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CELLS AND MODULAR REPRESENTATIONS OF HECKE ALGEBRAS

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0. Introduction

0.1. Let $\mathscr{A}=\boldsymbol{Z}[\sqrt{q},\sqrt{q}^{-1}]$, where $\sqrt{q}$ is an indeterminate, H be the Hecke algebra of a (finite) irreducible Weyl group W over $\mathscr{A}$, and for any $\mathscr{A}$-algebra $\mathscr{B}$, put

$$H(\mathscr{B}):=H\otimes_{\mathscr{A}}\mathscr{B}.$$

Theorem A. *Assume that W is not of type E_8. If two left cells, say Γ and Γ', of W give $H(\boldsymbol{Q}(\sqrt{q}))$-modules isomorphic to each other, then the W-graphs associated to these left cells* [9] *are isomorphic to each other. Especially, the H-modules afforded by Γ and Γ' are isomorphic to each other.*

(See (1.5) for the case of E_8.)

0.2. We use Theorem A to determine how an irreducible $H(\boldsymbol{Q}(\sqrt{q}))$-module decomposes into irreducible $H(\boldsymbol{F}_p(\sqrt{q}))$-modules (in the Grothendieck group) under the 'reduction modulo p' assuming that W is not of type E_8.

By [4, 6.3](cf. [8, (3.9) and (3.10)]), the decomposition matrix of $H(\boldsymbol{Q}(\sqrt{q}))$ with respect to the 'reduction modulo p' is the identity matrix except for the following prime numbers p (=bad primes).

(a) For type A_l: none.
(b) For types $B_l=C_l$ and $D_l:p=2$.
(c) For types E_6, E_7, F_4 and $G_2:p=2,3$.
(d) For types $E_8:p=2,3,5$.

Thus it is enough to determine the decomposition matrices for these exceptional cases.

Let $\mathscr{A}_p$ be the localization of $\mathscr{A}$ at the prime ideal $p\mathscr{A}$, $\mathscr{R}=\hat{\mathscr{A}}_p$ its completion (a complete discrete valuation ring), $\mathscr{K}$ the fractional field of $\mathscr{R}$, and $\bar{\mathscr{K}}$ $(=\boldsymbol{F}_p(\sqrt{q}))$ the residue field of $\mathscr{R}$. Let $\bar{L}_\iota$, $\iota\in I$ (resp. X_σ, $\sigma\in\Sigma$) be the totality of irreducible $H(\bar{\mathscr{K}})$-modules (resp. $H(\mathscr{K})$-modules) up to isomorphism, and $[\bar{X}_\sigma]$ the class in the Grothendieck group of the $H(\bar{\mathscr{K}})$-modules obtained from X_σ by the 'reduction modulo p'.

Theorem B. *Consider the case (b).*
(1) *$H(\bar{\mathscr{K}})$ splits over $\bar{\mathscr{K}}$.*
(2) *The constructible representations* ([12, 5.29], [16]) *are naturally in one-to-one correspondence with $\bar{L}_l$'s. Suppose that c_l is the constructible representation corresponding to $\bar{L}_l$, and $[c_l]=\Sigma_\sigma d_{\sigma,l}[X_\sigma]$ in the Grothendieck group of $H(\mathscr{K})$-modules. Then $[\bar{X}_\sigma]=\Sigma_l d_{\sigma,l}[\bar{L}_l]$ in the Grothendieck group of $H(\bar{\mathscr{K}})$-modules.*

Since $d_{\sigma,l}$'s are explicitly known [11] (where constructible representations are called *cells*), this theorem determines the decomposition matrix in the case (b). (Cf. Convention below.)

Theorem C. *Consider the case (c).*
(1) *$H(\bar{\mathscr{K}})$ splits over $\bar{\mathscr{K}}$.*
(2) *The partition of the irreducible $H(\bar{\mathscr{K}})$-modules into blocks is a refinement of their partition into families* [12, §4].
(3) *Let $\mathscr{F}$ be a family, $\mathscr{G}$ the associated finite group* ($=\mathfrak{S}_n$, $n\leq 4$) [12, §4].
Then the family $\mathscr{F}$ consists of

(I) $E^{g_1,\rho_1},\cdots,E^{g_s,\rho_s}$ ($s\geq 0$, $(g_i,\rho_i)\in\mathscr{M}_0(\mathscr{G})$ [17, 3.6]) *which are obtained from principal indecomposable $\mathscr{H}(\mathscr{R})$-modules by the scalar extension to $\mathscr{K}$,*

and the remaining $E^{\bar{g},\rho}$'s ($(\bar{g},\rho)\in\mathscr{M}_0(\mathscr{G})$), which form one block, say B (possibly $=\phi$). Each irreducible $H(\mathscr{K})$-module of (I) *forms its own block, and the corresponding diagonal block of the decomposition matrix is* 1.

(II) *The diagonal block of the decomposition matrix corresponding to B depends only on $\mathscr{G}$ and p (with a slight modification in the G_2 case, as is described below).*

We list (I) *and* (II) *in the following. (If $s=0$, we omit* (I). *If $B=\phi$, we omit* (II).) *Each row in the table of* (II) *corresponds to $E^{\bar{g},\rho}\in B$; $(\bar{g},\rho)$ is written at the left. Columns correspond to isomorphism classes of irreducible $H(\boldsymbol{F}_p(\sqrt{q}))$-modules.*

($\mathscr{G}=e$, $p=2,3$):

(I) (1,1).

($\mathscr{G}=\mathfrak{S}_2$, $p=2$):

(II)

1, 1	1	1
g_2, 1	1	·
1, ε	·	1

($\mathscr{G}=\mathfrak{S}_2$, $p=3$):

(I) $(1,1)$, $(g_2,1)$, $(1,\varepsilon)$.

$(\mathscr{G}=\mathfrak{S}_3,\ p=2)$:

(I) $(1,r)$, $(g_3,1)$.

(II)

$1, 1$	1	1
$g_2, 1$	1	·
$1, \varepsilon$	·	1

(*In the case of* G_2, *the* 3*rd row and the* 2*nd column should be removed.*)

$(\mathscr{G}=\mathfrak{S}_3,\ p=3)$:

(I) $(g_2,1)$.

(II)

$1, 1$	1	1	·
$1, r$	1	·	1
$g_3, 1$	·	1	·
$1, \varepsilon$	·	·	1

(*In the case of* G_2, *the* 4*th row and the* 3*rd column should be removed.*)

$(\mathscr{G}=\mathfrak{S}_4,\ p=2)$:

(I) $(g_3,1)$.

(II)

$1, 1$	1	1	1	·	·
$g_2, 1$	1	1	2	·	·
$g_2', 1$	1	·	1	1	·
$1, \lambda^1$	·	1	1	·	1
$g_4, 1$	1	·	·	·	·
g_2, ε''	·	1	·	·	·
$1, \sigma$	·	·	1	1	1
g_2', ε''	·	·	1	·	·
g_2', ε'	·	·	·	1	·
$1, \lambda^2$	·	·	·	·	1

$(\mathscr{G}=\mathfrak{S}_4,\ p=3)$:

(II) $(1,\lambda^1)$, $(1,\lambda^2)$, $(g_2,1)$, (g_2,ε''), $(g_2',1)$, (g_2',ε''), (g_2',ε'), $(g_4,1)$.

(II)

$1, 1$	1	1
$g_3, 1$	1	·
$1, \sigma$	·	1

(See (4.10) for the case of E_8).

0.3. Let us explain our motivation.

The concept of 'left cell' was introduced by Kazhdan and Lusztig [9]. Since then it has played important roles in various places in the representation theory.

Our best knowledge so far about left cells would be the following result of Lusztig [16]:

$$\{\text{the representations of } \boldsymbol{C}W \text{ afforded by the left cells}\}$$
$$=\{\text{the constructible representations}\},$$

where the right hand side is defined and explicitly determined in [12].

Inspite of this achievement, our understanding of the left cells are still insufficient. For example, each left cell contains a unique 'distinguished involution', whose understanding is desired but our present knowledge is very poor.

The main achievements of the present work are

(1) a certain uniqueness theorem concerning left cells (Theorem A),

(2) '{the left cells} = {the principal indecomposable modules}' in the case (b) (see Theorem B′ in (3.5), which is a refinement of Theorem B), and

(3) to explain why and how a deviation from the simple picture (2) occurs in the case (c) (see Theorem C′ in (3.6) and its proof).

Our explanation, which is unfortunately not satisfactory in the sense that we need a case study in §3, could be paraphrased as follows:

The common pattern of the blocks of the decomposition matrices (*i.e., the common pattern of the principal indecomposable modules*) *comes from the decomposition matrices* (= *the tables in* §2, *conjecturally*) *of the asymptotic Hecke algebras* $J_{\Gamma\cap\Gamma^{-1}}$ *via the mechanism explained in Step* 3 *of the proof of Theorem C′. More precisely, in the case* (b), *the decomposition matrices of* $J_{\Gamma\cap\Gamma^{-1}}$ *are always trivial and this is the reason why the picture* (2) *is so simple. In general, the decomposition matrices of* $J_{\Gamma\cap\Gamma^{-1}}$ *explains why and how the deviation from the simple picture* (2) *occurs* (*cf.* (3)).

As we have said above, our explanation is not intrinsic in some places. Thus our result can not be regarded as a final one, but it enables us to study left cells from the view point of the modular representation theory.

0.4. This paper consists of 4 sections. In section 1, we prove Theorem A. In section 2, we determine the decomposition matrix of the algebra $J_{\Gamma\cap\Gamma^{-1}}$ [17, 3.1] ($=\boldsymbol{Z}\cdot t(\Gamma\cap\Gamma^{-1})$ in the notation here) with respect to the 'reduction modulo p', except for some left cells Γ. Using these results, we prove Theorems B and C in section 3 and section 4. The proof goes as follows: first, we show our decomposition matrices are equal to those of the asymptotic Hecke algebra J [17, §3], and then we determine the latter using the result of section 2.

0.5. Reading the first draft of the present paper, G. Lusztig informed me that he has proved Conjecture 3.15 of [17] (see [18, 10.6], where this fact is mentioned) and that, based on this result, he has independently proved Theorem A of the present paper, including the case of E_8. Since his proof is not yet available, I leave the argument in its original form without assuming these results of Lusztig, especially in (1.4) and (2.5). (See (1.5) and (2.4).)

CONVENTION. If two objects, say X and Y, are *canonically* isomorphic, we sometimes write $X=Y$ for $X\simeq Y$, and identify them. One example of this convention is that if two algebras, say A and B, are naturally isomorphic, then we identify the isomorphism classes of A-modules and those of B-modules. Especially, we sometimes identify the Grothendieck groups of the modules over $\boldsymbol{Q}W$, $H(\mathscr{K})$, $J(\mathscr{K})$, etc. Another example is that if a K-algebra is split over the field K, and K' is an extension field of K, then we identify the isomorphism classes of irreducible A-modules and those of irreducible $A\otimes_K K'$-modules.

For a ring A, 'A-module' means 'unitary left A-module', i.e., a left A-module on which the identity element 1_A of A acts as the identity automorphism, unless otherwise stated.

Concerning the (ordinary) representation theory of Weyl groups and concerning the elements of $\mathscr{M}(\mathscr{G})$ (including $\mathscr{M}(\mathscr{G})$ itself), we follow the notation used in [12]. Concerning the subgroups of the symmetric group $\mathfrak{S}_n$, we follow the notation used in [17].

Concerning the modular representation theory, we follow the definitions given in [3, I]. In addition, we also use the word 'block' in the following way.

Let R be a complete discrete valuation ring, k the residue field, K the fractional field, A an R-algebra which is an R-free module of finite rank, and e a centrally primitive idempotent of A. Following [3, I, §7], we say that an A-module (resp. $A\otimes_R k$-module) M belongs to the block $B(e)$ if $eM=M$ (resp. $(e\otimes 1_k)M=M$). We also use the same terminology for an $A\otimes_R K$-module M; M is said to belong to the block $B(e)$ if $(e\otimes 1_K)M=M$, or equivalently, if M has an A-stable R-lattice belonging to $B(e)$. When e is not specified, this terminology becomes slightly abusive since the definition depends on A (not only on $A\otimes_R K$). However the meaning will be always clear from the context.

NOTATION. The complex number field, the rational number field and the rational integer ring are denoted by $\boldsymbol{C}$, $\boldsymbol{Q}$ and $\boldsymbol{Z}$, respectively. The p-adic completions of $\boldsymbol{Q}$ and $\boldsymbol{Z}$ are denoted by $\boldsymbol{Q}_p$ and $\boldsymbol{Z}_p$. The prime field of characteristic

$p>0$ is denoted by $\boldsymbol{F}_p$. We denote $\{n\in \boldsymbol{Z} \mid n\geq 0 \text{ (resp. } >0)\}$ by $\boldsymbol{Z}_{\geq 0}$ (resp. $\boldsymbol{Z}_{>0}$). For a set X, $|X|$ denotes the cardinality. For a ring A, $K(A)$ denotes the Grothendieck group of A-modules of finite type. Some other symbols are listed below.

List of Symbols

0.1. $\mathscr{A}$, $\sqrt{q}$, H, W, $H(\ )$

0.2. $\mathscr{R}$, $\mathscr{K}$, $\bar{\mathscr{K}}$, $E^{\bar{g},\rho}$, $\mathscr{M}_0(\mathscr{G})$

1.1. B, B_0, τ, $\tilde{}$

1.2. $C_w=C(w)$, $h_{x,y,z}$, $t_w=t(w)$, $\gamma_{x,y,z}=\gamma(x,y,z)$, $[C_w]=[C(w)]$, $C(W')$ ($W'\subset W$ a subset), $t(W')$, $[C(W')]$, $a(z)$, $\mathscr{D}$, d_Γ, W^i, $W^{\geq i}$, J^i, $J^{\geq i}$, $\mathrm{gr}^i(J)$, $\mathrm{gr}(J)$, $H^{\geq i}$, $\mathrm{gr}^i(H)$, $\mathrm{gr}(H)$

1.3. ϕ

2.3. $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})$, $\mathscr{M}_{\mathscr{G}/\mathscr{H}}(\mathscr{G})$

3.4. I, $\bar{L}(\iota)$, $U^0(\iota)$, $U(\iota)$, Σ, $X(\sigma)$, $[\bar{X}(\sigma)]$

3.5. $c(\iota)$, $d(\sigma,\iota)$

3.6.

