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# THE SLICE DETERMINED BY MODULI EQUATION xy=2z IN THE DEFORMATION SPACE OF ONCE PUNCTURED TORI

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## 1. Introduction and statement of results

We recall some terminology and results in [2], [3] and [6]. Let $A$ and $B$ be loxodromic elements of $PSL(2,\boldsymbol{C})$, that is, $A$ and $B$ are $2\times 2$ complex matrices with determinant 1 and their traces do not lie on the closed interval $[-2,2]$. By an obvious isomorphism we identify $PSL(2,\boldsymbol{C})$ with the Möbius transformations group. Denote by $G=\langle A,B\rangle$ the Möbius subgroup generated by $A$ and $B$. Let $x$, $y$ and $z$ be the traces of $A$, $B$ and $AB$, respectively. The triple $(x,y,z)$ is called a moduli triple of $G$. We restrict ourselves to the case in which triple $(x,y,z)$ satisfies the moduli equation

$$(*)\qquad x^2+y^2+z^2=xyz.$$

Let $A_0=\begin{pmatrix}\sqrt{2}+1 & 0\\ 0 & \sqrt{2}-1\end{pmatrix}$ and $B_0=\begin{pmatrix}\sqrt{2} & 1\\ 1 & \sqrt{2}\end{pmatrix}$. Put $G_0=\langle A_0,B_0\rangle$. Then $(\sqrt{8},\sqrt{8},4)$ is a moduli triple of $G_0$ satisfying $(*)$, $G_0$ is a Fuchsian group of the first kind and $\Omega(G_0)/G_0$ is a pair of once punctured tori, where $\Omega(G_0)$ denotes the region of discontinuity of $G_0$. For each quasi-Fuchsian group $G=\langle A,B\rangle$ such that $\Omega(G)/G$ is a pair of once punctured tori, there is a quasiconformal mapping $f$ of the extended plane such that $A=fA_0f^{-1}$ and $B=fB_0f^{-1}$. Hence $G$ is a quasiconformal deformation of $G_0$. The set of all such quasi-Fuchsian groups is called a deformation space of once punctured tori and denoted by $\boldsymbol{D}(G_0)$. Under a normalization each triple satisfying $(*)$ determines $A$ and $B$ uniquely so that a group $G=\langle A,B\rangle$ of $\boldsymbol{D}(G_0)$, too. We identify $\boldsymbol{D}(G_0)$ with the subset of $\boldsymbol{C}^3$ consisting of triples each of which satisfies $(*)$ and determines a quasi-Fuchsian group. We put

$$\boldsymbol{T}^*=\{(x,y,z)|x^2+y^2+z^2=xyz\}\subset\boldsymbol{C}^3.$$

Then, by stability of quasi-Fuchsian groups, $\boldsymbol{D}(G_0)$ is an open subset of $\boldsymbol{T}^*$. In [3] the following is shown.

**Theorem 1 ([3]).** *Let $G=\langle A,B\rangle$ be a group generated by loxodromic elements $A$ and $B$ satisfying* $(*)$. *If triple $(x,y,z)$ of $G$ also satisfies*

(1) $$x>2 \quad and \quad y>2,$$

*then $G\in \boldsymbol{D}(G_0)$.*

We put

$$S=\{(x,y,z)|x>2,\ y>2\}\cap \boldsymbol{T}^*.$$

Then Theorem 1 implies $S\subset \boldsymbol{D}(G_0)$. We denote by $\bar{S}$ the closure of $S$ in $\boldsymbol{T}^*$. Then $\bar{S}\backslash S$ consists of the following three sets:

$$b_1=\{(2,y,z)|y^2+z^2+4=2yz,\ y>2\}$$

$$b_2=\{(x,2,z)|x^2+z^2+4=2xz,\ x>2\}$$

$$b_3=\{(2,2,z)|z^2+8=4z\}.$$

For these sets the following are noted in [6].

