geq P_p(D_k^3)P_p(O\leftrightarrow a_k \text{ in } F_k)+P_p(D_k^4)P_p(O\leftrightarrow b_k \text{ in } F_k)\\
&\qquad+P_p(D_k^5)P_p(O\leftrightarrow a_k \text{ or } O\leftrightarrow b_k \text{ in } F_k)\\
&=P_p(O\leftrightarrow a_{k+1} \text{ or } O\leftrightarrow b_{k+1})
\end{aligned}$$

by assumption.

(ii) Suppose $a\in F_k'$. We see

$$\begin{aligned}
(30)\qquad &P_p(a\leftrightarrow a_{k+1} \text{ or } a\leftrightarrow b_{k+1})\\
&\geq P_p(a\leftrightarrow a_{k+1} \text{ in } F_k')\\
&\quad+P_p(a\not\leftrightarrow a_{k+1} \text{ or } a\leftrightarrow c_k \text{ in } F_k')P_p(c_k\leftrightarrow b_{k+1} \text{ in } F_k'').
\end{aligned}$$

Here we note that

$$\begin{aligned}
&P_p(a\not\leftrightarrow a_{k+1} \text{ and } a\leftrightarrow c_k \text{ in } F_k')\\
&=P_p(a\leftrightarrow a_{k+1} \text{ or } a\leftrightarrow c_k \text{ in } F_k')-P_p(a\leftrightarrow a_{k+1} \text{ in } F_k')\\
&\geq(2\Phi_k-\Theta_k)-P_p(a\leftrightarrow a_{k+1} \text{ in } F_k')
\end{aligned}$$

by assumption. Using this and (30), we have

$$P_p(a\leftrightarrow a_{k+1} \text{ or } a\leftrightarrow b_{k+1})$$

$$\geq P_p(a \leftrightarrow a_{k+1} \text{ in } F_k')$$
$$+\{(2\Phi_k - \Theta_k) - P_p(a \leftrightarrow a_{k+1} \text{ in } F_k')\} P_p(c_k \leftrightarrow b_{k+1} \text{ in } F_k'')$$
$$= P_p(a \leftrightarrow a_{k+1} \text{ in } F_k')(1-\Phi_k) + (2\Phi_k - \Theta_k)\Phi_k$$
$$\geq \Phi_k(1-\Phi_k) + 2\Phi_k^2 - \Phi_k\Theta_k = \Phi_k + \Phi_k^2 - \Phi_k\Theta_k.$$

Here we used assumption again. Now it is enough to show

$$\Phi_k + \Phi_k^2 - \Phi_k\Theta_k - P_p(\boldsymbol{O} \leftrightarrow a_{k+1} \text{ or } \boldsymbol{O} \leftrightarrow b_{k+1}) \geq 0. \tag{31}$$

The left-hand side of (31) equals

$$\Phi_k + \Phi_k^2 - \Phi_k\Theta_k - (2\Phi_{k+1} - \Theta_{k+1})$$
$$= (\Phi_k + \Phi_k^2 - \Phi_k\Theta_k) - 2(\Phi_k^2 + \Phi_k^3 - \Phi_k\Theta_k^2) + (3\Phi_k^2\Theta_k - 2\Theta_k^3)$$
$$= \Phi_k(1-\Theta_k)(1 - 3\Phi_k + 2\Theta_k) + 2(\Phi_k - \Theta_k)^2(1-\Phi_k)$$
$$+ 2\Theta_k(\Phi_k - \Theta_k)(1 - 2\Phi_k + \Theta_k).$$

By (26), we see all terms above are nonnegative. Hence the proof is completed. □

Proof of Lemma 4.2.

$$E_p(|C| \,|\, M=n) = \sum_{a \in V} P_p(\boldsymbol{O} \leftrightarrow a \,|\, M=n)$$
$$\geq \sum_{a \in F_n} P_p(\boldsymbol{O} \leftrightarrow a \text{ in } F_n \,|\, M=n)$$
$$\geq \sum_{a \in F_n} \frac{P_p(\boldsymbol{O} \leftrightarrow a, \boldsymbol{O} \leftrightarrow a_n, \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n, M=n)}{P_p(M=n)}.$$

