Nakano, T. and Tamai, K.
Osaka J. Math.
33 (1996), 347–364

ON SOME MAXIMAL GALOIS COVERINGS OVER AFFINE AND PROJECTIVE PLANES

TETSUO NAKANO and KEN-ICHI TAMAI

(Received March 26, 1995)

Introduction

In Namba [3; Capter 1], various examples of Galois coverings over affine and projective planes are constructed. Among them, the Galois coverings over $\boldsymbol{C}^2$ with branch locus $B_3:=\{(v,w)\in \boldsymbol{C}^2|v^3=w^2\}$ are studied in detail ([3; pp.43–52]), and as an application, the existence or non-existence of some maximal Galois coverings over $\boldsymbol{P}^2$ with branch locus $\overline{B_3}\cup l_\infty$ is shown, where $\overline{B_3}$ is the projective closure of B_3 and l_∞ is the infinite line ([3; Proposition 1.3.27, 1.3.29]).

In this note, we extend his results to the Galois coverings over $\boldsymbol{C}^2$ with branch locus $B_q:=\{(v,w)\in \boldsymbol{C}^2|v^q=w^2\}$, where q is a positive odd integer, under the condition that the maximal Galois group $G(\boldsymbol{C}^2,eB_q)$ of $(\boldsymbol{C}^2,eB_q)$ is finite. It turns out that we have five cases in all, three casess of which appear in [3; p.43]. As an application, we determine when there exists the maximal Galois coverings over $\boldsymbol{P}^2$ with branch locus $\overline{B_q}\cup l_\infty$, and also describe the explicit structure of $G(\boldsymbol{P}^2,e\overline{B_q}+ml_\infty)$ in these cases.

This note is organized as follows. In Section 1, we review some general facts from the Galois theory of branched coverings. We begin Section 2 with giving a simple presentation of $G(\boldsymbol{C}^2,eB_q)$ in Proposition 2.1, from which we can construct the maximal abelian coverings of $(\boldsymbol{C}^2,eB_q)$ easily (Proposition 2.3). Using this presentation, we determine when $G(\boldsymbol{C}^2,eB_q)$ is finite in Theorem 2.4 according to Coxeter-Moser [1]. Then we give an explicit structure of $G(\boldsymbol{C}^2,eB_q)$ in the cases where $G(\boldsymbol{C}^2,eB_q)$ is finite, which is our maim result (Theorem 2.6). When $G(\boldsymbol{C}^2,eB_q)$ is infinite, we give a sufficient condition for $G(\boldsymbol{C}^2,eB_q)$ to be unsolvable (Corollary 2.10). In Section 3, we describe the explicit structure of the maximal Galois group $G(\boldsymbol{P}^2,e\overline{B_q}+ml_\infty)$ and determine when the maximal Galois covering of $(\boldsymbol{P}^2,e\overline{B_q}+ml_\infty)$ exists in the cases where $G(\boldsymbol{C}^2,eB_q)$ is finite (Proposition 3.1, Corollary 3.3).

We note that the isomorphisms given in Theorem 2.6 are more or less known in abstract form (cf. Coxeter-Moser [1;6.7], Namba [3;p.50]) and our contribution is the explicit description of these isomorphisms, which is essentially used in Section 3.

The case where q is even seems more complicated, and will be studied in the forthcoming paper under the same title. We note that part of this note is taken from the master thesis Tamai [6].

NOTATIONS. (1) For a group G and $g_i \in G$ $(1 \leq i \leq n)$, we denote by $\langle g_1, g_2, \cdots, g_n \rangle$ the subgroup of G generated by g_i's and by $N\langle g_1, g_2, \cdots, g_n \rangle$ the smallest normal subgroup containing g_i's. $\boldsymbol{Z}[G]$ denotes the group ring of G over the ring of integers $\boldsymbol{Z}$. For a ring A, $M_n(A)$ is the set of all square matrices of degree n whose entries belong to A, and $I_n \in M_n(A)$ denotes the identity matrix.
(2) Given two groups H, N, we denote by $N \underset{\alpha}{\rtimes} H$ the semi-direct product of H and N. Here, $\alpha: H \to \mathrm{Aut}(N)$ is a homomorphism and the product is defined by $(h_1 \cdot n_1)(h_2 \cdot n_2) := h_1 h_2 \cdot n_1^{h_2} n_2$, where $h_i \in H$, $n_i \in N$, $n_1^{h_2} = \alpha(h_2)(n_1)$.
(3) For a pair (M, D) of a complex manifold M and an effective divisor D on M, we denote by $G(M, D)$ the maximal Galois group of (M, D) and by $\pi: X(M, D) \to M$ the maximal Galois covering of (M, D) if it exists (see Section 1 for the definitions).

1. Summary of the Galois theory of branched coverings

In this section, we summarize some general facts on the Galois theory of branched coverings in the category of complex analytic spaces, following Namba [3; Chapter 1].

Let M be a complex manifold and $D = \sum_{i=1}^{s} e_i D_i$ $(e_i > 0)$ an effective divisor on M, where D_i is an irreducible component of D. We set $B := C_1 \cup D_2 \cup \cdots \cup D_s$. Let $\pi: X \to \mathrm{M}$ be a branched covering over M, where X is a normal irreducible reduced complex analytic space, $R_\pi \subset X$ the ramification locus of π, and $B_\pi \subset M$ the branch locus of π. For an irreducible hypersurface $C \subset X$, we denote by $e_C(\pi)$ the ramification index of π at C (cf. Namba [3; p.10]). We say that π branches at D (resp. at most at D) if the following three conditions are satisfied:
(1) $B_\pi = B$ (resp. $B_\pi \subset B$).
(2) $R_\pi = \pi^{-1}(B)$ (resp. $R_\pi \subset \pi^{-1}(B)$).
(3) For any irreducible hypersurface $C \subset X$ such that $\pi(C) = D_j$, $e_C(\pi) = e_j$ (resp. $e_C(\pi) | e_j$).

We fix a base point $p_0 \in M - B$ and take a point $p_j \in D_j - \mathrm{Sing}(B)$, where $\mathrm{Sing}(B)$ is the singular locus of B. Take a local coordinate system $(z_1, z_2, \cdots, z_n)$ defined on a neighborhood U of p_j such that (1) p_j corresponds to the origin, (2) $B \cap U$ is given by $z_n = 0$. Take a loop δ_j around D_j in U defined by $\{(0, 0, \cdots, 0, \varepsilon e^{2\pi\sqrt{-1}t}) \in U \mid 0 \leq t \leq 1\}$, where $\varepsilon > 0$ is sufficiently small, and take a path ω_j in $M - B$ from p_0 to $q_j = (0, 0, \cdots, \varepsilon) \in U$. We define $\gamma_j = \omega_j^{-1} \delta_j \omega_j$, which is a loop around D_j starting from p_0.

Let J be the smallest normal subgroup $N\langle \gamma_1^{e_1}, \cdots, \gamma_s^{e_s} \rangle$ in $\pi_1(M - B, p_0)$ containing $\gamma_j^{e_j}$ $(1 \leq j \leq s)$, which is determined independent of the choices of γ_j's (we confuse a loop γ_j with its homotopy class in $\pi_1(M - B, p_0)$). We set $G(M, D) := \pi_1(M - B, p_0)/J$

and call this the maximal Galois group of (M,D) in this note. Then we have a Galois correspondence of the following type:

Theorem 1.1 (cf. Namba [3; Theorem 1.3.9]). (1) *There is a bijective map* Φ *from the set* $\{f: X \to M | f$ *is a finite Galois covering which branches at most at* $D\}/\simeq$ *to the set* $\{K \subset G(M,D) | K$ *is a normal subgroup of finite index*$\}$, *where* $\simeq$ *means the isomorphism between branched coverings over* M. $\Phi(f)$ *is defined by* $\Phi(f) = f_*(\pi_1(X - f^{-1}(B), q_0)) \bmod J$, *where* $q_0 \in X - f^{-1}(B)$ *is a base point over* p_0 *and* $f_*: \pi_1(X - f^{-1}(B), q_0) \to \pi_1(M - B, p_0)$ *is the injective homomorphism induced by* f.

(2) *This correspondence* Φ *satisfies the following properties*:

(a) $G_f \simeq G(M,D)/\Phi(f)$, *where* G_f *denotes the covering transformation group of* f.