**Theorem 2 ([6]).** *Let $G$ be a group whose moduli triple lies on $b_1\cup b_2$. Then $G$ is a boundary group of $\boldsymbol{D}(G_0)$. More precisely, $G$ is a regular b-group.*

**Theorem 3 ([6]).** *Let $G$ be a group whose moduli triple lies on $b_3$. Then $G$ is a boundary group of $\boldsymbol{D}(G_0)$. More precisely, $G$ is a web group.*

A *regular b-group* is a Kleinian group $G$ having only one simply connected invariant component $\Delta$ such that

$$2\mathrm{Area}(\Delta/G)=\mathrm{Area}(\Omega(G)/G),$$

where Area implies the hyperbolic area. A *web* group is a Kleinian group such that each component subgroup is a quasi-Fuchsian group of the first kind. We remark that the web group in Theorem 3 is not quasi-Fuchsian, but it has an infinite number of components. By Theorems 2 and 3 we see that

$$S=\{(x,y,z)|x>2,\ y>2\}\cap \boldsymbol{D}(G_0),$$

so we shall call $S$ the *slice determined by* (1).

Comparing with Theorems 1, 2 and 3, we shall investigate another slice $S'$ of $\boldsymbol{D}(G_0)$ determined by moduli equation

(2) $$xy=2z.$$

Explicitely,

$$S'=\{(x,y,z)|xy=2z\}\cap \boldsymbol{D}(G_0).$$

In comparison with Theorem 1 we prove the following.

**Theorem 4.** *Let*

$$E=\{(x,y,z)|xy=2z,\ |x|>2,\ |y|>2\}\cap \boldsymbol{T}^*.$$

*Then* $E\subset S'$.

The set $\bar{E}\backslash E$ consists of the following three sets:

$$e_1=\{(x,y,z)|xy=2z,\ |x|=2,\ |y|>2\}\cap \boldsymbol{T}^*$$
$$e_2=\{(x,y,z)|xy=2z,\ |x|>2,\ |y|=2\}\cap \boldsymbol{T}^*$$
$$e_3=\{(x,y,z)|xy=2z,\ |x|=|y|=2\}\cap \boldsymbol{T}^*.$$

In contrast to Theorem 2 the following holds.

**Theorem 5.** $e_1\cup e_2\subset S'$.

We also prove the following.

**Theorem 6.** *Let $G$ be a group whose moduli triple lies on $e_3$. Then $G$ is a boundary group of $\boldsymbol{D}(G_0)$.*

In §2 we make a normalization of generators and then show as Theorems 7 and 8 that each group of the slice $S'$ has a symmetric region of discontinuity and a symmetirc fundamental domain with respect to the origin and that none of the boundary groups of $S'$ is a $b$-group. In §3 we modify a criterion for discontinuity of [4] and then prove Theorems 4 and 5. Lastly, we prove Theorem 6 in §4.

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## 2. Normalization and symmetry

Let $A$ and $B$ be loxodromic elements of $PSL(2,\boldsymbol{C})$ and put $G=\langle A,B\rangle$. We assume that the moduli triple $(x,y,z)$ of $G$ satisfies $(*)$ and (2). Firstly, we shall normalize $A$ and $B$. Conjugating by a Möbius transformation, we normalize $A$ such that

$$A=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},\quad \alpha=re^{i\theta},\quad r>1\quad \textit{and}\quad -\frac{\pi}{2}<\theta\leq\frac{\pi}{2}.$$

Note that $\alpha\beta=1$ and $\beta=1/\alpha$. We write $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $x=\alpha+\beta$, $y=a+d$ and $z=\alpha a+\beta d$.