Let $D_n^6 = (D_n^1)^c \cap (D_n^2)^c$. Note that if $M=n$ and $\boldsymbol{O} \leftrightarrow a_n$, $\boldsymbol{O} \leftrightarrow b_n$, then $(D_n^6)^c$ occurs. For $a \in F_n$, we see

$$P_p(\boldsymbol{O} \leftrightarrow a,\ \boldsymbol{O} \leftrightarrow a_n,\ \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n,\ M=n)$$
$$= P_p(\boldsymbol{O} \leftrightarrow a,\ \boldsymbol{O} \leftrightarrow a_n,\ \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n,\ D_n^6 \text{ occurs})$$
$$= P_p(\boldsymbol{O} \leftrightarrow a,\ \boldsymbol{O} \leftrightarrow a_n,\ \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6)$$
$$\geq P_p(\boldsymbol{O} \leftrightarrow a \text{ in } F_n) P_p(\boldsymbol{O} \leftrightarrow a_n,\ \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n) P_p(D_n^6)$$
$$= P_p(\boldsymbol{O} \leftrightarrow a \text{ in } F_n) P_p(\boldsymbol{O} \leftrightarrow a_n,\ \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n,\ M=n).$$

Here we used FKG inequality for the forth line. Therefore

$$E_p(|C| \,|\, M=n) \geq \sum_{a \in F_n} P_p(\boldsymbol{O} \leftrightarrow a \text{ in } F_n) P_p(\boldsymbol{O} \leftrightarrow a_n,\ \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n \,|\, M=n)$$

$$\geq \frac{2}{9} \sum_{a \in F_n} P_p(O \leftrightarrow a \text{ in } F_n)$$

by Lemma 4.1. Note that $|\{a \in V : a \in F_n\}| = (3/2)(3^n + 1)$. By virtue of Lemma 4.3, we see

$$E_p(|C| | M = n) \geq \frac{2}{9} \cdot \frac{3}{2} \cdot 3^n \Phi_n(p)$$

$$\geq \frac{2}{9} \cdot 3^n \quad \text{for } n < n_0.$$

We used (23) and the fact that $\Phi_n(p) \geq \Theta_n(p)$ for the last inequality. □

We proceed to the estimate of $P_p((1/9) \cdot 3^n \leq |C| \leq (9/2) \cdot 3^n)$.

Lemma 4.4. $P_p\left(\frac{1}{9} \cdot 3^n \leq |C| \leq \frac{9}{2} \cdot 3^n\right) \geq \frac{2}{79} P_p(M = n)$ *if* $n < n_0$.

Proof. Note that $|C| \leq (9/2) \cdot 3^n$ if $M = n$. Then we see the following.

$$E_p(|C| | M = n)$$

$$= E_p(|C|\,; |C| \geq \frac{1}{9} \cdot 3^n | M = n) + E_p(|C|\,; |C| < \frac{1}{9} \cdot 3^n | M = n)$$

$$\leq \frac{9}{2} \cdot 3^n P_p(|C| \geq \frac{1}{9} \cdot 3^n | M = n) + \frac{1}{9} \cdot 3^n P_p(|C| < \frac{1}{9} \cdot 3^n | M = n).$$

By Lemma 4.2, we have

$$P_p(|C| \geq \frac{1}{9} \cdot 3^n | M = n) \geq \frac{2}{79},$$

thus the proof is completed. □

Lemma 4.5. $P_p(M = n) > \Phi_n(p)\{1 - \Phi_n(p)\}^2$ *if* $n < n_0$.

Proof. Recall (21), that is

$$P_p(M = n) = 2\Phi_n - \Theta_n - 2\Phi_n^2 - 2\Phi_n^3 + 2\Phi_n\Theta_n^2 + 3\Phi_n^2\Theta_n - 2\Theta_n^3.$$

Let $\pi(y) = 2x - y - 2x^2 - 2x^3 + 2xy^2 + 3x^2y - 2y^3$. It is enough to show that $\pi(y) > x(1-x)^2$ if $2/3 \leq x < 1$, $(3x-1)/2 < y < x$. Note that

$$\pi'(y) = -6y^2 + 4xy + 3x^2 - 1,$$

and that

$$\pi'\left(\frac{3x-1}{2}\right) = \frac{1}{2}(1-x)(9x-5) > 0, \ \pi'(x) = x^2 - 1 < 0.$$

Hence $\pi(y) > \min\{\pi((3x-1)/2), \ \pi(x)\}$, $\pi((3x-1)/2) = (1-x)^2(x+3)/4$ and $\pi(x) = x(1-x)^2$, so $\pi((3x-1)/2) > \pi(x)$ for $2/3 \le x < 1$. This completes the proof. □

