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TOWARD DETERMINATION OF THE SINGULAR FIBERS OF MINIMAL DEGENERATION OF SURFACES WITH $\kappa=0$

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1. Introduction

Let $f: X \to \mathcal{D}$ be a projective surjective morphism from a complex normal 3-fold X to a disk $\mathcal{D}: =\{z \in \boldsymbol{C} ; |z|<1\}$. Assume that f is a minimal degeneration of surfaces, i.e., X has only $\boldsymbol{Q}$-factorial terminal singularities with nef canonical divisor K_X, and that general fibers are smoooth minimal surfaces with $\kappa=0$. The standard way for studying this degeneration is to use the so called semistable reduction, but it is impracticable in general. Another way was suggested by Y. Kawamata in [7], which may be called a *log minimal reduction* and explained as follows. Put $\Theta: =f^*(0)_{\text{red}}$, take a log resolution for the log pair (X, Θ), μ: $(Y, \Theta_Y) \to (X, \Theta)$ and apply the log minimal model program for (Y, Θ_Y). Then after shrinking $\mathcal{D}$ with a projective surjective morphism $\hat{f}: \hat{X} \to \mathcal{D}$, where $\hat{X}$ is normal $\boldsymbol{Q}$-factorial 3-fold, $(\hat{X}, \hat{\Theta})$ is strictly log terminal in the sense of [20] and $K_{\hat{X}}+\Theta$ is $\hat{f}$-nef. We note here that $\hat{X} \backslash \operatorname{Supp} \hat{\Theta}$ is smooth, and Supp $\hat{\Theta}=$Supp $\hat{f}^*(0)$. We call this new degeneration *a log minimal degeneration*. Log minimal degenerations can be studied in the same way as usual semistable degeneration, for example, irreducible components of the special fiber are normal and cross normally (see [20], Corollary 3.8). We should note that the theory of the log minimal degeneration was predicted in [18], (8.9). The aim of this paper is to determine (up to flops) the singular fiber of a minimal degeneration of surfaces with $\kappa=0$ of type II (see Definition 4.1) in the special case as explained in the statement of Theorem 4.3 and of type I (see Definition 5.1) under the condition that an associated log minimal degeneration has an irreducible component which is a ν_0-log surface of abelian type (see Definition 5.3) by the above method. In the section 2, we firstly review degenerations of elliptic curves as warming-up. We classify ν_0-log surfaces of type *II* in the section 3 and apply these results to degenerations of type *II* in the section 4. In the section 5, we classify ν_0-log surfaces of abelian type which is an ideal generalization of a log Enriques surface whose log canonical cover is an

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abelian surface in the sense of D.-Q. Zhang [25] and apply these results to a classification of degenerations of type I associated with ν_0-log surfaces of abelian type in the section 5. So far our list in this section does not cover Iitaka-Ueno's work on the first kind of degenerations of principally polarlized abelian surfaces [23], [24], but out statement is made under weaker assumptions on the general fible, which is important for applications to 3-folds. Our method is simple but powerful, so we expect that this method would work in any characteristic.

NOTATIONS and CONVENTIONS

In what follows we shall use the following notations.

$A_{n,q}$: A surface singularity which is defined by the automorphism of C^2, σ : $(x, y) \rightarrow (\zeta x, \zeta^q y)$ were $n, q \in N$ and ζ is thce primitive n-th root of unity is called the quotient singularity of type $A_{n,q}$.

$(1/n)(w_1, w_2, w_3)$: A 3-dimensional singularity which is defined by the automorphism of C^3, σ : $(x, y, z) \rightarrow (\zeta^{w1}x, \zeta^{w2}y, \zeta^{w3}z)$ where n, $w_i \in N$ for $i=1$, 2, 3 and ζ is the primitive n-th root of unity is called the quotient singularity of type $(1/n)(w_1, w_2, w_3)$.

By $(C_3, \{xy=0\})/Z_2(1, 1, 1)$ for instance, we mean a pair;

$$(C^3/\langle\sigma\rangle, \{(x, y, z) \in C^3 ; xy=0\}/\langle\sigma\rangle),$$

where σ acts on C^3 such that $\sigma^*(x, y, z)=(-x, -y, -z)$.

Σ_d : Hirzebruch surface of degree d. ∞-section : A section on Σ_d with self-intersection number d.

n-section : An irreducible curve on a ruled surface whose intersection number with a fibre of the ruling is n.

$(-n)$-curve : A smooth connected rational curve on a surface with self intersection number $(-n)$, where $n \in N$.

$\sim$: linear equivalence.

$\sim_{\text{num}}$: numerical equivalence.

$\lfloor \Delta \rfloor$: reduced part of the boundary Δ.

$\{\Delta\}$: fractional part of the boundary Δ.

χ_{top} : topological Euler characteristic.

ν : $X^\nu \rightarrow X$: The normalization of a scheme X.

We use terminology such as strictly log terminal, purely log terminal and so on freely. For definition of these terminology, we refer the reader to [20] or [10].

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2 and to the referee for correcting typographical errors and simplifying the part of the proof of Theorem 5.1 in the cases that Cartier indices are 3 and 5.

2. Degeneration of elliptic curves

Firstly, let us work log minimal degeneration of elliptic curves as warming-up. The log minimal reductions of minimal degeneration of elliptic curves are well known. Conversely, we can classify log minimal degeneration of elliptic curves by using the adjunction theory, the classification of surface log cononical singularities and [5], Lemma 6.1 as follows. We note that this gives another easy proof of [5], Theorem 6.1.

Proposition 2.1. *Let $\hat{f}: \hat{S}\to\mathcal{D}$ be a proper surjective morphism from a normal surface $\hat{S}$ onto a disk $\mathcal{D}$. Assume that general fibers of $\hat{f}$ are smooth elliptic curves and $(\hat{S}, \hat{\Theta})$ is weak Kawamata log terminal and $K_{\hat{S}}+\hat{\Theta}$ is $\hat{f}$-nef, where $\hat{\Theta}: \hat{f}^*(0)_{\mathrm{red}}$. Then the special fiber $\hat{S}_0: =\hat{f}^*(0)$ is classified as follows. We note that they all exist.*

${}_mI_{0,\log}$: *$\hat{S}_0=m\hat{\Theta}$. where $m\in N$ and $\hat{\Theta}$ is a smooth elliptic curve.*

${}_mI_{b,\log}$: *$\hat{S}_0$ is the singular fiber of a degeneration obtained by blowing up successively some singular loci of the support of the singular fibre of a minimal degeneration of type ${}_mI_b(b\geq 2)$.*

$I^*_{0,\log}$: *$\hat{S}_0=2\hat{\Theta}$, where $\hat{\Theta}$ is an irreducible smooth rational curve on which lie four singular points of type $A_{2,1}$.*

$I^*_{b,\log}$: *$\hat{S}_0$, is the singular fiber obtained by blowing up singular points of the support of the singular fiber of a log minimal degeneration $g: (\bar{S}, \bar{\Theta})\to \mathcal{D}$. $\bar{S}_0: =g^*(0)=\sum_{i=0}^{b}2\bar{\Theta}_i(b\geq 1)$, where $\bar{\Theta}_i$ are irreducible smooth rational curves and $\bar{\Theta}_i\cdot\bar{\Theta}_{i+1}=1$ for $0\leq i\leq b-1$, $\bar{\Theta}_i\cdot\bar{\Theta}_j=0$ otherwise. And on each $\bar{\Theta}_0$, $\bar{\Theta}_b$ lie two quotient singular points of $\bar{S}$ of type $A_{2,1}$. Other singular points of $\bar{S}$ do not lie on $\bar{\Theta}_i, 1\leq i\leq b-1$.*

$II_{\log}$: *$\hat{S}_0=6\hat{\Theta}$, where $\hat{\Theta}$ is an irreducible smooth rational curve on which lie three quotient singular points of $\hat{S}$ of type $A_{6,1}, A_{2,1}$, $A_{3,1}$ respehtively.*

$II^*_{\log}$: *$\hat{S}_0=6\hat{\Theta}$, where $\hat{\Theta}$ is an irreducible smooth rational curve on which lie three quotient singular points of $\hat{S}$ of type $A_{6,5}$, $A_{2,1}$, $A_{3,2}$ respectively.*

$III_{\log}$: *$\hat{S}_0=4\hat{\Theta}$, where $\hat{\Theta}$ is an irreducible smoooth rational curve on which lie three quotient singular points of $\hat{S}$ of type $A_{4,1}$, $A_{4,1}$, $A_{2,1}$ respectively.*

$III^*_{\log}$: *$\hat{S}_0=4\hat{\Theta}$, where $\hat{\Theta}$ is an irreducible smooth rational curve on which lie three quotient singular points of $\hat{S}$ of type $A_{4,3}$, $A_{2,1}$, $A_{4,3}$ respectively.*

$IV_{\log}$: $\hat{S}_0=3\hat{\Theta}$, *where* $\hat{\Theta}$ *is an irreducible smooth rational curve on which lie three quotient singular points of* $\hat{S}$ *of type* $A_{3,1}$.

$IV^*_{\log}$: $\hat{S}_0=3\hat{\Theta}$, *where* $\hat{\Theta}$ *is an irreducible smooth rational curve on which lie three quotient singular points of* $\hat{S}$ *of type* $A_{3,2}$.

Proof. Take any irreducible component $\hat{\Theta}_0$ of $\hat{\Theta}$. Let $\{P_i\,;\ i\in I\}$ be all singular points of $\hat{S}$ which lie on $\hat{\Theta}_0$. Put $m_i:=|\mathrm{Weil}(\mathcal{O}_{\hat{S},P_i})|$ (=the order of the Weil local class group of $\mathcal{O}_{\hat{S},P_i}$) and $n(\hat{\Theta}_0):=(\hat{\Theta}-\hat{\Theta}_0)\cdot\hat{\Theta}_0$. Then

$$0=(K_{\hat{S}}+\hat{\Theta})\cdot\hat{\Theta}_0=2g(\hat{\Theta}_0)-2+\sum_{i\in I}\frac{m_i-1}{m_i}+n(\hat{\Theta}_0),$$

where $g(\hat{\Theta}_0)$ is the genus of $\hat{\Theta}_0$ (see [20] or [10]). From the above formula, we can derive $g(\hat{\Theta}_0)\leq 1$. When $g(\hat{\Theta}_0)=1$, S is smooth and singular fiber is of type ${}_mI_{0,\log}$. So we may assume $g(\hat{\Theta}_0)=0$ in what follows. Because we have $n(\hat{\Theta}_0)\leq 2$, we divide the proof into three cases $n(\hat{\Theta}_0)=0,\ 1,\ 2$.

Case $n(\hat{\Theta}_0)=0$. In this case we have $\sum_{i\in I}(m_i-1)/m_i=2$, hence $(m_i\,;\ i\in I)=(2,2,2,2),\ (2,3,6),\ (2,4,4),\ (3,3,3)$.

Subcase $(m_i\,;\ i\in I)=(2,2,2,2)$. In this case, we can deduce that the singular fibre $\hat{S}_0$ is of type $I^*_{0,\log}$.

Subcase $(m_i\,;\ i\in I)=(2,3,6)$. When the three singularities are of type $A_{2,1}$, $A_{3,1}$, $A_{6,1}$ respectively, the strict transform $\hat{\Theta}'_0$ of $\hat{\Theta}_0$ on the minimal resolution M is a (-1)-curve. After blowing down (-1)-curves, we get a singular fiber of type $II_{\log}$. When the three singularities are of type $A_{2,1}$, $A_{3,2}$, $A_{6,1}$ respectively, we have $K_M\cdot\hat{\Theta}'_0=-(2/3)$, which is contradiction. When the three singularities are of type $A_{2,1}$, $A_{3,1}$, $A_{6,5}$ respectively, we have $K_M\cdot\hat{\Theta}'_0=-(1/3)$, which is contradiction. When the three singularities are of type $A_{2,1}$, $A_{3,2}$, $A_{6,5}$ respectively, the strict transform $\hat{\Theta}'_0$ is a (-2)-curve, of type $A_{2,1}$, $A_{3,2}$, $A_{6,5}$ respectively, the strict transform $\hat{\Theta}'_0$ is a (-2)-curve, so we get a singular fiber of type II^*. Hence multiplicity of $\hat{\Theta}_0$ in the singular fiber is 6 and we obtained a singular fiber of type $II^*_{\log}$.

Subcase $(m_i\,;\ i\in I)=(2,4,4)$. When the three singularities are of type $A_{2,1}$, $A_{4,1}$, $A_{4,1}$ respectively, the strict transform $\hat{\Theta}'_0$ of $\hat{\Theta}_0$ on the minimal resolution is a (-1)-curve and after blowing down (-1)-curves we get a singular fiber of type III. Hence rultiplicity of $\hat{\Theta}_0$ in $\hat{S}_0$ is 4 and we obtain a singular fiber of type $III_{\log}$. When the three singularities are of type $A_{2,1}$, $A_{4,1}$, $A_{4,3}$ respectively, we have $K_M\cdot\hat{\Theta}'_0=-(1/2)$, which is a contradiction. When the three singularities are of type $A_{2,1}$, $A_{4,3}$, $A_{4,3}$ respectively, the strict transform $\hat{\Theta}'_0$ of $\hat{\Theta}_0$ on the minimal resolution is a (-2)-curve and we get a singular fiber of type III. Hence the multiplicity of $\hat{\Theta}_0$ in $\hat{S}_0$ is 4 and we obtain a singular fiber of type $III^*_{\log}$.

Subcase $(m_i\,;\ i\in I)=(3,3,3)$. When the three singularities are of type $A_{3,1}$, $A_{3,1}$, $A_{3,1}$ respectively, the strict transform $\hat{\Theta}'_0$ of $\hat{\Theta}_0$ on the minimal resolution is a (-1)-curve and after blowing down the (-1)-curve, we get a singular fiber of type

IV. Hence the multiplicity of $\hat{\Theta}_0$ in $\hat{S}_0$ is 3, so we obtained a singular fiber of type $IV_{\log}$. When the three singularities are of type $A_{3,1}$, $A_{3,1}$, $A_{3,2}$ respectively, we have $K_M\cdot\hat{\Theta}_0'=-(2/3)$, which is a contradiction. When the three singularities are of type $A_{3,1}$, $A_{3,2}$, $A_{3,2}$ respectively, we have $K_M\cdot\hat{\Theta}_0'=-(1/3)$, which is a contradiction. When the three singularities are of type $A_{3,2}$, the strict transform Θ_0' is a (-2)-curve, so we get a singular fiber of type IV^*. Hence multiplicity of $\hat{\Theta}_0$ in the singular fiber is 3 and we obtain a singular fiber of type $IV^*_{\log}$.

Case $n(\hat{\Theta}_0)=1$. In this case we have $(m_i\,;\, i\in I)=(2, 2)$, so two singularities of type $A_{2,1}$ lie on $\hat{\Theta}_0$. $\hat{\Theta}$ has a unique component of $\hat{S}_0$, say $\hat{\Theta}_1$, which has non empty intersection with $\hat{\Theta}_0$. $\hat{\Theta}_1$ has the same type as $\hat{\Theta}_0$ or $n(\hat{\Theta}_1)=2$. Thus we get a chain of rational curves $\hat{\Theta}_0$, $\hat{\Theta}_1$, ..., $\hat{\Theta}_n$ and $\hat{S}$ has only four singularities of type $A_{2,1}$, each two of which lie on $\hat{\Theta}_0$, $\hat{\Theta}_n$ respectively. After taking the minimal resolution, this chain must be blown down to a singular fiber of type I_b^*. So we obtain a singular fiber of type $I^*_{b,\log}$.

Case $n(\hat{\Theta}_0)=2$. Assume $\hat{S}_0$ is not a singular fiber of type $I^*_{b,\log}$. Then we have a cycle of rational curves, which must be blown down to a singular fiber of type ${}_mI_b$. Thus we obtained a singular fiber of type ${}_mI_{b,\log}$. ■

3. Classification of ν_0-log surfaces of type II

Let $\hat{f}:(\hat{X}, \hat{\Theta})\to\mathcal{D}$ be a log minimal degeneration of surfaces with $\chi=0$ and let $\hat{\Theta}_i$ be any irreducible component of $\hat{\Theta}$. Then $(\hat{\Theta}_i, \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i))$ is a *ν_0-log surface* in the following sense (see [20], (3.2.3)).

DEFINITION 3.1. A normal surface with a boundary (S, Δ) is called a ν_0-log surface, when the following conditions (1), (2), (3), (4) are satisfied.

(1) (S, Δ) is weak Kawamata log terminal.

(2) $K_S+\Delta\sim_{\mathrm{num}}0$,where $\sim_{\mathrm{num}}$ is the numerical equivalence.

(3) $\mathrm{Supp}\lfloor\Delta\rfloor\cap\mathrm{Supp}\{\Delta\}=\emptyset$, where $\lfloor\Delta\rfloor$ is the reduced part of Δ and $\{\Delta\}$ is the fractional part of Δ.

(4) All coefficients of Δ are elements of $\{(m-1)/m|m\in\boldsymbol{N}\cup\{\infty\}\}$

It is important to classify ν_0-log surfaces and the following is a key lemma to study ν_0-log surfaces which is proved essentially in the proof of Proposition 2.1.

Lemma 3.1. *Let (S, Δ) be a ν_0-log surface. Then a connected component D of $\lfloor\Delta\rfloor$ and the singularities of S in its neighborhood are one of the following 7 types.*

$I_{0,\log}$: *D is a smooth elliptic curve and S is smooth in its neighborhood.*

$I_{b,\log}$: *$D=\sum_{i=1}^{b} C_i (b\geq 2)$, where C_i's form a cycle of rational curves forming a cycle and S is smooth in its neighborhood.*

$I_{0,\log}^{*}$: *D is a smooth rational curve on which lie 4 quotient singularities of type $A_{2,1}$.*

$I_{b,\log}^{*}$: *D is a linear chain of rational curves, i.e., $D=\sum_{i=0}^{b} C_i (b\geq 1)$, where C_i' s are irreducible smooth rational curves an $C_i\cdot C_{i+1}=1 (0\leq i\leq b-1)$, $C_i\cdot C_j=0$ otherwise, and each of edge curves C_0, C_b contains two singular points of S of type $A_{2,1}$.*

$II_{\log}$: *D is an irreducible smooth rational curve on which lie three quotient singular points of S of type $A_{6,1}$ (or $A_{6,5}$), $A_{2,1}$, $A_{3,1}$ (or $A_{3,2}$), respectively.*

$III_{\log}$: *D is an irreducible smooth rational curve on which lie three quotient singular points of S of type $A_{4,1}$ (or $A_{4,2}$), $A_{4,1}$ (or $A_{4,2}$), $A_{2,1}$, respectively.*

$IV_{\log}$: *D is an irreducible smooth rational curve on which lie three quotient singular points of S of type $A_{3,1}$ or $A_{3,2}$.*

In what follows we shall classify ν_0-log surfaces in certain cases.

Lemma 3.2. (cf. [2], Lemma (6.1)) *Let (S, Δ) be a ν_0-log surface.*

(1) *Assume that there is a connected reduced curve $C_0\subset\lfloor \Delta\rfloor$ of type $I_{b,\log}$ ($b\leq 2$) as in Lemma* 3.1. *Then $\Delta=C_0$, $\mathcal{O}_S(K_S+\Delta)\simeq\mathcal{O}_S$, S is rational and $(S, 0)$ is canonical.*

(2) *Assume that there is a component of type $I_{0,\log}$ as in Lemma* 3.1,*i.e., a smooth elliptic curve $C_0\subset\lfloor \Delta\rfloor$. Then S is rational or birationally elliptic ruled and (S,Δ) satisfies one of the following* :

   (*a*) *$\Delta=C_0$ and (S, Δ) is cononical.*
   (*b*) *$\Delta=C_0+C_1$, where C_1 is a smooth elliptic curve. $\mathcal{O}_S(K_S+\Delta)\simeq\mathcal{O}_S$, S is a birationally elliptic ruled surface and (S, Δ) is canonical.*
   (*c*) *$\Delta=C_0+(1/2)C_1+(1/2)C_2$, where C_1, C_2 are smooth elliptic curves, S is birationally elliptic ruled, (S, Δ) is canonical and S is smooth in a neighborhood of* Supp Δ.
   (*d*) *$\Delta=C_0+(1/2)C_1$, where C_1 is an irreducible curve. S is birationally elliptic ruled, (S, Δ) is canonical and S is smooth in a neighborhood of* Supp Δ.

DEFINITION 3.2. A ν_0-log surface (S, Δ) is called a ν_0-log surface of type *III*, *II*, II_a, II_b, II_c, II_d, when the conditions in (1), (2), (2-*a*), (2-*b*), (2-*c*), (2-*d*) of Lemma 3.2 are satisfied respectively.