**Proposition.** *Equality* (2) *is equivalent to* $a=d$.

Proof. By the substitution of $x=\alpha+\beta$, $y=a+d$ and $z=\alpha a+\beta d$ into (2), equality (2) reduces to $(\alpha-\beta)(a-d)=0$. Since $\alpha\neq\beta$, we have $a=d$. □

Substituting $x=\alpha+\beta$, $y=2a$ and $z=(\alpha+\beta)a$ into $(*)$, we have $(\alpha-\beta)^2a^2=(\alpha+\beta)^2$. We shall choose a sign of $a$ such that $a=(\alpha+\beta)/(\alpha-\beta)$. Conjugating by a Möbius transformation having the same fixed points with that of $A$, which leaves $A$ invariant, we normalize $B$ such that $a=c$. Then we have

$$B=\begin{pmatrix} a & b \\ a & a \end{pmatrix}, \quad a=\frac{\alpha+\beta}{\alpha-\beta} \quad \text{and} \quad y=\frac{2(\alpha+\beta)}{\alpha-\beta}. \tag{3}$$

Under this normalization we remark the following.

**Theorem 7.** *If $G$ lies on the slice $S'$ determined by* (2), *then, under the normalization above, $\Omega(G)$ is symmetric with respect to the origin. Furthermore, there is a fundamental domain for $G$ which is symmetric with respect to the origin.*

Proof. By (3) we have $B^{-1}(-z)=-B(z)$. This implies that $B^{-n}(-z)=-B^n(z)$ for any $z$ and any integer $n$. We also have $A^n(-z)=-A^n(z)$. Let $M$ be an element of $G$ not being the identity. Then

$$M=A^{m_1}B^{n_1}A^{m_2}B^{n_2}\cdots A^{m_i}B^{n_i}$$

for some integers $m_j$, $n_j$ $(j=1,2,\cdots,i)$. Set

$$M^-=A^{m_1}B^{-n_1}A^{m_2}B^{-n_2}\cdots A^{m_i}B^{-n_i}.$$

By the successive use of the identities shown just above we obtain

$$M^-(-z)=-M(z).$$

Let $M$ be a loxodromic element of $G$ and let $z_1$ and $z_2$ be the fixed points of $M$. Then $M^-$ is also a loxodromic element of $G$ and equation $M^-(-z)=-M(z)$ implies that $-z_1$ and $-z_2$ are the fixed points of $M^-$. Since the limit set, $\Lambda(G)$, of $G$ is the closure of the fixed points of loxodromic elements of $G$, we obtain that $\Lambda(G)$ is symmetric with respect to the origin, so is the region of discontinuity, too. Since $G$ is a quasi-Fuchsian group of the first kind, there is a fundamental domain consisting of two pieces. Let $D^+$ be one of the two components of a fundamental domain for $G$ lying in a component of $G$. We put $D^-=\{z|-z\in D^+\}$. Now it is not difficult to see that $D^+\cup D^-$ is a funamantal domain for $G$. By construction, it is symmetric with respect to the origin. □

It is shown in [1] that each boundary group of any Bers slice has just one simply connected invariant component. Such a group is called a $b$-group. In contrast to the Bers slices we have the following.

**Theorem 8.** *None of the boundary groups of $S'$ is a b-group.*

Proof. Assume that $G$ is a group of $\bar{S}'\backslash S'$ and is a $b$-group, that is, $G$ has just one simply connected invariant component. Let $\Delta$ be the simply connected invariant component of $G$. Let $D^+$ be the piece of a fundamental domain for $G$ lying in $\Delta$. By Theorem 7 we see that the set $D^-=\{z|-z\in D^+\}$ is a subset of $\Omega(G)$. If $D^-$ lies in $\Delta$, then there is a curve $C\subset\Delta$ connecting a point $p\in D^+$ to $-p\in D^-$. Then the curve symmetric to $C$ with respect to the origin also lies in $\Delta$ and connects $p$ to $-p$. Then the closed curve $C\cup C^-\subset\Delta$ separates 0 from $\infty$. Since 0 and $\infty$ lie on $\Lambda(G)$, this contradicts our assumption that $\Delta$ is simply connected. Hence $D^-$ lies in another component, say $\Delta'$. By symmetry, $\Delta'$ is also an invariant component of $G$, a contradiction. □

## 3. Proof of Theorems 4 and 5

We recall a sufficient condition for $G=\langle A,B\rangle$ to be Kleinian.