(b) f_1 *dominates* f_2 *if and only if* $\Phi(f_1) \subset \Phi(f_2)$. *Here we say that a branched covering* $f_1: X_1 \to M$ *dominates another covering* $f_2: X_2 \to M$ *if there exixts a surjective holomorphic map* $g: X_1 \to X_2$ *such that* $f_2 \circ g = f_1$.

(3) f *branches at* D *if and only if the order of* $[\gamma_j]$ *is* e_j $(1 \leq j \leq s)$, *where* $[\gamma_j] \in G(M,D)/\Phi(f)$ *denotes the coset containing* γ_j.

We call the universal Galois covering among the branched coverings which branch at most at D the maximal Galois covering of (M,D) if it exists. More precisely,

DEFINITION 1.2. Let $\pi: X \to M$ be a Galois covering which branches at D. We say that π is the maximal Galois covering of (M,D) if π dominates any branched covering which branches at most at D.

Our next task is to give a criterion for the existence of the maximal Galois covering in terms of $G(M,D)$. Consider a point $p \in \mathrm{Sing}(B)$ and take a sufficiently small neighborhood W of p in M which is an open ball with respect to a local coordinate system with center p. Let $i_*: \pi_1(W - (W \cap B), p_0') \to \pi_1(M - B, p_0)$ be the homomorphism induced by the inclusion $i: W - (W \cap B) \to M - B$, where $p_0' \in W - (W \cap B)$ is a base point, and let $g: \pi_1(M - B, p_0) \to G(M,D)$ be the natural surjection. We have a composition map $g \circ i_*: \pi_1(W - (W \cap B), p_0') \to G(M,D)$. Consider the following condition on a subgroup $K \subset G(M,D)$:

CONDITION 1.3. For any point $p \in \mathrm{Sing}(B)$, $(g \circ i_*)^{-1}(K)$ is of finite index in $\pi_1(W - (W \cap B), p_0')$.

We set $\tilde{K} := \cap K$, where K runs over all the subgroups of $G(M,D)$ satisfying Condition 1.3. $\tilde{K}$ is a normal subgroup of $G(M,D)$.

Theorem 1.4 (cf. Namba [3; Theorem 1.3.10]). *There exists the maximal*

Galois covering $\pi: X(M,D) \to M$ of (M,D) if and only if the following two conditions are satisfied:
(1) $\tilde{K}$ *satisfies Condition* 1.3.
(2) $ord([\gamma_j]) = e_j$ $(1 \leq j \leq s)$, *where* $[\gamma_j] \in G(M,D)/\tilde{K}$ *means the coset containing* γ_j.
In this case, we have the following:
(a) $G_\pi \simeq G(M,D)/\tilde{K}$.
(b) $X(M,D)$ *is simply-connected.*

Asuume that $G(M,D)$ is finite. Then we have the following corollary to Theorem 1.4, since Condition 1.3 is satisfied for any subgroup of G and hence $\tilde{K} = \{1\}$ in this case.

Corollary 1.5. *If $G(M,D)$ is finite, then there exists the maximal Galois covering $\pi: X(M,D) \to M$ of (M,D) if and only if $ord([\gamma_j]) = e_j$ $(1 \leq j \leq s)$, where $[\gamma_j] \in G(M,D)$ is the coset containing γ_j. In this case, $G_\pi \simeq G(M,D)$.*

2. Calculation of $G(C^2, eB_q)$

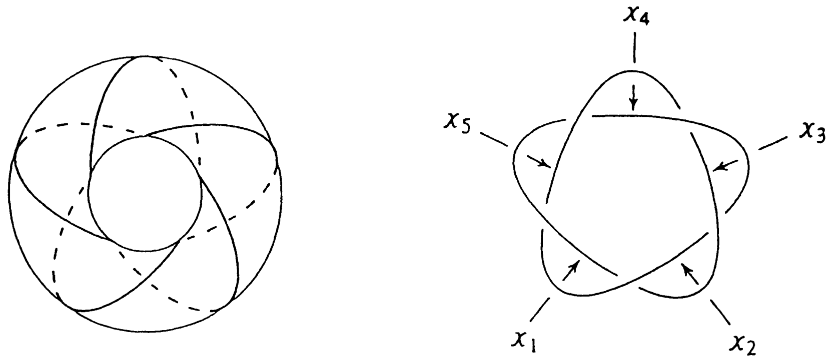
For an integer $q > 0$, we set $B_q = \{(v,w) \in C^2 | w^2 = v^q\}$. Suppose that q is odd. Let γ be a loop around B_q in $C^2 - B_q$ as in Section 1 and we define $G(e;q) := :G(C^2, eB_q) = \pi_1(C^2 - B_q,\ p_0)/N\langle\gamma^e\rangle$. Suppose that $q = 2r$ is even. We set $B_q^1 = \{(v,w) \in C^2 | w = v^r\}$ and $B_q^2 = \{(v,w) \in C^2 | w = -v^r\}$ so that $B_q = B_q^1 \cup B_q^2$. Let γ_i be a loop around B_q^i $(i = 1,2)$ and we set $G(e_1, e_2,; q) := G(C^2, e_1 B_q^1 + e_2 B_q^2) = \pi_1(C^2 - B_q,\ p_0)/N\langle\gamma_1^{e_1}, \gamma_2^{e_2}\rangle$. Then $G(e;q)$ and $G(e_1,e_2;q)$ have the following simple presentations.

Proposition 2.1.

$$
\begin{cases}
G(e;q) \simeq \langle a,b | a^e = 1,\ \underbrace{abab\cdots a}_{q} = \underbrace{baba\cdots b}_{q}\rangle & \text{if } q \text{ is odd} \\
G(e_1,e_2;q) \simeq \langle a,b | a^{e_1} = b^{e_2} = 1,\ \underbrace{abab\cdots ab}_{q} = \underbrace{baba\cdots ba}_{q}\rangle & \text{if } q \text{ is even.}
\end{cases}
$$

Proof. Put $S^3 := \{(v,w) \in C^2 |\ |v|^2 + |w|^2 = 1\}$, $T^2 := \{(v,w) \in C^2 |\ |v| = |w| = \dfrac{1}{\sqrt{2}}\}$, and $k(2,q) := \left\{\left(\dfrac{1}{\sqrt{2}} e^{4\pi i s}, \dfrac{1}{\sqrt{2}} e^{2\pi i s q}\right) \in C^2 |\ 0 \leq s \leq 1\right\}$ $(i = \sqrt{-1})$. Then $k(2,q) \subset T^2$ is the torus knot (or link) of type $(2,q)$. Let $C(k(2,q)) = \{(tv, tw) \in C^2 | t \geq 0,\ (v,w) \in k(2,q)\}$ be the cone over $k(2,q)$. Since (C^2, B_q) is homeomorphic to $(C^2, C(k(2,q)))$, it follows that $\pi_1(C^2 - B_q)$ is isomorphic to $\pi_1(S^3 - k(2,q))$ (we omit the base point of the fundamental group). Now, we take the Wirtinger generators $x_1, x_2, \cdots, x_q$ as in the

$k(2,5)$



Figure

figure above and obtain the Wirtinger presentation of the knot (or link) group $\pi_1(S^3-k(2,q))$ as follows (cf. Stillwell [5;4.2.3]):

$$\pi_1(S^3-k(2,q))\simeq\langle x_1,x_2,\cdots,x_q|x_1x_q=x_2x_1=x_3x_2=\cdots=x_qx_{q-1}\rangle.$$

From this presentation, we eliminate $x_3,\cdots,x_q$ and get the following presentations:

$$\pi_1(S^3-k(2,q))\simeq\begin{cases}\langle x_1,x_2|\underbrace{x_1x_2\cdots x_2x_1}_{q}=\underbrace{x_2x_1\cdots x_1x_2}_{q}\rangle & (q\text{: odd})\\[2ex] \langle x_1,x_2|\underbrace{x_1x_2\cdots x_1x_2}_{q}=\underbrace{x_2x_1\cdots x_2x_1}_{q}\rangle & (q\text{: even})\end{cases}$$

If q is odd, then we can take x_1 as γ. If q is even, we can take x_i as γ_i $(i=1,2)$. Hence $\pi_1(C^2-B_q)/N\langle\gamma^e\rangle$ (or $\pi_1(C^2-B_q)/N\langle\gamma_1^{e_1},\gamma_2^{e_2}\rangle$) has the desired presentation. $\square$

REMARK 2.2. The groups that have the same presentations as in Proposition 2.1 appear in Coxeter-Moser [1; 6.7], in which $G(e_1,e_2;q)$ is denoted by $e_1[q]e_2$ (q: even) and $G(e;q)$ by $e[q]e$ (q: odd). These groups also occur in the theory of regular complex polygons.