Proof. First we note that S is rational or birationally ruled.
(1) Assume $h^1(\mathcal{O}_S)>0$. Let μ: $M\to S$ be the minimal resolution and τ :

$M\to N$ a birational morphism to a relatively minimal model N. N has a $\boldsymbol{P}^1$-bundle structure $p: N\to\Gamma$ over a smooth curve of genus $h^1(\mathcal{O}_S)$. The assumption implies that there is a rational irreducible component of $\tau_*\mu^*C_0$ which dominates Γ or $\tau_*\mu^*C_0$ turns out to be a fibre of p, though both cases are absurd. Hence S is rational. From an exact sequence:

$$0\to\mathcal{O}_S(-\lfloor\Delta\rfloor)\to\mathcal{O}_S\to\mathcal{O}_{\lfloor\Delta\rfloor}\to 0,$$

we have the following exact sequence:

$$H^1(\mathcal{O}_S)\xrightarrow{\alpha}H^1(\mathcal{O}_{\lfloor\Delta\rfloor})\to H^2(\mathcal{O}_S(-\lfloor\Delta\rfloor))\to 0 \qquad (*)$$

Since S is rational, we have an injection $H^1(\mathcal{O}_{\lfloor\Delta\rfloor})\hookrightarrow H^2(\mathcal{O}_S(-\lfloor\Delta\rfloor))$. On the other hand, we have $1=h^1(\mathcal{O}_{C_0})\leq h^1(\mathcal{O}_{\lfloor\Delta\rfloor})$. Hence $h^2(\mathcal{O}_S(-\lfloor\Delta\rfloor))>0$. By the Serre duality theorem, we get

$$h^2(\mathcal{O}(-\lfloor\Delta\rfloor))=h^0(\mathcal{H}om(\mathcal{O}_S(-\lfloor\Delta\rfloor),\ \omega_S)).$$

Since $\mathcal{H}om(\mathcal{O}_S(-\lfloor\Delta\rfloor),\ \omega_S)$ is torsion-free, we have an injection

$$H^0(\mathcal{H}om(\mathcal{O}_S(-\lfloor\Delta\rfloor),\ \omega_S))\hookrightarrow H^0(\mathcal{O}_S(K_S+\lfloor\Delta\rfloor)).$$

Hence $h^0(\mathcal{O}_S(K_S+\lfloor\Delta\rfloor))>0$. Since $K_S+\Delta\sim_{\text{num}}0$, for an ample divisor H on S, we have

$$(K_S+\lfloor\Delta\rfloor)\cdot H=-\{\Delta\}\cdot H$$

and

$$(K_S+\lfloor\Delta\rfloor)\cdot H=\{\Delta\}\cdot H=0.$$

Thus

$$\{\Delta\}=0,\ \mathcal{O}_S(K_S+\Delta)\simeq\mathcal{O}_S.$$

From Lemma 3.1, we can deduce that any connected component of $\lfloor\Delta\rfloor$ other than D is of type $I_{0,\log}$ or $I_{b,\log}$ ($b\geq 2$), but since $h^1(\mathcal{O}_{\lfloor\Delta\rfloor})=1$, we have $\lfloor\Delta\rfloor=C_0$, i.e., $\Delta=C_0$. S has only quotient singularities which are all Gorenstein, so $(S, 0)$ is canonical.

(2) First we note that $h^1(\mathcal{O}_S)\leq 1$. For if $h^1(\mathcal{O}_S)>0$, C_0 dominates Γ for which we used the same notation as in (1). From $(*)$, we have the following exact sequence:

$$0\to\operatorname{Im}\alpha\to H^1(\mathcal{O}_{\lfloor\Delta\rfloor})\to H^2(\mathcal{O}_S(-\lfloor\Delta\rfloor))\to 0.$$

We note that $\dim\operatorname{Im}\alpha\leq 1$. First assume that $\dim\operatorname{Im}\alpha=0$. In this case we have an injection $H^1(\mathcal{O}_{\lfloor\Delta\rfloor})\hookrightarrow H^2(\mathcal{O}_S(-\lfloor\Delta\rfloor))$. In the same way as in the argument in (1), we can deduce that $\Delta=C_0$, $\mathcal{O}_S(K_S+\Delta)\simeq\mathcal{O}_S$ and $(S, 0)$ is cononical. So we are in the case (2-a). In what follows we assume that $\dim\operatorname{Im}\alpha=1$. Then, S is birationally elliptic ruled and we have $h^1(\mathcal{O}_{\lfloor \Delta \rfloor})\leq 2$.

Case $h^1(\mathcal{O}_{\lfloor \Delta \rfloor})=2$; In this case, we have $\{\Delta\}=0$, $\mathcal{O}_S(K_S+\Delta)\simeq\mathcal{O}_S$. Each connected component of $\lfloor \Delta \rfloor$ is of type $I_{0,\log}$ or $I_{b,\log}$ $(b\geq 2)$, but from (1), there are no components of type $I_{b,\log}(b\geq 2)$. Hence $\Delta=C_0+C_1$, where C_1 is a smooth elliptic curve and we are in the case (2-b).

Case $h^1(\mathcal{O}_{\lfloor \Delta \rfloor})=1$; Let $\mu: M\to S$, $\tau: M\to N$, $p: N\to\Gamma$ be as in the proof of (1). Since S has only rational singularities and Γ is an elliptic curve, there is a morphism $\pi: S\to\Gamma$ such that $p\circ\tau=\pi\circ\mu$. Let

$$\Delta=C_0+\tilde{C}+\sum_{i\in I}\frac{m_i-1}{m_i}C_v^{(i)}+\sum_{j\in J}\frac{n_j-1}{n_j}C_h^{(j)}$$

be the decomposition of Δ, where $\tilde{C}$ is a reduced curve and $C_v^{(i)}$ $(i\in I)$(resp. $C_h^{(j)}$ $(j\in J)$) are irreducible curves which are vertical (resp. horizontal) with respect to π. Let l be a general fibre of π. Then we have

$$0=(K_S+\Delta)\cdot l=-2+C_0\cdot l+\tilde{C}\cdot l+\sum_{j\in J}\frac{n_j-1}{n_j}C_h^{(j)}\cdot l,$$

from which we can deduce that $(A)(n_j;\ j\in J)=(2,\ 2)$, $\tilde{C}\cdot l=0$, $C_0\cdot l=1$, $C_j\cdot l=1(j=1,\ 2)$, where $C_j:\ =C_h^{(j)}$ or $(B)(n_j;\ \in J)=(2)$, $\tilde{C}\cdot l=0$, $C_0\cdot l=1$, $C_1\cdot l=2$, where $C_1:\ =C_h^{(j)}$
or (C) $J=\emptyset$, $\tilde{C}\cdot l=0$, $C_0\cdot l=2$ since $\tilde{C}$ does not contain any reduced curve of type $I_{0,\log}$ or $I_{b,\log}$ $(b\geq 2)$ in Lemma 3.1. Since μ is a minimal resolution, there is an effective $\boldsymbol{Q}$-divisor D such that $K_M+D=\mu^*(K_S+\Delta)$. Put $\bar{D}:\ \tau_*D$. Then we can write

$$\bar{D}=\begin{cases} C_0'+(1/2)C_1'+(1/2)C_2'+F, & \text{Case } (A)\\ C_0'+(1/2)C_1'+F, & \text{Case } (B)\\ C_0'+F, & \text{Case } (C)\end{cases}$$

where C_0', C_1', C_2' are the strict transform of C_0, C_1, C_2 respectively and F is an effective $\boldsymbol{Q}$-divisor composed of fibres of p. Note that and $K_N+\bar{D}=\tau_*(K_M+D)\sim_{\text{num}}0$. Theref re, we have

$$\begin{aligned}0&=(K_N+\bar{D})\cdot C_0'\\ &=\begin{cases}(K_N+C_0')\cdot C_0'+(1/2)(C_1'+C_2')\cdot C_0'+C_0'\cdot F, & \text{Case } (A)\\ (K_N+C_0')\cdot C_0'+(1/2)C_1'\cdot C_0'+C_0'\cdot F, & \text{Case } (B)\\ (K_N+C_0')\cdot C_0'+C_0'\cdot F. & \text{Case } (C)\end{cases}\end{aligned}$$

In the case (A)(resp. (B)), since C_0' is a section of p, we have $C_i'\cdot C_0'=0$ $(i=1,\ 2)$(resp. $C_1'\cdot C_0'=0$) and $F=0$. In the case (C), let $\nu:\ C_0'^{\nu}\to C_0'$ be the normalization of C_0'. Since

$$\frac{(K_N+C_0')\cdot C_0'}{2}+1=g(C_0'^{\nu})+\dim\ \nu_*\mathcal{O}_{C_0'^{\nu}}/\mathcal{O}_{C_0'},$$

we can deduce that $F=0$ and C_0' is a smooth elliptic curve. Since C_1', C_2' are smooth in the case (A) and C_1' is a 2-section (i.e., intersection number with a fibre of p is 2) in the case (B), we can see that $(N, \bar{D})$ is canonical. So we can write

$$K_M+\tau^{-1}\bar{D}=\tau^*(K_N+\bar{D})+E,$$

where E is an effective $\mathbf{Q}$-divisor which is τ-exceptional. Since

$$\begin{aligned}K_S+\Delta_h&=\mu_*(K_M+\tau_*^{-1}\bar{D}\\&=\mu_*\tau^*(K_N+\bar{D})+\mu_*E,\end{aligned}$$

where Δ_h is the horizontal component of Δ, and

$$0\sim_{\mathrm{num}}K_S+\Delta\sim_{\mathrm{num}}\mu_*E+\Delta_v,$$

where Δ_v is the vertical component of Δ, we have $\mu_*E=0$, $\Delta_v=0$. Hence

$$\Delta=\begin{cases}C_0+(1/2)C_1+(1/2)C_2, & \text{Case}(A)\\ C_0+(1/2)C_1, & \text{Case}(B)\\ C_0, & \text{Case}(C)\end{cases}$$

and

$$K_S+\Delta=\mu_*\tau^*(K_N+\bar{D}). \qquad (**)$$

Write

$$D=\mu_*^{-1}\Delta+\sum_{i\in I}a_iE_i,$$

where $\{E_i\,;\ i\in I\}$ are all μ-exceptional divisors and a_i's are non-negative rational numbers for $i\in I$. Since

$$0\sim_{\mathrm{num}}K_N+\bar{D}=\tau_*(K_M+\mu_*^{-1}\Delta)\sim_{\mathrm{num}}-\sum_{i\in I}a_i\tau_*E_i,$$

we have $\sum_{i\in I}a_i\tau_*E_i=0$. This implies that if $\mu(E_i)\cap\mathrm{supp}\Delta\neq\emptyset$ or $\mu(E_i)$ is a singular point of S other than a rational double point, then $a_i>0$ and E_i is τ-exceptional. So there is an open subset $U\subset S$ such that the rational map σ: $\tau\circ\mu^{-1}$ is a morphism on U and $S\backslash U$ consists of rational double points of S which do not lie in Supp Δ. Put $V:=\mu^{-1}(U)$. From $(**)$, we have $K_S+\Delta|_U=\sigma^*(K_N+\bar{D})$ and

$$K_M+\tau_*^{-1}\bar{D}|_V=K_M+\mu_*^{-1}\Delta|_V=(\tau|_V)^*(K_N+\bar{D})-\sum_{i\in I}a_iE_i|_V.$$

This implies $a_i\leq0$ if $\mu(E_i)\in U$. Thus we get $a_i=0$ for all $i\in I$, hence (S, Δ) is canonical and S is smooth in a neighborhood of $\mathrm{supp}\Delta$. In the case (A), C_1, C_2 are smooth elliptic curves and we are in the case (2-c). In the case (B)(resp. (C)), we know that we are in the case (2-d)(resp. (2-a)). ∎

DEFINITION 3.3. Let (S, Δ) be a ν_0-log surface of type II_c (resp. type II_d). If (S, Δ) is terminal in a neighborhood of Supp $\{\Delta\}$, we call (S,Δ), *a special ν_0-log surface of type II_c (resp. of type II_d).*

DEFINITION 3.4.

(*a*) A log surface (S,Δ) is called an *elliptic singular ν_0-log surface of type II_b (resp. II_c, resp. II_d)* if S has only one simple elliptic singular point $P\in S$ and $(\tilde{S}, \Delta_{\tilde{S}})$ is a ν_0-log surface of type II_b (resp. II_c, resp. II_d), where $\mu: \tilde{S}\to S$ is the minimal resolution of $P\in S$.

(*b*) Let S be a reduced irreducible surface which is Cohen-Macaulay and let Δ be a boundary on S. We call (S, Δ) a *degenerate ν_0-log surface of type II_b (resp. II_c, resp. II_d)*, if (S,Δ) is semi-canonical and (S^ν, Θ) is a ν_0-log surface of type II_b (resp. II_c, resp. II_d), where $\nu: S^\nu\to S$ is the normalization of S and Θ is defined by $K_{S^\nu}+\Theta=\nu^*(K_S+\Delta)$(For the definition of "semi-canonical", see [10] or [9].).

(*c*) A log surface (S, Δ) is called a *quasi $K3$ surface* if S has only two simple elliptic singular points $P_1, P_2\in S$ and $(\tilde{S}, \Delta_{\tilde{S}})$ is a ν_0-log surface of type II_b, where $\mu: \tilde{S}\to S$ is the minimal resolution of $P_1, P_2\in S$.

(*d*) Let S be a reduced irreducible surface which is Cohen-Macaulay and let Δ be a boundary on S. We call (S, Δ) an *elliptic singular degenerate ν_0-log surface* if S has only one simple elliptic singular point $P\in S$, $(S\backslash\{P\}, \Delta)$ is semi-canonical and $(\tilde{S}, \Theta_{\tilde{S}})$ is of type II_b, where Θ is defined as above and $\mu: \tilde{S}\to S^\nu$ is the minimal resolution of P.

4. Degeneration of type II

DEFINITION 4.1. A minimal degeneration of surfaces with $\kappa=0$ $f: X\to\mathfrak{D}$ is said to be of type II if f has a log minimal reduction $\hat{f}: (\hat{X}, \hat{\Theta})\to\mathfrak{D}$ such that there is at least one irreducible component $\hat{\Theta}_i$ of $\hat{\Theta}$ such that $\lfloor \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i)\rfloor$ contains a connected component of type $I_{0,\log}$ as in Lemma 3.1.

From the results in the previous section, we obtain the following theorem.

Theorem 4.1. *Let $(\hat{X}, \hat{\Theta})$ be a normal log 3-fold such that $(\hat{X}, \hat{\Theta})$ is strictly log terminal, $\lfloor \hat{\Theta}\rfloor=\hat{\Theta}$, Supp $\hat{\Theta}$ is connected and Sing $\hat{X}\subset$Supp $\hat{\Theta}$. Assume that $(K_{\hat{X}}+\hat{\Theta})|_{\hat{\Theta}}\sim_{\mathrm{num}}0$ and that there is at least one irreducible component $\hat{\Theta}_i$ of $\hat{\Theta}$ such that $\lfloor \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i)\rfloor$ contains a connected component of type $I_{0,\log}$ as in Lemma 3.1. Let $\hat{\Theta}=\sum_{i=1}^{b}\hat{\Theta}_i$ be the irreducible decomposition. Then one of the following holds.*

(1) *$\hat{X}$ has only terminal singularities. $(\hat{\Theta}_i, \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i))$ is a ν_0-log surface*

of type II_b *for all* i. $\widehat{\Theta}_i \cap \widehat{\Theta}_j \neq \emptyset$ *if* $|i-j|=1$ *or* $(i, j)=(1, b)$ *and* $\widehat{\Theta}_i \cap \widehat{\Theta}_j = \emptyset$ *if* $|i-j|>1$ *and* $(i, j) \neq (1, b)$.

(2) $\widehat{X}$ *has only canonical singularities and* $\widehat{X} \backslash \mathrm{Supp}\,\{\mathrm{Diff}_{\widehat{\Theta}}(0)\}$ *has only terminal singularities.* $(\widehat{\Theta}_i, \mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i))$ *is a* ν_0*-log surface of type* II_b *for* $2 \leq i \leq b-1$ *and of type* II_a, II_c *or* II_d *for* $i=1, b$. $\widehat{\Theta}_i \cap \widehat{\Theta}_j \neq \emptyset$ *if* $|i-j|=1$ *and* $\widehat{\Theta}_i \cap \widehat{\Theta}_j = \emptyset$ *if* $|i-j|>1$.

Proof. From the assumption and Lemma 3.2, there is an irreducible component $\widehat{\Theta}_1$ of $\widehat{\Theta}$ such that $(\widehat{\Theta}_1, \mathrm{Diff}_{\widehat{\Theta}_1}(\widehat{\Theta}-\widehat{\Theta}_1))$ is a ν_0-log surface of type II. Since $\widehat{\Theta}$ is connected, for any component $\widehat{\Theta}_i$ of $\widehat{\Theta}$, $(\widehat{\Theta}_i, \mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i))$ is a ν_0-log surface of type II. We note that in a neighborhood of $\cup_{\widehat{\Theta}_i \subset \widehat{\Theta}} \mathrm{Supp} \lfloor \mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i) \rfloor$, $\widehat{X}$ is smooth. For a component $\widehat{\Theta}_i$ of $\widehat{\Theta}$, if $(\widehat{\Theta}_i, \mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i))$ is not a ν_0-log surface of type II_c, we have $\{\mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i)\}=0$, so $(\widehat{X}, \widehat{\Theta})$ is canonical in a neighborhood of $\widehat{\Theta}_i \backslash \mathrm{Supp} \lfloor \mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i) \rfloor$ by the following theorem.

Theorem 4.2. ([10], Corollary 17.2) *Let* $(X, S+B)$ *be a normal log* 3*-fold with only* ***Q****-factorial singularities. Assume that* S *is reduced*, $(X, S+B)$ *is log canonical in codimension* 2. *Then we have*

$$\mathrm{totaldiscrep}\,(S^\nu, \mathrm{Diff}_{S^\nu}(B)) = \mathrm{discrep}\,(\mathrm{Center} \cap S \neq \emptyset, X, S+B),$$

where S^ν *is the normalization of* S. *For the definition of "totaldiscrep" and "discrep", see* [10].

Since Sing $\widehat{X} \subset \mathrm{Supp}\,\widehat{\Theta}$, we can deduce that $(\widehat{X}, 0)$ is terminal in a neighborhood of $\widehat{\Theta}_i \backslash \mathrm{Supp} \lfloor \mathrm{Diff}_{\widehat{\Theta}_i}(\widehat{\Theta}-\widehat{\Theta}_i) \rfloor$. From the following lemma, we get the desired result.

Lemma 4.1. *Let* X *be a normal* ***Q****-factorial complex* 3*-fold and let* S *be a reduced irreducible surface on* X. *Assume that* (X, S) *is purely log terminal*, Sing $X \subset S$, $(S, \mathrm{Diff}_S(0))$ *is canonical and all coefficients of the components of* $\mathrm{Diff}_S(0)$ *are* $1/2$. *Then* X *has only canonical singularities.*

Proof. Let $\mu_1 : X^c \to X$ be the canonical blowing up, i.e., μ_1 is a projective birational morphism from a normal 3-fold X^c with only canonical singularities to X and K_{X^c} is μ_1-ample. Let $\mu_2 : Y \to X^c$ be a $\boldsymbol{Q}$-factorization of X^c, i.e., μ_2 is a projective birational morphism from a normal $\boldsymbol{Q}$-factorial 3-fold Y with only canonical singularities to X^c and μ_2 is isomorphic in codimension 1. Put $\mu := \mu_1 \circ \mu_2$. Then we can write

$$K_Y + \sum_{j \in J} a_j E_j = \mu^* K_X, \; \mu^* S = \widetilde{S} + \sum_{j \in J} r_j E_j$$

where $\widetilde{S} := \mu_*^{-1} S$, the E_j $(j \in J)$ are all μ-exceptional divisors and the a_j and the

r_j ($j \in J$) are positive rational numbers. From the above, we have

$$K_Y + \tilde{S} + \sum_{j \in J} (a_j + r_j) E_j = \mu^*(K_X + S)$$

and we can show that $(Y, \tilde{S})$ is purely log terminal, in particular, $\tilde{S}$ is normal. Taking the adjunction of the above equality, we get

$$K_{\tilde{S}} + \mathrm{Diff}_{\tilde{S}}(\sum_{j \in J} (a_j + r_j) E_j) = \mu^*(K_S + \mathrm{Diff}_S(0)).$$

Since $(S, \mathrm{Diff}_S(0))$ is canonical, we have $\sum_{j \in J} (a_j + r_j) E_j|_{\tilde{S}} = 0$. By the $\boldsymbol{Q}$-factoriality of Y, we can deduce that $(\bigcup_{j \in J} E_j) \cap \tilde{S} = \emptyset$ which implies $J = \emptyset$ and X has only canonical singularities. ■

Starting with Theorem 4.1, we can have insight into the minimal degenerations of type II.