**Theorem 9 ([4]).** *Let $G=\langle A,B\rangle$ be a subgroup of $PSL(2,C)$ generated by loxodromic elements $A=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$ and $B=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $bc\neq 0$ and satisfying (*). If, for each integer n, the inequality*

$$\frac{|\alpha^n a|+|\beta^n d|}{|\alpha^n a+\beta^n d|}<\frac{|\alpha|+|\beta|}{|\alpha-\beta|} \tag{4}$$

*holds, then G is quasi-Fuchsian and represents a pair of once punctured tori.*

REMARK. The moduli equation (*) is equvalent to the trace equation $tr(ABA^{-1}B^{-1})=-2$ (see, for example, [2] or Lemma 3 in [5]). Since (*) is symmetric with respect to $x$ and $y$ or since $tr(ABA^{-1}B^{-1})=tr(BAB^{-1}A^{-1})$, Theorem 9 is true even if we interchange the mormalizations of $A$ and $B$.

Under the moduli equation (2), Theorem 9 reduces to the following.

**Theorem 10.** *Let $G=\langle A,B\rangle$ be as in Theorem* 9 *and assume that G also* *satisfies* (2). *If, for each positive integer n, the inequality*

$$\frac{|\alpha^n|+|\beta^n|}{|\alpha^n+\beta^n|}<\frac{|\alpha|+|\beta|}{|\alpha-\beta|} \tag{5}$$

*holds, then G is quasi-Fuchsian.*

Proof. By Proposition in §2 we see that (4) reduces to (5). We shall show that (5) holds for each integer $n$ whenever (5) holds for each positive integer $n$. Since $\beta=1/\alpha$, we have $|\alpha^n|+|\beta^n|=|\alpha^{-n}|+|\beta^{-n}|$ and $|\alpha^n+\beta^n|=|\alpha^{-n}+\beta^{-n}|$. Hence (5) holds for each negative integer $n$ whenever it holds for each positive integer $n$. To show (5) holds for $n=0$ observe that, for $n=0$ and $n=1$, (5) is equivalent to $|\alpha-\beta|<|\alpha|+|\beta|$ and to $|\alpha-\beta|<|\alpha+\beta|$, respectively. Since $|\alpha+\beta|\leq|\alpha|+|\beta|$, we see that (5) holds for $n=0$ whenever it holds for $n=1$. Thus we have Theorem 10 by Theorem 9. □

Now, we shall check (5) of Theorem 10 for each positive integer $n$, that will give us proofs of Theorems 4 and 5. By Remark under Theorem 9 we may assume that $(x,y,z)\in E\cup e_1$, so $|x|\geq 2$ and $|y|>2$.

For $n=1$, (5) reduces to

$$|\alpha-\beta|<|\alpha+\beta|.$$

By (3) we see that $|y|=|2(\alpha+\beta)/(\alpha-\beta)|$ and so condition $|y|>2$ implies $|\alpha-\beta|<|\alpha+\beta|$. Hence (5) holds for $n=1$.

In order to treat the case $n\geq 2$, we shall put

$$u=r^2+r^{-2} \quad \text{and} \quad v=\cos 2\theta,$$

where $\alpha=re^{i\theta}$. By the inequalities $|\alpha-\beta|<|\alpha+\beta|$, which is shown just above, and $|x|=|\alpha+\beta|\geq 2$ we have

$$0<v\leq 1 \quad \text{and} \quad u\geq 4-2v. \tag{6}$$

For $n=2$, in polar coordinate $\alpha=re^{i\theta}$, (5) reduces to

$$\frac{r^4+r^{-4}+2}{r^4+r^{-4}+2\cos 4\theta}<\frac{r^2+r^{-2}+2}{r^2+r^{-2}-2\cos 2\theta}.$$

Using the equalities $r^4+r^{-4}=u^2-2$ and $\cos 4\theta=2v^2-1$, one shows that it is equivalent to

$$(u-1+v)^2-(1-v)^2-4(1-v)>0.$$

Inequalities (6) imply that the left hand side is not smaller than

$$(3+v)^2-(1-v)^2-4(1-v)$$

and that it is greater than 4, so (5) holds for $n=2$.