Step 1. Λ, $c(\lambda)$, $b(\lambda)$, $\Gamma(\lambda,i)$, $g(\lambda,\iota)$, $U^0(\lambda,i,\iota,j)$, $\mathscr{F}$, $\mathscr{G}$, $\boldsymbol{H}$, $c(\mathscr{H})$, $b(\mathscr{H})$, $\Gamma(\mathscr{H},i)$, $g(\mathscr{H},\iota)$, $U^0(\mathscr{H},i,\iota,j)$

Step 2. $d'(\sigma,\mathscr{H})$

Step 3. $I(\mathscr{H})$, $\Gamma(\mathscr{H})$, $d(\mathscr{H})$, $U^0(\mathscr{H},\iota)$, $U(\mathscr{H},\iota)$, $\mu(\mathscr{H},m,k)$

4.1. J-res, J-ind, $\langle\rangle_A$

4.3. $a(E)$, $m(E')$, J-res, J-ind

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1.0. Here we prove Theorem A. Our argument depends heavily on the works of G. Lusztig, especially on [17]. Therefore, first we need to correct some misprints in [17] which affect our argument.

p.254, $\uparrow l.13$: $3.6\to 3.7$
p.259: Table $(\mathfrak{S}_3,\mathfrak{S}_e)\to$ Table $(\mathfrak{S}_3,\mathfrak{S}_2)$
p.259: Table $(\mathfrak{G}_3,2)\to$ Table $(\mathfrak{S}_3,e)$
p.259, Table $(\mathfrak{S}_4,\mathrm{Dyh}_8)$, row $(g_4,1)$, 4th column: $0\to 1$
p.261, Table $(\mathfrak{S}_5,\mathfrak{S}_2)$

row (g_2,r), 8th column: $-1\to 0$

row (g_2,r), 10th column: $1\to -1$

row (g_2,r), 12th column: $0\to 1$

row $(1,\lambda^2)$, 19th, 20th and 21st columns: $2\to -2$

1.1. Let R be an associative ring with 1 which is a free $\boldsymbol{Z}$-module with a fixed $\boldsymbol{Z}$-basis B. Assume that

(1) if $b,b'\in B$, then $bb'=\Sigma_{b''\in B}n_{b''}b''$, $n_{b''}\in\boldsymbol{Z}_{\geq 0}$,

and that

(2) $1=\Sigma_{b\in B_0}b$ for some $B_0\subset B$.

Let $\tau: R\to\boldsymbol{Z}$ be a $\boldsymbol{Z}$-linear function defined by

$$\tau(b)=\begin{cases}1 & \text{if } b\in B_0,\\ 0 & \text{if } b\notin B_0.\end{cases}$$

Assume that

(3) there exists an involution $x\to x^{\sim}$ of R as a $\boldsymbol{Z}$-module such that $(x\cdot x')^{\sim}=x'^{\sim}\cdot x^{\sim}$, $B^{\sim}=B$, and $\tau(b\cdot b'^{\sim})=\delta_{b,b'}$ $(b,b'\in B)$. Here δ denotes the Kronecker delta. The pair (R,B) is said to be a *based ring* [17, §1]. Note that the remaining data B_0, τ and $^{\sim}$ are uniquely determined by R and B. Note also that (1) does not imply (2). (For example, consider the ring $R=\boldsymbol{Z}b_0\oplus\boldsymbol{Z}b_1$ whose multiplication is defined by $b_0^2=b_0+b_1$, $b_0b_1=b_1b_0=b_0+2b_1$, and $b_1^2=2b_0+3b_1$, whose identity is $2b_0-b_1$.)

1.2. Let $\mathscr{A}$, $\sqrt{q}$, H, W and $H(-)$ be as in (0.1). Let $\{C_w\}_{w\in W}$ be the free $\mathscr{A}$-basis of H defined by D. Kazhdan and G. Lusztig [9, Theorem 1.1], and $C_xC_y=\Sigma_z h_{x,y,z}C_z$ $(h_{x,y,z}\in\mathscr{A})$. For each $z\in W$, define $a(z)\in\boldsymbol{Z}_{\geq 0}$ so that

$$(-\sqrt{q})^{a(z)}h_{x,y,z}\in\boldsymbol{Z}[\sqrt{q}] \quad \text{for all } x,y\in W, \text{ and}$$
$$(-\sqrt{q})^{a(z)-1}h_{x,y,z}\notin\boldsymbol{Z}[\sqrt{q}] \quad \text{for some } x,y\in W.$$

See [13, §2]. Then $a(z)=a(z^{-1})$ [13, 2.2]. Let $\gamma_{x,y,z}$ be the constant term of $(-\sqrt{q})^{a(z)}\cdot h_{x,y,z^{-1}}$. Let J be the free $\boldsymbol{Z}$-module with basis $\{t_w\}_{w\in W}$. Define a multplication of J by $t_xt_y=\Sigma_z\gamma_{x,y,z^{-1}}t_z$. Then $(J,\{t_w\}_{w\in W})$ is known to be based ring [17, 3.1]. In this case the set B_0 is $\{t_d\}_{d\in\mathscr{D}}$, where $\mathscr{D}$ is the set of distinguished involutions [14, 1.4]. The involution $^{\sim}$ is given by $t_w^{\sim}=t_{w^{-1}}$. For each left cell [9, p.167], $\Gamma\cap\mathscr{D}$ consists of exactly one element which is denoted by d_Γ [14, 1.10]. To avoid complicated subscript, we write $C(w)$, $t(w)$, $\gamma(x,y,z)$ for C_w, t_w, $\gamma_{x,y,z}$. For a subset $W'\subset W$, and for a ring $\mathscr{B}$, put $C(W'):=\{C(w)|w\in W'\}$,

$t(W'):=\{t(w)|w\in W'\}$, $\mathscr{B}\cdot C(W'):=\oplus_{w\in W'}\mathscr{B}C(w)$, and $\mathscr{B}\cdot t(W'):=\oplus_{w\in W'}\mathscr{B}t(w)$. Put $W^i:=\{w\in W|a(w)=i\}$ and $W^{\geq i}:=\bigcup_{j\geq i}W^j$. For a $\boldsymbol{Z}$-algebra $\mathscr{B}$, put $J(\mathscr{B}):=J\otimes_{\boldsymbol{Z}}\mathscr{B}$, $t(w):=t(w)\otimes 1\in J(\mathscr{B})$, $J^i(\mathscr{B}):=\mathscr{B}\cdot t(W^i)$, $J^{\geq i}(\mathscr{B}):=\mathscr{B}\cdot t(W^{\geq i})$, $\mathrm{gr}^i J(\mathscr{B})\ :\ =J^{\geq i}(\mathscr{B})/J^{\geq i+1}(\mathscr{B})$ $(=J^i(\mathscr{B}))$, and $\mathrm{gr}(J(\mathscr{B})):=\oplus_i\,\mathrm{gr}^i(J(\mathscr{B}))$. For an $\mathscr{A}$-algebra $\mathscr{B}$, put $H(\mathscr{B}):=H\otimes_{\mathscr{A}}\mathscr{B}, C(w):=C(w)\otimes 1\in H(\mathscr{B})$, $H^{\geq i}(\mathscr{B}):=\mathscr{B}\cdot C(W^{\geq i})$, $\mathrm{gr}^i(H(\mathscr{B}))\ :=\ H^{\geq i}(\mathscr{B})\ /\ H^{\geq i+1}(\mathscr{B})$, and $\mathrm{gr}(H(\mathscr{B})):=\oplus_i\,\mathrm{gr}^i(H(\mathscr{B}))$. Let $[C_w]=[C(w)]$ $(w\in W)$ be the image of $C_w=C(w)$ in $\mathrm{gr}^{a(w)}(H(\mathscr{B}))$. For a subset $W'\subset W$, put $[C(W')]:=\{[C(w)]|w\in W'\}$, and $\mathscr{B}\cdot[C(W')]:=\oplus_{w\in W'}\mathscr{B}[C(w)]$.

For a left cell Γ, $\boldsymbol{Z}\cdot t(\Gamma)$ is a left J-module, $\boldsymbol{Z}\cdot t(\Gamma)=Jt(d_\Gamma)$, and $\mathscr{A}\cdot[C(\Gamma)]$ is a left H-module.

1.3. Let Γ and Γ' be two left cells in W. Then

(1)
$$\begin{aligned}\mathrm{Hom}_J(\boldsymbol{Z}\cdot t(\Gamma),\ \boldsymbol{Z}\cdot t(\Gamma'))&=\mathrm{Hom}_J(Jt(d_\Gamma),\ Jt(d_{\Gamma'}))\\&=t(d_\Gamma)Jt(d_{\Gamma'})=\boldsymbol{Z}\cdot t(\Gamma'\cap\Gamma^{-1}).\end{aligned}$$

The following three assertions are obvious.

(2) By the multiplication of $J(\boldsymbol{C})$, $\boldsymbol{C}\cdot t(\Gamma\cap\Gamma^{-1})$ is a $\boldsymbol{C}$-algebra, and $\boldsymbol{C}\cdot t(\Gamma'\cap\Gamma^{-1})$ is a left $\boldsymbol{C}\cdot t(\Gamma\cap\Gamma^{-1})$-module.

(3) Define the multiplication by $(f\cdot g)(x):=g(f(x))$. Then $\mathrm{Hom}_{J(\boldsymbol{C})}(\boldsymbol{C}\cdot t(\Gamma),\ \boldsymbol{C}\cdot t(\Gamma))$ is a $\boldsymbol{C}$-algebra, and $\mathrm{Hom}_{J(\boldsymbol{C})}(\boldsymbol{C}\cdot t(\Gamma), \boldsymbol{C}\cdot t(\Gamma'))$ is a left $\mathrm{Hom}_{J(\boldsymbol{C})}(\boldsymbol{C}\cdot t(\Gamma), \boldsymbol{C}\cdot t(\Gamma))$-module.

(4) The algebra and module structures of (2) and (3) are identified by the isomorphism of (1).
Consider the following condition for $y\in\Gamma'\cap\Gamma^{-1}$.

(5) $\gamma(x,y,y^{-1})=0$ for any $x\in\Gamma\cap\Gamma^{-1}\backslash\{d\}$ $(d:=d_\Gamma)$.

(6) If $y\in\Gamma'\cap\Gamma^{-1}$ satisfies (5), then $t(y)t(y^{-1})=t(d)$.

Proof.

$$\begin{aligned}t(y)t(y^{-1})&=t(d)+\sum_{x\in\Gamma\cap\Gamma^{-1}\backslash\{d\}}\gamma(y,y^{-1},x)t(x^{-1}),\quad\text{by [14, 1.4]}\\&=t(d),\quad\text{by [14, 1.8] and (5).}\end{aligned}$$

(7) If $y\in\Gamma'\cap\Gamma^{-1}$ satisfies (5), then the right multiplication by $t(y)$ gives a bijection $t(\Gamma)\to t(\Gamma')$, whose inverse is the right multiplication by $t(y^{-1})$.

Proof. Let $z\in\Gamma$. By (6), $t(z)t(y)t(y^{-1})=t(z)t(d)=t(z)$. Since $(J,t(W))$ is a based ring, $t(z)t(y)=t(w)$ for some $w\in W$. Since $t(z)t(y)\in \boldsymbol{Z}\cdot t(\Gamma')$, $w\in\Gamma'$. Thus the right multiplication of $t(y)$ gives a mapping $t(\Gamma)\to t(\Gamma')$, whose inverse is the right multiplication by $t(y^{-1})$.

(8) If there exists $y\in\Gamma'\cap\Gamma^{-1}$ satisfying (5), then the W-graphs associated to Γ and Γ' [9] are isomorphic to each other.

Proof. Let ϕ: $H(\mathscr{A})\to J(\mathscr{A})$ be the $\mathscr{A}$-algebra homomorphism defined in [14, 2.4], which gives left $H(\mathscr{A})$-module structures on $\mathscr{A}\cdot t(\Gamma)$ and $\mathscr{A}\cdot t(\Gamma')$. By (7), we get an isomorphism between these two $H(\mathscr{A})$- modules preserving their bases $t(\Gamma)$ and $t(\Gamma')$. Note that ϕ is defined so that the $\mathscr{A}$-linear isomorphism $\mathrm{gr}(H(\mathscr{A}))\to J(\mathscr{A})$, $[C(w)]\mapsto t(w)$ is compatible with the left $H(\mathscr{A})$-action [15, 1.4]. (This can be proved using [14, 1.4, (b) and 2.4, (d)].) Thus we get an $H(\mathscr{A})$-isomorphism $\mathscr{A}\cdot[C(\Gamma)]\to\mathscr{A}\cdot[C(\Gamma')]$ preserving their natural bases, and we get the desired result.

1.4. Proof of Theorem A. Since the assertion is already verified for $W=W(F_4)$ [19, Theorem 3.2] (see also [6]) by a direct calculation of left cells and their W-graphs, we may exclude F_4. Hence it is enough to prove the following. Let Ω be the two-sided cell containing the left cell Γ, and $\mathscr{G}$ the finite group ($\simeq\mathfrak{S}_2^n$ for some n, $\mathfrak{S}_3$, $\mathfrak{S}_4$, or $\mathfrak{S}_5$) associated to Ω [12, §4].

(1) Assume that $\mathfrak{S}\not\simeq\mathfrak{S}_4$ or $\mathscr{G}_5$. If $\boldsymbol{C}\cdot t(\Gamma)\simeq\boldsymbol{C}\cdot t(\Gamma')$ as left $J(\boldsymbol{C})$-modules, then there exists $y\in\Gamma'\cap\Gamma^{-1}$ satisfying (1.3, (5)).