Theorem 4.3. *Let $f: X \to \mathcal{D}$ be a minimal projective degeneration of surfaces with $x = 0$ and let $\hat{f}: (\hat{X}, \hat{\Theta}) \to \mathcal{D}$ be a log minimal reduction of f with $\mathcal{D}$ shrunk if necessary. Assume that there is at least one irreducible component $\hat{\Theta}_i$ of $\hat{\Theta}$ such that $\lfloor \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta} + \hat{\Theta}_i) \rfloor$ contains a connected component of type $I_{0,\log}$ as in Lemma 3.1 and that $\hat{\Theta}$ does not contain non-special ν_0-log surfaces. Then after flopping X over $\mathcal{D}$ if necessary, the singular fibre $f^*(0)$ has one of the following types.*

I: $f^*(0) = m\Theta$, where $m \in \boldsymbol{N}$ and Θ is an irreducible reduced surface such that the non-normal locus of S is a smooth elliptic curve C and $(\Theta^\nu, \nu^{-1}(C))$ is a ν_0-log surface of type II_b, where $\nu: \Theta^\nu \to \Theta$ is the normalization of Θ.

II: $f^*(0) = \sum_{i=1}^b m\Theta_i$, where $m \in \boldsymbol{N}$, $(\Theta_i, \sum_{j \neq i} \Theta_j|_{\Theta_i})$ is a ν_0-log surface of type II_b for all i, $\Theta_i \cap \Theta_j \neq \emptyset$ if $|i-j| > 1$ and $(i, j) \neq (1, b)$.

III: $f^*(0) = \sum_{i=1}^b m\Theta_i$, where $m \in \boldsymbol{N}$. If $b \geq 2$, $(\Theta_i, \sum_{j \neq i} \Theta_j|_{\Theta_i})$ is a ν_0-log surface of type II_b for $2 \leq i \leq b-1$ and $(\Theta_i, \sum_{j \neq i} \Theta_j|_{\Theta_i})(i = 1, b)$ is either a ν_0-log surface of type II_a or an elliptic singular or degenerate ν_0-log surface of type II_b. $\Theta_i \cap \Theta_j \neq \emptyset$ if $|i-j| = 1$. $\Theta_i \cap \Theta_j = \emptyset$ if $|i-j| > 1$. If $b = 1$, Θ_1 is either a quasi $K3$ surface or an elliptic singular or degenerate ν_0-log surface.

IV:

$$f^*(0) = \begin{cases} \sum_{i=1}^b 2m\Theta_{1,i} + \sum_{j=1}^2 m\Theta_{2,j}, & \text{Case } (\alpha), \\ \sum_{i=1}^b 2m\Theta_{1,i} + \sum_{j=1}^4 m\Theta_{2,j}, & \text{Case } (\beta), \end{cases}$$

whero $m \in \boldsymbol{N}$ and the $\Theta_{2,j}$ are elliptic ruled surfaces. In the case (α), if $b \leq 2$, $(\Theta_{1,i}, \sum_{l \neq i} \Theta_{1,l} + (1/2)\sum_{j=1}^2 \Theta_{2,j}|_{\Theta_{1,i}})$ is a special ν_0-log surface of type II_c for $i = 1$, of type II_b for $2 \leq i \leq b-1$ and for $i = b$, a ν_0-log surface of type II_a or an elliptic singular or degenerate ν_0-log surface of type II_b. $\Theta_{1,i} \cap \Theta_{1,j}$

$\neq\emptyset$ if $|i-j|=1$. $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ if $|i-j|>1$. Moreover, if $b=1$, $(\Theta_{1,1}, (1/2)\sum_{j=1}^{2}\Theta_{2,j}|_{\Theta_{1,1}})$ is an elliptic singular or degenerate ν_0-log surface of type II_c. In the case (β), $b\geq 2$ and $(\Theta_{1,i}, \sum_{l\neq i}\Theta_{1,l}+(1/2)\sum_{j=2}^{2}\Theta_{2,j}|_{\Theta_{1,1}})$ is a ν_0-log surface of type II_b for $2\leq i\leq b-1$, a special ν_0-log surface of type II_c for $i=1,b$. $S_i\cap S_j\neq\emptyset$ if $|i-j|=1$ and $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ if $|i-j|>1$.

V:

$$f^*(0)=\begin{cases}\sum_{i=1}^{b}2m\Theta_{1,i}+m\Theta_2, & \text{Case } (\alpha),\\ \sum_{i=1}^{b}2m\Theta_{1,i}+\sum_{j=1}^{2}m\Theta_{2,j}, & \text{Case } (\beta),\end{cases}$$

where $m\in\boldsymbol{N}$, the Θ_2 and $\Theta_{2,j}$ are ruled surfaces. In the case (α), if $b\geq 2$, $(\Theta_{1,i},\sum_{l\neq i}\Theta_{1,l}+(1/2)\sum_{j=1}^{2}\Theta_{2,j}|_{\Theta_{1,i}})$ is a special ν_0-log surface of type II_d for $i=1$, of type II_b for $2\leq i\leq b-1$ and for $i=b$, a ν_0-log surface of type II_a or an elliptic singular or degenerate ν_0-log surface of type $II_b\cdot\Theta_{1,i}\cap\Theta_{1,i}\neq\emptyset$ if $|i-j|=1$ and $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ if $|i-j|>1$. Moreover, if $b=1$, $(\Theta_{1,1}, (1/2)\sum_{j=1}^{2}\Theta_{2,j}|_{\Theta_{1,1}})$ is an elliptic singular or degenerate ν_0-log surface of type II_d. In the case (β), $b\geq 2$ and $(\Theta_{1,i}, \sum_{l\neq i}\Theta_{1,l}+(1/2)\sum_{j=1}^{2}\Theta_{1,j}|_{\Theta_{1,i}})$ is a ν_0-log surface of type II_b for $2\leq i\leq b-1$, a special ν_0-log surface of type II_d for $i=1$, b. $\Theta_{1,i}\cap\Theta_{1,j}\neq\emptyset$ if $|i-j|=1$ and $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ if $|i-j|>1$.

VI:

$$f^*(0)=\sum_{i=1}^{b}2m\Theta_{1,i}+\sum_{j=1}^{3}m\Theta_{2,j},$$

*where $m\in\boldsymbol{N}$, $\Theta_{2,1}$, $\Theta_{2,2}$ are elliptic ruled surfaces, $\Theta_{2,3}$ is a ruled surface and $(\Theta_{1,i}, \sum_{l\neq i}\Theta_{1,l}+(1/2)\sum_{j=1}^{3}\Theta_{2,j}|_{\Theta_{1,i}})$ is a ν_0-log surface of type II_b for $2\leq i\leq b-1$, a ν_0-*log *surface of type II_c for $i=1$, of type II_d for $i=b$. $\Theta_{1,i}\cap\Theta_{2,j}\neq\emptyset$ if $|i-j|=1$ and $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ if $|i-j|>1$.*

EXAMPLE. There is a minimal degeneration whose special fibre has simple elliptic singularities even if the total space is smooth. For example, let X be a hypersurface in $\boldsymbol{P}^3\times\mathscr{D}$ which is defined by the equation $X^4+Y^4+X^2Y^2+Z^2W^2+t(Z^4+W^4)$, where X, Y, Z, W are homogeneous coordinates of $\boldsymbol{P}^3$ and $\mathscr{D}:=\{t\in\boldsymbol{C}\,;\,|t|<1/2\}$. Let $f: X\to\mathscr{D}$ be the morphism induced by the natural projection $p: \boldsymbol{P}^3\times\mathscr{D}\to\mathscr{D}$. Then, it is easy to verify that X is smooth and has trivial canonical bundle, $X_t:=f^*(t)$ is a smooth quartic surface for $t\neq 0$ and that X_0 is normal and has only two simple elliptic singularities of type $\tilde{E}_7$ as its singularites (see [19]).

Proof of Theorem 4.3. If $(\Theta_i, \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i))$ is not a ν_0-log surface of type II_c or II_d for any irreducible component $\hat{\Theta}_i$ of $\hat{\Theta}$, $f: X\to\mathscr{D}$ can be obtained from $\hat{f}:\hat{X}\to\mathscr{D}$ by running the minimal model program ($\mathscr{D}$shrunk if necessary). Let Θ be the strict transform of $\hat{\Theta}$ on X, then, there is a positive integer m such that $f_*(0)=m\hat{\Theta}$ since some multiples K_X and $K_X+\Theta$ are multiples of $f^*(0)$. If $I:=$

$\{i\ ;\ \hat{\Theta}\subset\hat{\Theta},\ (\hat{\Theta}_i,\ \mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i))$ is a special ν_0-log surface of type II_c or $II_d\}$ is not empty, then $|I|=1$ or 2 and $(\hat{X}, 0)$ is terminal outside of $Z:=U_{i\in I}\mathrm{Supp}\{\mathrm{Diff}_{\hat{\Theta}_i}(\hat{\Theta}-\hat{\Theta}_i)\}$. We can show that the singularities of $\hat{X}$ in a neighborhood of Z is $A_{2,1}\times Z$ by the following lemma.

Lemma 4.2. *Let $(p\in X,S)$ be a germ of 3-dimensional purely log terminal singularity, where S is a $\mathbf{Q}$-Cartier prime divisor. Assume that $p\in S$, $X\backslash S$ is smooth and that $(p\in S, \mathrm{Diff}_S(0))$ is terminal. Then $p\in X$ is a smooth point of X or there is a positive integer m such that X is isomorphic to $\boldsymbol{C}^3/\boldsymbol{Z}_m(1, q, 0)$ near $p\in X$ and $0\in\boldsymbol{C}^3/\boldsymbol{Z}_m(1, q, 0)$, where q is a positive integer such that $(m, q)=1$.*

Proof of Lemma 4.2. *Assume first that $p\in X$ does not lie on the support of* $\mathrm{Diff}_S(0)$. *From* [10], *Corollary* 17.12, *(X, S) is canonical outside the support of* $\mathrm{Diff}_S(0)$. *Since* Sing $X\subset S$, *$(p\in X, 0)$ is terminal but by the proof of* Lemma 5.3 *in* [10], *$p\in X$ is in fact smooth because $p\in S$ is smooth. Assume that $p\in$* Supp $\mathrm{Diff}_S(0)$. *Let* $\mathrm{Diff}_S(0)=\sum_i^l\{(m_i-1)/m_i\}\Gamma_i$ *be the irreducible decomposition, where the m_i are integers which are equal to or larger than* 2. *Since $(p\in S$,* $\mathrm{Diff}_S(0))$ *is terminal, we have $\sum_i^l(m_i-1)/m_i<1$, whence $l=1$. So we may write* $\mathrm{Diff}_S(0)=\{(m-1)/m\}\Gamma$, *where $m\geq 2$. Take the log canonical cover $\pi:\tilde{X}\to X$ with respect to K_X+S and put $\tilde{S}:=\pi^{-1}(S)$. Let mr be the local Cartier index of K_X+S at p, where r is a positive integer, and assume that $r<1$. From* [10], Lemma 16.13, *$(\tilde{X}, \tilde{S})$ is purely log terminal, hence canonical. So $\tilde{S}$ is a disjoint union of r-irreducible components, but this is absurd. Thus, we get $r=1$ and $\tilde{S}$ is irreducible. Since* Sing $\tilde{X}\subset\tilde{S}$, *$(\tilde{X}, 0)$ is terminal., We have*

$$K_{\tilde{S}}=(\pi|_{\tilde{S}})^*(K_S+\frac{m-1}{m}\Gamma)$$

and the proof of Corollary 2.2 in [20] shows that $\tilde{S}$ is smooth. We can also check this directly. Hence $\tilde{X}$ is smooth and X must be a cyclic quotient singularity. Thus the lemma follows from [6], Lemma 9.9. ■

We blow-up these singularities to obtain $\sigma:\tilde{X}\to\hat{X}$, where $(\tilde{X}, 0)$ is terminal. Let $\tilde{\Theta}_{2,j}(j\in J)$ be exceptional divisors of σ and put $\tilde{\Theta}:=\sigma_*^{-1}\hat{\Theta}+\sum_{j\in J}(1/2)\tilde{\Theta}_{2,j}$. Then we have $K_{\tilde{X}}+\tilde{\Theta}=\sigma^*(K_{\hat{X}}+\hat{\Theta})$ and $f: X\to\mathscr{D}$ can be obtained from $\hat{f}\circ\sigma$: $(\tilde{X}, \tilde{\Theta})\to\mathscr{D}$ by running the minimal model program with shrunk if necessary. Let Θ be the strict transform of $\tilde{\Theta}$ on X. Then some multiples of $K_X+\Theta$ and K_X are multiples of $f^*(0)$ and there is a positive integer m such that $f^*(0)=2m\Theta$. In the course of applying the minimal model program, we have to care about divisorial contractions which might produce bad degenerations. We know that irrational surfaces are not contracted to points by the contraction associated with an extremal ray.

Claim 1. If $(\tilde{\Theta}_1, (\tilde{\Theta}-\tilde{\Theta}_1)|_{\tilde{\Theta}_1})$ is a special ν_0-log surface of type II_c or II_d, $\tilde{\Theta}_1$ is not contracted in the course of running the minimal model program.

Proof of Claim 1. Let $\tilde{f}^{(i)}: \tilde{X}^{(i)}\to\mathcal{D}$ be a 3-fold over $\mathcal{D}$ which is obtained from $\tilde{X}$ by divisorial contractions and flips. Let $\rho: \tilde{X}^{(i)}\to\tilde{X}^{(i+1)}$ be a divisorial contraction which contracts $\tilde{\Theta}_1^{(i)}$, where $\tilde{\Theta}_1^{(i)}$ is the strict transform of $\tilde{\Theta}_1$ on $\tilde{X}^{(i)}$. First suppose that there is an irreducible component $\tilde{\Theta}_2^{(i)}$ of $\lfloor \tilde{\Theta}^{(i)} \rfloor$ other than $\tilde{\Theta}_1^{(i)}$ which has non-empty intersection with $\tilde{\Theta}_1^{(i)}$, where $\tilde{\Theta}^{(i)}$ is the strict transform of $\tilde{\Theta}$ on $\tilde{X}^{(i)}$. Let n_1, n_2 be multiplicities of $\tilde{\Theta}_1^{(i)}$, $\tilde{\Theta}_2^{(i)}$ respectively, and let l be a general fibre of the ruling of $\tilde{\Theta}_1^{(i)}$. Since we have $(K_{\tilde{X}^{(i)}}+\tilde{\Theta}^{(i)})\cdot l=0$, we have $\tilde{\Theta}_1^{(i)}\cdot l=-K_{\tilde{X}^{(i)}}\cdot l-2$. Since $H^1(\omega_{\tilde{X}^{(i)}}\otimes\mathcal{O}_l)=0$ and $l\subset\operatorname{Reg}\tilde{X}^{(i)}$, we have $-K_{\tilde{X}^{(i)}}\cdot l=1$ and $\tilde{\Theta}_1^{(i)}\cdot l=-1$. From this, we get

$$0=\tilde{f}^{(i)*}(0)\cdot l=n_1\tilde{\Theta}_1^{(i)}\cdot l+n_2+n_1=n_2,$$

which is a contradiction. If there is no irreducible component of $\lfloor \tilde{\Theta}^{(i)} \rfloor$, then $K_{\tilde{X}^{(i)}}$ is numerically trivial over $\mathcal{D}$ and this leads to a contradiction. ■

Claim 2. The divisorial contraction of a ν_0-log surface of type II_b does not change singularities locally on neighbouring surfaces.

Proof of Claim 2. Let $\rho: \tilde{X}^{(i)}\to\tilde{X}^{(i+1)}$ be a divisorial contraction associated with an extremal ray which contracts $\tilde{\Theta}_1^{(i)}$ to a curve, where $(\tilde{\Theta}_1^{(i)},(\tilde{\Theta}^{(i)}-\tilde{\Theta}_1^{(i)})|_{\tilde{\Theta}_1^{(i)}})$ is a ν_0-log surface of type II_b, and let l be a general fiber of the ruling of $\tilde{\Theta}_1^{(i)}$. Let $\tilde{\Theta}_2^{(i)}$ be one of the neighbouring surfaces. Since we have $(-\tilde{\Theta}_2^{(i)}-K_{\tilde{X}^{(i)}})\cdot l=0$, $-\tilde{\Theta}_2^{(i)}-K_{\tilde{X}^{(i)}}$ is ρ-trivial, hence $R^1\rho_*\mathcal{O}_{\tilde{X}^{(i)}}(-\tilde{\Theta}_2^{(i)})=0$ and $\mathcal{O}_{\tilde{\Theta}_2^{(i+1)}}\simeq\rho_*\mathcal{O}_{\tilde{\Theta}_2^{(i+1)}}$, where $\tilde{\Theta}_2^{(i+1)}=\rho_*\tilde{\Theta}_2^{(i)}$. So ρ induces an isomorphism $\tilde{\Theta}_2^{(i)}\simeq\tilde{\Theta}_2^{(i+1)}$. ■

Claim 3. When a ν_0-log surface of type II_a is contracted to a point by a divisorial contraction, this contraction produces a simple elliptic singularity on a neighbouring surface.

Proof of Claim 3. Let $\rho: \tilde{X}^{(i)}\to\tilde{X}^{(i+1)}$ be a divisorial contraction of an extremal ray which contracts $\tilde{\Theta}_1^{(i)}$ to a point, where $(\tilde{\Theta}_1^{(i)},(\tilde{\Theta}^{(i)}-\tilde{\Theta}_1^{(i)})|_{\tilde{\Theta}_1^{(i)}})$ is a ν_0-log surface of type II_a. Let $\tilde{\Theta}_2^{(i)}$ be a neighbouring surface. Since $-\tilde{\Theta}_1^{(i)}-\tilde{\Theta}_2^{(i)}-K_{\tilde{X}^{(i)}}$ is ρ-trivial, $R^1\rho_*\mathcal{O}_{\tilde{X}^{(i)}}(-\tilde{\Theta}_1^{(i)}-\tilde{\Theta}_2^{(i)})=0$. So we have a surjection

$$\mathcal{O}_{\tilde{\Theta}_2^{(i+1)}}\to\rho_*\mathcal{O}_{\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)}}. \tag{4.1}$$

From an exact sequence;

$$0\to\mathcal{O}_{\tilde{\Theta}_1^{(i)}}(-\tilde{\Theta}_2^{(i)})\to\mathcal{O}_{\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)}}\to\mathcal{O}_{\tilde{\Theta}_1^{(i)}}\to0,$$

we have the following exact sequence

$$\rho_*\mathcal{O}_{\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)}}\to\rho_*\mathcal{O}_{\tilde{\Theta}_1^{(i)}}\to R^1\rho_*\mathcal{O}_{\tilde{\Theta}_1^{(i)}}(-\tilde{\Theta}_2^{(i)}).$$

Put $\Gamma:=\tilde{\Theta}_2^{(i)}|_{\tilde{\Theta}_1^{(i)}}$. Since Γ is irreducible, we have an injection $H^1(\mathcal{O}_{\tilde{\Theta}_1^{(i)}}(-\Gamma))\hookrightarrow H^1(\mathcal{O}_{\tilde{\Theta}_1^{(i)}})$. But since $\tilde{\Theta}_1^{(i)}$ is rational, we deduce that $R^1\rho_*\mathcal{O}_{\tilde{\Theta}_1^{(i)}}(-\tilde{\Theta}_2^{(i)}=H^1(\mathcal{O}_{\tilde{\Theta}_1^{(i)}}$

$(-\Gamma))=0$. So we have a surjection

$$\rho_* \mathcal{O}_{\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)}} \to \rho_* \mathcal{O}_{\tilde{\Theta}_1^{(i)}}. \tag{4.2}$$

From (4.1) and (4.2), we deduce that $\mathcal{O}_{\tilde{\Theta}_2^{(i+1)}} \simeq \rho_* \mathcal{O}_{\tilde{\Theta}_2^{(i)}}$ and Γ is contracted to a simple elliptic singularity on $\tilde{\Theta}_2^{(i+1)}$. ■

Claim 4. When a ν_0-log surface of type II_a is contracted to a curve by a divisorial contraction, this contraction produces non-normal singularities on the neighbouring surfaces, but these singularities are Cohen-Macaulay and semi-canonical.