Proof of Theorem 1.2. First, we estimate $E_p|C|^k$ from below. By using Lemma 4.4 and 4.5, we see

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C| = l) \\ &\ge \sum_{n=4,8,12\cdots} \left(\frac{1}{9}\cdot 3^n\right)^k P_p\left(\frac{1}{9}\cdot 3^n \le |C| \le \frac{9}{2}\cdot 3^n\right) \\ &\ge \frac{1}{9^k}\cdot\frac{2}{79} \sum_{\substack{m\in N \\ 4m<n_0}} 3^{4km}\Phi_{4m}(p)\{1-\Phi_{4m}(p)\}^2. \end{aligned}$$

Let p be sufficiently large. Note that the function $\iota(x) = x(1-x)^2$ is decreasing in $2/3 \le x < 1$, and $\Phi_{4m}(p) \le e^{-2^{4m}/\xi(p)}$ by (7). We can see

$$\begin{aligned} &\sum_{\substack{m\in N \\ 4m<n_0}} 3^{4km}\Phi_{4m}(p)\{1-\Phi_{4m}(p)\}^2 \\ &\ge \sum_{\substack{m\in N \\ 4m<n_0}} 3^{4km} e^{-2^{4m}/\xi(p)}(1-e^{-2^{4m}/\xi(p)})^2 \\ &\ge \int_1^{\frac{n_0}{4}-1} 3^{4kx} e^{-2^{4x}/\xi(p)}(1-e^{-2^{4x}/\xi(p)})^2 dx \\ &= \frac{\{\xi(p)\}^{Dk}}{4\log 2}\int_{2^4/\xi(p)}^{2^{n_0-4}/\xi(p)} y^{Dk-1} e^{-y}(1-e^{-y})^2 dy. \end{aligned}$$

Here we set $y = 2^x/\xi(p)$ in the last line. Note that $\Theta_{n_0+1}(p) < 2/3$, hence $\Phi_{n_0+1}(p) < (1+2\Theta_{n_0+1}(p))/3 < 7/9$ by (24). From (29), if $\Phi_k(p) < 7/9$, then $\Phi_{k+1}(p)/\Phi_k(p) < 76/81$. We see

$$\Phi_{n_0+12}(p) < \left(\frac{76}{81}\right)^{11}\cdot\frac{7}{9} < \frac{1}{2}\cdot\frac{7}{9}.$$

Combining this with (7), we have

$$\frac{1}{2}e^{-2^{n_0+12}/\xi(p)} \leq \Phi_{n_0+12}(p) < \frac{1}{2}\cdot\frac{7}{9}.$$

Hence $2^{n_0-4}/\xi(p) > 2^{-16}\log(9/7)$. Since $\xi(p) \to \infty$ as $p \to 1$, $E_p|C|^k > K_1\{\xi(p)\}^{Dk}$ holds if we take

$$K_1(k) = \int_{2^{-17}\log(9/7)}^{2^{-16}\log(9/7)} y^{Dk-1}e^{-y}(1-e^{-y})^2 dy > 0.$$

Now we proceed to estimate from above. Note that $P_p(M \geq n) \leq 2\Phi_n(p) \leq 2e^{-2^n/\xi(p)}$, and we can see easily $P_p((3/2)\cdot 3^n < |C| \leq (3/2)\cdot 3^{n+1}) \leq P_p(M \geq n) \leq 2e^{-2^n/\xi(p)}$. Hence

$$\begin{aligned} E_p|C|^k &= \sum_{l=1}^{\infty} l^k P_p(|C| = l) \\ &\leq 1 + \sum_{n=0}^{\infty}\left(\frac{3}{2}\cdot 3^{n+1}\right)^k P_p\left(\frac{3}{2}\cdot 3^n < |C| \leq \frac{3}{2}\cdot 3^{n+1}\right) \\ &\leq 1 + 2\cdot\left(\frac{9}{2}\right)^k \sum_{n=0}^{\infty} 3^{kn}e^{-2^n/\xi(p)}. \end{aligned}$$

Now

$$\begin{aligned} \int_0^{\infty} 3^{kx}e^{-2^x/\xi(p)}dx &= \frac{\{\xi(p)\}^{Dk}}{\log 2}\int_{\xi(p)^{-1}}^{\infty} y^{Dk-1}e^{-y}dy \\ &\leq \frac{\Gamma(Dk)}{\log 2}\cdot\{\xi(p)\}^{Dk}. \end{aligned}$$