Proof. Let $\mathscr{H}$ be the subgroup of $\mathscr{G}$ (determined up to conjugacy) associated to Γ [17, 3.8]. Note that Ω, $\mathscr{G}$ and $\mathscr{H}$ are determined by the isomorphism class of the $J(\boldsymbol{C})$-module $\boldsymbol{C}\cdot t(\Gamma)$ ([1], [2]; cf. (4.6)). Hence the same $(\mathscr{G},\mathscr{H})$ is also associated to Γ'. Define $\mathscr{M}_0(\mathscr{G})\subset\mathscr{M}(\mathscr{G})$ and E^m ($m\in\mathscr{M}_0(\mathscr{G})$) as in [17, 3.6]. Put $M_m:=\dim t(d_\Gamma)E^m=\dim t(d_{\Gamma'})E^m$ [17, 3.8 and 3.9]. By (1.3, (2)–(4)), $\boldsymbol{C}\cdot t(\Gamma'\cap\Gamma^{-1})$, is a left regular representation of the $\boldsymbol{C}$-algebra $\boldsymbol{C}\cdot t(\Gamma\cap\Gamma^{-1})$, in which $t(d_\Gamma)E^{1,1}$ appears exactly once as an irreducible component. (Cf. [17, 1.5]. Note that $m=(1,1)$ is an element of $\mathscr{M}_0(\mathscr{G})$, and $E^{1,1}$ appears in $J(\boldsymbol{C})\cdot t(d_\Gamma)=\boldsymbol{C}\cdot t(\Gamma)$ exactly once.) Hence

$$(2)\qquad \sum_{x\in\Gamma\cap\Gamma^{-1}}\mathrm{Tr}(t(x),E^{1,1})\mathrm{Tr}(t(x),\boldsymbol{C}\cdot t(\Gamma'\cap\Gamma^{-1}))$$

$$=\sum_{x\in\Gamma\cap\Gamma^{-1}}\mathrm{Tr}(t(x),E^{1,1})\mathrm{Tr}(t(x),E^{1,1})\quad\text{by [17, 1.3,(c) and 1.5,(b)]}$$

$$=|\mathscr{G}|\quad\text{by [17, 1.3,(c) and 3.6,(b)]}$$

(cf. [17, 3.6, (c)]). On the other hand, the most left hand side of (2) is equal to

(3)
$$\sum_{x\in\Gamma\cap\Gamma^{-1}} \mathrm{Tr}(t(x),E^{1,1}) \sum_{y\in\Gamma'\cap\Gamma^{-1}} \gamma(x,y,y^{-1})$$
$$=\dim(\boldsymbol{C}\cdot t(\Gamma'\cap\Gamma^{-1}))+\sum_{y\in\Gamma'\cap\Gamma^{-1}}\left[\sum_{\substack{x\in\Gamma\cap\Gamma^{-1}\\ x\neq d_\Gamma}} \mathrm{Tr}(t(x),E^{1,1})\gamma(x,y,y^{-1})\right].$$

(Note that $\mathrm{Tr}(t(d),E^{1,1})=M_{1,1}=1$ [17, 3.10, (b) and Appendix].) Denote the inside of [ ] by $\Xi(y)$. Then (1.3, (5)) holds for some $y\in\Gamma'\cap\Gamma^{-1}$ if and only if

(4) $$\Xi(y)=0 \quad \text{for some } y\in\Gamma'\cap\Gamma^{-1},$$

since $\mathrm{Tr}(t(x),\ E^{1,1})\in \boldsymbol{Z}_{>0}$ [17, 3.10, (b) and Appendix] and $\gamma(x,y,y^{-1})\in \boldsymbol{Z}_{\geq 0}$. If (4) does not hold, then $\Xi(y)\geq 1$ for any y, and

(5)
$$|\mathscr{G}|=\dim\boldsymbol{C}\cdot t(\Gamma'\cap\Gamma^{-1})+\sum_{y\in\Gamma'\cap\Gamma^{-1}}\Xi(y) \quad \text{by (2) and (3)}$$
$$\geq 2\dim\boldsymbol{C}\cdot t(\Gamma'\cap\Gamma^{-1})$$
$$=2\dim \mathrm{Hom}_{J(\boldsymbol{C})}(\boldsymbol{C}\cdot t(\Gamma),\ \boldsymbol{C}\cdot t(\Gamma')) \quad \text{by (1.3, (1))}$$
$$=2\sum_m M_m^2 \quad \text{by [17, 1.2,(a), 3.8,(a) and 3.9,(a)].}$$

But if $\mathscr{G}$ is an elementary abelian 2-group, then $|\mathscr{G}|=\Sigma_m M_m^2<2\Sigma_m M_m^2$ by [17, 3.10]. Hence (4) holds in this case. If $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_3,e)$, then $|\mathscr{G}|=6$ and $2\Sigma_m M_m^2=12$ [17, Table $(\mathfrak{S}_3,e)$]. Hence (4) holds also in this case.

Next let us consider the case where $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_3,\mathfrak{S}_3)$. Then $|\mathscr{G}|=2\Sigma_m M_m^2=6$. If (4) does not hold, then $\Xi(y)=1$ for any $y\in\Gamma'\cap\Gamma^{-1}$ by (5). Let $z\in\Gamma\cap\Gamma^{-1}$ be the unique element such that $\mathrm{Tr}(t(z),E^{1,1})=2$, i.e., the element corresponding to the second column of [17, Table $(\mathfrak{S}_3,\mathfrak{S}_3)$]. Then $1=\Xi(y)\geq 2\gamma(z,y,y^{-1})$ and hence $\gamma(z,y,y^{-1})=0$ for any $y\in\Gamma'\cap\Gamma^{-1}$. Then the trace of the representing matrix of $t(z)$ for the left regular representation of $\boldsymbol{C}\cdot t(\Gamma\cap\Gamma^{-1})$ is zero. But the trace should be $2+0+(-1)=1$ as is seen from the second column of [17, Table $(\mathfrak{S}_3,\mathfrak{S}_3)$]. Hence (4) holds. The same argument works also for $(\mathfrak{S}_3,\mathfrak{S}_2)$. Thus we have completed the proof of Theorem A.

REMARK 1.5. Even in the E_8 case, if two left cells Γ and Γ' of W are contained in a two sided cell whose associated group is $\not\simeq\mathfrak{S}_5$, then the statement of Theorem A remains valid, as is seen from our argument.

If we assume Conjecture 3.15 of [17] together with the subsequent lines ($l.11$–$l.13$), then we can prove in general the existence of $y\in\Gamma'\cap\Gamma^{-1}$ satisfying (1.3, (5)), and hence Theorem A would hold including E_8.

2

Here we determine (except for some left cells Γ) the decomposition matrix of $\boldsymbol{Q}\cdot t(\Gamma\cap\Gamma^{-1})$ under the 'reduction modulo p', i.e., how an irreducible $\boldsymbol{Q}\cdot t(\Gamma\cap\Gamma^{-1})$-module decomposes into irreducible $\boldsymbol{F}_p\cdot t(\Gamma\cap\Gamma^{-1})$ module (in the Grothendieck group) under the 'reduction modulo p'.

Lemma 2.1. (1) *Let F be a field. Assume that a ring homomorphism $\mathscr{A}\to F$ is given, and $H(F)$ is semisimple. Then $\phi\otimes F: H(F)\xrightarrow{\sim} J(F)$.* (2) *Consider $\boldsymbol{Q}$ as an $\mathscr{A}$-algebra by $\mathscr{A}\to\boldsymbol{Q}$, $\sqrt{q}\mapsto 1$. Then $\phi\otimes\boldsymbol{Q}: H(\boldsymbol{Q})\xrightarrow{\sim} J(\boldsymbol{Q})$. ($H(\boldsymbol{Q})=\boldsymbol{Q}W$.) Especially $J(\boldsymbol{Q})$ is a split semisimple $\boldsymbol{Q}$-algebra.*

Proof. (1) If we consider $J(F)$ as a left $H(F)$-module by $h\cdot j=\phi(h)j$ ($h\in H(F)$, $j\in J(F)$), then $J(F)\simeq \mathrm{gr}(H(F))$ as left $H(F)$-modules [15, 1.4]. Assume that $H(F)$ is semisimple. If an irreducible $H(F)$-module E is contained in the kernel of $\phi\otimes F$, then E does not appear in the left $H(F)$-module $J(F)$. But in the left $H(F)$-module $\mathrm{gr}(H(F))$ ($\simeq H(F)$, since $H(F)$ is semisimple), every irreducible $H(F)$-module appears as an irreducible component. Hence $\phi\otimes F$ is injective. Comparing dimensions, we can see that $\phi\otimes F$ is an isomorphism. (2) follows from (1).

Lemma 2.2. *For any left cell Γ, $\boldsymbol{Q}\cdot t(\Gamma\cap\Gamma^{-1})$ is a split semisimple $\boldsymbol{Q}$-algebra.*

Proof. This follows from (2.1). Cf. [17, 1.5].

2.3. First we determine the decomposition matrix of $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{Q}$ [17, 2.2] under the 'reduction modulo p'. Since the irreducible $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{Q}$-modules are explicitly known [17, 2.1,(i) and 2.2,(b)], directly decomposing them after $\otimes\boldsymbol{F}_p$, we can calculate the decomposition matrix. (Here we need a lengthy calculation.) In the same time, we can also see that $\boldsymbol{Q}$ (resp. $\boldsymbol{F}_p$) is a splitting field for $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{Q}$ (resp. $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{F}_p$), i.e., every irreducible representation over $\boldsymbol{Q}$ (resp. $\boldsymbol{F}_p$) is absolutely irreducible. In the following, we write down the decomposition matrices for the pairs $(\mathscr{G},\mathscr{H})$ appearing in [17, 2.13,(a)] and for the prime numbers p such that the decomposition matrices are not the identity matrices. The rows correspond to (absolutely) irreducible $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{Q}$-modules. At the left of each row, we write the associated element $(\bar{g},\rho)\in\mathscr{M}_{\mathscr{G}/\mathscr{H}}(\mathscr{G})$ [17, 2.6], $|Z(\bar{g})|/\dim\rho$ and the dimension of the module, in this order. The columns correspond to absolutely irreducible $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{F}_p$-modules, whose dimension (over $\boldsymbol{F}_p$) is written at the top of the column. Each column also corresponds to a principal indecomposable $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{Z}_p$-module, say U^0. The number written at the top of each column is also equal to the number of appearance of direct summands $\simeq U^0$ in any fixed decomposition of $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes \boldsymbol{Z}_p$ into indecomposable left ideals. We write dot for 0.

Table $(\mathfrak{S}_3, \mathfrak{S}_3)$

			$p=2$		$p=3$	
			1	1	1	1
1, 1	6	1	1	·	1	·
g_2, 1	2	1	1	·	·	1
g_3, 1	3	1	·	1	1	·

Table $(\mathfrak{S}_3, \mathfrak{S}_2)$

			$p=2$		$p=3$	
			1	1	1	1
1, 1	6	1	1	·	1	·
1, r	3	1	·	1	1	·
g_2, 1	2	1	1	·	·	1

Table $(\mathfrak{S}_3, e)$

			$p=2$		$p=3$	
			1	2	1	1
1, 1	6	1	1	·	1	·
1, r	3	2	·	1	1	1
1, ε	6	1	1	·	·	1

Table $(\mathfrak{S}_4, \mathfrak{S}_4)$

			$p=2$		$p=3$			
			1	1	1	1	1	1
1, 1	24	1	1	·	1	·	·	·
g_2, 1	4	1	1	·	·	1	·	·
g'_2, 1	8	1	1	·	·	·	1	·
g_3, 1	3	1	·	1	1	·	·	·
g_4, 1	4	1	1	·	·	·	·	1

Table $(\mathfrak{S}_4, \mathfrak{S}_3)$

			$p=2$		$p=3$			
			1	1	1	1	1	1
$1, 1$	24	1	1	·	1	·	·	·
$1, \lambda^1$	8	1	1	·	·	1	·	·
$g_2, 1$	4	1	1	·	·	·	1	·
g_2, ε''	4	1	1	·	·	·	·	1
$g_3, 1$	3	1	·	1	1	·	·	·

Table $(\mathfrak{S}_4, \mathrm{Dyh}_8)$

			$p=2$		$p=3$				
			1	1	1	1	2	1	1
$1, 1$	24	1	1	·	1	·	·	·	·
$1, \sigma$	12	1	·	1	1	·	·	·	·
$g_2, 1$	4	1	1	·	·	1	·	·	·
$g'_2, 1$	8	2	1	1	·	·	1	·	·
g'_2, ε'	8	1	·	1	·	·	·	1	·
$g_4, 1$	4	1	1	·	·	·	·	·	1

Table $(\mathfrak{S}_4, \mathfrak{S}_2)$

			$p=2$		$p=3$				
			1	1	1	2	1	1	1
$1, 1$	24	1	1	·	1	·	·	·	·
$1, \lambda^1$	8	2	1	1	·	1	·	·	·
$1, \lambda^2$	8	1	·	1	·	·	1	·	·
$1, \sigma$	12	1	·	1	1	·	·	·	·
$g_2, 1$	4	1	1	·	·	·	·	1	·
g_2, ε''	4	1	1	·	·	·	·	·	1

Table $(\mathfrak{S}_4, \mathfrak{S}_2 \times \mathfrak{S}_2)$

			$p=2$	$p=3$				
			1	1	1	2	1	1
$1, 1$	24	1	1	1	·	·	·	·
$1, \lambda^1$	8	1	1	·	1	·	·	·
$1, \sigma$	12	1	1	1	·	·	·	·
$g_2, 1$	4	2	2	·	·	1	·	·
$g'_2, 1$	8	1	1	·	·	·	1	·
g'_2, ε''	8	1	1	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathfrak{S}_5)$

			$p=2$			$p=3$					$p=5$					
			1	1	1	1	1	1	1	1	1	1	1	1	1	1
$1, 1$	120	1	1	·	·	1	·	·	·	·	1	·	·	·	·	·
$g_3, 1$	6	1	·	1	·	1	·	·	·	·	·	1	·	·	·	·
$g'_2, 1$	8	1	1	·	·	·	1	·	·	·	·	·	1	·	·	·
$g_5, 1$	5	1	·	·	1	·	·	1	·	·	1	·	·	·	·	·
$g_2, 1$	12	1	1	·	·	·	·	·	1	·	·	·	·	1	·	·
$g_4, 1$	4	1	1	·	·	·	·	·	·	1	·	·	·	·	1	·
$g_6, 1$	6	1	·	1	·	·	·	·	1		·	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathfrak{S}_4)$