Proof of Claim 4. Let $\rho: \tilde{X}^{(i)} \to \tilde{X}^{(i+1)}$ be a divisorial contraction of an extremal ray which contracts $\tilde{\Theta}_1^{(i)}$ to a curve, where $(\tilde{\Theta}_1^{(i)}, (\tilde{\Theta}^{(i)}-\tilde{\Theta}_1^{(i)})|_{\tilde{\Theta}_1^{(i)}})$ is a ν_0-log surface of type II_a. Let $\tilde{\Theta}_2^{(i)}$ be a neighbouring surface. As in the above argument, we have a surjection

$$\mathcal{O}_{\tilde{\Theta}_2^{(i+1)}} \to \rho_* \mathcal{O}_{\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)}}. \tag{4.3}$$

Let $\pi: \mathcal{S}_2(\tilde{\Theta}_2^{(i+1)}) \to \tilde{\Theta}_2^{(i+1)}$ be the S_2-infication of $\tilde{\Theta}_2^{(i+1)}$ and put $\tilde{\Theta}' := \tilde{\Theta} \times_{\tilde{\Theta}_2^{(i+1)}} \mathcal{S}_2(\tilde{\Theta}_2^{(i+1)})$, where $\tilde{\Theta} := \tilde{\Theta}_1^{(i)} + \tilde{\Theta}_2^{(i)}$. Let $\pi': \tilde{\Theta}' \to \tilde{\Theta}$ be the first projection and let $\rho': \tilde{\Theta}' \to \mathcal{S}_2(\tilde{\Theta}_2^{(i+1)})$ be the second projection,

$$\begin{array}{ccc} \tilde{\Theta}' & \xrightarrow{\pi'} & \tilde{\Theta} \\ \rho' \downarrow & & \downarrow \rho \\ S_2(\tilde{S}_2^{(i+1)}) & \xrightarrow{\pi} & \tilde{\Theta}_2^{(i+1)} \end{array}$$

Since π is finite and isomorphic in codimension 1, π' is finite, birational on each component and isomorphic on the generic point of the double locus. So π' is also isomorphism in codimension 1 and $\mathcal{O}_{\tilde{\Theta}} \simeq \pi'_* \mathcal{O}_{\tilde{\Theta}'}$ since $\tilde{\Theta}$ is Cohen-Macaulay. Thus π' is an isomorphism. From (4.3), the natural inclusions

$$\mathcal{O}_{\tilde{\Theta}_2^{(i+1)}} \hookrightarrow \pi_* \mathcal{O}_{S_2(\tilde{\Theta}_2^{(i+1)})} \hookrightarrow \pi_* \rho'_* \mathcal{O}_{\tilde{\Theta}'} \simeq \rho_* \mathcal{O}_{\tilde{\Theta}_2^{(i+1)}}$$

are surjective. Therefore $\tilde{\Theta}_2^{(i+1)}$ satisfies Serre's condition S_2 i.e., $\tilde{\Theta}_2^{(i+1)}$ is Cohen-Macaulay. Since the normalization of the new singularity of $\tilde{\Theta}_2^{(i+1)}$ coincides with $\tilde{\Theta}_2^{(i)}$, it is easy to see that $\tilde{\Theta}_2^{(i+1)}$ is semi-canonical. ■

Claim 5. Any degenerate ν_0-log surface of type II_b is not contracted by a divisorial contraction.

Proof of Claim 5. Let $\rho: \tilde{X}^{(i)} \to X^{(i+1)}$ be a divisorial contraction associated with an extremal ray which contracts $\tilde{\Theta}_1^{(i)}$, where $(\tilde{\Theta}_1^{(i)}, (\tilde{\Theta}^{(i)}-\tilde{\Theta}_1^{(i)})|_{\tilde{\Theta}_1^{(i)}})$ is a degenerate ν_0-log surface of type II_b. Since $h^1(\mathcal{O}_{\tilde{\Theta}_1^{(i)}}) \neq 0$, $\rho(\tilde{\Theta}_1^{(i)})$ is not a point, but a curve. Let l be a general fibre of $\rho|_{\tilde{\Theta}_1^{(i)}}: \tilde{\Theta}_1^{(i)} \to \rho(\tilde{\Theta}_1^{(i)})$ and let $\tilde{\Theta}_2^{(i+1)}$ be a neighbouring surface. Then since $K_{\tilde{X}^{(i)}}\cdot l=-1$ and $\tilde{\Theta}_2^{(i)}\cdot l=1$, we have $\tilde{\Theta}_1^{(i)}\cdot l=(K_{\tilde{X}^{(i)}}+\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)}\cdot l=1$, we have $\tilde{\Theta}_1^{(i)}\cdot l=(K_{\tilde{X}^{(i)}}+\tilde{\Theta}_1^{(i)}+\tilde{\Theta}_2^{(i)})\cdot l=0$, which is a contradiction. ■

By the following lemma, we can see that essentially new singularities do not appear after flips and flops.

Lemma 4.3. *Let φ (resp. φ^+): X(resp. X^+)$\to Z$ be a projective birational morphism from a normal complex 3-fold X (resp. X^+) to a normal 3-fold Z, such tiat φ (resp. φ^+) is an isomorphism in codimension 1, $(X, 0)$(resp. $(X^+, 0)$) is Kawamata log terminal and $-K_X$ (resp. K_{X^+}) is φ-nef (resp. φ^+-nef). Furthermore, let S be a reduced surface on X such that (X, S) is log canonical and K_X+s is φ-numerically trivial. Assume that any irreducible component of the exceptional locus of φ is not contained in the non-normal locus of S. Let S^+ be the strict transform of S on X^+. In the above situation, if S is Cohen-Macaulay, S^+ is Cohen-Macaulay, too.*

Proof. *We have $\mathcal{O}_{\bar{S}}\simeq\varphi_*\mathcal{O}_S$ by the vanishing theorem, where $\bar{S}:=\varphi_*S$. Let $\pi: S_2(\bar{S})\to\bar{S}$ be the S_2-ification of $\bar{S}$ and let $S': S\times_{\bar{S}}\mathcal{S}_2(\bar{S})$. Let π': $S'\to S$ be the first projection and let φ': $S'\to\mathcal{S}_2(\bar{S})$ be the second projection,*

$$\begin{array}{ccc} S' & \xrightarrow{\pi'} & S \\ \varphi\downarrow & & \downarrow\varphi \\ S_2(\bar{S}) & \xrightarrow{\pi} & \bar{S} \end{array}$$

Since π is finite and isomorphic in codimension 1, π' is finite and birational on each component. By the assumption, π' is also an isomorphism in codimension 1. In the same way as in the proof of Claim 4, we can see that π' is an isomorphism and $\bar{S}$ is Cohen-Macaulay. Let π^+: $\mathcal{S}_2(S^+)\to S^+$ be the S_2-ification of S^+. From an exact sequence

$$0\to\mathcal{O}_{S^+}\to\pi_*^+\mathcal{O}_{\mathcal{S}_2(S^+)}\to\mathcal{N}\to 0,$$

where $\mathcal{N}$ is a sheaf such that dim Supp $\mathcal{N}=0$, we have the following exact sequence

$$0\to\varphi_*^+\mathcal{O}_{S^+}\xrightarrow{\alpha}\varphi_*^+\pi_*^+\mathcal{O}_{\mathcal{S}_2(S^+)}\to\varphi_*^+\mathcal{N}\to R^1\varphi_*^+\mathcal{O}_{S^+}.$$

We note that the last term of the above exact sequence is 0, because $R^1\varphi_*^+\mathcal{O}_{X^+}=0$ and $R^2\varphi_*^+\mathcal{O}_{X^+}(-S^+)=0$. Since $\bar{S}$ is Cohen-Macaulay, we have $\mathcal{O}_{\bar{S}}\simeq\varphi_*^+\pi_*^+\mathcal{O}_{\mathcal{S}_2(S^+)}$. So we have the inclusions

$$\mathcal{O}_{\bar{S}}\to\varphi_*^+\mathcal{O}_{S^+}\to\varphi_*^+\pi_*^+\mathcal{O}_{\mathcal{S}_2(S^+)},$$

hence $\mathcal{N}=0$, which implies that S^+ is Cohen-Macaulay.

Claim 6. The non-normal locus of a degenerate ν_0-log surface is not contracted by a flipping (or flopping) contraction.

Proof of Claim 6. Let $\varphi: \tilde{X}^{(i)} \to \tilde{X}^{(i+1)}$ be a flipping (or flopping) contraction of an extremal ray which contracts the non-normal locus, say C, of $\tilde{S}^{(i)}$, where $(\tilde{S}^{(i)}, (\tilde{\Theta}^{(i)} - \tilde{S}^{(i)})|_{\tilde{S}^{(i)}})$ is a degenerate ν_0-log surface. From an exact sequence;

$$0 \to \mathcal{O}_{\tilde{S}^{(i)}} \to \nu_* \mathcal{O}_{\tilde{S}^{(i)\nu}} \to \mathcal{O}_C \to 0,$$

we have an exact sequence;

$$R^1 \varphi_* \mathcal{O}_{\tilde{S}^{(i)}} \to R^1 \varphi_* (\nu_* \mathcal{O}_{\tilde{S}^{(i)\nu}}) \to R^1 \varphi_* \mathcal{O}_C,$$

where the last term is 0 since C is a rational curve. We can show that the first term is also 0 since $R^1 \varphi_* \mathcal{O}_{\tilde{X}^{(i)}} = R^2 \varphi_* \mathcal{O}_{\tilde{X}^{(i)}}(-\tilde{S}^{(i)}) = 0$. Hence $R^1 \varphi_* (\nu_* \mathcal{O}_{\tilde{S}^{(i)\nu}}) = 0$. Let $\varphi^\nu: \tilde{S}^{(i)\nu} \to \tilde{S}^{(i+1)\nu}$ be the morphism induced by φ. Since

$$\begin{aligned} 0 &= R^1 \varphi_* (\nu_* \mathcal{O}_{\tilde{S}^{(i)\nu}}) = R^1 (\varphi \circ \nu)_* \mathcal{O}_{\tilde{S}^{(i)\nu}} = R^1 (\nu \circ \varphi^\nu)_* \mathcal{O}_{\tilde{S}^{(i)\nu}} \\ &= \nu_* \mathrm{R}^1 \varphi^\nu_* \mathcal{O}_{\tilde{S}^{(i)\nu}}, \end{aligned}$$

we have $R^1 \varphi^\nu_* \mathcal{O}_{\tilde{S}^{(i)\nu}} = 0$, which is a contradiction because φ^ν contracts an elliptic curve.

Flips and flops may produce new non-normal singularities, but if the non-normal locus contains a curve, we can show that this assumption leads to a contradiction by the classification of ν_0-log surfaces. Recalling that Serre's conditions S_2 and R_1 are equivalent to the normality, we can deduce that new non-normal points do not appear. We note by easy observation that the speciality of ν_0-log surfaces is preserved under flips but not under flops. Thus we have proved Theorem 4.3. ■

5. Classification of ν_0-log surfaces of abelian type

Let $\hat{f}: (\hat{X}, \hat{\Theta}) \to \mathcal{D}$ be a log minimal degeneration of surfaces with $\kappa = 0$ and assume that $\hat{\Theta}$ is irreducible. Then $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is a ν_0-log surface of type I in the following sense.

DEFINITION 5.1. Let (S, Δ) be a ν_0-log surface. (S, Δ) is called a *ν_0-log surface of type I*, if $\lfloor \Delta \rfloor = 0$.

We note that a ν_0-log surface of type I is a Log Enriques surface in the sense of De-Qi Zhang [25], if $\Delta = 0$ and $q(S) = 0$.

DEFINITION 5.2. Let (S, Δ) be a ν_0-log surface of type I. A number defined by

$$\mathrm{CI}(S, \Delta) := \mathrm{Min}\{n \in \boldsymbol{N}\,;\ n(K_S+\Delta) \text{ is Cartier}\}$$

is called *the Cartier index* of (S, Δ).
Let (S, Δ) be as above and let r the minimum value such that $r(K_S+\Delta)\sim 0$. We define the *log canonical cover* of (S, Δ) as

$$\pi : \tilde{S} := \mathrm{Spec}_S \oplus_{i=0}^{r-1} \mathcal{O}_S(\lfloor -i(K_S+\Delta) \rfloor) \to S,$$

where the $\mathcal{O}_S$-algebra structure of $\oplus_{i=0}^{r-1} \mathcal{O}_S(\lfloor -i(K_S+\Delta) \rfloor)$ is given by a nowhere vanishing section of $\mathcal{O}_S(r(K_S+\Delta))$. This definition does not depend on the choice of the nowhere vanishing sections up to isomorphisms. By the definition and [20], Corollary 2.2, $\tilde{S}$ is a normal surface with only rational double points and has trivial canonical bundle. So $\tilde{S}$ is a K3 surface with only rational double points or an abelian surface by the classification theory of surfaces.

DEFINITION 5.3. Let (S,Δ) be a ν_0-log surface of type I, and $\pi : \tilde{S} \to S$ be the log canonical cover. When $\tilde{S}$ is K3 surface with only rational double points (resp. abelian surface), (S, Δ) is called *ν_0-log surface of type* K3 (resp. *ν_0-log surface of abelian type*).

The next lemma gives us a hope of classifying ν_0-log surfaces. We refer the reader to [16] Theorem 3.1 or [25] Lemma 2.3

Lemma 5.1. Let $\tilde{\rho}$ be the Picard number of the minimal resolution of the log canonical cover $\tilde{S}$ and let φ be the Euler function. Then $\varphi(\mathrm{CI}(S, \Delta))|(22-\tilde{\rho})$ if (S, Δ) is a ν_0-log surface of type K3 and $\varphi(\mathrm{CI}(S, \Delta))|(6-\tilde{\rho})$ under the assumption that (S, Δ) is a ν_0-log surface of abelian type and that $\mathrm{CI}(S, \Delta)(K_S+\Delta)\sim 0$.

In what follows, we mean by writing Sing $S=\sum_{n,q} m_{n,q} A_{n,q}$, that the singular locus of S is composed of $m_{n,q}$ singular points of type $A_{n,q}$.

Theorem 5.1. *ν_0-log surfaces of abelian type (S, Δ) can be classified as follows. In the list below, we mean by writing C', the strict transform of a curve $C \subset S$ on the minimal resolution of S.*

I : S is an abelian surface or a hyperelliptic surface and $\Delta=0$.

II : $S \simeq \boldsymbol{P}_E(\mathcal{O}_E \oplus \mathcal{L})$, where E is a smooth elliptic curve and $\mathcal{L} \in Pic^0 E$. Moreover, Supp Δ *is smooth and Δ has one of the following types.*

II_α : $\Delta=\sum_i^4 (1/2) C_i$, where C_i is a section and $C_i^2=0$ for every i.

II_β : $\Delta=\sum_i^2 (1/2) C_i$, where C_1 is a 3-section and C_2 is a section. C_i is a smooth elliptic curve and $C_i^2=0$ for every i.

II_γ : $\Delta=\sum_i^2 (1/2) C_i$, where C_i is a 2-section which is a smooth elliptic curve

and $C_i^2=0$ *for every* i.

II_δ : $\Delta=\sum_i^3(1/2)C_i$, *where* C_1 *is a* 2-*section which is a smooth elliptic curve,* C_i *is a section for* $i=2$, 3 *and* $C_i^2=0$ *for every* i.

II_ε : $\Delta=(1/2)C$, *where* C *is a* 4-*section which is a smooth elliptic curve and* $C^2=0$.

III_α : $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, *where* E *is a smooth elliptic curve and* $\mathcal{L}\in Pic^0E$. *Moreover,* $\Delta=\sum_{i=3}^c(2/3)C_i$, *where* $C_i(i=1, 2, 3)$ *are sections with self-intersection number* 0 *and they are disjoint from each other.*

III_β : $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, *where* E *is a smooth elliptic curve and* $\mathcal{L}\in Pic^0E$. *Moreover,* $\Delta=\sum_{i=1}^2(2/3)C_i$, *where* C_1 *is a* 2-*section which is a smooth elliptic curve and* C_2 *is a section. Moreover,* $C_i(i=1, 2)$ *are disjoint from each other and have self-intersection number* 0.

III_γ : $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, *where* E *is a smooth elliptic curve and* $\mathcal{L}\in Pic^0E$, *and* $\Delta=(2/3)C$, *where* C *is a* 3-*section which is a smooth elliptic curve and* $C^2=0$.

III_δ : S *is a normal rational surface with* $\rho(S)=4$, $\mathrm{Sing}S=9A_{3,1}$ *and* $\Delta=0$. *The minimal resolution* M *of* S *is obtained by blowing up* Σ_d $(d\leq 3)$.

IV_α : $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, *where* E *is a smooth elliptic curve and* $\mathcal{L}\in Pic^0E$, *and* $\Delta=\sum_{i=1}^2(3/4)C_{1,i}+(1/2)C_2$, *where* $C_{1,i}$, C_2 *are sections with self-intersection numbers* 0.

IV_β : $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, *where* E *is a smooth elliptic curve and* $\mathcal{L}\in Pic^0E$, *and* $\Delta=(3/4)C_1+(1/2)C_2$, *where* C_1 *is a* 2-*section which is a smooth elliptic curve,* C_2 *is a section and* $C_i^2=0$ *for* $i=1, 2$.

IV_γ : S *is normal rational surface with* $\rho(S)=2$ *and* $\mathrm{Sing}\ S=8A_{2,1}$. *The minimal resolution* M *of* S *is obtained by blowing up* $\Sigma_d(i\leq 4)$. *Moreover,* $\Delta=\sum_{i=1}^3(1/2)C_{2,i}$, *where* $C_{2,1}$ *is a smooth elliptic curve with* $C_{2,1}'^2=0$, $C_{2,i}\simeq \boldsymbol{P}^1$ *with* $C_{2,i}'^2=-2$ *for* $i=2, 3$, *and* $C_{2,i}\cap\mathrm{Sing}\ S=4A_{2,1}$ *for* $i=1, 2$.

IV_δ : S *is a normal rational surface with* $\rho(S)=2$ *and* $\mathrm{Sing}\ S=8A_{2,1}$. *The minimal resolution* M *of* S *is obtained by blowing up* $\Sigma_d(d\leq 4)$. *Moreover,* $\Delta=\sum_{i=1}^2(1/2)C_{2,i}$, *where* $C_{2,i}\simeq \boldsymbol{P}^1$ *with* $C_{2,i}'^2=-2$ *for* $i=1, 2$, *and* $C_{2,i}\cap\mathrm{Sing}\ S=4A_{2,1}$ *for* $i=1, 2$.

V : S *is a rational surface with* $\rho(S)=2$, $\mathrm{Sing}\ S=5A_{5,2}$ *and* $\Delta=0$. *The minimal resolution* M *of* S *is obtained by blowing up* $\Sigma_d(d\leq 3)$.

VI_α : $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, *where* E *is a smooth elliptic curve and* $\mathcal{L}\in Pic^0E$, *and* $\Delta=(5/6)C_1+(2/3)C_2+(1/2)C_3$, *where* $C_i(i=1, 2, 3)$ *are sections with* *self-intersection number* 0 *and disjoint from each other.*

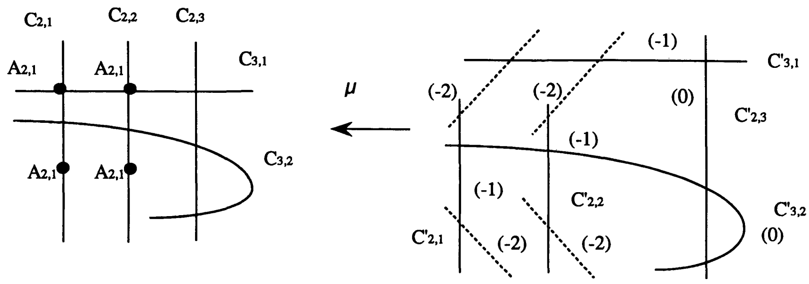
VI_β: $S\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$ *and* $\Delta=\sum_{i=1}^3(2/3)C_{2,i}+\sum_{j=1}^4(1/2)C_{3,j}$, *where* $C_{2,i}(i=1, 2, 3)$ *are fibres of the first projection* $S\to\boldsymbol{P}^1$ *and* $C_{3,j}(j=1, 2, 3, 4)$ *are fibres of the second projection* $S\to\boldsymbol{P}^1$.

VI_γ: S *is a rational surface with* $\rho(S)=2$, *and* Sing $S=3A_{3,1}+3A_{3,2}$. *The minimal resolution* M *of* S *is obtained by blowing up* $\sum_d(i\leq 4)$. *Furthermore,* $\Delta=\sum_{i=1}^{2}(1/2)C_{3,i}$, *where* $C_{3,1}$ *is a smooth elliptic curve with self-intersection number* 0, $C_{3,2}\simeq\boldsymbol{P}^1$ *with* $C_{3,2}'^2=-2$, $C_{3,1}\cap$Sing $S=\emptyset$, $C_{3,2}\cap$Sing $S=3A_{3,2}$ *and* $C_{3,1}\cap C_{3,2}=\emptyset$.