We recall that an abelian covering $\pi:X\to M$ of a complex manifold M which branches at D is called maximal if π dominates any abelian covering of M which branches at most at D. The maximal abelian covering of C^2 which branches at eB_q (q: odd) or eB_q (q: even) can be obtained easily as follows.

Proposition 2.3. (1) *Assume that q is odd. Set* $X:=\{(u,v,w)\in C^3|u^e+v^q-w^2$

$=0\}$ *and define* $\pi: X \to \boldsymbol{C}^2$ *by* $\pi((u,v,w))=(v,w)$. *Then* $\pi: X \to \boldsymbol{C}^2$ *is the maximal abelian covering of* $\boldsymbol{C}^2$ *which branches at* eB_q. G_π *is isomorphic to* $\boldsymbol{Z}/e\boldsymbol{Z}$.
(2) *Assume that* $q=2r$ *is even. For a pair of positive integers* e_1,e_2, *set* $Y:=\{(u_1,u_2,v)\in \boldsymbol{C}^3 | u_1^{e_1}+2v^r-u_2^{e_2}=0\}$ *and define* $\rho: Y \to \boldsymbol{C}^2$ *by* $\rho((u_1,u_2,v)):=(v,u_1^{e_1}+v^r)$. *Then* $\rho: Y \to \boldsymbol{C}^2$ *is the maximal abelian covering of* $\boldsymbol{C}^2$ *which branches at* $e_1B_q^1+e_2B_q^2$. G_ρ *is isomorphic to* $\boldsymbol{Z}/e_1\boldsymbol{Z}\oplus\boldsymbol{Z}/e_2\boldsymbol{Z}$.

Proof. (1) X is a normal irreducible surface, and it is easy to see that $\pi: X \to \boldsymbol{C}^2$ is a cyclic covering of degree e which branches at eB_q. We show that $\pi: X \to \boldsymbol{C}^2$ is the maximal abelian covering which branches at eB_q. Let $\mu: Z \to \boldsymbol{C}^2$ be any abelian covering which branches at most at eB_q. By Theorem 1.1, it is enough to show $\Phi(\pi)\subset\Phi(\mu)$. For a group G, we set $G^{ab}:=G/G'$, where G' is the commutator subgroup of G. Since

$$\begin{aligned} G(e;q)^{ab} &\simeq \langle a,b | a^e=1, \underbrace{ab\cdots aba}_{q}=\underbrace{ba\cdots bab}_{q}, ab=ba\rangle \\ &\simeq \boldsymbol{Z}/e\boldsymbol{Z}, \end{aligned}$$

the index of $G(e;q)'$ in $G(e;q)$ is e. Since $G(e;q)/\Phi(\pi)\simeq G_\pi\simeq\boldsymbol{Z}/e\boldsymbol{Z}$, we conclude $\Phi(\pi)=G(e;q)'$. Now, $G(e;q)/\Phi(\mu)\simeq G_\mu$ is abelian and hence $\Phi(\mu)\supset G(e;q)'=\Phi(\pi)$.
(2) Set $X_1:=\{(u_1,v,w)\in\boldsymbol{C}^3 | u_1^{e_1}=w-v^r\}$, $X_2:=\{(u_2,v,w)\in\boldsymbol{C}^2 | u_2^{e_2}=w+v^r\}$, and define $\pi_i: X_i\to\boldsymbol{C}^2$ by $\pi_i((u_i,v,w))=(v,w)$ $(i=1,2)$. π_i is a cyclic covering of degree e_i over $\boldsymbol{C}^2$. We form the fibred product $X_1\times_{\boldsymbol{C}^2}X_2$ of π_1 and π_2, which is isomorphic to Y over $\boldsymbol{C}^2$. Y is an abelian covering over $\boldsymbol{C}^2$ which branches at $e_1B_q^1+e_2B_q^2$ with Galois group isomorphic to $\boldsymbol{Z}/e_1\boldsymbol{Z}\oplus\boldsymbol{Z}/e_2\boldsymbol{Z}$. On the other hand, it is easy to see $G(e_1,e_2;q)^{ab}\simeq\boldsymbol{Z}/e_1\boldsymbol{Z}\oplus\boldsymbol{Z}/e_2\boldsymbol{Z}$, from which it follows that Y is the maximal abelian covering which branches at $e_1B_q^1+e_2B_q^1$ as in (1). $\square$

In the rest of this note, we are concerned with the explicit structure of $G(e;q)$, assuming q is odd. The following theorem is essentially due to Coxeter-Moser [1].

Theorem 2.4. *Let* $e\geq 2$ *and* $q\geq 3$ *be integers with* q *odd. Then* $G(e;q)$ *is a finite group if and only if* $e=2$ *or* $(e,q)=(3,3)$, $(4,3)$, $(5,3)$, $(3,5)$.

We need a lemma for the proof of Theorem 2.4.

Lemma 2.5 (cf. Coxeter-Moser[1;p.79]). *$G(2,e;2q)$ contains a subgroup of index* 2 *which is isomorphic to* $G(e;q)$.

Proof of Lemma (2.5). By adjoining a new letter $c:=aba$ to $G(2,e;2q)$

$=\langle a,b|a^2=b^e=1, (ab)^q=(ba)^q\rangle$, we have $G(2,e;2q)\simeq\langle a,b,c|b^e=1, \underbrace{bcb\cdots b}_{q}=\underbrace{cbc\cdots c}_{q},$ $a^2=1, aba=c\rangle$. Let $H:=<b,c>$ be a subgroup of $G=G(2,e;2q)$ generated by b and c. Then the index $[G:H]$ of H in G is 2. Indeed, any word $w=w(a,b,c)\in G$ can be rewritten as $w=\varphi(b,c)$ or $a\varphi(b,c)$, where $\varphi(b,c)$ is word not containing a, since $a^2=1$, $ab=ca$ and $ba=ac$. Hence we have $[G:H]\leq 2$. Assume $a\in H$. We have $G(2,e;2q)\simeq F<a,b,c>/N<Y>$, where $F<a,b,c>$ is a free group generated by $\{a,b,c\}$ and $Y=\{b^e, \underbrace{bcb\cdots b}_{q}\underbrace{c^{-1}b^{-1}c^{-1}\cdots c^{-1}}_{q}, a^2, abac^{-1}\}$ is the relation set. Then $a=\eta(b,c)\lambda$ in $F<a,b,c>$, where λ is a finite product of conjugates of words or inverses of words in Y. This gives a contradiction since the sum of the exponents of a in λ is even. Therefore $a\notin H$ and $[G:H]=2$.

Next, we calculate a presentation of H according to Johnson [2;Chapter 9]. Let $X=\{a,b,c\}$ be a generator set of G and $U=\{1,a\}$ a Schreier transversal for H in G. The $B\hat{R}$-table is given as follows:

Table 1

	b	c	a	b^e	$bcb\cdots bc^{-1}\cdots c^{-1}$	a^2	$abac^{-1}$
1	b	c	1	b^e	$bcb\cdots bc^{-1}\cdots c^{-1}$	a^2	$abac^{-1}$
a	aba^{-1}	aca^{-1}	a^2	ab^ea^{-1}	$abcb\cdots bc^{-1}\cdots c^{-1}a^{-1}$	a^2	$a^2bac^{-1}b^{-1}$

Here the rows are indexed by U and the columns by (X,Y). In the left-hand half of the table, the (u,x)-entry is $ux\overline{ux}^{-1}$, where $\overline{ux}\in U$ is the element which belongs to the same coset modulo H as ux, and in the right-hand half of the table, the (u,y)-entry is uyu^{-1}. Hence the $B\hat{S}$-table is given as follows:

Table 2

d_1	d_3	-	d_1^e	$d_1d_3\cdots d_1d_3^{-1}\cdots d_3^{-1}$	d_5	$d_2d_5d_3^{-1}$
d_2	d_4	d_5	d_2^e	$d_2d_4\cdots d_2d_4^{-1}\cdots d_4^{-1}$	d_5	$d_5d_1d_4^{-1}$

Here $d_1=b$, $d_2=aba^{-1}$, $d_3=c$ etc., and the elements in the right-hand half of the $B\hat{S}$-table are those in the $B\hat{R}$-table rewritten in terms of d_i's. It follows that H is presented as $\langle d_i\ (1\leq i\leq 5)\mid$ eight relations in the $B\hat{S}$-table$\rangle$. By eliminating d_2, d_4 and d_5, we have $H\simeq\langle d_1, d_3|d_1^e=d_3^e=1, \underbrace{d_1d_3d_1\cdots d_1}_{q}=\underbrace{d_3d_1d_3\cdots d_3}_{q}\rangle$.