For $n=3$, in polar coordinate, (5) reduces to

$$\frac{r^6+r^{-6}+2}{r^6+r^{-6}+2\cos 6\theta}<\frac{r^2+r^{-2}+2}{r^2+r^{-2}-2\cos 2\theta}.$$

Making use of the equalities $r^6+r^{-6}=u^3-3u$ and $\cos 6\theta=4v^3-3v$, we have

$$\frac{u^3-3u+2}{u^3-3u+8v^3-6v}<\frac{u+2}{u-2v}.$$

A calculation shows that this is equivalent to

$$f(u,v)=(1+v)u^3-2(2+3v-2v^3)u+8v^3-4v>0. \tag{7}$$

Since

$$\frac{\partial f(u,v)}{\partial u}=3(1+v)u^2-2(2+3v-2v^3)>3u^2-10>0,$$

$f(u,v)$ is an increasing function of $u$. Hence by (6), in order to show (7), it suffices to show that $f(4-2v,v)>0$. A calculation shows that

$$f(4-2v,v)=4(2+(1-v)(10-3v-12v^2+4v^3)).$$

We put $g(v)=10-3v-12v^2+4v^3$. If $g(v)\geq 0$, then $f(4-2v,v)>0$. If $g(v)<0$, then we have

$$2+(1-v)g(v)\geq 2+g(v)=3(1-v^2)(4-v)+v^3>0.$$

Thus, in both cases we have $f(4-2v,v)>0$ so that (7) holds. Therefore we have shown that (5) holds for $n=3$.

For $n\geq 4$, in polar coodinate, (5) reduces to

$$\frac{r^{2n}+r^{-2n}+2}{r^{2n}+r^{-2n}+2\cos 2n\theta}<\frac{r^2+r^{-2}+2}{r^2+r^{-2}-2\cos 2\theta}=\frac{u+2}{u-2v}.$$

Since

$$\frac{r^{2n}+r^{-2n}+2}{r^{2n}+r^{-2n}+2\cos 2n\theta}\leq\frac{r^8+r^{-8}+2}{r^8+r^{-8}-2}=\frac{u^4-4u^2+4}{u^4-4u^2},$$

to show (5) it suffices to show

$$\frac{u^4-4u^2+4}{u^4-4u^2}<\frac{u+2}{u-2v}$$

or, equivalently,

(8) $$h(u,v)=(1+v)u^4-4(1+v)u^2-2u+4v>0.$$

Making use of inequalities $u>2$ and $v>0$, one obtains

$$\frac{\partial h(u,v)}{\partial u}=4(1+v)(u^2-2)u-2>14.$$

Hence $h(u,v)$ is an increasing function of $u$. There are two cases to consider. Case I: $v<1$. By (6) we see that, in order to show (8), it suffices to show $h(4-2v,v)>0$. A calculation shows that

$$h(4-2v,v)=8(1-v)(2(1+v)(2-v)(3-v)-1)>0.$$

Case II: $v=1$. Since $r>1$, we have $u>2$. Hence

$$h(u,1)=2(u-2)(u^3+2u^2-1)>0.$$

Thus we have shown (8). Hence (5) holds for $n\geq 4$.

Therefore we have shown that (5) holds for all positive integer $n$. Then Theorem 10 implies Theorems 4 and 5. □

## 4. Proof of Theorem 6

We shall prove the theorem in a sequence of lemmas. Let $(x,y,z)\in e_3$.