VI_δ: S *is a rational surface with* $\rho(S)=2$, *and* Sing $S=3A_{3,1}+3A_{3,2}$. *The minimal resolution* M *of* S *is obtained by blowing up* $\sum_d(d\leq 4)$. *Furthermore,* $\Delta=(1/2)C_3$, *where* $C_3\simeq\boldsymbol{P}^1$ *with* $C_3'^2=-2$, $C_{3,2}\cap$Sing $S=3A_{3,2}$ *and* $C_{3,1}\cap C_{3,2}=\emptyset$.

XII_α: $S\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$ *and* $\Delta=\sum_{i=1}^2(3/4)C_{1,i}+\sum_{j=1}^3(2/3)C_{2,j}+(1/2)C_3$, *where* $C_{1,i}(i=1, 2)$ *and* C_3 *are fibres of the first projection* $S\to\boldsymbol{P}^1$ *and* $C_{2,j}(j=1, 2, 3)$ *are fibres of the second projection* $S\to\boldsymbol{P}^1$.

XII_β: S *is a normal rational surface with* $\rho(S)=2$ *and* Sing $S=4A_{2,1}$. *The minimal resolution* M *of* S *is obtained by blowing up* $\sum_d(d\leq 6)$. *Furthermore,* $\Delta=\sum_{i=1}^2(2/3)C_{2,i}+\sum_{j=1}^3(1/2)C_{3,j}$, *where* $C_{2,i}$, $C_{3,j}\simeq\boldsymbol{P}^1$, $C_{3,1}'^2=C_{3,2}'^2=C_{2,1}'^2=-1$ *and* $C_{3,3}'^2=C_{2,2}'^2=0$. *The configuration of* Supp Δ *and the singular loci of* S *are given as follows.*



Proof. Let $M\to S$ be the minimal resolution and let $\pi: \tilde{S}\to S$ be the global log canonical cover with respect to the pair (S, Δ). By σ, we signify a generator of the covering transformation group Gal $(\tilde{S}/S)$. Put $\rho=\rho(S)$. We fix these notations in what follows. From Lemma 5.1, the possible value of CI (S, Δ) is 1, 2, 3, 4, 5, 6, 8, 10 or 12. We may assume that CI $(S, \Delta)\geq 2$.

Case CI $(S, \Delta)=2$. After taking an ētale cover of S, we may assume that $2(K_S+\Delta)\sim 0$ since the ētale quotient of an elliptic ruled surface is also an elliptic ruled surface. Let $p\in\tilde{S}$ be any fixed point of under the action of Gal $(\tilde{S}/S)$. The generator σ of Gal $(\tilde{S}/S)$ acts on m_p/m_p^2 in such a way that $\sigma^*(x, y)=(x, -y)$ for a suitable basis x, y. Therefore S is a smooth surface with $q=1$ and $\Delta=(1/2)C$, where C is a not necessarily connected smooth curve. Thus $S\simeq \boldsymbol{P}_E(\mathcal{O}_E\oplus\mathcal{L})$, where E is a smooth elliptic curve and $\mathcal{L}\in\mathrm{Pic}^0E$ (see [17], 4.2.1 or [1], Lemma 6.2).

Case CI $(S, \Delta)=3$. If S is not rational, we are in one of the cases III_α, III_β or III_γ. Assume that S is rational.

Take $p\in\tilde{S}$ as above. Then $(a)\sigma^*(x, y)=(\zeta x, y)$, $(b)(\zeta x, \zeta^2 y)$ or $(c)(\zeta x, \zeta y)$ for a suitable basis x, y of m_p/m_p^2, where ζ is a primitive cubic root of unity. But the case (a) is excluded from the assumption that S is rational and (b) is also excluded since K_S is Cartier at $\pi(p)$. Therefore we are in the case (c). Put $\tilde{S}=V/L$, where $V=T_{\tilde{S},p}$ and L is a rank 4 free $\boldsymbol{Z}$-module. Since the action of $\langle\sigma\rangle$ on L is faithfull and torsion free and $\boldsymbol{Z}[\langle\sigma\rangle]\simeq\boldsymbol{Z}[\zeta]$ is a principal ideal domain, we have $L\simeq\boldsymbol{Z}[\zeta]^{\oplus 2}$ as $\boldsymbol{Z}[\zeta]$-module. From the assumption that $\sigma^*(x, y)=(\zeta x, \zeta y)$, V is unique as the 2-dimensional eigen vector space of $T_{\tilde{S},p}\oplus\bar{T}_{\tilde{S},p}\simeq\boldsymbol{Z}[\zeta]^{\oplus 2}\otimes\boldsymbol{C}$ associated with the eigen value ζ under the action of $\zeta\otimes id$. Therefore V/L and the action of $\langle\sigma\rangle$ is unique up to isomorphism. Hence $\tilde{S}\simeq E_\zeta\times E_\zeta$, where $E_\zeta=\boldsymbol{C}^2/\boldsymbol{Z}+\zeta\boldsymbol{Z}$ and $\sigma([z, w)]=[\zeta z, \zeta w]$ for $[z, w]\in E_\zeta\times E_\zeta$. Thus S has 9 singular points of type $A_{3,1}$ and $\Delta=0$. Let $Z:=\mathrm{Sing}\ S$. Since $\pi|_{\tilde{S}\backslash\pi^{-1}(Z)}: \tilde{S}\backslash\pi^{-1}(Z)\to S\backslash Z$ is ētale, we have

$$\chi_{\mathrm{top}}(\tilde{S})-9=3(\chi_{\mathrm{top}}(M)-18). \tag{5.1}$$

Nothing that $\chi_{\mathrm{top}}(\tilde{S})=0$ and that $\chi_{\mathrm{top}}(M)=2+\rho+s$, we obtain that $\rho=4$. Thus we are in the case III_δ.

Case CI $(S, \Delta)=4$. If S is not rational, we are in the case IV_α or IV_β. Assume S is rational. Let $p\in\tilde{S}$ be as above. Then $(a)\sigma^*(x, y)=(\sqrt{-1}x, y)$,$(b)(\sqrt{-1}x, \sqrt{-1}y)$, $(c)(-x, \sqrt{-1}y)$ or $(d)(-\sqrt{-1}x, \sqrt{-1}y)$ for a suitable basis x, y of m_p/m_p^2. But case (a) is excluded by the assumption that S is rational and cases (b) and(d) are also excluded since $2K_S$ and K_S are Cartier at $\pi(p)$ respectively. Therefore all singular points of S are of type $A_{2,1}$ and Δ can be written as $\Delta=(1/2)C$, where C is a smooth reduced curve such that Sing $S\subset$Supp C. Let $\mu: M\to S$ be the minimal resolution and put $C':=\mu_*^{-1}C$. Since $\pi|_{\tilde{S}\backslash\pi^{-1}(\mathrm{Supp}\ \Delta)}: \tilde{S}\backslash\pi^{-1}(\mathrm{Supp}\ \Delta)\to S\backslash\mathrm{Supp}\ \Delta$ is ētale, we have

$$\chi_{\mathrm{top}}(\tilde{S})-\chi_{\mathrm{top}}(\pi^{-1}(C\backslash Z))-s=4(\chi_{\mathrm{top}}(M)-\chi_{\mathrm{top}}(C')-2s+s), \tag{5.2}$$

where $Z:=\mathrm{Sing}\ S$ and s is the number of the singular points of S. Since $\chi_{\mathrm{top}}(\tilde{S})=0$, $\chi_{\mathrm{top}}(\pi^{-1}(C\backslash Z))=2\chi_{\mathrm{top}}(C\backslash Z)=-2(K_M\cdot C'+C'^2)-2s$ and $\chi_{\mathrm{top}}(M)=2+\rho+s$, we obtain that

$$K_M\cdot C'+C'^2=(1/2)s-2\rho-4. \tag{5.3}$$

On the other hand, we have

$$K_M+(1/2)C'+(1/4)E\sim_{\mathrm{num}}0, \tag{5.4}$$

where $E:=\sum_{i=1}^{s}E_j$ and $E_j(1\le j\le s)$ are (-2)-curves. From (5.4), we get $K_M^2+(1/2)K_M\cdot C'=0$ and $K_M\cdot C'+(1/2)C'^2+(1/4)s=0$. Hence $K_M\cdot C'=2\rho+2s-20$ and $C'^2=40-4\rho-(9/2)s$, since $K_M^2=10-\rho-s$. These two equations plugged into (5.3) yield $s=8$. Let C_i be any irreducible component of Supp $\varDelta$. Since $\pi^{-1}(C_i)$ is a disjoint union of elliptic curves, C_i is (1) an elliptic curve or (2) isomorphic to $\boldsymbol{P}^1$, and the number of singular points of S which is contained in C_i is 4. In the case (1), we have $C_i^2=0$and in the case (2), $C_i^2=-2$. Assume that there are two or more elliptic components of C. Let $\tau: M\to N$ be a birational morphism from M to a relatively minimal model N and let $\bar{C}:=\tau_*C$ and $\bar{E}:=\tau_*E$. If $N\simeq\boldsymbol{P}^2$, then $\bar{C}$ is a union of two smooth cubic curves and $\bar{E}=0$. If $N\simeq\sum_d$, then $\bar{C}$ is a union of two smooth elliptic curves and $\bar{E}=0$. Hence $E=0$ and this is a contradiction. So we are in the cases IV_γ or IV_δ.

Case CI $(S, \varDelta)=5$. If S is not rational, then S is an elliptic ruled surface. Let f be a fibre of the ruling. Then we have $(K_S+(4/5)C, f)=0$, hence $C\cdot f=5/2$, which is absurd. Assume S is rational. Let $p\in$ be as above. Then $(a)\sigma^*(x, y)=(\zeta x, y)$, $(b)(\zeta x, \zeta y)$, $(c)(\zeta x, \zeta^2 y)$ or $(d)(\zeta x, \zeta^4 y)$ for a suitable basis x, y of m_p/m_p^2, where ζ is a primitive fifth root of unity. But the case (a) is excluded by the assumption that S is rational and the case $(d)$is also excluded since K_S is Cartier at $\pi(p)$. Put $\tilde{S}=V/L$, where $V=T_{\tilde{S},p}$ and L is a rank 4 free $\boldsymbol{Z}$-module. Since the action of $\langle\sigma\rangle$ on L is faithfull and torsion free and $\boldsymbol{Z}[\langle\sigma\rangle]\simeq\boldsymbol{Z}[\zeta]$ is a principal ideal domain, we have $L\simeq\boldsymbol{Z}[\zeta]$ as $\boldsymbol{Z}[\zeta]$-module. Assume we are in the case (b). Then the eigen vector space of $T_{\tilde{S},p}\simeq\boldsymbol{Z}[\zeta]\otimes\boldsymbol{C}$ associated with the eigen value ζ under the action of $\zeta\otimes id$ has dimension 1, which is absurd. Hence we are in the case (c). From the assumption that $\sigma^*(x, y)=(\zeta x, \zeta^2 y)$, V is unique as the direct summand of the two eigen vector spaces of $T_{\tilde{S},p}\oplus\bar{T}_{\tilde{S},p}\simeq\boldsymbol{Z}[\zeta]\otimes\boldsymbol{C}$ associated with the eigen values ζ and ζ^2 under the action of $\zeta\otimes id$. Therefore V/L and the action of $\langle\sigma\rangle$ is unique up to isomorphism. Hence $S\simeq\boldsymbol{C}^2/L$, where $L:=\{(n_1+\zeta n_3+(\zeta+\zeta^3)n_4, n_2+(\zeta+\zeta^3)n_3-\zeta^4 n_4) | n_i\in\boldsymbol{Z}(i=1, 2, 3, 4)\}$ and $\sigma([z, w])=[(\zeta^3+\zeta)z+w, -\zeta^4 z]$ for $[z, w]\in\boldsymbol{C}^2/L$(see [23]). Thus S has 5 singular points of type $A_{5,2}$ and $\varDelta=0$. Let $Z:=\mathrm{Sing}\ S$. Since $\pi|_{\tilde{S}\setminus\pi^{-1}(Z)}: \tilde{S}\setminus\pi^{-1}(Z)\to S\setminus Z$ is étale, we have

$$\chi_{\mathrm{top}}(\tilde{S})-5=5(\chi_{\mathrm{top}}(M)-15). \tag{5.5}$$

Since $\chi_{\mathrm{top}}(\tilde{S})=0$ and $\chi_{\mathrm{top}}(M)=\rho+12$, we obtain that $\rho=2$. Thus we are in the case V.

Case CI$(S, \varDelta)=6$. If S is not rational, we are in the case VI_α. Assume that S is rational. Let $p\in\tilde{S}$ be as above. Then $(a)\sigma^*(x, y)=(\zeta x, y)$, $(b)(\zeta^3 x, \zeta^2 y)$, $(c)(\zeta^2 x, \zeta^5 y)$ for a suitable basis x, y of m_p/m_p^2. In the same way as in the argument in the case CI$(S, \varDelta)=4$, we can exclude the case (a). Therefore $\varDelta$ can be written as $\Delta=(2/3)C_1+(1/2)C_2$, where $C_i(i=1, 2)$ is smoothe reduced curve such that C_1 and C_2 meet transversely and singular points of S are of type $A_{3,1}$. Moreover, if $p\in S$ is a singular point of type $A_{3,1}$ (resp. $A_{3,2}$), then $p\notin \mathrm{Supp}\,\Delta$(resp. $p\in C_2\backslash C_1$). Put $C'_i: =\mu_*^{-1}C_i(i=1, 2)$, $Z: =\mathrm{Sing}\ S\cup \mathrm{Sing\ Supp}\ \Delta$. Let s (resp. s_2) be the number of the singular point of S of type $A_{3,2}$ (resp. $A_{3,1}$) and s_3 be the intersection number of C_1 and C_2. Since $\pi|_{\tilde{S}\backslash\pi^{-1}(Z)}: \tilde{S}\backslash\pi^{-1}(Z)\to S\backslash Z$ is ētale, we have

$$\begin{aligned}&\chi_{\mathrm{top}}(\tilde{S})-\chi_{\mathrm{top}}(\pi^{-1}(C_1\backslash Z))-\chi_{\mathrm{top}}(\pi^{-1}(C_2\backslash Z))-s_1-2s_2-s_3\\&\quad=6(\chi_{\mathrm{top}}(M)-\chi_{\mathrm{top}}(C'_1\backslash\mu^{-1}Z)-\chi_{\mathrm{top}}(C'_2\backslash\mu^{-1}Z)\\&\qquad-3s_1-2s_2-s_3),\end{aligned}\tag{5.6}$$

Since $\chi_{\mathrm{top}}(\tilde{S})=0$, $\chi_{\mathrm{top}}(\pi^{-1}(C_1\backslash Z))=2\chi_{\mathrm{top}}C_1\backslash Z)=-2(K_M\cdot C'_1+C'^2_1)-2s_3$, $\chi_{\mathrm{top}}(\pi^{-1}(C_2\backslash Z))=3\chi_{\mathrm{top}}(C_2\backslash Z)=-3(K_M\cdot C'_1+C'^2_1)-3s_1-3s_3$ and $\chi_{\mathrm{top}}(M)=2+\rho+2s_1+s_2$, we obtain that

$$4(K_M\cdot C'_1+C'^2_1)+3(K_M\cdot C'_2+C'^2_2)=2s_1+4s_2-2s_3-6\rho-12.\tag{5.7}$$

On the other hand, we have

$$K_M+(2/3)C'_1+(1/2)C'_2+(1/3)E_1+(1/6)E_2+(1/3)E_3\sim_{\mathrm{num}}0,\tag{5.8}$$

where $E_k: =\sum_{i=1}^{s_1}E_{k,i}(k=1, 2)$, $E_{k,i}(1\leq j\leq s_1)$ are (-2)-curves, $E_3: =\sum_{j=1}^{s_2}E_{3,j}(1\leq j\leq s_2)$ are (-3)-curves. Form (5.8), we get $K_M^2+(2/3)K_M\cdot C'_1+(1/2)K_M\cdot C'_2+(1/3)s_2=0$, hence

$$4K_M\cdot C'_1+3K_M\cdot C'_2=12s_1+4s_2+6\rho-60,\tag{5.9}$$

since $K_M^2=10-2s_1-s_2-\rho$. And we have

$$K_N\cdot C'_1+(2/3)C'^2_1=-(1/2)s_3,\tag{5.10}$$

$$K_M\cdot C'_2+(1/2)C'^2_1=-(1/3)s_1(2/3)s_3.\tag{5.11}$$

Let $H_1\subset\mathrm{Gal}(\tilde{S}/S)$ be the subgroup of order 3 and put $\tilde{S}_1: =\tilde{S}/H_1$. Let $\pi_1: \tilde{S}_1\to S$ be the induced morphism. Define the boundary Δ_1 on S_1 such that $K_{S_1}+\Delta_1=\pi_1^*(K_S+\Delta)$. Then (S_1, Δ_1) is a ν_0-log surface of abelian type with $\mathrm{CI}(S_1, \Delta_1)=3$. Since $\pi_1^{-1}(C_1)$ is a disjoint union of smooth elliptic curves, we have

$$K_M\cdot C'_1+C'^2_1+(1/2)s_3=0\tag{5.12}$$

by the Hurwitz formula. By the same argument as above, we have

$$K_M\cdot C'_2+C'^2_2+(2/3)(s_1+s_3)=0.\tag{5.13}$$

From (5.10), (5.11), (5.12) and (5.13), we have

$$K_M\cdot C'_1=-(1/2)s_3,\ C'^2_1=0,\ K_M\cdot C'_2=-(2/3)s_3,\ C'^2_2=-(2/3)s_1.\tag{5.14}$$

From (5.9) and (5.14), we obtain

$$2(3s_1+s_2+s_3)=3(10-\rho), \tag{5.15}$$

hence $2|\rho$. Since $\rho\leq\rho(\tilde{S})$ and $\rho(\tilde{S})=2$ or 4, we have $\rho=2$ or 4. Noting that $s_1\equiv0(\mathrm{mod}\ 3)$ and $s_3\equiv0(\mathrm{mod}\ 12)$ from (5.14) and the fact that $K_M\cdot C_1'+C_1'^2\equiv0(\mathrm{mod}\ 2)$, we have the following possibilities ; (1) $\rho=2$ and $s_1=s_2=3$, $s_3=0$, (2) $\rho=2$ and $s_1=0$, $s_2=12$, $s_3=0$, (3) $\rho=2$ and $s_1=s_2=0$, $s_2=12$, (4) $\rho=4$ and $s_1=3$, $s_2=s_3=0$, (5) $\rho=4$ and $s_1=0$, $s_2=9$, $s_3=0$. On the other hand, from (5.7) and (5.14), we have

$$2s_1+2s_2+s_3=3\rho+6, \tag{5.16}$$

hence the cases (2) and (4) are excluded. Let $\tau: M\to N$ be a birational morphism from M to $N\simeq\sum_d$. For $i=1, 2$, Put $\bar{C}_i:=\tau_*C_i'$, $\bar{E}_i:=\tau_*E_i$ and $\bar{C}_i\sim n_i\theta+l_if$, where θ *is* a section such that $\theta^2\leq0$ and f is a fibre of N. We note that $4n_1+3n_2\leq12$ and $4l_1+3l_2\leq6d+12$ from $K_N+(2/3)\bar{C}_1+(1/2)\bar{C}^2\sim_{\mathrm{num}}0$. Assume that we are in the case (1). Let $H_1\subset\mathrm{Gal}(\tilde{S}/S)$ be the subgroup of order 3 and put $\tilde{S}_1:=\tilde{S}/H_1$. We note that $(\tilde{S}_1, (2/3)\pi_1^{-1}C_1)$ is a ν_0-log surface of abelian type with $\mathrm{CI}(\tilde{S}, (2/3)\pi_1^{-1}C_1)=3$, where $\pi_1: \tilde{S}_1\to S$ is the induced morphism. If $C_1\neq0$, then $\tilde{S}_1$ is an elliptic ruled surface, which contradicts $s_2=3$. Hence $C_1=0$. Assume that C_2 contains at least two elliptic components. Since $n_2\leq4$, we have $\bar{C}_2=\bar{C}_{2,1}+\bar{C}_{2,2}$, where $\bar{C}_{2,i}(i=1, 2)$ is a 2-section. Put $\bar{C}_{2,i}\sim2\theta+l_{2,i}f$ for $i=1, 2$. Since $\pi(\bar{C}_{2,i})=l_{2,i}-i-1\geq1(i=1, 2)$ and $3l_2=3(l_{2,1}+l_{2,2})\leq6d+12$, we have $E_j=0$ for $j=1, 2, 3$ and $C_{2,i}$ is an elliptic curve for $i=1, 2$. Hence $E_j=0$ for $j=1, 2, 3$, which is absurd. Nothing that $K_M\cdot C_2+C_2^2=-2$, we conclude that we are in the cases VI_γ or VI_δ. Assume that we are in the case (3). Since $4n_1+3n_2=12$, we have $(3\text{-}a)n_1=3$, $n_2=0$ or $(3\text{-}b)n_1=0$, $n_2=4$. Consider the case $(3\text{-}a)$. From the equation $K_S\cdot C_2+C_2^2=-8$, we have $l_2=4$ and $l_1=(3/2)d$. On the other hand, we have $l_1\geq2d$ by the assumption, hence $l_1=d=0$ and $C_1\cdot\theta=0$. Thus we are in the case VI_β. Under the assumption in the case $(3\text{-}b)$, we conclude that we are also in the case VI_β by the same way as above. Assume that we are in the case (5). If $C_2=0$, then $3(K_S+\Delta)$ is Cartier, which contradicts the assumption. Let $H_2\subset\mathrm{Gal}(\tilde{S}/S)$ be the subgroup of order 2 and put $\tilde{S}_2:=\tilde{S}/H_2$. We note that $(\tilde{S}_2, (1/2)\pi_2^{-1}C_2)$ is a ν_0-log surface of abelian type with $\mathrm{CI}(\tilde{S}_2, (1/2)\pi_2^{-1}C_2)=2$, where $\pi_2: \tilde{S}_2\to S$ is the induced morphism. Let $p\in S$ be a singular point of S and put $\bar{p}:=\pi_2^{-1}(p)$. Let L be the fibre of the ruling on $\tilde{S}_2$ which goes through the point $\bar{p}$. By construction, we have a faithful and fixed point free group action on the set $\pi_2^{-1}(C_2)\cap L$ but this set is composed of exactly four points, which is absurd.