So we can take $K_2(k) < \infty$ such that $E_p|C|^k < K_2\{\xi(p)\}^{Dk}$. □

5. Site percolation on the pre-Sierpinski gasket

We define the Bernoulli site percolation on the pre-Sierpinski gasket; each vertices in V are open with probability p and closed with $1-p$ independently. Let $\tilde{P}_p$ denote its distribution. We write $x \leftrightarrow y$ if there exists a sequence of open vertices $x = x_0, x_1, \cdots, x_{n-1}, x_n = y$ such that there is a bond in E which connects x_j with x_{j+1} for $0 \leq j \leq n-1$. We define another notations in the same manner as before. We introduce connectivity functions;

$$\tilde{\Phi}_n(p) = \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ in } \Delta \boldsymbol{O} a_n b_n),$$

$$\tilde{\Theta}_n(p) = \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ and } \boldsymbol{O} \leftrightarrow b_n \text{ in } \Delta \boldsymbol{O} a_n b_n).$$

We see $\tilde{\Phi}_0(p)=p^2$ and $\tilde{\Theta}_0(p)=p^3$ by definition.

Proposition 5.1. *For each* $n\geq 0$ *and* $0\leq p\leq 1$,

$$\tilde{\Phi}_{n+1}(p)=p^{-1}\{\tilde{\Phi}_n(p)\}^2+p^{-2}\{\tilde{\Phi}_n(p)\}^3-p^{-3}\tilde{\Phi}_n(p)\{\tilde{\Theta}_n(p)\}^2, \tag{32}$$

$$\tilde{\Theta}_{n+1}(p)=3p^{-2}\{\tilde{\Phi}_n(p)\}^2\tilde{\Theta}_n(p)-2p^{-3}\{\tilde{\Theta}_n(p)\}^3. \tag{33}$$

Proof. We prove (32). Let $\widetilde{A_n^1}$ and $\widetilde{A_n^2}$ be events given by

$$\widetilde{A_n^1}=\{\boldsymbol{O}\leftrightarrow a_n \text{ in } F_n\}\cap\{a_n\leftrightarrow a_{n+1} \text{ in } F_n'\},$$

$$\widetilde{A_n^2}=\{\boldsymbol{O}\leftrightarrow b_n \text{ in } F_n\}\cap\{b_n\leftrightarrow c_n \text{ in } F_n''\}\cap\{c_n\leftrightarrow a_{n+1} \text{ in } F_n'\}.$$

Then we have

$$\tilde{\Theta}_{n+1}(p)=\tilde{P}_p(\widetilde{A_n^1})+\tilde{P}_p(\widetilde{A_n^2})-\tilde{P}_p(\widetilde{A_n^1}\cap\widetilde{A_n^2}). \tag{34}$$

Remark that $F_n\cap F_n'=\{a_n\}$. So we see $\tilde{P}_p(\widetilde{A_n^1})=p^{-1}\{\tilde{\Phi}_n(p)\}^2$. Similarly, we have $\tilde{P}_p(\widetilde{A_n^2})=p^{-2}\{\tilde{\Phi}_n(p)\}^3$, $\tilde{P}_p(\widetilde{A_n^1}\cap\widetilde{A_n^2})=p^{-3}\{\tilde{\Theta}_n(p)\}^2\tilde{\Phi}_n(p)$. Thus (32) follows from (34) immediately. (33) is proved in the same way. □

Let $\hat{\Phi}_n(p)=p^{-1}\tilde{\Phi}_n(p)$ and $\hat{\Theta}_n(p)=p^{-\frac{3}{2}}\tilde{\Theta}_n(p)$. Then we have the same recursions as (4), (5):

$$\hat{\Phi}_{n+1}(p)=\{\hat{\Phi}_n(p)\}^2+\{\hat{\Phi}_n(p)\}^3-\hat{\Phi}_n(p)\{\hat{\Theta}_n(p)\}^2, \tag{35}$$

$$\hat{\Theta}_{n+1}(p)=3\{\hat{\Phi}_n(p)\}^2\hat{\Theta}_n(p)-2\{\hat{\Theta}_n(p)\}^3. \tag{36}$$