Since $Q^{-1}d_1Q=d_3$, where $Q=\underbrace{d_1d_3d_1\cdots d_1}_{q}=\underbrace{d_3d_1d_3\cdots d_3}_{q}$, we conclude $H\simeq G(e;q)$. □

Proof of Theorem 2.4. Assume that $G(e;q)$ is finite. By Lemma 2.5, $G(2,e;2q)$ is also finite. We recall that the polyhedral group $P(x,y,z):=\langle a,b|a^x=b^y=(ab)^z=1\rangle$ $(x,y,z\geq 2)$ is finite if and only if $(x,y,z)=(2,2,z)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ and their permutations (cf. Coxeter-Moser [1;6.4]). Since $P(2,e,q)$ is a homomorphic image of $G(2,e;2q)$, we have $(e,q)=(2,q)$, $(3,3)$, $(4,3)$, $(5,3)$, $(3,5)$, where q is odd and ≥ 3. The converse part of the proof follows from Theorem 2.6 below, or Coxeter-Moser [1;p,79]. □

The following theorem is the main ingredient of this note. We note that the isomorphisms φ_2, φ_4 and φ_1 for $q=3$ in the theorem are given in Namba [3; p.50] in abstract form, and φ_5 is found in Coxeter-Moser [1;p.78]. We make a detailed calculation of group presentations for this theorem since it gives the explicit form of φ_i, which is essentially used in section 3.

We denote by C_n the cyclic group of order n, by $D_{2q}:=\langle x,y|x^q=y^2=1, y^{-1}xy=x^{-1}\rangle$ the dihedral group of order $2q$, by $Q_8:=\langle x,y|x^2=y^2, x^4=1, y^{-1}xy=x^{-1}\rangle$ the quaternion group of order 8, and by $\boldsymbol{SL}(2,\boldsymbol{Z}_n)$ the special linear group of degree 2 whose entries belong to $\boldsymbol{Z}_n=\boldsymbol{Z}/n\boldsymbol{Z}$.

Theorem 2.6. *We have the following isomorphisms*:

(1) $\varphi_1: G(2;q)\simeq D_{2q}$; $\varphi_1(a)=y$, $\varphi_1(b)=y^{-1}x$.

(2) $\varphi_2: G(3;3)\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_3)$; $\varphi_2(a)=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, , $\varphi_2(b)=\begin{bmatrix}1&0\\2&1\end{bmatrix}$.

(3) $\varphi_3: G(4;3)\simeq (Q_8\underset{\alpha}{\rtimes} C_3)\underset{\beta}{\rtimes} C_4$; $\varphi_3(a)=s\cdot 1\cdot 1$, $\varphi_3(b)=s\cdot t\cdot x$ $(C_3=\langle t\rangle,\ c_4=\langle s\rangle)$.

$\psi_3: G(4;3)\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_3)\underset{\gamma}{\rtimes} C_4$; $\psi_3(a)=s\cdot I_2$, $\psi_3(b)=s\cdot\begin{bmatrix}0&1\\2&1\end{bmatrix}$ $(C_4=\langle s\rangle)$. *α, β, γ are described in the proof.*

(4) $\varphi_4: G(5;3)\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5)\times C_5$; $\varphi_4(a)=\left(\begin{bmatrix}4&1\\1&3\end{bmatrix}, w^3\right)$, $\varphi_4(b)=\left(\begin{bmatrix}1&4\\0&1\end{bmatrix}, w^3\right)(C_5=\langle w\rangle)$.

(5) $\varphi_5: G(3;5)\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5)\times C_3$; $\varphi_5(a)=\left(\begin{bmatrix}0&2\\2&4\end{bmatrix}, w^2\right)$, $\varphi_5(b)=\left(\begin{bmatrix}3&4\\3&1\end{bmatrix}, w^2\right)(C_3=\langle w\rangle)$.

Proof. In this proof, we set $G:=G(e;q)$ for short. (1) is clear from the definition of D_{2q}.

(2) Set $s=\begin{bmatrix}1&1\\0&1\end{bmatrix}$, $t=\begin{bmatrix}1&0\\2&1\end{bmatrix}\in \boldsymbol{SL}(2,\boldsymbol{Z}_3)$. Then we have $s^3=1$, sts$=$tst. Since $\{s,t\}$

generates $SL(2,Z_3)$ we have a surjective homomorphism $\varphi_2 : G \to SL(2,Z_3)$ defined by $\varphi_2(a)=s$, $\varphi_2(b)=t$. Since $\mathrm{ord}(SL(2,Z_3))=24$, it is enough to show $\mathrm{ord}(G)=24$.

Consider the following exact sequence

$$1 \to G' \to G \xrightarrow[f]{} C_3 \to 1,$$

where $f(a)=f(b)=w$, $C_3=\langle w\rangle$, and G' is the commutator subgroup of G. We take $U:=\{1,a,a^2\}$ as a Schreier transveral for G' in G. Then the $B\hat{R}$ and $B\hat{S}$ tables for G' are given as follows:

Table 3

	a	b	$abab^{-1}a^{-1}b^{-1}$	a^3
1	1	ba^{-1}	$abab^{-1}a^{-1}b^{-1}$	a^3
a	1	aba^{-2}	$a^2bab^{-1}a^{-1}b^{-1}a^{-1}$	a^3
a^2	a^3	a^2b	$a^3bab^{-1}a^{-1}b^{-1}a^{-2}$	a^3

Table 4

-	c_2	$c_3c_1c_4^{-1}c_2^{-1}$	c_1
-	c_3	$c_4c_2^{-1}c_1^{-1}c_3^{-1}$	c_1
c_1	c_4	$c_1c_2c_3^{-1}c_4^{-1}$	c_1

Hence $G' \simeq \langle c_i\ (1 \le i \le 4) \mid \text{six relations in the } B\hat{S}\text{-table}\rangle \simeq \langle c_2,c_3 \mid c_3=c_2c_3c_2,\ c_2=c_3c_2c_3\rangle$, which is isomorphic to Q_8 by the correspondence $c_2 \to x$, $c_3 \to y$. Thus $\mathrm{ord}(G')=8$ so that $\mathrm{ord}(G)=24$.