Case $\mathrm{CI}(S, \Delta)=8$. We claim that this case does not occur. Decompose Δ as $\Delta=(7/8)C_1+(3/4)C_2+(1/2)C_3$, where $C_i(i=1, 2, 3)$ is a reduced curves. If S is not rational, then S is an elliptic ruled surface. Let f be a fibre of the ruling of S and put $n_i:=(C_i, f)$, then we have $7n_1+6n_2+4n_3=16$, hence $n_1=0$ and $4(K_S+\Delta)$ is Cartier, which is absurd. Assume that S is rational. Let $p\in\tilde{S}$ be as above. Then $(a)\sigma^*(x, y)=(\zeta x, y)$, $(b)(\zeta x, \zeta^2y)$, $(c)(\zeta^2x, \zeta^3y)$, or $(d)(\zeta x, \zeta^4y)$, where ζ is a primitive eighth root of unity, for a suitable basis x, $\circ$ of m_p/m_p^2. Therefore

Supp $\varDelta$ is smooth and all singular points of S are of type $A_{4,1}$, $A_{4,3}$ or $A_{2,1}$ and if $p\in S$ is a singular point of S, then $p\in C_2$ and p is of type $A_{2,1}$ or $p\in C_3$ and p is type $A_{4,1}$, $A_{4,3}$ or $A_{2,1}$. We can get $C_1=0$ by the same way as in the argument in the case $\mathrm{CI}(S, \varDelta)=4$. Let $H_1\subset \mathrm{Gal}(\tilde{S}/S)$ be the subgroup of order 2 and put $\tilde{S}_1:=\tilde{S}/H_1$. Let $\pi_1: \tilde{S}_1\to S$ be the induced morphism and define the boundary $\tilde{\varDelta}_1$ on $\tilde{S}_1$ such that $K_{\tilde{S}_1}+\tilde{\varDelta}_1=\pi_1^*(K_S+\varDelta)$. Then $(\tilde{S}_1, \tilde{\varDelta}_1)$ is a ν_0-log surface of abelian type with $\mathrm{CI}(\tilde{S}_1, \tilde{\varDelta}_1)=2$. Let $p\in \tilde{S}_1$ be a fixed point of the action of σ^2 on $\tilde{S}_1$ and L be the fibre of the ruling on $\tilde{S}_1$which passes through p. Since the cyclic group of order four acts on the set $L\cap \mathrm{Supp}\,\tilde{\varDelta}_1$ which is composed of exactly four points, this set decomposes to a disjoint union of orbits whose cardinality is 1, 1, 1, 1 or 1, 1, 2 respectively. But in the first case, σ^2 acts trivially on L and in the second case, σ^4 acts trivially on L, which is absurd.

Case $\mathrm{CI}(S, \varDelta)=10$. We claim that this case does not occur. We may assume that $10(K_S+\varDelta)\sim 0$. Let $H_1\subset \mathrm{Gal}(\tilde{S}/S)$ be the subgroup of order 2 and put $\tilde{S}_1:=\tilde{S}/H_1$. Let $\pi_1: \tilde{S}_1\to S$ be the induced morphism and define the boundary $\tilde{\varDelta}_1$ on $\tilde{S}_1$ such that $K_{\tilde{S}_1}+\tilde{\varDelta}_1=\pi_1^*(K_S+\varDelta)$. Then $(\tilde{S}_1, \tilde{\varDelta}_1)$ is a ν_0-log surface of abelian type with $\mathrm{CI}(\tilde{S}_1, \tilde{\varDelta}_1)=2$. σ^2 acts on $\tilde{S}_1$, hence on Alb $\tilde{S}_1$, but since it is well klown that group action of order 5 on an elliptic curve is trivial or fixed point free, the action of σ^2 on $\tilde{S}_1$ is fixed point free, which contradicts the assumption.

Case $\mathrm{CI}(S, \varDelta)=12$. If S is not rational, then S is an elliptic ruled surface and Supp $\varDelta$ is a disjoint union of smooth elliptic curves. Let $\varDelta=(11/12)C_1+(5/6)C_2+(3/4)C_3+(2/3)C_4+(1/2)C_5$ be the decomposition of $\varDelta$ and f be a fibre of the ruling of S. Put $n_i:=(C_i, f)$. We have $11n_1+10n_2+9n_3+8n_4+6n_5=24$ from the assumption, hence $(n_i;\ 1\leq i\leq 5)=(0, 1, 0, 1, 1)$, $(0, 0, 2, 0, 1)$, $(0, 0, 0, 0, 4)$ or $(0, 0, 0, 3, 0)$ and $4(K_S+\varDelta)$ or $6(K_S+\varDelta)$ is Cartier, which is absurd. Therefore S is rational. Let $\pi: \tilde{S}\to S$ and $p\in \tilde{S}$ be as above. We have $(a)\sigma^*(x, y)=(\zeta x, y)$, $(b)(\zeta^2 x, \zeta^3 y)$, $(c)(\zeta^2 x, \zeta^5 y)$, $(d)(\zeta x, \zeta^4 y)$, $(e)(\zeta^3 x, \zeta^4 y)$, $(f)(\zeta x, \zeta^6 y)$, where ζ is a primitive twelfth root of unity, for a suitable basis x, y of m_p/m_p^2. Therefore, C_i is smooth for $1\leq i\leq 5$, Supp $C_i\cap$ Supp $C_j=\emptyset$ for $i<j$ except for $(i, j)=(3, 4)$, $(4, 5)$ and each components of C_3 and C_4, C_4 and C_5 intersect transversely. If p is any singular point of S, then p is of type $A_{3,1}$ and $p\in S\backslash \mathrm{Supp}\varDelta$ or $p\in C_3$, of type $A_{3,2}$ and $p\in C_5$, of type $A_{6,5}$ and $p\in C_5$, of type $A_{2,1}$ and $p\in C_5$ or $p\in C_2$. Let s_1 be the number of the singular points $p\in S$ of type $A_{2,1}$ such that $p\in C_2$, s_2 be the number of the singular points $p\in S$ of type $A_{3,1}$ such that $p\in C_3$, s_3 be the number of singular points $p\in S$ of type $A_{2,1}$such that $p\in C_4\cap C_5$, s_4 be the number of the singular points of $p\in S$ of type $A_{6,5}$ such that $p\in C_5$, s_5 be the number of the singular points $p\in S$ of type $A_{3,2}$ such that $p\in C_5$, s_6 be the number of the singular points of $p\in S$ of type $A_{2,1}$ such that $p\in C\backslash C_4$, s_7 be the number of the point $p\in S$ such that $p\in C_3\cap C_4$, s_8 be the number of the point $p\in S$ such that $p\in C_4\cap C_5$ and S is smooth at p and s_9 be the number of the singular points $p\in S$ of type $A_{3,1}$ such that $p\notin \mathrm{Supp}\varDelta$. Let $H_1\subset \mathrm{Gal}\,(\tilde{S}/S)$ be the subgroup of orber 6 and put $\tilde{S}_1:=\tilde{S}/H_1$ and let $\pi_1: \tilde{S}_1\to S$ be the induced morphism. Assume that $C_1\neq 0$ or $C_2\neq$

0. Define the boundary $\tilde{\Delta}_1$ as

$$\tilde{\Delta}_1:=(5/6)\pi_1^{-1}(C_1\cup C_2)+(2/3)\pi_1^{-1}(C_4)+(1/2)\pi_1^{-1}(C_3\cup C_5).$$

We note that $K_{\tilde{S}_1}+\tilde{\Delta}_1=\pi_1^*(K_S+\Delta)$ and $(\tilde{S}_1, \tilde{\Delta}^{\cdot})$ is a ν_0-log surface of type VI_α by construction. The induced group action on $\tilde{S}_1$ has fixed point $p\in\tilde{S}_1$ by assumption. Let L be the fibre of the ruling on $\tilde{S}_1$ which passes through p. Since the sets $L\cap\pi_1^{-1}(C_1\cup C_2)$, $L\cap\pi_1^{-1}(C_4)$ and $L\cap\pi_1^{-1}(C_3\cup C_5)$ are Gal $(\tilde{S}_1/S)$-invariant, Gal $(\tilde{S}_1/S)$ acts on L trivially, which is absurd. Thus we get $C_1=0$, $C_2=0$ and $s_1=0$. Assume that $s_2\neq 0$. Then there is a singular point $p\in S$ of type $A_{3,1}$ such that $p\in C_3$. Let $H_2\subset\mathrm{Gal}(\tilde{S}/S)$ be the subgroup of order 4 and put $\tilde{S}_2:=\tilde{S}/H_2$ and let $\pi_2:\tilde{S}_2\to S$ be the induced morphism. Define the boundary $\tilde{\Delta}_2$ as $\tilde{\Delta}_2:=(3/4)\pi_1^{-1}(C_3)+(1/2)\pi_1^{-1}(C_5)$. We note that $K_{\tilde{S}_2}+\tilde{\Delta}_2=\pi_2^*(K_S+\Delta)$ and $(\tilde{S}_2, \tilde{\Delta}_2)$ is a ν_0-log surface of type IV_α or IV_β by construction. The induced group action on $\tilde{S}_2$ has a fixed point $\tilde{p}\in\pi_2^{-1}(p)$. Let L be the fibre of the ruling on $\tilde{S}_2$ which passes trough $\tilde{p}$. Since the sets $L\cap\pi_2^{-1}(C_3)$ and $L\cap\pi_2^{-1}(C_5)$ is Gal $(\tilde{S}_2/S)$-invariant, the action of Gal $(\tilde{S}_2/S)$ on L has three fixed points, hence trivial,which is absurd. Therefore we obtain $s_2=0$. Put $Z:=\mathrm{Sing\,Supp}\,\Delta\cup\mathrm{Sing}\,S$, $C_i':=\mu_*^{-1}C_i(3\leq i\leq 5)$. Since $\pi|_{\tilde{S}\setminus\pi^{-1}(\mathrm{Supp}\,\Delta\cup\mathrm{Sing}\,S)}:\ \tilde{S}\setminus\pi^{-1}(\mathrm{Supp}\,\Delta\cup\mathrm{Sing}\,S)\to S\setminus(\mathrm{Supp}\,\Delta\cup\mathrm{Sing}\,S)$ is ètale, we have

$$\begin{aligned}&\chi_{\mathrm{top}}(\tilde{S})-\sum_{i=3}^{5}\chi_{\mathrm{top}}(\pi^{-1}(C_i\setminus Z))-s_3-s_4-2s_5-3s_6-2s_8-4s_9\\&=12(\chi_{\mathrm{top}}(M)-\sum_{i=3}^{5}\chi_{\mathrm{top}}(C_i'\setminus\mu^{-1}Z)-2s_3-6s_4-3s_5-2s_6-s_7\\&\quad-s_8-2s_9).\end{aligned}\tag{5.17}$$

Since we have $\chi_{\mathrm{top}}(\tilde{S})=0$,

$$\begin{aligned}\chi_{\mathrm{top}}(\pi^{-1}(C_3\setminus Z))&=3\chi_{\mathrm{top}}(C_3\setminus Z)=3(\chi_{\mathrm{top}}(C_3)-s_7),\\ \chi_{\mathrm{top}}(\pi^{-1}(C_4\setminus Z))&=4\chi_{\mathrm{top}}(C_4\setminus Z)=4(\chi_{\mathrm{top}}(C_4)-s_3-s_7-s_8),\\ \chi_{\mathrm{top}}(\pi^{-1}(C_5\setminus Z))&=6\chi_{\mathrm{top}}(C_5\setminus Z)=6(\chi_{\mathrm{top}}(C_5)-s_3-s_4-s_5-s_6-s_8)\end{aligned}$$

and

$$\chi_{\mathrm{top}}(M)=2+\rho+s_3+5s_4+2s_5+s_5+s_9,$$

we obtain

$$\begin{aligned}&9(K_M\cdot C_3'+C_3'^2)+8(K_M\cdot C_4'+C_4'^2)+6(K_M\cdot C_5'+C_5'^2)\\&\quad=-3s_3+5s_4+4s_5+3s_6-6s_7-4s_8+8s_9-12\rho-24.\end{aligned}\tag{5.18}$$

On the other hand, we have

$$\begin{aligned}&K_M+(3/3)C_3'+(2/3)C_4'+(1/2)C_5'\\&\quad+(7/12)E_1+(5/12)E_{2,1}+(1/3)E_{2,2}+(1/4)E_{2,3}+(1/6)E_{2,4}\\&\quad+(1/12)E_{2,5}+(1/3)E_{3,1}+(1/6)E_{3,2}+(1/4)E_4+(1/3)E_5\\&=\mu^*(K_S+\Delta)\sim\mathrm{num}\,0,\end{aligned}\tag{5.19}$$

where $E_1:=\sum_{i=1}^{s_3}E_1(i)$, $E_{2,j}:=\sum_{i=1}^{s_4}E_{2,j}(i)$ $(1\le j\le 5)$, $E_{3,j}:=\sum_{i=1}^{s_5}E_{3,j}(i)$ $(j=1, 2)$, $E_4:=\sum_{i=1}^{s_6}E_4(i)$, $E_5:=\sum_{i=1}^{s_9}E_5(i)$, $E_1(i)$ $(1\le i\le s_3)$, $E_{2,j}(i)$ $(1\le i\le s_4, 1\le j\le 5)$, $E_{3,j}(i)$ $(1\le i\le s_5, j=1, 2)$ and $E_4(i)$ $(1\le i\le s_6)$ are (-2)-curves and $E_5(i)$ $(1\le i\le s_9)$ are (-3)-curves. From (5.18), we have

$$K_M^2+(3/4)K_M\cdot C_3'+(2/3)K_M\cdot C_4'+(1/2)K_M\cdot C_5'+(7/12)s_2+(1/3)s_9=0, \tag{5.20}$$
$$K_M\cdot C_3'+(3/4)C_3'^2+(2/3)s_7=0, \tag{5.21}$$
$$K_M\cdot C_4'+(1/3)C_4'^2+(7/12)s_3+(3/4)s_7+(1/2)s_8=0, \tag{5.22}$$

and

$$K_M\cdot C_5'+(2/3)C_5'^2+(7/12)s_3+(5/12)s_4+(1/3)s_5+(1/4)s_6+(2/3)s_8=0. \tag{5.23}$$

Since we have

$$K_M^2=12-\chi_{\mathrm{top}}(M)=10-\rho-s_3-5s_4-2s_5-s_6-s_9, \tag{5.24}$$

we get

$$\begin{aligned}&9K_M\cdot C_3'+8K_M\cdot C_4'+6K_M\cdot C_5'\\&=12\rho-120+12s_3+60s_4+24s_5+12s_6+8s_9.\end{aligned} \tag{5.25}$$

Let $\pi_2:\ \widetilde{S}_2\to S$ as above. Since $\pi_2^{-1}(C_3)$ is 0 or a disjoint union of elliptic curves, we have

$$K_M\cdot C_3'+C_3'^2+(2/3)s_7=0. \tag{5.26}$$

Let $H_3\subset\mathrm{Gal}(\widetilde{S}/S)$ be the subgroup of order 3 and put $\widetilde{S}_3:=\widetilde{S}/H_3$ and let $\pi_3:\widetilde{S}_3\to S$ be the induced morphism. Define the boundary $\widetilde{\Delta}_3$ as $\widetilde{\Delta}_3:=(2/3)\pi_3^{-1}(C_4)$. We note that $K_{\widetilde{S}_3}+\widetilde{\Delta}_3=\pi_3^*(K_S+\Delta)$ and $(\widetilde{S}_3, \widetilde{\Delta}_3)$ is a ν_0-log surface with $\mathrm{CI}(\widetilde{S}_3, \widetilde{\Delta}_3)=3$. Since $\pi_3^{-1}(C_4)$ is 0 or a disjoint union of elliptic curves, we have

$$K_M\cdot C_4'+C_4'^2+(3/4)s_3+(3/4)s_7+(1/2)s_8=0. \tag{5.27}$$

Let $H_4\subset\mathrm{Gal}(\widetilde{S}/S)$ be the subgroup of order 2 and put $\widetilde{S}_4:=\widetilde{S}/H_4$ and let $\pi_4:\widetilde{S}_4\to S$ be the induced morphism. Define the boundary $\widetilde{\Delta}_4$ as $\widetilde{\Delta}_4:=(1/2)\pi_4^{-1}(C_3\cup C_5)$. We note that $K_{\widetilde{S}_4}+\widetilde{\Delta}_4=\pi_4^*(K_S+\Delta)$ and $(\widetilde{S}_4, \widetilde{\Delta}_4)$ is a ν_0-log surface with $\mathrm{CI}(\widetilde{S}_4, \widetilde{\Delta}_4)=2$. Since $\pi_4^{-1}(C_4)$ is 0 or a disjoint union of elliptic curves, we have

$$K_M\cdot C_5'+C_5'^2+(5/6)s_3+(5/6)s_4+(2/3)s_5+(1/2)s_6+(2/3)s_8=0. \tag{5.28}$$

From (5.18), (5.26), (5.27) and (5.28), we obtain

$$4s_3+5s_4+4s_5+3s_6+3s_7+2s_8+4s_9=6(\rho+2). \tag{5.29}$$

From (5.21) and (5.26), we get

$$K_M\cdot C_3'=-(2/3)s_7,\ C_3'^2=0. \tag{5.30}$$

From (5.22) and (5.27), we get

$$K_M\cdot C_4'=-(1/4)s_3-(3/4)s_7-(1/2)s_8,\ C_4'^2=-(1/2)s_3. \tag{5.31}$$

From (5.23) and (5.28), we get

$$K_M\cdot C_5'=-(1/3)s_3-(2/3)s_8,\ C_5'^2=-(1/2)s_3-(5/6)s_4-(2/3)s_5-(1/2)s_6. \tag{5.32}$$

From (5.25), (5.30), (5.31) and (5.32), we obtain

$$4s_3+15s_4+6s_5+3s_6+3s_7+2s_8+2s_9=3(10-\rho). \tag{5.33}$$

Since $4|6-\rho(\widetilde{S})$, we have $\rho(\widetilde{S})=2$, hence $\rho\leq\rho(\widetilde{S})=2$. From (5.29) and (5.33), we get $\rho\equiv 0 \pmod 2$, hence $\rho=2$ and

$$2(2s_3+4s_5+s_8)+3(s_6+s_7)=24,\ s_4=0,\ s_9=s_5. \tag{5.34}$$

We note that since $K_M\cdot C_3'$, $C_4'^2$, $K_M\cdot C_5'\in\boldsymbol{Z}$ and $K_M\cdot C_4'+C_4'^2\equiv 0 \pmod 2$, we have

$$s_7\equiv 0 \pmod 3, \tag{5.35}$$

$$s_3\equiv 0 \pmod 2, \tag{5.36}$$

$$3s_3+3s_7+2s_8\equiv 0 \pmod 8 \tag{5.37}$$

and

$$s_3+2s_8\equiv 0 \pmod 3 \tag{5.38}$$

from (5.30), (5.31) and (5.32) and that in fact, we have

$$s_7\equiv 0 \pmod 6, \tag{5.39}$$

from (5.35), (5.36) and (5.37). From (5.34), (5.36), (5.37), (5.38) and (5.39), we obtain that (1) $s_5=s_9=3$, $s_i=0$ for $i=3, 6, 7, 8$, (2) $s_8=12$, $s_i=0$ for $i=3, 5, 6, 7, 9$, (3) $s_{33}=s_6=2$, $s_8=5$, $s_i=0$ for $i=5, 7, 9$, (4) $s_7=6$, $s_8=3$, $s_i=0$ for $i=3, 5, 6, 9$ or (5) $s_6=8$, $s_i=0$ for $i=3, 5, 7, 8, 9$.