**Lemma 1.** $x=\bar{y}=\sqrt{3}\pm i$.

Proof. By (3) and $|x|=|y|=2$ we have $|\alpha-\beta|=2$. Hence we have $|\alpha+\beta|=|\alpha-\beta|=2$ or, in polar coordinate $\alpha=re^{i\theta}$,

$$r^2+r^{-2}+2\cos 2\theta=r^2+r^{-2}-2\cos 2\theta=4.$$

It follows that $\cos 2\theta=0$ and $r^2+r^{-2}=4$. We obtain $\theta=\pm\pi/4$ and $r=(\sqrt{3}+1)/\sqrt{2}$. Hence

$$\alpha=\frac{\sqrt{3}+1}{2}(1\pm i)\quad\text{and}\quad\beta=\frac{\sqrt{3}-1}{2}(1\mp i).$$

Therefore we have $x=\alpha+\beta=\sqrt{3}\pm i$ and, by (3), $y=\sqrt{3}\mp i$. □

We choose the sign such that $x=\bar{y}=\sqrt{3}+i$. The proof for the case $x=\bar{y}=\sqrt{3}-i$ is similar. By (3) we see that $a=(\sqrt{3}-i)/2$ and $b=-i$. Hence we have

$$A=\begin{pmatrix} \frac{(\sqrt{3}+1)(1+i)}{2} & 0 \\ 0 & \frac{(\sqrt{3}-1)(1-i)}{2} \end{pmatrix} \text{ and } \quad B=\begin{pmatrix} \frac{\sqrt{3}-i}{2} & -i \\ \frac{\sqrt{3}-i}{2} & \frac{\sqrt{3}-i}{2} \end{pmatrix}.$$

**Lemma 2.** *Let* $F=\langle AB,BA\rangle$ *and* $C=\{z\in \mathbf{C} \mid |z-(\sqrt{3}+3i)/2|=\sqrt{2}\}$. *Then* $F$ *is a Fuchsian group of the first kind with* $C$ *as the invariant circle.*

Proof. Since

$$AB=\begin{pmatrix} \frac{2+\sqrt{3}+i}{2} & \frac{(\sqrt{3}+1)(1-i)}{2} \\ \frac{2-\sqrt{3}-i}{2} & \frac{2-\sqrt{3}-i}{2} \end{pmatrix}$$

and

$$BA=\begin{pmatrix} \frac{2+\sqrt{3}+i}{2} & \frac{(\sqrt{3}-1)(1+i)}{2} \\ \frac{2+\sqrt{3}+i}{2} & \frac{2-\sqrt{3}-i}{2} \end{pmatrix},$$

putting

$$T=\begin{pmatrix} (\sqrt{3}+1)(1-i) & -\sqrt{3}+i \\ 4 & 2(1-(2+\sqrt{3})i) \end{pmatrix},$$

we calculate and obtain

$$TABT^{-1}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad TBAT^{-1}=\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}.$$

It is well known and is also easy to see that the group $TFT^{-1}$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ is a Fuchsian group of the first kind with the extended real axis as the invariant circle. A calculation shows that $T(C)$ is identical with the extended real axis. □

**Lemma 3.** *G is not quasi-Fuchsian.*

Proof. Since $F$ is a subgroup of $G$ so that $C \subset \Lambda(G)$, since 0 is a fixed point of the loxodromic element $A$ so that $0 \in \Lambda(G)$, and since 0 does not lie on $C$, it is clear that $G$ is not quasi-Fuchsian. □

This lemma tells us that $G$ does not lie in $\boldsymbol{D}(G_0)$. Theorem 4 tells us that $G \in \bar{E} \subset \bar{S}' \subset \bar{\boldsymbol{D}}(G_0)$ so that $G$ lies on $\bar{\boldsymbol{D}}(G_0)$. These two facts imply Theorem 6.

REMARK. An argument similar to one in [6] will show that $G$ is a web group with two symmetric non-equvalent components; one is bounded by $C$ and the other is symmetric to it with respect to the origin.

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