			$p=2$			$p=3$					$p=5$						
			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$1, 1$	120	1	1	·	·	1	·	·	·	·	1	·	·	·	·	·	
$g_3, 1$	6	1	·	1	·	1	·	·	·	·	·	1	·	·	·	·	·
$g'_2, 1$	8	1	1	·	·	·	1	·	·	·	·	·	1	·	·	·	·
$1, \lambda^1$	30	1	·	·	1	·	·	1	·	·	1	·	·	·	·	·	·
g_3, ε	6	1	·	1	·	·	·	1	·	·	·	·	·	1	·	·	·
$g_2, 1$	12	1	1	·	·	·	·	·	1	·	·	·	·	·	1	·	·
$g_4, 1$	4	1	1	·	·	·	·	·	·	1	·	·	·	·	·	1	·
g_2, r	6	1	·	·	1	·	·	·	1	·	·	·	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathfrak{S}_3 \times \mathfrak{S}_2)$

			$p=2$			$p=3$						$p=5$							
			1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	1	1
$1, 1$	120	1	1	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·
$g_3, 1$	6	1	·	1	·	1	·	·	·	·	·	·	1	·	·	·	·	·	·
$g_2', 1$	8	1	1	·	·	·	1	·	·	·	·	·	·	1	·	·	·	·	·
$1, \nu$	24	1	1	·	·	·	·	1	·	·	·	·	·	·	1	·	·	·	·
$1, \lambda^1$	30	1	·	·	1	·	·	1	·	·	·	1	·	·	·	·	·	·	·
g_2', ε''	8	1	1	·	·	·	·	·	1	·	·	·	·	·	·	1	·	·	·
$g_2, 1$	12	2	2	·	·	·	·	·	·	1	1	·	·	·	·	·	1	·	·
$g_6, 1$	6	1	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·	1	·
g_2, r	6	1	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathrm{Dyh}_8)$

			$p=2$			$p=3$						$p=5$							
			1	1	1	1	2	1	1	1	1	1	2	1	1	1	1	1	1
$1, 1$	120	1	1	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·
$g_2', 1$	8	2	1	1	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·
$1, \nu$	24	1	·	1	·	·	·	1	·	·	·	·	·	1	·	·	·	·	·
$1, \lambda^1$	30	1	·	·	1	·	·	1	·	·	·	1	·	·	·	·	·	·	·
$1, \nu'$	24	1	·	1	·	1	·	·	·	·	·	·	·	·	1	·	·	·	·
g_2', ε'	8	1	·	1	·	·	·	·	1	·	·	·	·	·	·	1	·	·	·
$g_2, 1$	12	1	1	·	·	·	·	·	·	1	·	·	·	·	·	·	1	·	·
$g_4, 1$	4	1	1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	1	·
g_2, r	6	1	·	·	1	·	·	·	·	1	·	·	·	·	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathfrak{S}_3)$

			$p=2$			$p=3$						$p=5$							
			1	1	2	1	1	1	1	1	1	1	1	1	1	1	1	2	1
$1, 1$	120	1	1	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·
$g_3, 1$	6	1	·	1	·	1	·	·	·	·	·	·	1	·	·	·	·	·	·
$1, \nu$	24	1	1	·	·	·	1	·	·	·	·	·	·	1	·	·	·	·	·
$1, \lambda^1$	30	2	·	·	1	·	1	1	·	·	·	1	·	·	1	·	·	·	·
g_3, ε	6	1	·	1	·	·	·	1	·	·	·	·	·	·	·	1	·	·	·
$1, \lambda^2$	20	1	1	·	·	·	·	·	1	·	·	·	·	·	1	·	·	·	·
$g_2, 1$	12	1	1	·	·	·	·	·	·	1	·	·	·	·	·	·	1	·	·
g_2, r	6	2	·	·	1	·	·	·	·	1	1	·	·	·	·	·	·	1	·
g_2, ε	12	1	1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathfrak{S}_2 \times \mathfrak{S}_2)$

			$p=2$			$p=3$						$p=5$							
			1	1	2	1	1	2	1	1	2	1	1	2	1	1	1	2	2
$1, 1$	120	1	1	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·
$g'_2, 1$	8	1	1	·	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·
$1, \nu$	24	2	1	1	·	·	·	1	·	·	·	·	·	1	·	·	·	·	·
$1, \lambda^1$	30	2	·	·	1	·	·	1	·	·	·	1	·	·	1	·	·	·	·
$1, \nu'$	24	1	·	1	·	1	·	·	·	·	·	·	·	·	·	1	·	·	·
g'_2, ε''	8	1	1	·	·	·	·	·	1	·	·	·	·	·	·	·	1	·	·
$1, \lambda^2$	20	1	·	1	·	·	·	·	·	1	·	·	·	·	1	·	·	·	·
$g_2, 1$	12	2	2	·	·	·	·	·	·	·	1	·	·	·	·	·	·	1	·
g_2, r	6	2	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·	1

Table $(\mathfrak{S}_5, \mathfrak{S}_2)$

			$p=2$				$p=3$						$p=5$							
			1	2	1	2	1	3	1	3	1	1	1	3	2	2	1	1	2	1
$1, 1$	120	1	1	·	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·	·
$1, \nu$	24	3	1	1	·	·	·	1	·	·	·	·	·	1	·	·	·	·	·	·
$1, \lambda^1$	30	3	·	·	1	1	·	1	·	·	·	·	1	·	1	·	·	·	·	·
$1, \nu'$	24	2	·	1	·	·	1	·	1	·	·		·	·	·	1	·	·	·	·
$1, \lambda^2$	20	3	1	1	·	·	·	·	·	1	·	·	·	·	1	·	1	·	·	·
$1, \lambda^3$	30	1	·	·	1	·	·	·	1	·	·	·	·	·	·	·	1	·	·	·
$g_2, 1$	12	1	1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	1	·	·
g_2, r	6	2	·	·	·	1	·	·	·	·	1	1	·	·	·	·	·	·	1	·
g_2, ε	12	1	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	1

2.4. Let Γ be a left cell of W, and $(\mathscr{G},\mathscr{H})$ the pair of finite groups associated to Γ as in (1.4). Let us determine the decomposition matrix of $\boldsymbol{Q}\cdot t(\Gamma\cap\Gamma^{-1})$ under the 'reduction modulo p', except for the 6 cases $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_5,\mathfrak{S}_3)$ $(p=2)$, $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_5,\mathfrak{S}_2\times\mathfrak{S}_2)$ $(p=2,3)$, and $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_5,\mathfrak{S}_2)$ $(p=2,3,5)$. At present the author can not determine the decomposition matrices in the 6 cases excluded here. Without to say, if we assume Conjecture 3.15 of [17], these decomposition matrices should coincide with those given in (2.3).

Lemma 2.5. *Possibly except for the above* 6 *cases,*
(1) $\boldsymbol{F}_p$ *is a splitting field for* $\boldsymbol{F}_p\cdot t(\Gamma\cap\Gamma^{-1})$,
(2) *the decomposition matrix of* $\boldsymbol{Q}\cdot t(\Gamma\cap\Gamma^{-1})$ *under the 'reduction modulo p' coincides with the decomposition matrix of* $K_{\mathscr{G}}(\mathscr{G}/\mathscr{H}\times\mathscr{G}/\mathscr{H})\otimes\boldsymbol{Q}$ *under the 'reduction modulo p', and*
(3) *two principal indecomposable* $\boldsymbol{Z}_p\cdot t(\Gamma\cap\Gamma^{-1})$*-modules, say* P^0 *and* Q^0, *are isomorphic if and only if* $P^0\otimes_{\boldsymbol{Z}_p}\boldsymbol{Q}_p\simeq Q^0\otimes_{\boldsymbol{Z}_p}\boldsymbol{Q}_p$ *as* $\boldsymbol{Q}_p\cdot t(\Gamma\cap\Gamma^{-1})$*-modules.*

Proof. (I) Let us consider the case $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_5,\mathfrak{S}_3)$ and $p=3$. Since the character table of $\boldsymbol{C}\cdot t(\Gamma\cap\Gamma^{-1})$ is known [17, Appendix], the rows of the decomposition matrix corresponding to 1-dimensional characters of $\boldsymbol{Q}\cdot t(\Gamma\cap\Gamma^{-1})$ can be easily determined. Thus we can see that the decomposition matrix is of the following form.

$1,1$	120	1	1	·	·	·	·	·	
$g_3,1$	6	1	1	·	·	·	·	·	
$1,\nu$	24	1	·	1	·	·	·	·	
$1,\lambda^1$	30	2	a_1	a_2	a_3	a_4	a_5	a_6	$\cdots$
g_3,ε	6	1	·	·	1	·	·	·	
$1,\lambda^2$	20	1	·	·	·	1	·	·	
$g_2,1$	12	1	·	·	·	·	1	·	
g_2,r	6	2	b_1	b_2	b_3	b_4	b_5	b_6	$\cdots$
g_2,ε	12	1	·	·	·	·	·	1	

Here the first column contains the elements $(\bar{g},\rho)\in\mathscr{M}_0(\mathscr{G})$ associated to the $\boldsymbol{C}\cdot t(\Gamma\cap\Gamma^{-1})$-modules, the second column contains $|Z(\bar{g})|/\dim\rho$, and the third column contains the dimensions of the modules. The irreducible $\boldsymbol{F}_p\cdot t(\Gamma\cap\Gamma^{-1})$-modules, say $\bar{L}_1,\cdots,\bar{L}_6$, corresponding to the first 6 columns (in this order) are one dimensional and hence absolutely irreducible. Let $U_i^0(1\leq i\leq 6)$ be the principal indecomposable $\boldsymbol{Z}_p\cdot t(\Gamma\cap\Gamma^{-1})$-modules corresponding to $\bar{L}_i$ $(1\leq i\leq 6)$ [3,I,(13.6)]. Then ith column also gives the multiplicity of $E^{\bar{g},\rho}$ appearing in $U_i^0\otimes\boldsymbol{Q}_p$. See [3,I, (17.8)] and (2.2).

For $E=E^{\bar{g},\rho}$, put $f_E:=|Z(\bar{g})|/\dim\rho$, and $\tau(x):=\Sigma_E f_E^{-1}\mathrm{Tr}(x,E)$. Then $A_0^0:=\boldsymbol{Z}\cdot t(\Gamma\cap\Gamma^{-1})$ and $\tau_0:=\tau|A_0^0$ satisfy [8, (2.5.1)–(2.5.3)]. (See [17, 2.7,(b)].) Hence by

[8, (2.16.1)],

$$\frac{1}{24}+\frac{a_2}{30}+\frac{b_2}{6}\in \boldsymbol{Z}_3 \ (=3\text{-adic integer ring}). \tag{4}$$

Since $a_i,b_i\geq 0$ and $\Sigma_i a_i=\Sigma_i b_i=2$, we have $(a_2,b_2)=(0,2)$, $(1,0)$ or $(2,1)$. If $(a_2,b_2)=(0,2)$, then $b_i=0$ for $i\neq 2$. Using relations for the 3rd, 5th, and 6th columns similar to (4), we can show that $a_3\neq 0$, $a_5\neq 0$ and $a_6\neq 0$. Thus we get a contradiction. If $(a_2,b_2)=(2,1)$, we get contradiction in the same way. Hence $(a_2,b_2)=(1,0)$.

Applying the same argument to the 3rd, 5th, and 6th columns, we get $(a_3,b_3)=(1,0)$, $(a_5,b_5)=(0,1)$ and $(a_6,b_6)=(0,1)$. The remaining a_i's and b_i's are equal to zero. Thus this decomposition matrix coincides with the corresponding one given in (2.3). (3) follows from (2).

(II) Let us consider the case $(\mathscr{G},\mathscr{H})=(\mathfrak{S}_3,e)$ and $p=2$. The determination of the rows of the decomposition matrix corresponding to one-dimensional characters is easy. For $E=E^{1,r}$, $1/f_E=1/3\in\boldsymbol{Z}_2$. By [8, (2.17)], the 'reduction modulo 2' of E is an absolutely irreducible principal indecomposable $\boldsymbol{F}_2\cdot t(\Gamma\cap\Gamma^{-1})$-module, and forms its own block. Thus the assertion is settled in this case.

The remaining cases can be settled by the same argument as in (I) and / or (II).

3

Here we determine how an irreducible $H(\boldsymbol{Q}(\sqrt{q}))$-module decomposes into irreducible $H(\boldsymbol{F}_p(\sqrt{q}))$-modules (in the Grothendieck group) under the 'reduction modulo p'.

3.1. By the definition of $\phi: H(\mathscr{A})\to J(\mathscr{A})$ [14, 2.4], and by [13, 5.4], $\phi(H^{\geq i}(\mathscr{A}))\subset J^{\geq i}(\mathscr{A})$. Hence ϕ induces $\mathrm{gr}(\phi\otimes\mathscr{B}):\mathrm{gr}(H(\mathscr{B}))\to\mathrm{gr}(J(\mathscr{B}))$ for any $\mathscr{A}$-algebra $\mathscr{B}$. The determinant of the matrix representing $\mathrm{gr}^i(\phi\otimes\mathscr{R})$ with respect to the bases $\{(-\sqrt{q})^i[C(w)]|w\in W^i\}$ and $t(W^i)$ is of the form $1+\sqrt{q}\alpha$ with some $\alpha\in\boldsymbol{Z}[\sqrt{q}]$ [15, 1.3,(c)]. Hence the determinant is invertible in $\mathscr{R}$, $\mathrm{gr}^i(\phi\otimes\mathscr{R})$ is an isomorphism, and consequently $\phi\otimes\mathscr{R}: H(\mathscr{R})\to J(\mathscr{R})$ is an isomorphism.