Case (1). If $C_3=0$, then $6(K_S+\varDelta)$ is Cartier, which is absurd. Therefore, we have $C_3\neq 0$ and $(\widetilde{S}_2, \widetilde{\varDelta}_2)$ is a ν_0-log surface of type IV_α or IV_β. Let $p\in S$ be a singular point of S such that $p\in C_5$ and L be a fibre of the ruling on $\widetilde{S}_1$ which passes through $\tilde{p}\in\pi_2^{-1}(p)$. By construction, $L\cap\pi_2^{-1}(C_3)$ admits fixed point free action of Gal $(\widetilde{S}_2/S)$, which is absurd since $L\cap\pi_2^{-1}(C_3)$ is composed of exactly two points and the order of Gal $(\widetilde{S}_2/S)$ is three.

Case (2). If $C_3\neq 0$, then C_3 is a disjoint union of elliptic curves. But since $(\widetilde{S}_1, \widetilde{\varDelta}_1)$ is a ν_0-log surface of type VI_β, each component of $\pi_1^{-1}(C_3)$ is a rational curve, which is absurd. Therefore $C_3=0$ and $6(K_S+\varDelta)$ is Cartier. Thus we get a contradiction.

Case (3). From the assumption, $(\widetilde{S}_1, \widetilde{\varDelta}_1)$ is a ν_0-log surface of type VI_β. Since each component of Supp $\widetilde{\varDelta}_1=\pi_1^{-1}(\text{Supp }\varDelta)$ is a rational curve, we have $C_3=0$ and each components of C_4 and C_5 is a rational curve. Since $K_M\cdot C_4'+C_4'^2=-4$ and $K_M\cdot C_5'+C_5'^2=-6$, we have irreducible decompositions of C_4 and C_5, $C_4=C_{4,1}+C_{4,2}$ and $C_5=C_{5,1}+C_{5,2}+C_{5,3}$, where $C_{4,i}$, $C_{5,j}\simeq\boldsymbol{P}^1$ for $i=1, 2$, $j=1, 2, 3$. Let

$s_3^{(4,i)}$ be the number of the singular points $p\in S$ of type $A_{2,1}$ such that $p\in C_{4,i}\cap C_5$, $s_3^{(5,j)}$ be the number of the singular points $p\in S$ of type $A_{2,1}$ such that $p\in C_4\cap C_{5,j}$, $s_8^{(4,i)}$ be the number of the points $p\in C_{4,i}\cap C_5$ such that S is smooth at p, $s_8^{(5,j)}$ be the number of the points $p\in C_4\cap C_{5,j}$ such that S is smooth at p, $s_6^{(j)}$ be the number of the points $p\in C_{5,j}$ such that $p\in S$ is a singular point of type $A_{2,1}$. In the same way as above, we have

$$K_M\cdot C'_{4,i}=-(1/4)s_3^{(4,i)}-(1/2)s_8^{(4,i)},\ C'^2_{4,i}=-(1/2)s_3^{(4,i)} \tag{5.40}$$

and

$$K_M\cdot C'_{5,j}=-(1/3)s_3^{(5,j)}-(2/3)s_8^{(5,j)},\ C'^2_5=-(1/2)s_3^{(5,j)}-(1/2)s_6^{(j)}, \tag{5.41}$$

where $C'_{4,i}:=\mu_*^{-1}C_{4,i}$ and $C'_{5,j}:=\mu_*^{-1}C_{5,j}$. From (5.40), we have

$$(s_3^{4,i)},\ s_8^{(4,i)},\ C'^2_{4,i})=(0,\ 4,\ 0) \text{ or } (2,\ 1,\ -1).$$

Since we have $C'^2_4=-1$, we obtain

$$(s_3^{(4,1)},\ s_8^{(4,1)},\ C'^2_{4,1})=(0,\ 4,\ 0),$$

and

$$(s_3^{(4,2)},\ s_8^{(4,2)},\ C'^2_{4,2})=(2,\ 1,\ -1).$$

From (5.41), we have

$$(s_3^{(5,j)},\ s_6^{(j)},\ s_8^{(5,j)},\ C'^2_{5,j})=(1,\ 1,\ 1,\ -1),\ (0,\ 4,\ 0,\ -2) \text{ or } (0,\ 0,\ 3,\ 0).$$

Since we have $K_M\cdot C'_5=-4$ and $C'^2_5=-2$, we obtain

$$(s_3^{(5,j)},\ s_6^{(j)},\ s_8^{(5,j)},\ C'^2_{5,j})=(1,\ 1,\ 1,\ -1) \text{ for } j=1,\ 2$$

and

$$(s_3^{(5,3)},\ s_6^{(3)},\ s_8^{(5,3)},\ C'^2_{5,3})=(0,\ 0,\ 3,\ 0).$$

For any i, j, we have $(\pi_1^*C_{4,i},\ \pi_1^*C_{5,j})=1$, 2 or 4, hence $(C_{4,i},\ C_{5,\ j})=1/2$, 1 or 2. Thus we conclude that we are in the case XII_β.

Case (4). Since S is nonsingular and $\rho=2$, we have $S\simeq\sum_d$ for some $d\geq 0$. We note that $K_S\cdot C_3=-4$, $C_3^2=0$, $K_S\cdot C_4=-6$, $C_4^2=0$, $K_S\cdot C_5=-2$, $C_5^2=0$ by assumption. Assume that $C_i\sim n_i\theta+l_if$ for $i=3$, 4, 5, where θ is a section such that $\theta^2\leq 0$ and f is a fibre of the ruling on S. Since we have $9n_3+8n_4+6n_5=24$, we have $(n_3,\ n_4,\ n_5)=(2,\ 0,\ 1)$, $(0,\ 3,\ 0)$ or $(0,\ 0,\ 4)$. If $(n_3,\ n_4,\ n_5)=(2,\ 0,\ 1)$, then we have $(l_3,\ l_4,\ l_5)=(d,\ 3,\ (1/2)d)$. Since $(C_3,\ \theta)\geq 0$ or $(C_5,\ \theta)\geq 0$, we have $l_3=l_5=d=0$. Thus we are in the case XII_β. If $(n_3,\ n_4,\ n_5)=(0,\ 3,\ 0)$, then we have $(l_3,\ l_4,\ l_5)=(2,\ (3/2)d,\ 1)$. Since $l_4\geq 2d$, we have $l_4=d=0$. Thus we are in the case XII_β again. If $(n_3,\ n_4,\ n_5)=(0,\ 0,\ 4)$, then we have $(l_3,\ l_4,\ l_5)=(2,\ 3,\ 2d-3)$ but we have $l_5\geq 3d$, which is absurd.

Case (5). Since we have $C_4\neq 0$ by assumption, $(\widetilde{S}_1,\ \widetilde{\Delta}_1)$ is a ν_0-log surface of type VI_β, which is absurd.

EXAMPLES. (1) Put $E_\zeta := \boldsymbol{C}/\boldsymbol{Z}+\zeta\boldsymbol{Z}$, where ζ is a primitive third root of unity and $A := E_\zeta\times E_\zeta$. Consider the action σ on A defined as $\sigma([z_1],[z_2])=([\zeta^2 z_2],[\zeta z_1])$ for $([z_1],[z_2])\in A$. Put $S := A/\langle\sigma\rangle$ and $\varDelta := (1/2)\{([z],[\zeta z])|z\in\boldsymbol{C}\}$. This log surface $(S,\varDelta)$ gives an example of ν_0-log surface of type II_ε.

(2) Let ζ and E_ζ be as is (1). Consider the action σ on $E_\zeta\times\boldsymbol{P}^1$ such that $\sigma([z],[w_1:w_2])=([\zeta z],[\zeta w_1:w_2])$. Put $S := E_\zeta\times\boldsymbol{P}^1/\langle\sigma\rangle$ and $\varDelta := (1/2)E_\zeta\times\{[1:0],[\zeta^2:1],[\zeta:1],[1:1]\}/\langle\sigma\rangle$. This log surface $(S,\varDelta)$ gives an example of ν_0-log surface of type VI_γ.

(3) Examples of ν_0-log surface of type IV_γ, VI_δ and XII_β are known. We refer the reader to [23].

6. Degeneration of type I associated with ν_0-log surface of abelian type

DEFINITION 6.1 A minimal degeneration of surfaces $f: X\to\mathcal{D}$ with $\chi=0$ is said to be *of type I* if f has a log minimal reduction $\hat{f}:(\hat{X},\hat{\Theta})\to\mathcal{D}$ such that $(\hat{\Theta},\mathrm{Diff}_{\hat{\Theta}}(0))$ is a ν_0-log surface of type I.

In this section, we study the singular fibres by using the results in the previous section.

Theorem 6.1. *Let $\hat{f}:(\hat{X},\hat{\Theta})\to\mathcal{D}$ be a projective log minimal degeneration of surfaces with $\chi=0$ and assume that $(\hat{\Theta},\mathrm{Diff}_{\hat{\Theta}}(0))$ is a ν_0-log surface of abelian type then the generic fibre is an abelian or hyperelliptic surface and there is a projective degeneration $f: X\to\mathcal{D}$ which is bimeromorphically equivalent to $\hat{f}:\hat{X}\to\mathcal{D}$ (we shrink $\mathcal{D}$ if necessary) such that X is a normal $\boldsymbol{Q}$-factorial 3-fold with only terminal singularities and one of the following holds.*

I: X is smooth and $f^*(0)=m\Theta$, Θ is an abelian surface or a hyperelliptic surface.

II_α: X is smooth and $f^*(0)=2m\Theta_0+\sum_{i=1}^4 m\Theta_{1,i}$, where $m\in\boldsymbol{N}$, $\Theta_{1,i}$ is an elliptic ruled surface for any i. $\Theta_{1,i}\cdot\Theta_0$ is a section whose self-intersection number 0 on each of the two components for $i\geq1$. $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ for $i>j\geq1$.

II_β: X is smooth and $f^*(0)=2m\Theta_0+\sum_{i=1}^2 m\Theta_{1,i}$, where $m\in\boldsymbol{N}$, Θ_0 and $\Theta_{1,i}$ are elliptic ruled surfaces for $i=1,2,\Theta_{1,2}\cdot\Theta_0$ is a 3-section on Θ_0 which is a smooth elliptic curve with the self-intersection number 0 and is a section on $\Theta_{1,2}$, $\Theta_{1,2}\cdot\Theta$ is a section whose self-intersection number 0 on each of the two components. $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ for $i>j\geq1$.

II_γ: X is smooth and $f^*(0)=2m\Theta_0+\sum_{i=1}^2 m\Theta_{1,i}$, where $m\in\boldsymbol{N}$, $\Theta_{1,i}$ is an elliptic ruled surface for $i=0,1,2$. $\Theta_{1,i}\cdot\Theta_0$ is a 2-section on Θ_0 which is a smooth

elliptic curve with the self-intersection number 0 *and is a section on* $\Theta_{1,i}$ *for* $i=1,2$. $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ *for* $i>j\geq 1$.

II_δ: *X is smooth and* $f^*(0)=2m\Theta_0+\sum_{i=1}^{3} m\Theta_{1,i}$, *where* $m\in\boldsymbol{N}$, Θ_0 *and* $\Theta_{1,i}$ *are elliptic ruled surfaces for any i.* $\Theta_{1,1}\cdot\Theta_0$ *is a* 2*-section on* Θ_0 *which is a smooth elliptic curve with the self-intersection number* 0 *and is a section on* $\Theta_{1,2}\cdot\Theta_{1,i}\cdot\Theta_0$ *is a section whose self-intersection number* 0 *on each of the two components.* $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ *for* $i>j\geq 1$.

II_ε: *X is smooth and* $f^*(0)=2m\Theta_0+m\Theta_1$, *where* $m\in\boldsymbol{N}$, Θ_i *is an elliptic ruled surface for* $i=0,1,\Theta_1\cdot\Theta_0$ *is a* 4*-section on* Θ_0 *which is a smooth elliptic curve with the self-intersection number* 0 *and is a section on* Θ_1.

III_α-1: *X is smooth and* $f^*(0)=3m\Theta_0+\sum_{i=1}^{3}(2m\Theta_{1,i,1}+m\Theta_{1,i,2})$, *where* $m\in\boldsymbol{N}$, Θ_0 *and* $\Theta_{1,i,j}$ *are elliptic ruled surfaces for any i, j.* $\Theta_{1,i,1}\cdot\Theta_0$ *and* $\Theta_{1,i,2}\cdot\Theta_{1,i,j}$ *are sections with the self-intersection number* 0 *on each of the two components for* $i=1,2,3$. $\Theta_{1,i,j}\cap\Theta_{1,k,l}=\emptyset$ *if* $i\neq k$ *and* $\Theta_{1,i,2}\cap\Theta_0=\emptyset$ *for* $i=1,2,3$ (*see Figure* III_α-1).

III_α-2: *X is smooth and* $f^*(0)=\sum_{i=1}^{3} m\Theta_{1,i}$, *where* $m\in\boldsymbol{N}$, $\Theta_{1,i}$ *is an elliptic ruled surface for* $i=1,2,3$. $\Theta_{1,1}\cdot\Theta_{1,2}=\Theta_{1,2}\cdot\Theta_{1,3}=\Theta_{1,3}\cdot\Theta_{1,1}$ *is a smooth elliptic curve which is a section on each* $\Theta_{1,i}$ (*see Figure* III_α-2).

III_β-1: *X is smooth and* $f^*(0)=3m\Theta_0+\sum_{i=1}^{2}(2m\Theta_{1,i,1}+m\Theta_{1,i,2})$, *where* $m\in\boldsymbol{N}$, Θ_0 *and* $\Theta_{1,i,j}$ *are elliptic ruled surfaces for any i, j.* $\Theta_{1,1,1}\cdot\Theta_0$ *is a* 2*-section on* Θ_0 *which is a smooth elliptic cusve with the self intersection number* 0 *and is a section on* $\Theta_{1,1,1}$. $\Theta_{1,2,1}\cdot\Theta_0$ *and* $\Theta_{1,i,2}\cdot\Theta_{1,i,1}$ *are sections with the self-intersection number* 0 *on each of the two components for* $i=1,2$. $\Theta_{1,i,j}\cap\Theta_{1,k,l}=\emptyset$ *if* $i\neq k$ *and* $\Theta_{1,i,2}\cap\Theta_0=\emptyset$ *for* $i=1,2$ (*see Figure* III_β-1).

III_β-2: *X is smooth and* $f^*(0)=\sum_{i=1}^{2} m\Theta_{1,i}$, *where* $m\in\boldsymbol{N}$. *There is a projective birational morphism* μ: $Y\to X$ *from a smooth* 3*-fold Y such that* $\tilde{f}^*(0)=3m\tilde{\Theta}_0+m\tilde{\Theta}_{1,1}+m\tilde{\Theta}_{1,2}$, *where* $\tilde{f}:=f\circ\mu$, $\tilde{\Theta}_i:=\mu_*^{-1}\Theta_i$ *for* $i=1,2$. $\tilde{\Theta}_{1,i}$ *is an elliptic ruled surface for* $i=0,1,2$. $\tilde{\Theta}_{1,1}\cdot\tilde{\Theta}_0$ *is a* 2*-section on* $\tilde{\Theta}_0$ *which is a smooth elliptic curve with the self-intersection number* 0 *and is a section on* $\tilde{\Theta}_{1,1}$. $\tilde{\Theta}_{1,2}\cdot\tilde{\Theta}_0$ *is a section whose self-intersection number* 0 *on each of the two components.* $\tilde{\Theta}_{1,1}\cap\tilde{\Theta}_{1,2}=\emptyset$ (*see Figure* III_β-2).

III_γ-1: *X is smooth and* $f^*(0)=3m\Theta_0+2m\Theta_{1,1}+m\Theta_{1,2}$, *where* $m\in\boldsymbol{N}$, $\Theta_{1,i}$ *is an elliptic ruled surface for any i.* $\Theta_{1,1}\cdot\Theta_0$ *is a* 3*-section on* Θ_0 *which is a smooth elliptic curve with the self-intersection number* 0 *and is a section on* $\Theta_{1,1}$. $\Theta_{1,2}\cdot\Theta_{1,1}$ *is a section with the self-intersection number* 0 *on each of the two components.* $\Theta_0\cap\Theta_{1,2}=\emptyset$ (*see Figure* III_γ-1).

III_γ-2: *X is smooth and* $f^*(0)=m\Theta$, *where* $m\in\boldsymbol{N}$. *There is a projective birational morphism* μ: $\to X$ *from a smooth* 3*-fold Y such that* $\tilde{f}^*(0)=3m\tilde{\Theta}_0$

$+m\tilde{\Theta}_1$, *where* $\tilde{f}:=f\circ\mu$, $\tilde{\Theta}_1:=\mu_*^{-1}\Theta_1$. $\tilde{\Theta}_i$ *is an elliptic ruled surface for* $i=0, 1$. $\tilde{\Theta}_1\cdot\tilde{\Theta}_0$ *is a* 3-*section on* $\tilde{\Theta}_0$ *with the self-intersection number* 0 *which is a smooth elliptic curve* (*see Figure* III_γ-2).

III_δ : $f^*(0)=3\Theta_0+\sum_{i=1}^{t}\Theta_{1,i}$, *where* Θ_0 *is a normal rational surface with* $\rho(\Theta_0)=4+t$ *and* $\Theta_{1,i}\simeq \boldsymbol{P}^2$ *for* $i\geq 1$. *Sing* $\Theta_0=\{p_i\,;\ 1\leq i\leq s\}$, *where* $p_i\in\Theta_0(1\leq i\leq s)$ *are singular points of type* $A_{3,1}$ *and* $s:=9-t$. $\Theta_{1,i}\cdot\Theta_0$ *is a* (-3)-*curve on* Θ_0 *and is a line on* $\Theta_{1,i}$ *for* $i\geq 1$. *If* $\{p_i\,;\ 1\leq i\leq s\}:=\mathrm{Sing}\ \Theta_0$, *then* Sing $X=\{p_i\,;\ 1\leq i\leq s\}$ *and analytic locally around* p_i, $(p_i\in X, \Theta)$ *is isomorphic to* $(0\in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 2)$. *Moreover, if* X_t *is an abelian surface for* $t\in\mathscr{D}^*$, *then* $t=0$ *or* 9 (*see Figure* III_δ).

IV_α-1 : X *is smooth and* $f^*(0)=4m\Theta_0+\sum_{i=1}^{2}(3m\Theta_{1,i,1}+2m\Theta_{1,i,2}+m\Theta_{1,i,3})+2m\Theta_2$, *where* Θ_0, $\Theta_{1,i,j}$, Θ_2 *are elliptic ruled surfaces.* $\Theta_{1,i,1}\cdot\Theta_0$ $(i=1, 2)$, $\Theta_2\cdot\Theta_0$, $\Theta_{1,i,3}\cdot\Theta_{1,i,2}$ *and* $\Theta_{1,i,2}\cdot\Theta_{1,i,1}$ *are sections of with the self-intersection number* 0 *on each of the two components.* $\Theta_{1,i,j}\cap\Theta_{1,k,l}=\emptyset$ *if* $i\neq k$, $\Theta_{1,i,3}\cap\Theta_{1,i,1}=\emptyset$ *for* $i=1, 2$ *and* $\Theta_2\cap\Theta_{1,i,j}=\emptyset$ *for any* i, j (*see Figure* IV_α-1).

IV_α-2 : X *is smooth and* $f^*(0)=m\Theta_{1,1}+m\Theta_{1,2}$, *where* $m\in N$, Θ_i *is an elliptic ruled surface for* $i=1, 2$. $\Theta_{1,1}\cdot\Theta_{1,2}=2\Gamma$ *where* Γ *is a section with the self-intersection number* 0 *on each of the two components* (*see Figure* IV_α-2).

