Koshitani, S. and Külshammer, B.
Osaka J. Math.
33 (1996), 343–346

A SPLITTING THEOREM FOR BLOCKS

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(Received October 4, 1993)

Let F be an algebraically closed field of prime characteristic p, let G be a finite group, and let H be a normal subgroup of G such that G/H is a p-group. Moreover, let B be a block of the group algebra FH of H over F.

By Osima's theorem, there is a unique block A of FG covering B. We are interested in the structure of A. As usual, the general case reduces to the special one where B is G-stable. Thus we assume in the following that B is G-stable and denote by P a defect group of A. Then $Q := P \cap H$ is a defect group of B, and $G = PH$ (see [3, V]).

If P is abelian then the character theory of A is described in a paper by R. Knörr [5]. We are interested in the structure of A as a ring under the additional hypothesis that Q has a complement in P. We prove that such a splitting of defect groups implies a splitting of blocks:

Theorem. *Let F be an algebraically closed field of prime characteristic p, and let H be a normal subgroup of a finite group G such that the factor group G/H is a p-group. Let B be a G-stable block of FH, and let A be the unique block of FG covering B. Suppose that A has an abelian defect group P and that $Q := P \cap H$ has a complement R in P. Then A and the tensor product $FR \otimes_F B$ are isomorphic F-algebras.*

Proof. As observed above, we have $G = PH = RH$ and $R \cap H = 1$. We consider the group algebra FG as a crossed product of FH with $G/H \cong R$, as usual (see [6] for crossed products). Since $1_A = 1_B$ for the block idempotents 1_A and 1_B of A and B, respectively, the block $A = 1_A FG = 1_B FG$ then becomes a crossed product of $1_B FH = B$ with $G/H \cong R$.

Arguing by induction on $q := |G:H| = |R|$ we may assume that $G/H \cong R$ is cyclic. We write $R = \langle r \rangle$. Then it suffices to show that the center ZA of A contains a graded unit x of A of degree r and order q; for, in that case, we will have $A = \oplus_{i=0}^{q-1} x^i B \cong FR \otimes_F B$.

From the main result in [5], we obtain $k(A) = q \cdot k(B)$, where $k(A)$ is the number of all irreducible complex characters of G in A. Hence $\dim ZA = q \dim ZB$. On the other hand, ZA is contained in the centralizer $C_A(B) =: C$ of B in A which is an algebra graded by R, with 1-component $C_1 = ZB$. We want to show that in our situation C is a crossed product of ZB with R. Thus we look at the subgroup

$$G[B] := \{g \in G : C_{gH} C_{g^{-1}H} = C_1\}$$

of G. This subgroup plays an important role in Dade's theory of block extensions ([1] and [2]).

Let $\mathscr{A}$ be a root of A in $F\mathrm{C}_G(P)$. Then $\mathscr{A}$ has defect group P, and $a := \mathscr{A}^{\mathrm{C}_G(Q)}$ is a well-defined block of $F\mathrm{C}_G(Q)$ with defect group P. Since $a^G = A$ we have $\mathrm{Br}_Q(1_A)1_a \neq 0$, and since $\mathrm{C}_G(Q) = \mathrm{C}_G(Q) \cap PH = P\mathrm{C}_H(Q)$ we have $1_a \in F\mathrm{C}_H(Q)$ by Osima's theorem. We choose a block b of $F\mathrm{C}_H(Q)$ covered by a such that $\mathrm{Br}_Q(1_B)1_b \neq 0$. Then b is a block of $F\mathrm{C}_H(Q)$ with defect group Q such that $b^H = B$.

Let $\mathrm{C}_G(Q)_b$, $\mathrm{N}_G(Q)_b$, $\mathrm{N}_H(Q)_b$ denote the stabilizers of b in $\mathrm{C}_G(Q)$, $\mathrm{N}_G(Q)$, $\mathrm{N}_H(Q)$, respectively. Since a is the unique block of $F\mathrm{C}_G(Q)$ covering b by Osima's theorem, it follows from Fong's theorems [3, V Theorems 3.12 and 3.14] that P is conjugate in $\mathrm{C}_G(Q)$ to a subgroup of $\mathrm{C}_G(Q)_b$. But $\mathrm{C}_G(Q) = P\mathrm{C}_H(Q)$, so P is conjugate in $\mathrm{C}_H(Q)$ to a subgroup of $\mathrm{C}_G(Q)_b$, which means that $P \subseteq \mathrm{C}_G(Q)_b$. Thus $\mathrm{C}_G(Q)_b = P\mathrm{C}_H(Q) = \mathrm{C}_G(Q)$.