(3) We take a Schreier transversal $U:=\{1,a,a^2,a^3\}$ for G' in G. The $B\hat{R}$- and $B\hat{S}$-tables are given as follows:

Table 5

	a	b	$abab^{-1}a^{-1}b^{-1}$	a^4
1	1	ba^{-1}	$abab^{-1}a^{-1}b^{-1}$	a^4
a	1	aba^{-2}	$a^2bab^{-1}a^{-1}b^{-1}a^{-1}$	a^4
a^2	1	a^2ba^{-3}	$a^3bab^{-1}a^{-1}b^{-1}a^{-2}$	a^4
a^3	a^4	a^3b	$a^4bab^{-1}a^{-1}b^{-1}a^{-3}$	a^4

Table 6

-	c_2	$c_3c_4^{-1}c_2^{-1}$	c_1
-	c_3	$c_4c_1c_5^{-1}c_3^{-1}$	c_1
-	c_4	$c_5c_2^{-1}c_1^{-1}c_4^{-1}$	c_1
c_1	c_5	$c_1c_2c_3^{-1}c_5^{-1}$	c_1

Hence $G'\simeq\langle c_i\ (1\leq i\leq 5)|$ eight relations in the $B\hat{S}$-table$\rangle\simeq\langle c_2,c_4|\ c_2=c_4c_2^2c_4,$ $c_4=c_2c_4^2c_2\rangle$. Next, we take $U:=:\{1,c_2,c_2^2\}$ as a Schreier transversal for $(G')'$ in G' and the $B\hat{R}$- and $B\hat{S}$-tables are given as follows: (we set $c_2=p$, $c_4=q$ in the $B\hat{R}$-table)

Table 7

	p	q	pq^2pq^{-1}	qp^2qp^{-1}
1	1	qp^{-1}	pq^2pq^{-1}	qp^2qp^{-1}
p	q	pqp^{-2}	$p^2q^2pq^{-1}p^{-1}$	pqp^2qp^{-2}
p^2	p^3	p^2q	$p^3q^2pq^{-1}p^{-2}$	$p^2qp^2qp^{-3}$

Table 8

-	d_2	$d_3d_4d_2^{-1}$	$d_2d_1d_2$
-	d_3	$d_4d_2d_3^{-1}$	$d_3d_1d_3$
d_1	d_4	$d_1d_2d_3d_1d_4^{-1}$	$d_4^2d_1^{-1}$

Hence $(G')'\simeq\langle d_i\ (1\leq i\leq 4)\ |$ six relations in the $B\hat{S}$-table$\rangle\simeq\langle d_3,d_4\,|\,d_3^2d_4^2=1,$ $d_3=d_4d_3d_4\rangle$, which is isomorphic to Q_8 by the correspondence $d_3\to x$, $d_4\to y$. We have $c_2^3=d_4^2\in(G')'\simeq Q_8$ and hence $\mathrm{ord}(c_2)=6$. Set $H:=:\langle c_2^2\rangle\simeq C_3$. Then $G'=(G')'\rtimes H\simeq Q_8\rtimes_\alpha C_3$, where $\alpha: C_3=\langle t\rangle\to\mathrm{Aut}(Q_8)$ is given by $\alpha(t)(x)=x^t=yx^2$ and $\alpha(t)(y)=y^t=yx$ since $c_2^{-2}d_3c_2^2=d_4^{-1}$ and $c_2^{-2}d_4c_2^2=d_3d_4^3$. If we set $L:=\langle a\rangle\subset G$, then $L\simeq C_4$ and $G=G'\rtimes L\simeq(Q_8\rtimes_\alpha C_3)\rtimes_\beta C_4$, where $\beta: C_4=\langle s\rangle\to\mathrm{Aut}(Q_8\rtimes_\alpha C_3)$ is given by $t^s=t^2\cdot yx^3$, $x^s=x$, $y^s=yx$. $a\in G$ corresponds to $s\cdot 1\cdot 1\in(Q_8\rtimes C_3)\rtimes C_4$, and b to $s\cdot t\cdot x$ under this isomorphism.

Next, we show $\boldsymbol{SL}(2,\boldsymbol{Z}_3)\simeq Q_8\rtimes_\alpha C_3$. Set $X:=\begin{bmatrix}2&1\\1&1\end{bmatrix}$, $Y:=\begin{bmatrix}0&1\\2&0\end{bmatrix}$, $T:=\begin{bmatrix}2&2\\1&0\end{bmatrix}$ $\in\boldsymbol{SL}(2,\boldsymbol{Z}_3)$. Then we have $Y^2=X^2$, $X^4=1$, $Y^{-1}XY=X^{-1}$ and $\{X,Y\}$ generates a subgroup N isomorphic to Q_8. We also have $\mathrm{ord}(T)=3$ and set $M:=\langle T\rangle$. Then $SL(2,Z_3)=N \rtimes M \simeq Q_8 \rtimes_{\alpha} C_3$ since $T^{-1}XT=YX^2$, $T^{-1}YT=YX$. Hence $G\simeq SL(2,Z_3)\rtimes_{\gamma} C_4$, where $\gamma: C_4=\langle s\rangle \to \mathrm{Aut}(SL(2,Z_3))$ is given by $X^s=X$, $Y^s=YX$, $T^s=T^2YX^3$. $a\in G$ corresponds to $s\cdot I_2$ and b to $s\cdot\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}$.

(4) We have $G^{ab}\simeq C_5$ and take $U=\{1,a,a^2,a^3,a^4\}$ as a Schreier transversal for G' in G. Then the $B\hat{R}$- and $B\hat{S}$-tables are given as follows:

Table 9

	a	b	$abab^{-1}a^{-1}b^{-1}$	a^5
1	1	ba^{-1}	$abab^{-1}a^{-1}b^{-1}$	a^5
a	1	aba^{-2}	$a^2bab^{-1}a^{-1}b^{-1}a^{-1}$	a^5
a^2	1	a^2ba^{-3}	$a^3bab^{-1}a^{-1}b^{-1}a^{-2}$	a^5
a^3	1	a^3ba^{-4}	$a^4bab^{-1}a^{-1}b^{-1}a^{-3}$	a^5
a^4	a^5	a^4b	$a^5bab^{-1}a^{-1}b^{-1}a^{-4}$	a^5

Table 10

-	c_2	$c_3c_4^{-1}c_2^{-1}$	c_1
-	c_3	$c_4c_5^{-1}c_3^{-1}$	c_1
-	c_4	$c_5c_1c_6^{-1}c_4^{-1}$	c_1
-	c_5	$c_6c_2c_1^{-1}c_5^{-1}$	c_1
c_1	c_6	$c_1c_2c_3^{-1}c_6^{-1}$	c_1

Hence $G'\simeq\langle c_i\,(1\le i\le 6)\mid$ ten relations in the $B\hat{S}$-table$\rangle\simeq\langle c_2,c_4\mid c_4c_2^2c_4=c_2c_4c_2,\ c_4^{-1}c_2^{-1}c_4=c_2^{-1}c_4c_2\rangle$. Now, if we set $d_2=\begin{bmatrix}4 & 0\\ 4 & 4\end{bmatrix}$, $d_4=\begin{bmatrix}4 & 4\\ 0 & 4\end{bmatrix}\in SL(2,Z_5)$, then we have $d_4d_2^2d_4=d_2d_4d_2$ and $d_4^{-1}d_2^{-1}d_4=d_2^{-1}d_4d_2$. Since $\{d_2,d_4\}$ generates $SL(2,Z_5)$, we have a surjective homomorphism $F: G'\to SL(2,Z_5)$ defined by $F(c_i)=d_i$ $(i=2,4)$. By applying the same argument as in the case (5) in Remark 2.7 below, we find $G'\simeq SL(2,Z_5)$ and hence F is an isomorphism.

Next, we show $G\simeq SL(2,Z_5)\times C_5$. Set $x:=\mathrm{aba}\in G$. Then $x^{10}\in G'$ and we have $x^{10}=(ab)^{15}=(c_3c_5c_1c_2c_4c_6)^3=(c_4^2c_2)^3$ which corresponds to $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\in SL(2,Z_5)$. Hence we have $\mathrm{ord}(x)=20$. If we set $w:=x^4$, then $\mathrm{ord}(w)=5$ and w

belongs to the center of G. Since $\{w^i \mid 0 \leq i \leq 4\}$ is a transversal for G' in G, we conclude $G \simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_5$. We have $G' \ni w^2 a = (c_4{}^2 c_2)^2 c_4$ which corresponds to $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \in \boldsymbol{SL}(2,\boldsymbol{Z}_5)$. Hence $a \in G$ corresponds to $\left(\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}, w^3\right) \in \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_5$, where $C_5 = \langle w \rangle$. Similarly, $b \in G$ corresponds to $\left(\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, w^3\right) \in \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_5$.