Hence we see that there exists $\hat{\xi}(p)>0$ such that

$$\lim_{n\to\infty}\frac{\hat{\Phi}_n(p)}{\exp\{-2^n/\hat{\xi}(p)\}}=1, \quad \text{that is} \quad \lim_{n\to\infty}\frac{\tilde{P}_p(\boldsymbol{O}\leftrightarrow a_n)}{p\exp\{-2^n/\hat{\xi}(p)\}}=1.$$

Lemma 5.2. *Let* $\sqrt{\tilde{p}}=\sqrt{p}+6(1-\sqrt{p})^2$. *Then there exists* $\varepsilon>0$ *such that*

$$2\leq\frac{\hat{\xi}(\tilde{p})}{\hat{\xi}(p)}\leq 4 \quad \textit{for} \quad 1-\varepsilon<p<1.$$

Proof. We use the same method as in Section 3 again. Let

$$\hat{\Psi}_n(p)=3\hat{\Phi}_n(p)-2\hat{\Theta}_n(p). \tag{37}$$

To apply (9), first we prove $(\hat{\Theta}_n(p),\hat{\Psi}_n(p))\in D$. (Recall $D=\{(x,y):0<x\leq y<1\}$.) Since $\hat{\Psi}_n(p)=\hat{\Theta}_n(p)+3\{\hat{\Phi}_n(p)-\hat{\Theta}_n(p)\}$, it is enough to prove $\hat{\Phi}_n(p)\geq\hat{\Theta}_n(p)$. Now

$$\hat{\Phi}_n(p)=p^{-1}\times\tilde{P}_p(\boldsymbol{O}\leftrightarrow a_n \text{ in } F_n)$$

$$
\begin{aligned}
&= \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ in } F_n \mid a_n \text{ is open}) \\
&= \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ in } F_n \mid a_n,\ b_n \text{ are open}), \\
\hat{\Theta}_n(p) &= p^{-\frac{3}{2}} \times \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ and } \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n) \\
&\leq p^{-2} \times \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ and } \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n) \\
&= \tilde{P}_p(\boldsymbol{O} \leftrightarrow a_n \text{ and } \boldsymbol{O} \leftrightarrow b_n \text{ in } F_n \mid a_n,\ b_n \text{ are open}).
\end{aligned}
$$

Hence we have $\hat{\Phi}_n(p) \geq \hat{\Theta}_n(p)$, which implies $(\hat{\Theta}_n(p), \hat{\Psi}_n(p)) \in D$.

A direct calculation from (35) and (36) shows

$$
\begin{aligned}
&\hat{\Theta}_2(\tilde{p}) - \hat{\Theta}_1(p) = 6(1-\sqrt{p})^2 + 204(1-\sqrt{p})^3 + \cdots, \\
&\hat{\Psi}_2(\tilde{p}) - \hat{\Psi}_1(p) = 12(1-\sqrt{p})^3 + \cdots, \\
&\hat{\Theta}_3(\tilde{p}) - \hat{\Theta}_1(p) = -6(1-\sqrt{p})^2 + 204(1-\sqrt{p})^3 + \cdots, \\
&\hat{\Psi}_3(\tilde{p}) - \hat{\Psi}_1(p) = -12(1-\sqrt{p})^3 + \cdots.
\end{aligned}
$$

We can take $\varepsilon > 0$ such that

$$
\hat{\Theta}_3(\tilde{p}) < \hat{\Theta}_1(p) < \hat{\Theta}_2(\tilde{p}),\ \hat{\Psi}_3(\tilde{p}) < \hat{\Psi}_1(p) < \hat{\Psi}_2(\tilde{p})
$$

for $1-\varepsilon < p < 1$.

Now we apply (9). We have for $n \geq 1$ and $1-\varepsilon < p < 1$,

$$
\hat{\Theta}_{n+2}(\tilde{p}) < \hat{\Theta}_n(p) < \hat{\Theta}_{n+1}(\tilde{p}), \quad \text{and} \quad \hat{\Psi}_{n+2}(\tilde{p}) < \hat{\Psi}_n(p) < \hat{\Psi}_{n+1}(\tilde{p}).
$$

We see $\hat{\Phi}_{n+2}(\tilde{p}) < \hat{\Phi}_n(p) < \hat{\Phi}_{n+1}(\tilde{p})$ by (37), so we have the conclusion. □

Proof of Theorem 1.3. Note that $\tilde{p} = \{\sqrt{p} + 6(1-\sqrt{p})^2\}^2 = p + 3(1-p)^2 + o((1-p)^2)$ as $p \to 1$. We have Theorem 1.3 in the same way as in Section 3.

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