In [2], Dade has defined a natural bilinear map

$$\omega : \mathrm{N}_H(Q)_b / \mathrm{C}_H(Q) \times \mathrm{C}_G(Q)_b / \mathrm{C}_H(Q) \to F^\times$$

and shown that $G[B] = \mathrm{C}_G(Q)_\omega H$ where

$$\mathrm{C}_G(Q)_\omega := \{g \in \mathrm{C}_G(Q)_b : \omega(\mathrm{N}_H(Q)_b / \mathrm{C}_H(Q), g\mathrm{C}_H(Q)) = 1\}$$

(see [2, (0.3b) and Corollary 12.6]). By definition, $\mathrm{C}_G(Q)_b / \mathrm{C}_G(Q)_\omega$ is isomorphic to a subgroup of $\mathrm{Hom}(\mathrm{N}_H(Q)_b / \mathrm{C}_H(Q), F^\times)$ and thus a p'-group (see [2, (11.13)]). On the other hand, in our situation $\mathrm{C}_G(Q)_b / \mathrm{C}_H(Q) \cong \mathrm{C}_G(Q)_b H / H$ is a p-group. Thus $\mathrm{C}_G(Q)_\omega = \mathrm{C}_G(Q)_b = \mathrm{C}_G(Q)$ and

$$G = PH = \mathrm{C}_G(Q)_b H = \mathrm{C}_G(Q)_\omega H = G[B].$$

It follows easily that C is a crossed product of the local algebra ZB with R (see [6, p.149]); in particular, $\dim C = q \dim ZB = \dim ZA$. Since $ZA \subseteq C$ we conclude that $ZA = C$.

The inertial group $\mathrm{N}_G(P)_{\mathscr{A}}$ acts on P, and $Q = P \cap H$ is an $\mathrm{N}_G(P)_{\mathscr{A}}$-stable subgroup of P. Since $\mathrm{N}_G(P)_{\mathscr{A}} / \mathrm{C}_G(P)$ is a p'-group, Maschke's theorem [4, Theorem 3.3.2] implies that Q has an $\mathrm{N}_G(P)_{\mathscr{A}}$-stable complement in P. We may assume that our notation is such that R is $\mathrm{N}_G(P)_{\mathscr{A}}$-stable. Since $G/H \cong R$ is abelian we obtain $[R, \mathrm{N}_G(P)_{\mathscr{A}}] \subseteq R \cap H = 1$. Thus $R \subseteq \mathrm{C}_P(\mathrm{N}_G(P)_{\mathscr{A}})$.

Let $\alpha = \mathscr{A}^{\mathrm{C}_G(R)}$. By Watanabe's result [8, Theorem 2 (ii)], the map

$$f : ZA \to Z\alpha, \quad z \mapsto \mathrm{Br}_R(z)1_\alpha$$

is an isomorphism of F-algebras. But we have $\mathrm{C}_G(R) = R \times \mathrm{C}_H(R)$; in particular,

$1_\alpha \in FC_H(R)$. This implies that f is R-graded.

There is a unique block β of $FC_H(R)$ covered by α, and we have $\alpha \cong FR\otimes_F\beta$ by multiplication; in particular, $Z\alpha \cong FR\otimes_F Z\beta$ by multiplication. Obviously, $r1_\beta$ is a graded unit of degree r and order q in $Z\alpha$. Thus $x:=f^{-1}(r1_\beta)$ is a graded unit of degree r and order q in ZA, and we are done.

REMARKS. (i) The condition that Q has a complement in P is essential. Take G the cyclic group of order p^2 and H its subgroup of index p, for instance.

(ii) The condition that P is abelian is also essential. Take G the extra-special group of order p^3 of exponent p and H its subgroup of index p for p odd, for example.

(iii) The theorem above is related to the main result of [7] where a similar splitting of blocks occurs.

(iv) It seems likely that our result holds also when the field F is replaced by a suitable complete discrete valuation ring $\mathcal{O}$. However, since Watanabe's result, on which we lean heavily, does not immediately lift to $\mathcal{O}$, a different proof would have to be found.

ACKNOWLEDGEMENTS. The authors would like to thank a referee for his or her advice.

This work was supported in part by the Alexander von Humboldt Foundation while the first author was staying at the Institute for Experimental Mathematics, Essen University (Jul.-Oct.,1992). He would like to thank the Humboldt Foundation for its financial support and Professor G.Michler and the people around him for their great hospitality.

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