(5) Similar to (4). We give some data for convenience. The $B\hat{R}$- and $B\hat{S}$-tables for G' in G are given as follows:

Table 11

	a	b	$ababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$	a^3	b^3
1	1	ba^{-1}	$ababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}$	a^3	b^3
a	1	aba^{-2}	$a^2babab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-1}$	a^3	ab^3a^{-1}
a^2	a^3	a^2b	$a^3babab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}a^{-2}$	a^3	$a^2b^3a^{-2}$

Table 12

-	c_2	$c_3c_1c_2c_3^{-1}c_4^{-1}c_2^{-1}$	c_1	$c_2c_3c_4$
-	c_3	$c_4c_3c_1c_4^{-1}c_2^{-1}c_1^{-1}c_3^{-1}$	c_1	$c_3c_4c_2$
c_1	c_4	$c_1c_2c_4c_2^{-1}c_1^{-1}c_3^{-1}c_4^{-1}$	c_1	$c_4c_2c_3$

$$\begin{aligned} G' &\simeq \langle c_i \ (1 \leq i \leq 4) \mid \text{nine relations in the } B\hat{S}\text{-table} \rangle \\ &\simeq \langle c_2, c_3 \mid c_2 c_3{}^2 c_2 = c_3 c_2 c_3,\ c_3 c_2{}^2 c_3 = c_2 c_3 c_2 \rangle \\ &\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5), \end{aligned}$$

where c_2 corresponds to $\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ and c_3 to $\begin{bmatrix} 4 & 0 \\ 4 & 4 \end{bmatrix}$. We have $G = G' \times \langle w \rangle \simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_3$, where $w = (ababa)^4 \in G$ (ord$(ababa) = 12$). $a \in G$ corresponds to $\left(\begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}, w^2\right) \in \boldsymbol{SL}(2,\boldsymbol{Z}_5)$ and $b \in G$ to $\left(\begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix}, w^2\right)$. □

REMARK 2.7. By Corollary 1.5, there exists the maximal Galois covering $\pi: X(\boldsymbol{C}^2, eB_q) \to \boldsymbol{C}^2$ in the five cases above. We have $X(\boldsymbol{C}^2, eB_q) \simeq \boldsymbol{C}^2$ in these cases. Since the cases (2),(3),(4) are studied in Namba [3;p.50], we briefly discuss the remaining two cases. In the case (1), the maximal Galois covering $\pi_1: \boldsymbol{C}^2 \to \boldsymbol{C}^2$

of $(C^2, 2B_q)$ is given by $(v,w)=\pi_1((s,t))=(st, (1/2)(s^q+t^q))$. Indeed, if we set $N=\{(x,y,z)\in C^3 | xy=z^q\}$ and $M=\{(u,v,w)\in C^3 | w^2-u^2=v^q\}$, then π_1 is decomposed as $\pi_1=f\circ g\circ h$, where $C^2 \xrightarrow{h} N \overset{\sim}{\underset{g}{\to}} M \xrightarrow{f} C^2$, and f, g, h are defined as follows: $h((s,t))=(s^q,t^q,st)$, $g((x,y,z))=((1/2)(x-y), z, (1/2)(x+y))$, and $f((u,v,w))=(v,w)$. Since h is unramified outside $(0,0,0)\in N$ and f branches at $2B_q$, we conclude that π_1 is a covering which branches at $2B_q$. It is easy to see that G_{π_1} is isomorphic to D_{2q} and generated by σ, $\tau\in\mathrm{Aut}(C^2)$, where $\sigma((s,t)):=(\zeta s,\zeta^{-1}t)$ $(\zeta:=e^{2\pi\sqrt{-1}/q})$ and $\tau((s,t)):=(t,s)$. Hence we conclude that π_1 is the maximal Galois covering of $(C^2, 2B_q)$. In the case (5), let $G: C^2 \to C^2/SL(2,Z_5)\simeq L:=\{(u,v,w)\in C^2 | u^3+v^5-w^2=0\}$ be the quotient map giving the binary icosahedral kleinian singularity (cf. Pinkham [4]), and define $F: L\to C^2$ by $F((u,v,w))=(v,w)$. Then $\pi_5:=F\circ G: C^2\to C^2$ is a covering which branches at $3B_5$. Since $C^2-\pi_5^{-1}((0,0))=C^2-\{(0,0)\}$ is simply-connected, π_5 is the maximal Galois covering of $(C^2,3B_5)$ by Namba [3; Corollary 1.3.12]. We also find that the maximal Galois group $G(C^2,3B_5)\simeq G_{\pi_5}$ is an extension of $SL(2,Z_5)$ by C_3 by this geometric argument without group-theoretic computation.

In the case where $G(e;q)$ is an infinite group, to determine whether $G(e;q)$ is solvable or not is a fundamental problem in the Galois theory of branched coverings. As for this, we have the following result:

Theorem 2.8. $G(e;q)'=(G(e;q)')'$ if and only if e is odd and $\mathrm{GCD}(e,q)=1$.

Proof. We set $G=G(e;q)$ and $G(e;q)'=N$ for short. According to Johnson [2; Chapter 12], let $F(X)$ be a free group generated by $X=\{S,T\}$ and $J:=\begin{bmatrix} \frac{\partial S^e}{\partial S} & \frac{\partial S^e}{\partial T} \\ \frac{\partial w}{\partial S} & \frac{\partial w}{\partial T}\end{bmatrix}\in M_2(Z[F(X)])$ be the jacobian of G, where $w=\underbrace{STS\cdots S}_{q}\underbrace{T^{-1}S^{-1}\cdots T^{-1}}_{q}\in F(X)$ and $\frac{\partial}{\partial S}, \frac{\partial}{\partial T}$ are the Fox derivations. Let $F(X)\xrightarrow[\phi]{} G\xrightarrow[\psi]{} G/N\simeq C_e=\langle s\rangle$ are the natural surjections and we denote by the same symbol the map $M_2(Z[F(X)])\xrightarrow[\phi]{} M_2(Z[G])\xrightarrow[\psi]{} M_2(Z[C_e])$ induced by them. We set $D:=\mu\circ\psi\circ\phi(J)\in M_{2e}(Z)$, where $\mu: Z[C_e]\to M_e(Z)$ (or $\mu: M_2(Z[C_e])\to M_{2e}(Z)$) is the blowing-up map (cf. Johnson [2;12.1]).

Now, we have

$$\psi\circ\phi(J)=\begin{bmatrix} e+s+s^2+\cdots+s^{e-1} & 0 \\ s^{-2}+s^{-4}+\cdots+s^{-(q-1)} & s^{-1}+s^{-3}+\cdots+s^{-q} \\ \quad -s^{-1}-s^{-3}-\cdots-s^{-q} & -s^{-2}-s^{-4}-\cdots-s^{-(q-1)}\end{bmatrix},$$

which is equivalent to

$$L:=\begin{bmatrix} e+s+s^2+\cdots+s^{e-1} & 0 \\ 0 & s^{-1}+s^{-3}+\cdots+s^{-q}-s^{-2}-s^{-4}-\cdots-s^{-(q-1)} \end{bmatrix}.$$

Here we say that two matrices are equivalent if they can be transformed to each other by elementary transformations.

We have $\mu\left(\sum_{i=0}^{e-1} s^i\right)=\begin{bmatrix} 1 & 1\cdots & 1 \\ 1 & 1\cdots & 1 \\ \vdots & & \\ 1 & 1\cdots & 1 \end{bmatrix}\in M_e(\boldsymbol{Z})$, which is equivalent to

$\begin{bmatrix} 1 & 0\cdots & 0 \\ 0 & 0\cdots & 0 \\ \vdots & & \\ 0 & 0\cdots & 0 \end{bmatrix}$. Since $N/N'\times \boldsymbol{Z}^{e-1}\simeq \boldsymbol{Z}^{2e}/\mathrm{Im}\ \mu(L)$ (Johnson [2;Proposition 1, p 161]), we conclude $N=N'$ if and only if e is odd and $\mathrm{GCD}(e,q)=1$ by the following lemma. □

Lemma 2.9. *For an odd integer $q\geq 3$, set $z:=\sum_{i=1}^{q}(-1)^{i-1}s^{-i}\in \boldsymbol{Z}[C_e]$, $C_e=\langle s\rangle$. Then z is a unit in $\boldsymbol{Z}[C_e]$ if and only if e is odd and* $\mathrm{GCD}(e,q)=1$.