Lemma 3.2. *If $J(\boldsymbol{F}_p)$ is split over $\boldsymbol{F}_p$, i.e., if the irreducible $J(\boldsymbol{F}_p)$-modules are absolutely irreducible, then the decomposition matrix of $J(\mathscr{K})$ with respect to the 'reduction modulo p' and that of $J(\boldsymbol{Q}_p)$ with respect to the 'reduction modulo p' are the same.*

Proof. This is obvious from the following diagram.

$$\begin{array}{ccc} K(J(\boldsymbol{Q}_p)) & \underset{\simeq}{\overset{\otimes_{\boldsymbol{Q}_p}\mathscr{K}}{\to}} & K(J(\mathscr{K})) \\ \downarrow \bmod p & & \downarrow \bmod p \\ K(J(\boldsymbol{F}_p)) & \overset{\otimes_{\boldsymbol{F}_p}\boldsymbol{F}_p(\sqrt{q})}{\to} & K(J(\boldsymbol{F}_p(\sqrt{q}))). \end{array}$$

Lemma 3.3. *Let Γ be a left cell contained in a two-sided cell Ω, and $\{e_i\}_i$ mutually orthogonal primitive idempotents of $\boldsymbol{Z}_p \cdot t(\Gamma \cap \Gamma^{-1})$ such that $t(d) = \Sigma_i e_i$ $(d := d_\Gamma)$.*
(1) $\boldsymbol{Z}_p \cdot t(\Gamma) = \oplus_i J(\boldsymbol{Z}_p) e_i$.
(2) *Each $J(\boldsymbol{Z}_p)e_i$ is a (non-zero) principal indecomposable $J(\boldsymbol{Z}_p)$-module, i.e., each e_i is a primitive idempotent of $J(\boldsymbol{Z}_p)$.*
(3) *$(\boldsymbol{Z}_p \cdot t(\Gamma \cap \Gamma^{-1}))e_i \simeq (\boldsymbol{Z}_p \cdot t(\Gamma \cap \Gamma^{-1}))e_j$ as $\boldsymbol{Z}_p \cdot t(\Gamma \cap \Gamma^{-1})$-modules if and only if $J(\boldsymbol{Z}_p)e_i \simeq J(\boldsymbol{Z}_p)e_j$ as $J(\boldsymbol{Z}_p)$-modules.*
(4) *If $J(\boldsymbol{Q}_p)e_i = \oplus_{m \in \mathscr{M}_X(\mathscr{G})} \mu(m) E^m$, where $X = \mathscr{G}/\mathscr{H}$ and $\mu(m)$'s are multiplicities (see [17, 2.6] for $\mathscr{M}_X(\mathscr{G})$ and [17, 3.6] for E^m), then $(\boldsymbol{Q}_p \cdot t(\Gamma \cap \Gamma^{-1}))e_i = \oplus_{m \in \mathscr{M}_X(\mathscr{G})} \mu(m)(t(d)E^m)$.*

Proof. Put $A := J(\boldsymbol{Z}_p)$ and $e := t(d)$. Note that $eAe = \boldsymbol{Z}_p \cdot t(\Gamma \cap \Gamma^{-1})$, and e is its identity element.
(2) Assume that $M \oplus N = Ae_i$ with A-modules M and N. Then $eM \oplus eN = eAe_i = (eAe)e_i$. Hence we may assume that $eN = 0$. Let m and n be the images of e_i by the projections to M and N, respectively. Then $N = An = A(en) = 0$.
(3) We can identify $\oplus_i (Ae \otimes_{eAe} (eAe)e_i) = Ae \otimes_{eAe} eAe = Ae = \oplus_i Ae_i$. Then $Ae \otimes_{eAe} (eAe)e_i = Ae_i$, and we get the 'only if part'. Since $e(Ae_i) = (eAe)e_i$, we get the 'if part'.
(4) $(\boldsymbol{Q}_p \cdot t(\Gamma \cap \Gamma^{-1}))e_i = t(d)J(\boldsymbol{Q}_p)t(d)e_i = t(d)J(\boldsymbol{Q}_p)e_i = \oplus_m \mu(m)(t(d)E^m)$.

3.4. By (3.1) and (3.2), the decomposition matrices (with respect to the 'reduction modulo p') of $H(\mathscr{K})$, $J(\mathscr{K})$ and $J(\boldsymbol{Q}_p)$ are the same, if

(1) $$J(\boldsymbol{F}_p) \text{ splits over } \boldsymbol{F}_p.$$

Therefore, we prove (1), we interprete Theorems B and C in terms of $J(\boldsymbol{Q}_p)$ etc., and we work exclusively with $J(\boldsymbol{Q}_p)$ etc. For this purpose, we need to modify some notation introduced in (0.2).

Let $\bar{L}(\iota)$ $(\iota \in I)$ be the totality of the irreducible $J(\boldsymbol{F}_p)$-modules up to isomorphism, $U^0(\iota)$ the corresponding principal indecomposable $J(\boldsymbol{Z}_p)$-modules [3.I,(13.6)], $U(\iota) := U^0(\iota) \otimes_{\boldsymbol{Z}_p} \boldsymbol{Q}_p$, $X(\sigma)$ $(\sigma \in \Sigma)$ the totality of the irreducible $J(\boldsymbol{Q}_p)$-modules up to isomorphism, and $[\bar{X}(\sigma)]$ the class in $K(J(\boldsymbol{F}_p))$ obtained from $X(\sigma)$ by the 'reduction modulo p'.

3.5. Theorem B follows from the following theorem.

Theorem B′. *Consider case* (*b*) *of* (0.2).
(1) *For any left cell* Γ *of* W, $\boldsymbol{Z}_2\cdot t(\Gamma)$ *is a principal indecomposable* $J(\boldsymbol{Z}_2)$*-module.*
(2) *For left cells* Γ *and* Γ' *of* W, *the following conditions are equivalent.*
(i) $\boldsymbol{Z}_2\cdot t(\Gamma)\simeq\boldsymbol{Z}_2\cdot t(\Gamma')$ *as left* $J(\boldsymbol{Z}_2)$*-modules.*
(ii) $\boldsymbol{Q}_2\cdot t(\Gamma)\simeq\boldsymbol{Q}_2\cdot t(\Gamma')$ *as left* $J(\boldsymbol{Q}_2)$*-modules.*
(iii) *The left* $\boldsymbol{C}W$*-modules associated to the left cells* Γ *and* Γ' [9] *are isomorphic to each other.*
(3) *The irreducible* $J(\boldsymbol{F}_2)$*-modules are absolutely irreducible.*
If $U^0(\iota)\simeq\boldsymbol{Z}_2\cdot t(\Gamma)$, *let* $c(\iota)$ *be the constructible representation* $\boldsymbol{C}\cdot t(\Gamma)$.
(4) *If* $c(\iota)=\Sigma d(\sigma,\iota)[X(\sigma)]$ *in* $K(J(\boldsymbol{Q}_2))$, *then* $[\bar{X}(\sigma)]=\Sigma_\iota d(\sigma,\iota)[\bar{L}(\iota)]$ *in* $K(J(\boldsymbol{F}_2))$.

Proof. In this case, $t(\Gamma\cap\Gamma^{-1})$ is multiplicatively closed in the ring J, and forms an elementary abelian 2-group [17, 3.11]. Hence the identity element $t(d)\in t(\Gamma\cap\Gamma^{-1})$ is a primitive idempotent of the group ring $\boldsymbol{Z}_2\cdot t(\Gamma\cap\Gamma^{-1})$. Thus (1) follows from (3.3,(2)). (2) follows from Theorem A. Since the $J(\boldsymbol{Q}_2)$-module $\boldsymbol{Q}_2\cdot t(\Gamma)$ contains a special representation $(=E^{1,1})$ as an irreducible component with multiplicity one, (3) and (4) follow from (2.1,(2)) and [3,I, (17.8)].

3.6. Theorem C follows from the following theorem.

Theorem C′. *Consider the case* (*c*) *of* (0.2).
(1) *For* $\iota,\iota'\in I$, *the following conditions are equivalent.*
(*i*) $U^0(\iota)\simeq U^0(\iota')$ *as left* $J(\boldsymbol{Z}_p)$*-modules.*
(*ii*) $U(\iota)\simeq U(\iota')$ *as left* $J(\boldsymbol{Q}_p)$*-modules.*
(2) *The irreducible* $J(\boldsymbol{F}_p)$*-modules are absolutely irreducible.*
(3) *The partition of the irreducible* $J(\boldsymbol{Q}_p)$*-modules into blocks is a refinement of their partition into families* [12, 4.2].
(4) *Let* $\mathscr{F}$ *be a family, and* $\mathscr{G}$ *the associated finite group* $(=\mathfrak{S}_n,\ n\le 4)$. *Then the family* $\mathscr{F}$ *consists of*

(I) $E^{g_1,\rho_1},\cdots,E^{g_s,\rho_s}$ *which are obtained from principal indecomposable* $J(\boldsymbol{Z}_p)$*-modules by the scalar extension to* $\boldsymbol{Q}_p$,

and the remaining $E^{\bar{g},\rho}$*'s, which form one block, say* B. (*We admit* $B=\phi$.) *Each irreducible* $J(\boldsymbol{Q}_p)$*-module of* (I) *forms its own block, and the corresponding diagonal block of the decomposition matrix is* 1.

(II) *The diagonal block of the decomposition matrix corresponding to* B

depends only on $\mathscr{G}$ *and* p (*with a slight modification in the* G_2 *case, as is remarked in Theorem C*). *The lists of* (I) *and* (II) *are the same as those given in Theorem C.*

Proof. We prove in several steps.

Step 1. Let $c(\lambda)$ $(\lambda \in \Lambda)$ be the totality of the constructible representations of W, and for each $\lambda \in \Lambda$, $\Gamma(\lambda,i)$ $(1<i\leq b(\lambda))$ the totality of the left cells of W such that $\boldsymbol{C}\cdot t(\Gamma(\lambda,i))\simeq c(\lambda)$. By [16],

$$J(\boldsymbol{Z}_p)=\bigoplus_{\lambda\in\Lambda}\bigoplus_{i=1}^{b(\lambda)}\boldsymbol{Z}_p\cdot t(\Gamma(\lambda,i)). \tag{5}$$

Let

$$\boldsymbol{Z}_p\cdot t(\Gamma(\lambda,i))=\bigoplus_{\substack{\imath\in I\\ 1\leq j\leq g(\lambda,\imath)}} U^0(\lambda,i,\imath,j) \tag{6}$$

be a decomposition into principal indecomposable left $J(\boldsymbol{Z}_p)$-modules such that $U^0(\lambda,i,\imath,j)\simeq U^0(\imath)$. (By Theorem A, $g(\lambda,\imath)$ is independent of i.) By the decompositions (5) and (6), by [3.I, (13.11)], and by the definition of family [12, 4.2] (cf. [11, p.225, l.5–l.8]), we get (3).

By (3), the decomposition matrix decomposes into diagonal blocks corresponding to families of the irreducible $J(\boldsymbol{Q}_p)$-modules. (Cf. Convention.) Thus it is enough to determine such diagonal block corresponding to each family $\mathscr{F}$. If the group $\mathscr{G}$ associated to $\mathscr{F}$ is e or $\mathfrak{S}_2$, the argument is the same as in the proof of Theorem B. Therefore, we take a family $\mathscr{F}$ whose associated group $\mathscr{G}$ is $\mathfrak{S}_3$ or $\mathfrak{S}_4$. Then only the constructible representations $c(\lambda)$ whose irreducible components belong to $\mathscr{F}$ are relevant to the remaining argument. Since such $c(\lambda)$'s are parametrized by a certain family $\boldsymbol{H}$ of subgroups of $\mathscr{G}$ [17, 3.8], we change the parameter set from (a subset of) Λ to $\boldsymbol{H}$, and write $c(\mathscr{H})$, $b(\mathscr{H})$, $\Gamma(\mathscr{H},i)$, $g(\mathscr{H},\imath)$, $U^0(\mathscr{H},i,\imath,j)$ $(\mathscr{H}\in\boldsymbol{H})$ for $c(\lambda)$, $b(\lambda)$, $\Gamma(\lambda,i)$, $g(\lambda,\imath)$, $U^0(\lambda,i,\imath,j)$ $(\lambda\in\Lambda)$, respectively.

Step 2. We know

(7) the explicit description of the irreducible decomposition

$$c(\mathscr{H})\simeq \bigoplus_{\substack{\sigma\in\Sigma\\ X(\sigma)\in\mathscr{F}}} d'(\sigma,\mathscr{H})X(\sigma)\ [11].$$

By (5) and (7),

$$\bigoplus_{\substack{\sigma\in\Sigma\\ X(\sigma)\in\mathscr{F}}}(\dim X(\sigma))X(\sigma)\simeq \bigoplus_{\substack{\mathscr{H}\in\boldsymbol{H}\\ \sigma\in\Sigma\\ X(\sigma)\in\mathscr{F}}} b(\mathscr{H})d'(\sigma,\mathscr{H})X(\sigma),$$

and hence

$$\sum_{\mathscr{H}\in\boldsymbol{H}} b(\mathscr{H})d'(\sigma,\mathscr{H})=\dim X(\sigma). \tag{8}$$

By (7) and (8), we can explicitly calculate the values of $b(\mathscr{H})$'s. We shall write down some of them in the following. We specify a family by the special representation contained in it. (Cf. [12, §4].) We list $\mathscr{H}$'s associated to constructible representations in the same order as in [11], and we write down $b(\mathscr{H})$ beside $\mathscr{H}$. We follow the notation used in [12]. We also include some data concerning E_8.