Proof. We identify $\boldsymbol{Z}[C_e]$ with $A:=\boldsymbol{Z}[x]/(x^e-1)$, where s^{-1} corresponds to the coset $\bar{x}$ of x. Since $\bar{x}$ is a unit in A, we show $\bar{z}=\sum_{i=0}^{q-1}(-1)^i\bar{x}^i\in A$ is a unit if and only if e is odd and $\mathrm{GCD}(e,q)=1$. Now, assume e is even. If there exist $F(x)$, $G(x)\in \boldsymbol{Z}[x]$ such that $\left(\sum_{i=0}^{q-1}(-1)^ix^i\right)F(x)+(x^e-1)G(x)=1$ in $\boldsymbol{Z}[x]$, then we have $qF(-1)=1$ by setting $x=-1$, contradiction. Assume that e is odd and $\mathrm{GCD}(e,q)=1$. By replacing x with $-x$, we may assume that $\bar{z}=\sum_{i=0}^{q-1}\bar{x}^i\in B:=\boldsymbol{Z}[x]/(x^e+1)$. Since $\mathrm{GCD}(2e,q)=1$, there exist $P(x)$, $Q(x)\in \boldsymbol{Z}[x]$ such that $(x^q-1)P(x)+(x^{2e}-1)Q(x)=x-1$ by the Euclidian division algorithm. Thus $\left(\sum_{i=0}^{q-1}x^i\right)P(x)+(x^e+1)\left(\sum_{i=0}^{e-1}x^i\right)Q(x)=1$ and hence $\bar{z}\in B$ is a unit. Assume that e is odd and $\mathrm{GCD}(e,q)=d>1$. Since $\sum_{i=0}^{d-1}\bar{x}^i$ divides $\sum_{i=0}^{q-1}\bar{x}^i$ in B, it is enough to show $\sum_{i=0}^{d-1}\bar{x}^i$ is a non-unit in B. Suppose that there exist $f(x)$, $g(x) \in \boldsymbol{Z}[x]$ such that

$$f(x)(x^e+1)+g(x)\left(\sum_{i=0}^{d-1} x^i\right)=1 \cdots (1).$$

We have $x^e+1=\left(\sum_{i=0}^{d-1} x^i\right)h(x)+2$ for some $h(x) \in \boldsymbol{Z}[x]$ by a direct division. By subsitituting this to (1), we obtain

$$(h(x)f(x)+g(x))\left(\sum_{i=0}^{d-1} x^i\right)+2f(x)=1.$$

Then $\sum_{i=0}^{d-1} x^i$ is a unit in $\boldsymbol{Z}_2[x]$, contradiction. □

Corollary 2.10. *If (e,q) is not one of those in Theorem 2.4 and $\mathrm{GCD}(e,q)=1$ with e odd, then $G(e;q)$ is an infinite unsolvable group.*

3. Calculation of $G(\boldsymbol{P}^2, e\overline{B_q}+ml_\infty)$

Let $G[e,m;q]:=G(\boldsymbol{P}^2, e\overline{B_q}+ml_\infty)$ be the maximal Galois group of $(\boldsymbol{P}^2, e\overline{B_q}+ml_\infty)$, where $\overline{B_q}=\{(x_0,x_1,x_2)\in \boldsymbol{P}^2 |\ x_0^q=x_1^2x_2^{q-2}\}$ and $l_\infty=\{(x_0,x_1,x_2)\in \boldsymbol{P}^2 |\ x_2=0\}$ (the infinite line). Let δ be a loop around l_∞ in $\boldsymbol{P}^2-(\overline{B_q}\cup l_\infty)$ and $[\delta]\in \pi_1(\boldsymbol{P}^2-(\overline{B_q}\cup l_\infty))$ $=\pi_1(\boldsymbol{C}^2-B_q)$ the homotopy class of δ. It is easy to see $[\delta^{-1}]=[\underbrace{x_1x_2x_1\cdots x_1}_{q}]$ $=[\underbrace{x_2x_1x_2\cdots x_2}_{q}]$ (cf. Proposition 2.1), and hence we have $G[e,m;q]\simeq\langle c,a,b|$ $a^e=c^m=1,\ \underbrace{aba\cdots a}_{q}=\underbrace{bab\cdots b}_{q}=c^{-1}\rangle \simeq G(e;q)/N<Q^m>$ $(Q:=\underbrace{aba\cdots a}_{q}=\underbrace{bab\cdots b}_{q})$. The explicit structure of $G[e,m;q]$ for the (e,q)'s given in Theorem 2.4 is as follows:

Proposition 3.1.

(1) $G[2,m;q]\simeq\begin{cases} D_{2q} & \text{if } m \text{ is even} \\ \{1\} & \text{if } m \text{ is odd}\end{cases}.$

(2) $G[3,m;3]\simeq\begin{cases} C_3 & \text{if } m\equiv 1,3 \pmod 4 \\ \boldsymbol{PSL}(2,\boldsymbol{Z}_3) & \text{if } m\equiv 2 \pmod 4 \\ \boldsymbol{SL}(2,\boldsymbol{Z}_3) & \text{if } m\equiv 0 \pmod 4\end{cases}$

$$(3)\quad G[4,m;3] \simeq \begin{cases} \{1\} & \text{if } m \equiv 1,3,5,7 \pmod 8 \\ (Q_8 \underset{\alpha}{\rtimes} C_3) \underset{\beta}{\rtimes} C_4 / \langle s^2 \cdot 1 \cdot x \rangle & \text{if } m \equiv 2,6 \pmod 8 \\ ((C_2 \times C_2) \underset{\bar{\alpha}}{\rtimes} C_3) \underset{\bar{\beta}}{\rtimes} C_4 & \text{if } m \equiv 4 \pmod 8 \\ G(4;3) & \text{if } m \equiv 0 \pmod 8 \end{cases}$$

$$(4)\quad G[5,m;3] \simeq \begin{cases} \{1\} & \text{if } m \equiv 1,3,7,9,11,13,17,19 \pmod{20} \\ \boldsymbol{PSL}(2,\boldsymbol{Z}_5) & \text{if } m \equiv 2,6,14,18 \pmod{20} \\ \boldsymbol{SL}(2,\boldsymbol{Z}_5) & \text{if } m \equiv 4,8,12,16 \pmod{20} \\ C_5 & \text{if } m \equiv 5,15 \pmod{20} \\ \boldsymbol{PSL}(2,\boldsymbol{Z}_5) \times C_5 & \text{if } m \equiv 10 \pmod{20} \\ \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_5 & \text{if } m \equiv 0 \pmod{20} \end{cases}$$

$$(5)\quad G[3,m;5] \simeq \begin{cases} \{1\} & \text{if } m \equiv 1,5,7,11 \pmod{12} \\ \boldsymbol{PSL}(2,\boldsymbol{Z}_5) & \text{if } m \equiv 2,10 \pmod{12} \\ C_3 & \text{if } m \equiv 3,9 \pmod{12} \\ \boldsymbol{SL}(2,\boldsymbol{Z}_5) & \text{if } m \equiv 4,8 \pmod{12} \\ \boldsymbol{PSL}(2,\boldsymbol{Z}_5) \times C_3 & \text{if } m \equiv 6 \pmod{12} \\ \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_3 & \text{if } m \equiv 0 \pmod{12} \end{cases}$$

Proof. (1) We have $G[2,m;q] \simeq \langle b,d \mid b^2 = d^q = 1, b^{-1}db = d^{-1} \rangle / N\langle x^m \rangle$, where $x = bd^{(q-1)/2}$, by setting $d = ab$. Since $\operatorname{ord}(x) = 2$, we have $G[2,m;q] \simeq D_{2q}$ if m is even. If m is odd, then $N\langle x^m \rangle = D_{2q}$ and hence $G[2,m;q] = \{1\}$.

(2) We have $\operatorname{ord}(Q) = 4$ since $\varphi_2(Q) = \varphi_2(aba) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \in \boldsymbol{SL}(2,\boldsymbol{Z}_3)$ (cf. Theorem 2.6). If $m \equiv 1,3 \pmod 4$, then $G[2,m;q] \simeq \langle a,b \mid a^3 = 1,\ aba = bab = 1 \rangle \simeq \langle a \mid a^3 = 1 \rangle \simeq C_3$. If $m \equiv 2 \pmod 4$, then $G[3,m;q] \simeq G(3;3)/N\langle Q^2 \rangle \simeq \boldsymbol{SL}(2,\boldsymbol{Z}_3)/\{\pm I_2\} \simeq \boldsymbol{PSL}(2,\boldsymbol{Z}_3)$. If $m \equiv 0 \pmod 4$, then $G[3,m;q] \simeq G(3;3) \simeq \boldsymbol{SL}(2,\boldsymbol{Z}_3)$.

(3) We have $\varphi_3(Q) = s^3 \cdot t^2 \cdot y \in (Q_8 \underset{\alpha}{\rtimes} C_3) \underset{\beta}{\rtimes} C_4$, and hence $\operatorname{ord}(Q) = 8$. If $m \equiv 1,3,5,7 \pmod 8$, then $G[4,m;3] \simeq \langle a,b \mid a^4 = aba = bab = 1 \rangle \simeq \{1\}$. If $m \equiv 2,6 \pmod 8$, then $\varphi_3(Q^2) = s^2 \cdot 1 \cdot x$ and hence $G[4,m;3] \simeq (Q_8 \underset{\alpha}{\rtimes} C_3) \underset{\beta}{\rtimes} C_4 / \langle s^2 \cdot 1 \cdot x \rangle$. If $m \equiv 4 \pmod 8$, then $\varphi_3(Q^4) = 1 \cdot 1 \cdot x^2$ and hence $G[4,m;3] \simeq ((C_2 \times C_2) \underset{\bar{\alpha}}{\rtimes} C_3) \underset{\bar{\beta}}{\rtimes} C_4$, where $\bar{\alpha}$, $\bar{\beta}$ are the homomorphisms induced by α, β. If $m \equiv 0 \pmod 8$, then $G[4,m;3] \simeq G(4;3)$.

(4) We have $\varphi_4(Q) = \left(\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}, w^4 \right) \in \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_5$ so that $\operatorname{ord}(Q) = 20$. If $m \equiv 1,3,7,9,11,13,17,19 \pmod{20}$, then $G[5,m;3] \simeq \langle a,b \mid a^5 = 1,\ aba = bab = 1 \rangle \simeq \langle a \mid a^5 = a^{-3} = 1 \rangle \simeq \{1\}$. If $m \equiv 2,6,14,18 \pmod{20}$, then $G[5,m;3] \simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5) \times C_5 / \{\pm I_2\} \times C_5 \simeq$

$\boldsymbol{PSL}(2,\boldsymbol{Z}_5)$. If $m\equiv 4,8,12,16 \pmod{20}$, then $G[5,m;3]\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5)\times C_5/C_5\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5)$. If $m\equiv 5,15 \pmod{20}$, then we have $\varphi_4(Q^5)=\left(\begin{bmatrix}3 & 0\\ 1 & 2\end{bmatrix}, 1\right)$. Hence $G[5,m;3]\simeq C_5$ since $N\langle\begin{bmatrix}3 & 0\\ 1 & 2\end{bmatrix}\rangle=\boldsymbol{SL}(2,\boldsymbol{Z}_5)$. If $m\equiv 10 \pmod{20}$, then we have $\varphi_4(Q^{10})=\left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}, 1\right)$ so that $G[5,m;3]\simeq \boldsymbol{PSL}(2,\boldsymbol{Z}_5)\times C_5$. If $m\equiv 0 \pmod{20}$, then $G[5,m;3]\simeq G(5;3)\simeq \boldsymbol{SL}(2,\boldsymbol{Z}_5)\times C_5$.

(5) We have $\varphi_5(Q)=\left(\begin{bmatrix}2 & 3\\ 0 & 3\end{bmatrix}, s\right)$ and the rest is similar to (4). □

REMARK 3.2. In the case (3), $G[4,m;3]$ ($m\equiv 2,6 \pmod 8$) is isomorphic to the symmetric group S_4 of degree 4 (cf. Namba [3;p.50]).

Corollary 3.3. *Suppose that (e,q) is one of those given in Theorem 2.4 and let $D=e\overline{B_q}+ml_\infty$. Then there exists the maximal covering $\pi: X(\boldsymbol{P}^2,D)\to \boldsymbol{P}^2$ if and only if*
(1) *$m=2$ in the case where $(e,q)=(2,q)$ (q:odd),*
(2) *$m=1,2,4$ in the case where $(e,q)=(3,3)$,*
(3) *$m=2,4,8$ in the case where $(e,q)=(4,3)$,*
(4) *$m=2,4,5,10,20$ in the case where $(e,q)=(5,3)$,*
(5) *$m=2,3,4,6,12$ in the case where $(e,q)=(3,5)$.*
In these cases, the Galois group G_π of π is isomorphic to $G[e,m;q]$.

Proof. Since $\mathrm{ord}(a)=2$ and $\mathrm{ord}(Q)=2$ in $G[2,m;q]\simeq D_{2q}$ (m: even), (1) follows from Corollary 1.5. By calculating $\mathrm{ord}(a)$ and $\mathrm{ord}(Q)$ using Theorem 2.6 and Proposition 3.1, the other assertions follow from Corollary 1.5 similarly. □

Let $S(e,m;q)=X(\boldsymbol{P}^2,e\overline{B_q}+ml_\infty)$ be the maximal Galois covering for the $(e,m;q)$'s given in Corollary 3.3. Then $S(e,m;q)$ is a normal projective irreducible rational surface since it is a compactification of $\boldsymbol{C}^2/$finite group. One of the singularities of $S(e,m;q)$ lies over (0,0,1), which is a quotient singularity. To detemine the structure of $S(e,m;q)$ (especially the singularities lying over (0,1,0)) will be an interesting poblem, which will be discussed elsewhere. We give a list of $S(e,m;q)$ and $G[e,m;q]$ for convenience.

Table 13

$S(e,m;q)$	$S(2,2;q)$
$G[e,m;q]$	D_{2q}

$S(3,1;3)$	$S(3,2;3)$	$S(3,4;3)$
C_3	$\boldsymbol{PSL}(2,\boldsymbol{Z}_3)$	$\boldsymbol{SL}(2,\boldsymbol{Z}_3)$

$S(4,2;3)$	$S(4,4;3)$	$S(4,8;3)$
S_4	$((C_2\times C_2)\rtimes C_3)\rtimes C_4$	$(Q_8\rtimes C_3)\rtimes C_4$

S(5,2;3)	S(5,4;3)	S(5,5;3)	S(5,10;3)	S(5,20;3)
$\boldsymbol{PSL}(2,\boldsymbol{Z}_5)$	$\boldsymbol{SL}(2,\boldsymbol{Z}_5)$	C_5	$\boldsymbol{PSL}(2,\boldsymbol{Z}_5)\times C_5$	$\boldsymbol{SL}(2,\boldsymbol{Z}_5)\times C_5$

$S(3,2;5)$	$S(3,3;5)$	$S(3,4;5)$	$S(3,6;5)$	$S(3,12;5)$
$\boldsymbol{PSL}(2,\boldsymbol{Z}_5)$	C_3	$\boldsymbol{SL}(2,\boldsymbol{Z}_5)$	$\boldsymbol{PSL}(2,\boldsymbol{Z}_5)\times C_3$	$\boldsymbol{SL}(2,\boldsymbol{Z}_5)\times C_3$

References

[1] H.S.M. Coxeter and W.O.J. Moser: *Generators and relations for discrete groups* (third edition), Springer-Verlag, Berlin-Heidelberg-New York, 1972.
[2] D.L. Johnson: *Presentations of Groups*, London Math. Soc. Student Texts **15**, Cambridge Univ. Press, Cambridge-New York-Port Chester-Melbourne-Sydney, 1990.
[3] M. Namba: *Branched coverings and algebraic functions*, Pitman Research Notes in Math. Series **161**, Longman Scientific and Technical, Harlow-New York, 1987.
[4] H. Pinkham: *Singularités de Klein* I, II, in Séminaire sur les Singularités des Surfaces, Springer Lec. Notes **777**, Springer-Verlag, Berlin-Heidelberg-New York, 1980, 1–20.
[5] J. Stillwell: *Classical Topology and Combinatorial Group Theory* (second edition), GTM **72**, Springer-Verlag, New York-Berlin-Heidelberg, 1993.
[6] K. Tamai: *On the Galois coverings of* $\boldsymbol{C}^2$ with branch locus $w^2=v^q$ (in Japanese), Master Thesis, Tokyo Denki Univ., 1994.

Department of Mathematical Sciences
College of Science and Engineering
Tokyo Denki University
Hatoyama, Hiki-gun, Saitama-ken
350–03 Japan