$W: F_4$

$\mathscr{F}: 12_1$

$\mathfrak{S}_4$	3
$\mathfrak{S}_3$	3
Dyh_8	1
$\mathfrak{S}_2$	1
$\mathfrak{S}_2 \times \mathfrak{S}_2$	4

$W: E_6$

$\mathscr{F}: 80_s$

e	20
$\mathfrak{S}_3$	10
$\mathfrak{S}_2$	50

$W: E_7$

$\mathscr{F}: 315'_a$

e	35
$\mathfrak{S}_3$	70
$\mathfrak{S}_2$	210

$W: E_8$

$\mathscr{F}: 1400_z$

e	56
$\mathfrak{S}_3$	448
$\mathfrak{S}_2$	896

$W: E_8$

$\mathscr{F}: 1400_x$

e	350
$\mathfrak{S}_3$	175
$\mathfrak{S}_2$	875

$W: E_8$

$\mathscr{F}: 4480_y$

$\mathfrak{S}_2 \times \mathfrak{S}_2$	1092
$\mathfrak{S}_3 \times \mathfrak{S}_2$	1596
$\mathfrak{S}_2$	70
$\mathfrak{S}_3$	378
Dyh_8	168
$\mathfrak{S}_4$	756
$\mathfrak{S}_5$	420

Step 3. Put $I(\mathscr{H}):=\{\iota\in I | g(\mathscr{H},\iota)>0\}$, $\Gamma(\mathscr{H}):=\Gamma(\mathscr{H},1)$, $d(\mathscr{H}):=d(\Gamma(\mathscr{H}))$, and for $\iota\in I(\mathscr{H})$, $U^0(\mathscr{H},\iota):=U^0(\mathscr{H},1,\iota,1)$, and $U(\mathscr{H},\iota):=U^0(\mathscr{H},\iota)\otimes_{\boldsymbol{Z}_p}\boldsymbol{Q}_p$. Then $\{U^0(\mathscr{H},\iota)|\iota\in I(\mathscr{H})\}$ are the totality (without repetition) of the principal indecomposable $J(\boldsymbol{Z}_p)$-modules up to isomorphism which appear as direct summands of $\boldsymbol{Z}_p\cdot t(\Gamma(\mathscr{H}))$. By (3.3), $\{t(d(\mathscr{H}))U^0(\mathscr{H},\iota)|\iota\in I(\mathscr{H})\}$ are the totality (without repetition) of the principal indecomposable $\boldsymbol{Z}_p\cdot t(\Gamma(\mathscr{H})\cap\Gamma(\mathscr{H})^{-1})$-modules up to isomorphism, and they are naturally in one-to-one correspondence with the columns of the table $(\mathscr{G},\mathscr{H})$ in (2.3). (Cf. (2.2), (2.5) and [3, I, (17.8)].) More precisely, the table $(\mathscr{G},\mathscr{H})$ in (2.3) has $|I(\mathscr{H})|$ columns in number, and if we denote by $\mu(m,k)=\mu(\mathscr{H},m,k)$ ($m\in\mathscr{M}_{\mathscr{G}|\mathscr{H}}(\mathscr{G})$ [17, 2.6], $1\leq k\leq|I(\mathscr{H})|$) the entries in the kth column, then there is a unique $\iota\in I(\mathscr{H})$ such that

$$(t(d(\mathscr{H}))U^0(\mathscr{H},\iota))\otimes_{\boldsymbol{Z}_p}\boldsymbol{Q}_p\simeq\bigoplus_m\mu(\mathscr{H},m,k)(t(d(\mathscr{H}))E^m),$$

or equivalently, such that

(9) $$U^0(\mathscr{H},\iota)\otimes_{Z_p}Q_p \simeq \bigoplus_m \mu(\mathscr{H},m,k)E^m.$$

Cf. (3.3). (Note that the column vectors of each table are different from each other. Note also that $(t(d(\mathscr{H}))U^0(\mathscr{H},\iota))\otimes_{Z_p}Q_p = t(d(\mathscr{H}))(U^0(\mathscr{H},\iota)\otimes_{Z_p}Q_p)$.

Thus we may take $I(\mathscr{H})=\{\iota \mid 1\le\iota\le|I(\mathscr{H})|\}$ and we may assume that ιth column is associated with $\iota\in I(\mathscr{H})$. By (6),

$$Z_p\cdot t(\Gamma(\mathscr{H}))\simeq \bigoplus_{\iota\in I(\mathscr{H})} g(\mathscr{H},\iota)U^0(\iota)$$

and

$$Z_p\cdot t(\Gamma(\mathscr{H})\cap\Gamma(\mathscr{H})^{-1})\simeq \bigoplus_{\iota\in I(\mathscr{H})} g(\mathscr{H},\iota)(t(d(\mathscr{H}))U^0(\iota)).$$

Hence the number written at the top of the ιth column of each table in (2.3) is $g(\mathscr{H},\iota)$, and hence the values of $b(\mathscr{H})g(\mathscr{H},\iota)$ (= the number of direct summands $U^0(\mathscr{H},i,\iota',j)$ $(1\le i\le b(\mathscr{H}),\ \iota'\in I,\ 1\le j\le g(\mathscr{H},\iota))$ with the given $\mathscr{H}$, and which are isomorphic to $U^0(\iota)$) are explicitly determined, which we shall write down in the following. (We exclude $(\mathscr{H},\iota)$ if $U(\mathscr{H},\iota)$ is irreducible.) The first column contains pairs $(\mathscr{H},\iota)$ $(\mathscr{H}\in \boldsymbol{H},\ \iota\in I(\mathscr{H}))$, which are divided into several groups so that the corresponding $J(Q_p)$-modules (or Q_pW-modules) $U(\mathscr{H},\iota)=U^0(\mathscr{H},\iota)\otimes_{Z_p}Q_p$ are mutually isomorphic to each other among each group. We write the isomorphism class of $U(\iota)=U^0(\iota)\otimes_{Z_p}Q_p$ at the most right column. The remaining columns give the values of $b(\mathscr{H})g(\mathscr{H},\iota)$. Without to say, these values depend on W and $\mathscr{F}$, which we write at the top of the column. We specify by the special representation contained in it. (Cf. [12, §4].) We also include some data concerning E_8.

Table ($\mathscr{G}=\mathfrak{S}_3$, $p=2$)

$(\mathscr{H},\iota)$	E_6, 80_s	E_7, $315'_a$	E_8, 1400_z	E_8, 1400_x	$U(\mathscr{H},\iota)$
$(\mathfrak{S}_3,1)$	10	70	448	175	$(1,1)+(g_2,1)$
$(\mathfrak{S}_2,1)$	50	210	896	875	
$(e,1)$	20	35	56	350	$(1,1)+(1,\varepsilon)$

Table ($\mathscr{G}=\mathfrak{S}_3$, $p=3$)

$(\mathscr{H},\iota)$	E_6, 80_s	E_7, $315'_a$	E_8, 1400_z	E_8, 1400_x	$U(\mathscr{H},\iota)$
$(\mathfrak{S}_3,1)$	10	70	448	175	$(1,1)+(g_3,1)$
$(\mathfrak{S}_2,1)$	50	210	896	875	$(1,1)+(1,r)$
$(e,1)$	20	35	56	350	
$(e,2)$	20	35	56	350	$(1,r)+(1,\varepsilon)$

Table ($\mathscr{G}=\mathfrak{S}_4$, $p=2$)

$(\mathscr{H},\iota)$	F_4, 12_1	$U(\mathscr{H},\iota)$
$(\mathfrak{S}_4,1)$ $(\mathrm{Dyh}_8,1)$	3 1	$12_1+16_1+9_2+4_3$
$(\mathfrak{S}_3,1)$ $(\mathfrak{S}_2,1)$	3 1	$12_1+9_3+16_1+4_4$
$(\mathrm{Dyh}_8,2)$	1	$6_2+9_2+1_2$
$(\mathfrak{S}_2,2)$	1	$9_3+1_3+6_2$
$(\mathfrak{S}_2+\mathfrak{S}_2,1)$	4	$12_1+9_3+6_2+2\cdot 16_1+9_2+4_1$

Table ($\mathscr{G}=\mathfrak{S}_4$, $p=3$)

$(\mathscr{H},\iota)$	F_4, 12_1	$U(\mathscr{H},\iota)$
$(\mathfrak{S}_4,1)$ $(\mathfrak{S}_3,1)$	3 3	12_1+6_1
$(\mathrm{Dyh}_8,1)$ $(\mathfrak{S}_2,1)$ $(\mathfrak{S}_2\times\mathfrak{S}_2,1)$	1 1 4	12_1+6_2

Note that we can now get (2), using (2.1), (9), [3,I, (17.8)], and noting that any column of any table $(\mathscr{G},\mathscr{H})$ in (2.3) (i.e., $\{\mu(\mathscr{H},m,k)|m\in\mathscr{M}_{\mathscr{G}/\mathscr{H}}(\mathscr{G})\}$ for various $\mathscr{G},\mathscr{H},k$) does not have a common divisor >1. Since the statement concerning (I) of (4) follows from [8, (2.17)], it remains to prove (1) and the part of (4) concerning (II).

Step 4. Let us consider the case where W is of type G_2, and $\mathscr{F}$ is the family containing the reflection representation. Then $\mathscr{G}=\mathfrak{S}_3$ and each E^m ($m\in\mathscr{M}_0(\mathscr{G})$) can be explicitly realized using W-graphs [9] as follows.

$1,\ 1$	$\boxed{s_2}\Lleftarrow\boxed{s_1}$
$g_2,\ 1$	$\boxed{s_2}$—$\boxed{s_1}$
$1,\ r$	$\boxed{s_2}$ $(=\varepsilon_1)$
$g_3,\ 1$	$\boxed{s_1}$ $(=\varepsilon_2)$

See [12, 4.8] for ε_1 and ε_2. Hence we can prove by a direct calculation.

Step 5a. Let W be of type E_6, $\mathscr{F}$ the family containing 80_s, and $p=2$. To prove (1), it is enough to show that

$$U(\mathscr{H},\iota)\simeq U(\mathscr{H}',\iota') \Rightarrow U^0(\mathscr{H},\iota)\simeq U^0(\mathscr{H}',\iota') \quad (\Leftrightarrow \iota=\iota'). \tag{10}$$

We have already shown that among all the direct summands $\{U^0(\mathscr{H},i,\iota,j)\}_{\mathscr{H},i,\iota,j}$ of $J(\boldsymbol{Z}_2)$, 10 are mutually isomorphic ($\simeq U^0(\mathfrak{G}_3,1)$), other 50 are also mutually isomorphic ($\simeq U^0(\mathfrak{S}_2,1)$), but members of these two different groups are possibly non-isomorphic to each other (even if they become isomorphic to $E^{1,1}+E^{g_2,1}$ after $\otimes_{\boldsymbol{Z}_2}\boldsymbol{Q}_2$). We have also shown that the remaining 20 are mutually isomorphic ($\simeq U^0(e,1)$), and become isomorphic to $E^{1,1}+E^{1,\varepsilon}$ after $\otimes_{\boldsymbol{Z}_2}\boldsymbol{Q}_2$. Cf. the column $(E_6,80_s)$ of Table ($\mathscr{G}=\mathfrak{S}_3, p=2$) in Step 3. Let us express these possibilities as follows.

(11a)

		10+50	20
80_s	1, 1	1	1
60_s	g_2, 1	1	·
20_s	1, ε	·	1

From this table, we can see that the irreducible $J(\boldsymbol{Q}_2)$-module 60_s decomposes, after the 'reduction modulo 2', as follows.

$$([60_s] \bmod 2)=[\![60]\!], \quad \text{or} \tag{12a}$$

$$([60_s] \bmod 2)=[\![10]\!]+[\![50]\!] \tag{13a}$$

in $K(J(\boldsymbol{F}_2))$. Here, for example, 10 means some (absolutely) irreducible $J(\boldsymbol{F}_2)$-module of dimension 10, and $[\![10]\!]$ means the corresponding element of the Grothendieck group. (Cf. (2.1), (2), and [3,I, (17.8)].) Assume that (13a) holds. In the same way as in (3.1), we can show that $H(\boldsymbol{F}_2(\sqrt{q}))\xrightarrow{\sim} J(\boldsymbol{F}_2(\sqrt{q}))$. Hence 10 etc. can be regarded as absolutely irreducible $H(\boldsymbol{F}_2(\sqrt{q}))$-modules. Thus we can consider the specialization

$$([60_s] \bmod 2)|_{\sqrt{q}\to 1}=[\![10]\!]+[\![50]\!]|_{\sqrt{q}\to 1} \qquad (\in K(\boldsymbol{F}_2 W)),$$

which should coincide with

(15a) $(([60_s]|_{\sqrt{q}\to 1} \bmod 2)=2[\![6]\!]+[\![8]\!]+[\![40]\!]$ (cf. the reference [5]; $60_s=\phi_{60,8}$),

since the following diagram is commutative.

$$\begin{array}{ccc}
K(J(\boldsymbol{Q}_2)) & \xrightarrow{\bmod 2} & K(J(\boldsymbol{F}_2)) \\
\| & & \| \\
K(H(\boldsymbol{Q}_2(\sqrt{q}))) & \xrightarrow{\bmod 2} & K(H(\boldsymbol{F}_2(\sqrt{q}))) \\
\downarrow{\scriptstyle \sqrt{q}\to 1} & & \downarrow{\scriptstyle \sqrt{q}\to 1} \\
K(\boldsymbol{Q}_2 W) & \xrightarrow{\bmod 2} & K(\boldsymbol{F}_2 W)
\end{array}$$

(The commutativity can be proved as follows. For $E\in K(H(\boldsymbol{Q}_2(\sqrt{q})))$, take a $\boldsymbol{Z}[\sqrt{q}]$-module E^0 such that $E^0\otimes \boldsymbol{Q}_2(\sqrt{q})=E$. Cf. the reference [7]. Then we can define $E^0\otimes_{\boldsymbol{Z}[\sqrt{q}]}\boldsymbol{F}_2[\sqrt{q}]$, $E^0\otimes \boldsymbol{Z}$ $(=E^0|_{\sqrt{q}\to 1})$, $(E^0\otimes \boldsymbol{F}_2[\sqrt{q}])\otimes \boldsymbol{F}_2$ $(=E^0\otimes \boldsymbol{F}_2[\sqrt{q}]|_{\sqrt{q}\to 1})$, and $(E^0\otimes \boldsymbol{Z})\otimes \boldsymbol{F}_2$, where the last two modules are isomorphic to each other. Their scalar extensions give $([E] \bmod 2)$, $[E]|_{\sqrt{q}\to 1}$, $([E] \bmod 2)|_{\sqrt{q}\to 1}$, and $(([E]|_{\sqrt{q}\to 1}) \bmod 2)$, respectively. Hence the diagram is commutative.)

But comparing dimensions, we can see that

$$[\![10]\!]+[\![50]\!]|_{\sqrt{q}\to 1}=2[\![6]\!]+[\![8]\!]+[\![40]\!]$$

is impossible. Hence (12a) holds, which implies (10) and also (4) in this case.

Step 5b. Let W and $\mathscr{F}$ be the same as in Step 5a, and $p=3$. The argument is almost the same as in Step 5a. (11a)–(13a) are replaced with the following.

(11b)

		10	50+20	20
80_s	1, 1	1	1	·
90_s	1, r	·	1	1
10_s	g_3, 1	1	·	·
20_s	1, ε	·	·	1

(12b) $\quad ([80_s] \bmod 3)=[\![10]\!]+[\![70]\!],\quad$ or

(13b) $\quad ([80_s] \bmod 3)=[\![10]\!]+[\![50]\!]+[\![20]\!].$

Assume that (13b) holds. Consider

(14b) $[\![10]\!]+[\![50]\!]+[\![20]\!]|_{\sqrt{q}\to\sqrt{-1}}$ $(\in K(H(\boldsymbol{F}_3(\sqrt{-1}))))$, where $\boldsymbol{F}_3(\sqrt{-1})$ is considered as an $\mathscr{A}$-algebra by $\sqrt{q}\to\sqrt{-1}\in \boldsymbol{F}_3(\sqrt{-1})$.

Then (14b) should coincide with

(15b) $((([80_s]|_{\sqrt{q}\to\sqrt{-1}}) \bmod 3)=2[\![1]\!]+2[\![13]\!]+2[\![10]\!]+[\![32]\!]$ in $K(H(\boldsymbol{F}_3(\sqrt{-1})))$ (cf.

the reference [5]; $80_s = \phi_{80,7}$),

which is impossible. Hence (12b) holds, and the proof is over in this case.

Step 5c. Let W be of type F_4, $\mathscr{F}$ the family containing 12_1, and $p=2$. (11a) is replaced with

(11c)

		3+1	3+1	4	1	1
12_1	$1, 1$	1	1	1	·	·
16_1	$g_2, 1$	1	1	2	·	·
9_2	$g'_2, 1$	1	·	1	1	·
9_3	$1, \lambda^1$	·	1	1	·	1
4_3	$g_4, 1$	1	·	·	·	·
4_4	g_2, ε''	·	1	·	·	·
6_2	$1, \sigma$	·	·	1	1	1
4_1	g'_2, ε''	·	·	1	·	·
1_2	g'_2, ε'	·	·	·	1	·
1_3	$1, \lambda^2$	·	·	·	·	1

Concerning the first (resp. the second) '3+1' in (11c), we should consider the two possibilities

$$([4_i] \bmod 2) = [\![4]\!] \text{ or } [\![3]\!] + [\![1]\!],$$

where $i=3$ (resp. $i=4$). By [6, Theorem C] ($4_3 = \phi''_{4,7}$, $4_4 = \phi'_{4,7}$),

$$([4_i] \bmod 2)|_{\sqrt{q}\to 1} = (([4_i]|_{\sqrt{q}\to 1}) \bmod 2) = [\![2]\!] + [\![2']\!],$$

where 2 and 2′ mean some irreducible F_2W-modules. Hence $([4_i] \bmod 2) = [\![4]\!]$, and the proof is over in this case.

Step 5d. Let W and $\mathscr{F}$ be the same as in Step 5c, and $p=3$. (11a) is replaced with

(11d)

		3+3	4+1+1
12_1	$1, 1$	1	1
6_1	$g_3, 1$	1	·
6_2	$1, \sigma$	·	1

Then we have possibilities

$$([6_1] \bmod 3) = [\![6]\!], \text{ or } [\![3]\!] + [\![3]\!],$$
$$([6_2] \bmod 3) = [\![6]\!], [\![5]\!] + [\![1]\!], [\![4]\!] + [\![2]\!], \text{ or } [\![4]\!] + [\![1]\!] + [\![1]\!].$$

Since

$$([6_i] \bmod 3)|_{\sqrt{q}\to 1} = (([6_i]|_{\sqrt{q}\to 1}) \bmod 3) = [\![6]\!]$$

[6, Theorem C] ($6_1=\phi'_{6,6}$, $6_2=\phi''_{6,6}$; one of $\phi'_{6,6}$'s in the most right table in [6, Theorem C] should be read as $\phi''_{6,6}$), $([6_i] \bmod 3)=[\![6]\!]$, and the proof is over in this case.

Step 5e. Let W be of type E_7, $\mathscr{F}$ the family containing $315'_a$, and $p=3$. (11a) is replaced with

(11e)

		70	210+35	35
$315'_a$	1, 1	1	1	·
$280'_a$	1, r	·	1	1
$70'_a$	g_3, 1	1	·	·
$35'_a$	1, ε	·	·	1

Then we have possibilities

$$([315'_a] \bmod 3)=[\![70]\!]+[\![245]\!] \quad \text{or} \quad [\![70]\!]+[\![210]\!]+[\![35]\!].$$

Since

$$([315'_a] \bmod 3)|_{\sqrt{q}\to 1}=([315'_a]|_{\sqrt{q}\to 1} \bmod 3)=[\![21]\!]+2[\![49]\!]+[\![196]\!]$$

[10, Table XI-i] ($315'_a=315_n$), $([315'_a] \bmod 3)=[\![70]\!]+[\![245]\!]$, and the proof is over in this case.

Step 5f. Let us consider the case where W is of type E_7, $\mathscr{F}$ is the family containing $315'_a$, and $p=2$. The argument is the same as in Step 5a, halfway. (11a) is replaced with

(11f)

		70+210	35
$315'_a$	1, 1	1	1
280_b	g_2, 1	1	·
$35'_a$	1, ε	·	1

Then

(12f) $\qquad ([280_b] \bmod 2)=[\![280]\!]$, or

(13f) $\qquad ([280_b] \bmod 2)=[\![70]\!]+[\![210]\!]$.

To show that (13f) does not occur, similar argument as in Step 5a does not work. Since we need some preliminary, we postpone the remainder of the proof to (4.9).

4

4.1. Let A be an associative ring with the identity element 1_A. We assume that A is a finite dimensional vector space over a field k, and $k \ni x \to x \cdot 1_A \in A$ gives a k-algebra structure in A. Let M be an A-module. We assume that M is a k-vector space of finite dimension, and that $(\alpha a)m = a(\alpha m) = \alpha(am)$ for $\alpha \in k$, $a \in A$, $m \in M$, but we *do not* assume that the A-module structure on M is unitary. In other words, possibly $1_A \cdot m \neq m$ for some $m \in M$. Let $\tilde{K}(A)$ be the Grothendieck group of such (possibly non-unitary) A-modules of finite k-dimension. Then $\tilde{K}(A)$ can be described in terms of the Grothendieck group of (unitary) A-modules of finite type as follows.

Define an A-module ω so that $\omega = k$ as a k-vector space, and $ax = 0$ for any $a \in A$ and $x \in \omega$. Then $\tilde{K}(A) = K(A) \oplus \boldsymbol{Z} \cdot [\omega]$, where $[\omega]$ is the class in $\tilde{K}(A)$ of ω. In fact, for any A-module M in $\tilde{K}(A)$, the decomposition $m = 1_A \cdot m + (m - 1_A \cdot m)$ gives the direct sum decomposition

$$M = \{m \in M \mid 1_A \cdot m = m\} \oplus \{m \in M \mid 1_A \cdot m = 0\},$$

where in the right hand side, the class of the first factor belongs to $K(A)$, and the class of the second factor belongs to $\boldsymbol{Z} \cdot [\omega]$. Let $j : \tilde{K}(A) \to K(A)$ be the projection, and $j^* : K(A) \to \tilde{K}(A)$ the inclusion.

Linearly extending $M \mapsto (A \ni a \to \mathrm{Tr}(a, M))$ for any A-module M in $\tilde{K}(A)$, we can define $\mathrm{Tr}(a, M)$ $(a \in M)$ for any $M \in \tilde{K}(A)$. Then, for $M \in \tilde{K}(A)$,

(1) $j(M) = 0$ if and only if $\mathrm{Tr}(a, M) = 0$ for any $a \in A$.

Let A' be a subalgebra of A satisfying the same assumption as A. Especially A' has the identity element $1_{A'}$, but possibly $1_A \neq 1_{A'}$. Define $\text{J-res}^A_{A'} : K(A) \to K(A')$ as the composition of

$$K(A) \xrightarrow{\textit{restriction}} \tilde{K}(A') \xrightarrow{j} K(A').$$

Define $\text{J-ind}^A_{A'} : K(A') \to K(A)$ as the composition of

$$K(A') \xrightarrow{j^*} \tilde{K}(A') \xrightarrow{A \otimes_{A'}} K(A).$$

(We define

$$A \otimes_{A'} M' = \mathrm{Hom}_{\text{right } A'\text{-module}}(A, M'^{\vee})^{\vee}$$

for an A'-module M' in $\tilde{K}(A')$, where $^{\vee}$ denotes the dual k-vector space.) For a unitary A-module M and a unitary A'-module M', put $\text{J-res}^A_{A'} M = 1_{A'} \cdot M$ and $\text{J-ind}^A_{A'} M' = A \otimes_{A'} M$. Then $\text{J-res}^A_{A'}[M] = [\text{J-res}^A_{A'} M]$ and $\text{J-ind}^A_{A'}[M'] = [\text{J-ind}^A_{A'} M]$.

If A is a semisimple k-algebra, we define a non-degenerate bilinear form $\langle\rangle_A$ on $K(A)$ so that for (unitary) simple A-modules E_1 and E_2, $\langle [E_1], [E_2] \rangle_A = 1$ if

$E_1 \simeq E_2$, and $=0$ if $E_1 \not\simeq E_2$.

If A and A' are semisimple k-algebras, then

$$(2) \qquad \langle \text{J-ind}_{A'}^{A}[M'],[M]\rangle_A = \langle [M'],\text{J-res}_{A'}^{A}[M']\rangle_{A'}$$

for any $[M] \in K(A)$ and $[M'] \in K(A')$.

Proof. We have natural identification

$$\mathrm{Hom}_A(\text{J-ind}_{A'}^{A}M',M) = \mathrm{Hom}_A(A \otimes_{A'} M',M) = \mathrm{Hom}_A((A \cdot 1_{A'}) \otimes_{A'} M',M)$$

$$= \mathrm{Hom}_{A'}(M',M|A') = \mathrm{Hom}_{A'}(M',\text{J-res}_{A'}^{A}M).$$

(Note that $A \cdot 1_{A'}$ is a unitary right A'-module.)

4.2. Let W' be a standard parabolic subgroup of W. For a $\boldsymbol{Z}$-algebra (resp. an $\mathscr{A}$-algebra) $\mathscr{B}$, put $J'(\mathscr{B}) := \mathscr{B} \cdot t(W')$ (resp. $H'(\mathscr{B}) := \mathscr{B} \cdot C(W')$). Note that $J'(\mathscr{B})$ (resp. $H'(\mathscr{B})$) is naturally isomorphic to the asymptotic Hecke algebra (resp. the Hecke algebra) of W' over $\mathscr{B}$. When $\mathscr{B}$ is a field, we consider $\text{J-res}_{J'(\mathscr{B})}^{J(\mathscr{B})}$ and $\text{J-ind}_{J'(\mathscr{B})}^{J(\mathscr{B})}$ by taking $A = J(\mathscr{B})$, $A' = J'(\mathscr{B})$ and $k = \mathscr{B}$ in (4.1). If there is no fear of confusion, we write $\text{J-res}_{W'}^{W}$ for $\text{J-res}_{J'(\mathscr{B})}^{J(\mathscr{B})}$, and $\text{J-ind}_{W'}^{W}$ for $\text{J-ind}_{J'(\mathscr{B})}^{J(\mathscr{B})}$.

4.3. For an irreducible $H(\boldsymbol{Q}(\sqrt{q}))$-module E, define an integer $a(E) \geq 0$ so that

$$\sqrt{q}^{a(E)}\mathrm{Tr}(C(x),E) \in \boldsymbol{Q}[\sqrt{q}] \text{ for all } x \in W, \text{ and}$$

$$\sqrt{q}^{a(E)-1}\mathrm{Tr}(C(x),E) \notin \boldsymbol{Q}[\sqrt{q}] \text{ for some } x \in W.$$

Suppose that $[E]|H'(\boldsymbol{Q}(\sqrt{q})) = \Sigma_{E'} m(E')[E']$ in $K(H'(\boldsymbol{Q}(\sqrt{q})))$, where E' runs over the set of isomorphism classes of the irreducible $H'(\boldsymbol{Q}(\sqrt{q}))$-modules. Define

$$\text{J-res}_{W'}^{W}[E] := \sum_{\substack{E' \\ a(E')=a(E)}} m(E')[E'],$$

and linearly extend it to $\text{J-res}_{W'}^{W} : K(H(\boldsymbol{Q}(\sqrt{q}))) \to K(H'(\boldsymbol{Q}(\sqrt{q})))$. (In general, $a(E') \leq a(E)$ if $m(E') \neq 0$.)

Let E' be an irreducible $H'(\boldsymbol{Q}(\sqrt{q}))$-module, and $[H(\boldsymbol{Q}(\sqrt{q})) \otimes_{H'(\boldsymbol{Q}(\sqrt{q}))} E'] = \Sigma_E m(E)[E]$ in $K(H(\boldsymbol{Q}(\sqrt{q})))$, where E runs over the set of isomorphism classes of the irreducible $H(\boldsymbol{Q}(\sqrt{q}))$-module. In [11], G.Lusztig has defined

$$\text{J-ind}_{W'}^{W}[E'] := \sum_{\substack{E \\ a(E')=a(E)}} m(E)[E].$$

Linearly extend it to $\text{J-ind}_{W'}^{W} : K(H'(\boldsymbol{Q}(\sqrt{q}))) \to K(H(\boldsymbol{Q}(\sqrt{q})))$. (To get along with

J-res$^W_{W'}$, we write here J-ind$^W_{W'}$for $J^W_{W'}$.)

Lemma 4.4. *For any prime number p, the following diagrams are commutative.*

(1)
$$\begin{array}{ccc}
K(H(\mathbf{Q}(\sqrt{q}))) & \xrightarrow{\text{J-res}^W_{W'}} & K(H'(\mathbf{Q}(\sqrt{q}))) \\
\| & & \| \\
K(J(\mathbf{Q})) & \xrightarrow{\text{J-res}^W_{W'}} & K(J'(\mathbf{Q})) \\
{\scriptstyle \bmod p}\downarrow & & {\scriptstyle \bmod p}\downarrow \\
K(J(\mathbf{F}_p)) & \xrightarrow{\text{J-res}^W_{W'}} & K(J'(\mathbf{F}_p)),
\end{array}$$

(2)
$$\begin{array}{ccc}
K(H(\mathbf{Q}(\sqrt{q}))) & \xleftarrow{\text{J-ind}^W_{W'}} & K(H'(\mathbf{Q}(\sqrt{q}))) \\
\| & & \| \\
K(J(\mathbf{Q})) & \xleftarrow{\text{J-ind}^W_{W'}} & K(J'(\mathbf{Q})).
\end{array}$$

Proof. Let E be an irreducible $H(\mathbf{Q}(\sqrt{q}))$-module. Then by the proof of [17, 3.3],

$$\mathrm{Tr}(C(x),E)=\mathrm{Tr}(t(x),E)\cdot(-\sqrt{q})^{-a(E)} + \text{higher power of } (-\sqrt{q})$$

for any $x\in W$. Let 1_J (resp. $1_{J'}$) be the identity element of J (resp. J'). (By [13, 6.3,(e)] and [14, 2.3], $1_J=\Sigma_{d\in\mathscr{D}}t(d)$ and $1_{J'}=\Sigma_{d\in\mathscr{D}\cap W'}t(d)$, where $\mathscr{D}$ is the set of distinguished involutions of W [14, 1.4].) For $x\in W'$

$$\mathrm{Tr}(t(x),E)=\mathrm{Tr}(t(x),\ 1_{J'}\cdot E)=\mathrm{Tr}(t(x),\text{J-res}^{J(\mathbf{Q})}_{J'(\mathbf{Q})}E)$$

and, in the notation of (4.3),

$$\begin{aligned}\mathrm{Tr}(C(x),E)&=\sum_{E'}m(E')\mathrm{Tr}(C(x),E')\\
&=\sum_{\substack{E'\\ a(E')=a(E)}} m(E')\mathrm{Tr}(t(x),E')\cdot(-\sqrt{q})^{-a(E)} + \text{higher powers of } (-\sqrt{q}).\end{aligned}$$

Hence

$$\mathrm{Tr}(t(x),\text{J-res}^{J(\mathbf{Q})}_{J'(\mathbf{Q})}E)=\sum_{\substack{E'\\ a(E')=a(E)}} m(E')\mathrm{Tr}(t(x),E')=\mathrm{Tr}(t(x),\text{J-res}^{H(\mathbf{Q})}_{H'(\mathbf{Q})}E),$$

and the commutativity of the upper square of (1) follows from (4.1, (1)).

Considering its adjoint, we get the commutative diagram (2). Cf. (4.1,(2)).

Next identify $K(J(\mathbf{Q}))=K(J(\mathbf{Q}_p))$ and $K(J('\mathbf{Q}))=K(J'(\mathbf{Q}_p))$. Let E be an irreducible

$J(\boldsymbol{Q}_p)$-module, E^0 a $J(\boldsymbol{Z}_p)$-stable lattice of E, $E_1^0 := 1_{J'} \cdot E^0$, and $E_2^0 := (1_J - 1_{J'}) \cdot E^0$. Then $E^0 = E_1^0 \oplus E_2^0$, E_1^0 is a $J'(\boldsymbol{Z}_p)$-stable $\boldsymbol{Z}_p$-lattice of $E_1 := 1_{J'} \cdot E$, and $1_{J'}$ acts as the identity automorphism on $E_1^0 \otimes_{\boldsymbol{Z}_p} \boldsymbol{F}_p$, and as zero on $E_2^0 \otimes_{\boldsymbol{Z}_p} \boldsymbol{F}_p$. Hence

$$\text{J-res}_{W'}^{W}[E] := [E_1],$$

$$((\text{J-res}_{W'}^{W}[E] \ \operatorname{mod} p) = [E_1^0 \otimes_{\boldsymbol{Z}_p} \boldsymbol{F}_p],$$

$$([E] \ \operatorname{mod} p) = [E^0 \otimes_{\boldsymbol{Z}_p} \boldsymbol{F}_p], \text{ and}$$

$$\text{J-res}_{W'}^{W}([E] \ \operatorname{mod} p) = [E_1^0 \otimes_{\boldsymbol{Z}_p} \boldsymbol{F}_p].$$

Hence we get the commutativity of the lower square of (1).

Lemma 4.6. [1],[2], ([12, 5.15]). *There is a one-to-one correspondence $\mathscr{F} \leftrightarrow \Omega(\mathscr{F})$ between the set of families for W [12, 4.2] and the set of two-sided cells of W [9], characterized by the following property. For a two-sided cells Ω, put $1_\Omega := \Sigma_{d \in \mathscr{D} \cap \Omega} t(d)$. Consider $\mathscr{F}$ as a set of isomorphism classes of (absolutely) irreducible $J(\boldsymbol{Q})$-modules. (Cf. Convention.) Then for an irreducible $J(\boldsymbol{Q})$-module E,*

$$E \in \mathscr{F} \Leftrightarrow 1_{\Omega(\mathscr{F})} \cdot E = E.$$

Lemma 4.7. *Let $\mathscr{F}$ be a family of W whose associated group $\mathscr{G}$ is not isomorphic to $\mathfrak{S}_4$ or $\mathfrak{S}_5$. Assume that for any constructible representation c of W whose irreducible constituents are all contained in $\mathscr{F}$, there exists a constructible representation c' of W' such that* $\text{J-ind}_{W'}^{W}(c') = c$. *Then for any $\boldsymbol{Z}$-algebra $\mathscr{B}$,*

(1) $\mathscr{B} \cdot t(\Omega) \subset$ (*the two sided ideal of $J(\mathscr{B})$ generated by* $1_{J'}$) $\quad (\Omega := \Omega(\mathscr{F}))$.

Proof. Let $d \in \mathscr{D} \cap \Omega$, and Γ the left cell of W containing d. Then $\boldsymbol{C} \cdot t(\Gamma)$ is a constructible representation of W [16], whose irreducible constituents are all contained in $\mathscr{F}$, and hence, by our assumption, there exists a left cell Γ' of W' such that

$$\text{J-ind}_{W'}^{W}(\boldsymbol{C} \cdot t(\Gamma')) \simeq \boldsymbol{C} \cdot t(\Gamma) \tag{2}$$

Let Γ_1 be the left cell of W containing Γ', and $d_1 := d(\Gamma') = d(\Gamma_1)$. Then $d_1 \in \mathscr{D} \cap W'$, $t(\Gamma_1) = t(W) \cdot t(d_1) \cdot 1_{J'} \subset t(W) \cdot 1_{J'}$, and

$$\text{J-ind}_{W'}^{W}(\boldsymbol{C} \cdot t(\Gamma')) \simeq \boldsymbol{C} \cdot t(\Gamma_1). \tag{3}$$

Here (3) is proved as follows. The natural homomorphism

$$\text{J-ind}_{W'}^{W}(\boldsymbol{C} \cdot t(\Gamma')) = J(\boldsymbol{C}) \otimes_{J'(\boldsymbol{C})} (J'(\boldsymbol{C}) \cdot t(d_1)) \to J(\boldsymbol{C}) \cdot t(d_1) = \boldsymbol{C} \cdot t(\Gamma_1)$$

gives a surjection between constructible representations [16]. On the other hand, a surjection between constructible representations is necessarily an isomorphism, as is seen from their explicit description [11].

By (2),(3),(1.3,(7)), and (1.4,(1)), there exists $y \in \Gamma_1 \cap \Gamma^{-1}$ such that

$$t(d) \in t(\Gamma) = t(\Gamma_1)t(y^{-1}) \subset t(W) \cdot 1_{J'} \cdot t(W).$$

Hence the right hand side of (1) contains the two-sided ideal generated by $1_\Omega = \Sigma_{d \in \mathscr{D} \cap \Omega} t(d)$, i.e., $\mathscr{B} \cdot t(\Omega)$.

Lemma 4.8. *Let E be an irreducible $J(\boldsymbol{Q}_p)$-module, E^0 a $J(\boldsymbol{Z}_p)$-stable $\boldsymbol{Z}_p$-lattice of E, M ($\neq 0$) an irreducible $J(\boldsymbol{F}_p)$-module appearing as a composition factor of $E^0 \otimes_{\boldsymbol{Z}_p} \boldsymbol{F}_p$, and $\mathscr{F}$ the family for W* [12, 4.2] *containing E (cf. Convention). If $\mathscr{F}$ satisfies the same assumption as in* (4.7), *then*

$$1_{J'} \cdot M \neq 0, \textit{ i.e., } \text{J-res}^W_{W'}[M] \neq 0 \textit{ in } K(J'(\boldsymbol{F}_p)).$$

Proof. Assume that $1_{J'} \cdot M = 0$. Then the annihilator $\mathrm{Ann}_{J(\boldsymbol{F}_p)} M$ contains the two-sided ideal of $J(\boldsymbol{F}_p)$ generated by $1_{J'}$, and hence contains $\boldsymbol{F}_p \cdot t(\Omega)$ ($\Omega := \Omega(\mathscr{F})$). Especially $1_\Omega \cdot M = 0$. On the other hand, since $E \in \mathscr{F}$, 1_Ω acts as the identity automorphism on E (cf. (4.6)), on E^0, and on M. Thus we get a contradiction.

4.9. End of the proof of Theorem C$'$. The remaining is the case where W is of type E_7, $\mathscr{F}$ is the family of W containing $315'_a$, and $p = 2$, and we should prove that (13f) in Step 5f does not hold. Assume that (13f) holds. Let W' ($W' \subset W$) be the standard parabolic subgroup of type E_6. Then by [12, (4.12.2)], by (13f), and by (12a),

$$\begin{array}{ccc} [280_b] & \xrightarrow{\text{J-res}^W_{W'}} & [60_s] \\ {\scriptstyle \text{mod } 2}\downarrow & & {\scriptstyle \text{mod } 2}\downarrow \\ [\![70]\!] + [\![210]\!] & \xrightarrow{\text{J-res}^W_{W'}} & [\![60]\!]. \end{array}$$

Hence $\text{J-res}^W_{W'}[\![70]\!] = 0$ or $\text{J-res}^W_{W'}[\![210]\!] = 0$. But by (4.8) either possibility can not occur. (The assumption in (4.7) is satisfied, as is seen from the explicit description of the constructible representations, cf. the reference [11], and from $315'_a = \text{J-ind}^W_{W'}(80_s)$, $280'_a = \text{J-ind}^W_{W'}(90_s)$, $70'_a = \text{J-ind}^W_{W'}(10_s)$, and $35'_a = \text{J-ind}^W_{W'}(20_s)$.)

REMARK 4.10. As is seen from our proof, even in the case (d) in (0.2), the statement of (3) of Theorem C' still holds. Hence each block of irreducible $J(\boldsymbol{Q}_p)$-modules is contained in a unique family. Fix a family $\mathscr{F}$ whose associated group $\mathscr{G}$ is not isomorphic to $\mathfrak{S}_5$, and restrict our attention to the inside of this family. Then the remaining statements of Theorem C$'$ also hold. As for (2) of

Theorem C′, the argument is the same as before even in the case (d). Let us give an outline of the proof of '(1) and (4) of Theorem C′ in the case (d)'.

(1) If $\mathscr{G}$ is isomorphic to e or $\mathfrak{S}_2$, the argument is the same as the proof of Theorem B′.

(2) Let $\mathscr{F}$ be the family containing 1400_z (resp. 1400_x). Then the argument is the same as in (4.9). Take the standard parabolic subgroup of type E_6 (resp. $E_6 \times A_1$) as W'.

(3) Let $\mathscr{F}$ be the family containing $1400'_z$ (resp. $1400'_x$). By (3.4) and '(2) of Theorem C′ in the case (d)', it is enough to determine the diagonal block of the decomposition matrix of $H(\mathscr{K})$ corresponding to the family $\mathscr{F}$. Noting that the involutive $\mathscr{A}$-algebra automorphism of $H(\mathscr{A})$ defined by $C(s) \to -C(s)-(\sqrt{q}-\sqrt{q}^{-1})$ ($s \in$ {simple reflections}) interchanges 1400_z and $1400'_z$ (resp. 1400_x and $1400'_x$), we can reduce the present case to the case (2).

Remark 4.11. We have worked with *complete* discrete valuation rings, simply because the completeness is assumed in our main reference [3]. In fact, this assumption is not necessary (cf. [8, §1]), and we may replace $\mathscr{R}$ with $\mathscr{A}_p$, $\boldsymbol{Z}_p$ with the localization of $\boldsymbol{Z}$ at p, and $\boldsymbol{Q}_p$ with $\boldsymbol{Q}$.

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References

[1] D. Barbasch, D. Vogan: *Primitive ideals and orbital integrals in complex classical groups*, Math. Ann. **259** (1982), 153–199.
[2] D. Barbasch, D. Vogan: *Primitive ideals and orbital integrals in complex exceptional groups*, J. Algebra **80** (1983), 350–382.
[3] W. Feit: The representation theory of finite groups, North-Holland, Amsterdam, 1982.
[4] P. Fleischmann: *Projective simple modules of symmetric algebras and their specializations with applications to Hecke algebras*, Arch. Math. **55** (1990), 247–258.
[5] M. Geck: *The decomposition numbers of the Hecke algebra of type E_6*, to appear in Math. Comp.
[6] M. Geck, K. Lux: *The decomposition numbers of the Hecke algebra of type F_4*, Manuscripta Math. **70** (1991), 285–306.
[7] A. Gyoja: *On the existence of a W-graph for an irreducible representation of a Coxeter group*, J. Algebra **86** (1984), 422–438.
[8] A. Gyoja: *Modular representation theory over a ring of higher dimension with applications to Hecke alpebras*, J. Algebra **174** (1995), 553–572.
[9] D. Kazhdan, G. Lusztig: *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.
[10] F. Khosraviyani: *Decomposition numbers of exceptional Weyl groups, I*, J. Algebra **91** (1984), 390–409.
[11] G. Lusztig: *A class of irreducible representations of a Weyl group. II*, Indagationes Math. **44** (1982), 219–226.
[12] G. Lusztig: *Characters of reductive groups over a finite field*, Ann. of Math. Studies **107** (1984), Princeton Univ. Press, Princeton.

[13] G. Lusztig: *Cells in affine Weyl groups*, Advanced Studies in Pure Math. **6** (1985), 255–287.
[14] G. Lusztig: *Cells in affine Weyl groups, II*, J. Algebra **109** (1987), 536–548.
[15] G. Lusztig: *Cells in affine Weyl groups, III*, J. Fac. Sci. Univ. Tokyo **34** (1987), 223–243.
[16] G. Lusztig: *Sur les cellules gauches des groupes de Weyl*, C.R. Acad. Sci. Paris **302** (1986), 5–8.
[17] G. Lusztig: *Leading coefficients of character values of Hecke algebras*, Proc. Symposia in Pure Math. **47** (1987), 235–262.
[18] G. Lusztig: *Cells in affine Weyl groups, IV*, J. Fac. Sci. Univ. Tokyo **36** (1989), 297–328.
[19] K. Takahashi: *The left cells and their W-graphs of Weyl group of type F_4*, Tokyo J. Math. **13** (1990), 327–340.

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