IV_β-1 : X *is smooth and* $f^*(0)=4m\Theta_0+3m\Theta_{1,1}+2m\Theta_{1,2}+m\Theta_{1,3}+2m\Theta_2$, *where* $m\in N$, $\Theta_{1,i}$ *and* Θ_2 *are elliptic ruled surfaces for any* i. $\Theta_{1,1}\cdot\Theta_0$ *is a* 2-section with the self-intersection number 0 which is a smooth elliptic curve and is a section on $\Theta_{1,1}$. $\Theta_2\cdot\Theta_0$, $\Theta_{1,2}\cdot\Theta_{1,1}$ and $\Theta_3\cdot\Theta_2$ are sections with the self-intersection number 0 on each of the two components. $\Theta_{1,1}\cap\Theta_{1,3}=\emptyset$, $\Theta_0\cap\Theta_{1,i}=\emptyset$ for $i=2, 3$ *and* $\Theta_2\cap\Theta_{1,i}=\emptyset$ *for* $i=1, 2, 3$ (*see Figure* IV_β-1).

IV_β-2 : X *is smooth and* $f^*(0)=m\Theta$, *where* $m\in N$ *and* Θ *is irreducible. there is a projective birational morphism* $\mu: Y\to X$ *from a smooth* 3-*fold* Y *such that* $\tilde{f}^*(0)=4m\tilde{\Theta}_0+m\tilde{\Theta}_{1,1}+m\tilde{\Theta}_{1,2}$, *where* $\tilde{f}:=f\circ\mu$, $\Theta_{1,i}$ *is an elliptic ruled surface,* $\tilde{\Theta}_{1,1}=\mu_*^{-1}\Theta$. $\tilde{\Theta}_{1,1}\cdot\tilde{\Theta}_0$ *is a* 2-*section with the self-intersection number* 0 *on* $\tilde{\Theta}_0$ *which is a smooth elliptic curve and is a section on* $\tilde{\Theta}_{1,1}\cdot\tilde{\Theta}_{1,2}\cdot\tilde{\Theta}_0$ *is a section with the self-intersection number* 0 *on each of the two components.* $\tilde{\Theta}_1\cap\tilde{\Theta}_{1,2}=\emptyset$ (*see Figure* IV_β-2).

IV_γ : $f^*(0)=4\Theta_0+\sum_{i=1}^{3}2\Theta_{1,i}+\sum_{i=1}^{2}\sum_{j_i=1}^{t_i}\Theta_{(i,j)}$, *where* Θ_0 *and* $\Theta_{1,i}$ *are normal rational surfaces with* $\rho(\Theta_0)=2+t_1+t_2$ *and* $\rho(\Theta_{1,i})=2$ *for* $i=1, 2$, $\Theta_{1,3}$ *is an elliptic ruled surface and* $\Theta_{(i,j_i)}\simeq\sum_2$. $t_i=0$ *or* 2 *or* 4 *for* $i=2, 3$ *and* $s:=8-\sum_{i=1}^{2}t_i$. $\Theta_{1,3}\cdot\Theta_0$ *is a smooth elliptic curve whose self intersection number* 0 *on each of the two components. The strict transform of* $\Theta_{1,i}\cdot\Theta_0$ *is a* (-2)-*curve on the minimal resolution of* Θ_0 *and is a* $((1/2)t_i-2)$-*curve on the minimal resolution of*$\Theta_{1,i}$ *for* $i=1, 2$. $\Theta_{(i,j_i)}\cdot\Theta_0$ *is a* (-2)-*curve on* Θ_0 *and is a fibre of the ruling on* $\Theta_{(i,j_i)}$ *for any* (i, j_i). $\Theta_{(i,j_i)}\cdot\Theta_{1,i}$ *is a* 0-*curve on* $\Theta_{1,i}$

and is a (-2)*-curve on* $\Theta_{(i,j_i)}$ *for* $i=1, 2$, $1\le j_i\le t_i$. $\Theta_{1,i}\cap\Theta_{1,j}=\emptyset$ *for* $i>j$, $\Theta_{(i,j_i)}\cap\Theta_{1,k}=\emptyset$ *if* $i\neq k$ *and* $\Theta_{(i,j_i)}$*'s are disjoint from each other. Putting* $\mathrm{Sing}\ \Theta_0=\{p^{(i)}_{1,j_i}\in\Theta_{1,i}\ ;\ 0\le j_i\le 8-t_i\ i=1, 2)\}$ *and* $\mathrm{Sing}\ \Theta_{1,i}=\{p^{(i)}_{1,j_i}, p^{(i)}_{2,j_i},;\ 0\le j_i\le 8-t_i\ (i=1, 2)\}$, *we have* $\mathrm{Sing}\ X=\{p^{(i)}_{1,j_i}, p^{(i)}_{2,j_i}\ ;\ 0\le j_i\le 8-t_i\ (i=1, 2)\}$ *and analytic locally around each* $p^{(i)}_{1,j_i}$, $(p^{(i)}_{1,j_i}\in X, \Theta)$ *is isomorphic to* $(0\in \boldsymbol{C}^3, \{xy=0\})/\boldsymbol{Z}_2(1, 1, 1)$, *around each* $p^{(i)}_{2,j_i}$, $(p^{(i)}_{2,j_i}\in X, \Theta)$ *is isomorphic to* $(0\in \boldsymbol{C}_3, \{x=0\})/\boldsymbol{Z}_2(1, 1, 1)$. *Moreover, if* X_t *is an abelian surface for* $t\in\mathcal{D}^*$, *then* $(t_1, t_2)=(0, 0)$, *or* $(4, 4)$ (*see Figure* IV_γ).

IV_δ: $f^*(0)=4\Theta_0+\sum_{i=1}^2 2\Theta_{1,i}+\sum_{i=1}^2\sum_{j_i=1}^{t_i}\Theta_{(i,j_i)}$, *where* Θ_0 *and* $\Theta_{1,i}$ *are normal rational surfaces with* $\rho(\Theta_0)=2+t_1+t_2$ *and* $\rho(\Theta_{1,i})=2$ *for* $i=1, 2$ *and* $\Theta_{(i,j_i)}\simeq\sum_2$. $t_i=0$ *or* 2 *or* 4 *for* $i=2, 3$ *and* $s:=8-\sum_{i=1}^2 t_i$. *The strict transform of* $\Theta_{1,i}\cdot\Theta_0$ *is a* (-2)*-curve on the minimal resolution of* Θ_0 *and is a*$((1/2)t_j-2)$*-curve on the minimal resolution of* $\Theta_{1,i}$ *for* $i=1, 2$. $\Theta_{(i,j_i)}\cdot\Theta_0$ *is a* (-2)*-curve on* Θ_0 *and is a fibre of the ruling on* $\Theta_{(i,j_i)}$ *for any* (i, j_i). $\Theta_{(i,j_i)}\cdot\Theta_{1,i}$ *is a* 0*-curve on* $\Theta_{1,i}$ *and is a* (-2)*-curve on* $\Theta_{(i,j_i)}$ *for* $i=1, 2$, $1\le j_i\le t_i$. $\Theta_{1,1}\cap\Theta_{1,2}=\emptyset$. $\Theta_{(i,j_i)}\cap\Theta_{1,k}=\emptyset$ *if* $i\neq k$ *and* $\Theta_{(i,j_i)}$*'s are disjoint from each other. Putting* $\mathrm{Sing}\ \Theta_0=\{p^{(i)}_{1,j_i}\in\Theta_{1,i}\ ;\ 0\le J_i\le 8-t_i(i=1, 2)\}$ *and* $\mathrm{Sing}\ \Theta_{1,i}=\{p^{(i)}_{1,j_i}, p^{(i)}_{2,j_i}\ ;\ 0\le j_i\le 8-t_i(i=1, 2)\}$, *we have* $\mathrm{Sing}\ X=\{p^{(i)}_{1,j_i}, p^{(i)}_{2,j_i}\ ;\ 0\le j_i\le 8-t_d(i=1, 2)\}$ *and analytic locally around each* $p^{(i)}_{1,j_i}$, $(p^{(i)}_{1,j_i}\in X, \Theta)$ *is isomorphic to* $(0\in \boldsymbol{C}^3, \{xy=0\})/\boldsymbol{Z}_2(1,1,1)$, *around each* $p^{(i)}_{2,j_i}$, $(p^{(i)}_{2,j_i}\in X, \Theta)$ *is isomorphic to* $(0\in \boldsymbol{C}^3, \{x=0\})/\boldsymbol{Z}_2(1, 1, 1)$. *Moreover, if* X_t *is an abelian surface for* $t\in\mathcal{D}^*$, *then* $(t_1, t_2)=(0, 0)$, *or* $(4, 4)$(*see Figure* IV_δ).

V-1: X *is smooth and* $f^*(0)=5\Theta_0+\sum_{i=1}^5(\Theta_{1,i}+2\Theta_{2,i})$, *where* Θ_0 *is a smooth rational surface with* $\rho(\Theta)=12$, $\Theta_{1,i}\simeq\sum_3$ *and* $\Theta_{2,i}\simeq\boldsymbol{P}^2$ *for* $1\le i\le 5$. $\Theta_{i,j}\cap\Theta_{k,l}=\emptyset$ *if* $j\neq l$. $\Theta_{1,i}\cdot\Theta_{2,i}$ *is a* (-3)*-curve on* $\Theta_{1,i}$ *and is a line on* $\Theta_{2,i}$. $\Theta_0\cdot\Theta_{1,i}$ *is a* (-2)*-curve on* Θ_0 *and is a fibre of the ruling on* $\Theta_{1,i}$. $\Theta_0\cdot\Theta_{2,i}$ *is a* (-3)*-curve on* Θ_0 *and is a line on* $\Theta_{1,i}$ (*see Figure* V-1).

V-2: $f^*(0)=5\Theta$, *where* Θ *is a normal rational surface with* $\rho(\Theta)=2$ *and has five quotient singularities* $\{p_j\ ;\ 1\le j\le 5\}$ *of type* $A_{5,2}$. $\mathrm{Sing}\ X=\{p_j\ ;\ 1\le j\le 5\}$ *and around each* p_j, $(p_j\in X, \Theta)$ *is isomorphic to* $(0\in\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_5(1, 2, 3)$.

V-3: X *is smooth and* $f^*(0)=\sum_{i=1}^5\Theta_{1,i}$, *where* $\Theta_{1,i}$ *is a smooth rational surface for* $1\le i\le 5$ *and* $\sum_{i=1}^5\rho(\Theta_{1,i})=20$. *There is projective birational morphism* μ: $Y\to X$ *from a smooth* 3*-fold* Y *to* X *such that* $g^*(0)=5\tilde{\Theta}_0+\sum_{i=1}^5\tilde{\Theta}_{1,i}$, *where* g *is the induced morphism from* Y *to* $\mathcal{D}$ *and* $\tilde{\Theta}_0\simeq\sum_d$ $(d\le 3)$, $\tilde{\Theta}_{1,i}$: $\mu_*^{-1}\Theta_{1,i}$, *is a smooth rational surface which is obtained by blowing up* $\sum_2$ *for* $1\le i\le 5$. $\tilde{\Theta}_0\cdot\tilde{\Theta}_{1,i}$ *is a section on* $\tilde{\Theta}_0$ *and is the strict transform of a fibre of the ruling on* $\tilde{\Theta}_{1,i}$ *for* $1\le i\le 5$. *For* $i>j>1$, $\tilde{\Theta}_{1,i}\cdot\tilde{\Theta}_{1,j}=\sum_{i=1}^5 m(i, j\ ;\ k)\Gamma_k$, *where* $\{\Gamma_k\ ;\ 1\le k\le 5\}$ *are rational curves which are disjoint from each other such that* $\tilde{\Theta}_0\cdot\Gamma_k=1$ *for all* k. $\sum_{i>j}\sum_k m(i, j\ ;\ k)=20$ *and either* (1) $m(i, j\ ;\ k)=1$, Γ_k *is a* (-1)*-curve on* $\tilde{\Theta}_{1,i}$ (*or on* $\tilde{\Theta}_{1,j}$) *and* (-2)*-curve on* $\tilde{\Theta}_{1,j}$ (*or on*

$\tilde{\Theta}_{1,i}$) *or (2)* $m(i, j; k)=2$, Γ_k *is a* (-1)*-curve on* $\tilde{\Theta}_{1,i}$ *and* $\tilde{\Theta}_{1,j}$ (*see Figure* V-3).

VI_α-1: X *is smooth and* $f^*(0)=6m\Theta_0+5m\Theta_{1,1}+4m\Theta_{1,2}+3m\Theta_{1,3}+2m\Theta_{1,4}+m\Theta_{1,5}+4m\Theta_{2,1}+2m\Theta_{2,2}+3m\Theta_3$, *where* $m\in \boldsymbol{N}$, Θ_0, $\Theta_{i,j}$ *and* Θ_3 *are elliptic ruled surfaces for any* i, j. $\Theta_0\cdot\Theta_{i,1}(i=1, 2)$, $\Theta_{1,j}\cdot\Theta_{1,j+1}(1\leq j\leq 4)$, $\Theta_{2,1}\cdot\Theta_{2,2}$ *and* $\Theta_0\cdot\Theta_3$ *are sections with the self-intersection number* 0 *on each component.* $\Theta_{i,j}\cap\Theta_{k,l}=\emptyset$ *if* $i\neq k$ *or* $i=k=1$ *and* $|j-l|>1$, $\Theta_3\cap\Theta_{i,j}=\emptyset$ *for any* i, j *and* $\Theta_0\cap\Theta_{i,j}$ *if* $i=j=2$ *or* $i=1$, $j\geq 2$ (*see Figure* VI_α-1).

VI_α-2: X *is smooth and* $f^*(0)=m\Theta$, *where* Θ^ν *is an elliptic ruled surface. The non-normal locus of* Θ *is an smooth elliptic curve* (*say* Γ) *and around any point* $p\in\operatorname{Sing}\Theta$, Θ *is defined by the equation* $y^2-x^3=0$ *analytic locally.* $\nu^{-1}\Gamma$ *is a section of* Θ^ν *with the self-intersection number* 0 (*see Figure* VI_α-2).

VI_β-1: X *is smooth and*

$$f^*(0)=6\Theta_0+\sum_{i=1}^{3}(4\Theta_{1,i,1}+2\Theta_{1,i,2})+\sum_{j=1}^{3}3\Theta_{2,j}+\sum_{1\leq k,l\leq 3}\Theta_{(k,l)},$$

where $\Theta_0\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$, $\Theta_{1,i,1}\simeq\Sigma_2$, $\Theta_{1,i,2}\simeq\Sigma_4$, $\Theta_{2,j}$ *is a smooth rational surface with* $\rho(\Theta_{2,j})=11$ *for* $j=1, 2, 3$ *and* $\Theta_{(k,l)}\simeq\boldsymbol{P}^2$ *for* $1\leq k\leq 3$, $1\leq l\leq 3$. $\Theta_{1,i,1}\cdot\Theta_0$ *is a fibre of the first projection* $\Theta_0\to\boldsymbol{P}^1$ *for* $i=1, 2, 3$ *and* $\Theta_{2,j}\cdot\Theta_0$ *is a fibre of the second projection* $\Theta_0\to\boldsymbol{P}^1$ *for* $j=1, 2, 3$. $\Theta_{1,i,1}\cdot\Theta_{1,i,2}$ *is a section on* $\Theta_{1,i,1}$ *which is disjoint fromthe negative section and is a* (-4)*-curve on* $\Theta_{1,i,2}$ *for* $i=1, 2, 3$, $\Theta_{1,i,j}\cdot\Theta_{2,k}$ *is a fibre of the ruling on* $\Theta_{1,i,j}$ *and is a* (-2)*-curve on* $\Theta_{2,k}$ *for any* i, j, k *and* $\Theta_{2,l}\cdot\Theta_{(k,l)}$ *is a* (-3)*-curve on* $\Theta_{2,l}$ *and is a line on* $\Theta_{(k,l)}\cdot\Theta_{1,i,j}\cap\Theta_{1,i',j'}=\emptyset$ *if* $i\neq i'$, $\Theta_{1,i,j}\cap\Theta_{(k,l)}=\emptyset$ *for any* i, j, k, l, $\Theta_{2,j}\cdot\Theta_{(k,l)}=\emptyset$ *if* $j\neq l$, $\{\Theta_{(k,l)}\}_{k,l}$ *are disjoint from each other* (*see Figure* VI_β-1).

VI_β-2: $f^*(0)=2\Theta_{1,1}+2\Theta_{1,2}+2\Theta_{1,3}$, *where* $\Theta_{1,i}$ *is a normal rational surface with* $\rho(\Theta_{1,i})=2$ *such that* $\operatorname{Sing}\Theta_{1,i}=6A_{2,1}$. $\Theta_{1,1}\cdot\Theta_{1,2}=\Theta_{1,2}\cdot\Theta_{1,3}=\Theta_{1,3}\cdot\Theta_{1,1}=:\Gamma$ *and the strict transform of* Γ *on the minimal resolution of each component is a* (-2) *curve for any* i, j. *The singular locus of* X *consists of the points* $p_i\in\Gamma$ $(i=1, 2, 3)$ *and the points* $p_j^{(k)}\in\Theta_{1,k}\backslash\Gamma$ $(j=1, 2, 3, k=1, 2, 3)$ *and analytic locally around* p_i, $(p_i\in X, \Theta)\simeq(0\in\boldsymbol{C}^3, \{xy(x-y)=0\})/\boldsymbol{Z}_2(1, 1, 1)$ *for* $i=1, 2, 3$, *around* $p_j^{(k)}$, $(p_j^{(k)}\in X, \Theta)\simeq(0\in\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_2(1, 1, 1)$ *for any* j, k (*see Figure* VI_β-2).

VI_γ-1: X *is smooth and*

$$f^*(0)=6\Theta_0+3\Theta_1+3\Theta_2+\sum_{i=1}^{3}(2\Theta_{1,i,1}+\Theta_{1,i,2}+2\Theta_{3,i}),$$

where Θ_0 *is a smooth rational surface with* $\rho(\Theta_0)=11$, $\Theta_1\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$, Θ_2 *is an elliptic ruled surface,* $\Theta_{1,i,1}\simeq\Sigma_2$, $\Theta_{1,i,2}\simeq\Sigma_4$ *for* $i=1, 2, 3$ *and* $\Theta_{3,i}\simeq\boldsymbol{P}^2$ *for* i

$=1, 2, 3$. *$\Theta_1 \cdot \Theta_0$ is a (-2)-curve on Θ_0 and is a fibre of the first projection $\Theta_1 \to \boldsymbol{P}^1$, $\Theta_2 \cdot \Theta_0$ is an elliptic curve with the self-intersection number* 0 *on each component, $\Theta_{1,i,j} \cdot \Theta_0$ is a (-2)-curve on Θ_0 and is a fibre of the ruling on $\Theta_{1,i,j}$ for $i=1, 2, 3$, $j=1, 2$, $\Theta_{3,i} \cdot \Theta_0$ is a (-3)-curve on Θ_0 and is a line on $\Theta_{3,i}$ for $i=1, 2, 3$. $\Theta_1 \cap \Theta_2 = \emptyset$, $\Theta_1 \cap \Theta_{1,i,2} = \emptyset$, $\Theta_1 \cap \Theta_{3,j} = \emptyset$, $\Theta_2 \cap \Theta_{1,i,j} = \emptyset$ and $\Theta_2 \cap \Theta_{3,j} = \emptyset$ for any i, j, $\Theta_{1,i,j} \cap \Theta_{3,i'} = \emptyset$ for any i, j, i' (see Figure VI_γ-1).*

VI_γ-2 : *$f^*(0) = 6\Theta_0 + 3\Theta_1 + 3\Theta_2$, where Θ_i is a normal rational surface with $\rho(\Theta_i) = 2$ such that* $\mathrm{Sing}\, \Theta_i = 3A_{3,1} + 3A_{3,2}$ *for $i=0, 1$ and Θ_2 is an elliptic ruled surface. The strict transform of $\Theta_1 \cdot \Theta_0$ is (-2)-curve on the minimal resolution of Θ_0, (-1)-curve on the minimal resolution of Θ_1, $\Theta_2 \cdot \Theta_0$ is an elliptic curve with the self-intersection number* 0 *on each component. $\Theta_1 \cap \Theta_2 = \emptyset$. The singular locus of X is consists of the points $p_i \in \Theta_1 \cap \Theta_0$ $(i=1, 2, 3)$ and the points $p_i^{(j)} \in \Theta_j \backslash (\Theta_1 \cap \Theta_0)$ $(i=1, 2, 3, j=0, 1)$ and analytic locally around p_i, $(p_i \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{xz=0\})/\boldsymbol{Z}_3(1, 2, 2)$ for $i=1, 2, 3$, where $\{x=0\}$ corresponds to Θ_0 and $\{z=0\}$ corresponds to Θ_1, around $p_i^{(0)}$, $(p_i^{(0)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 2)$, around $p_i^{(1)}$, $(p_i^{(1)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{x=0\})/\boldsymbol{Z}_3(1, 1, 2)$ for $i=1, 2, 3$ (see Figure VI_γ-2).*

VI_δ-1 : *X is smooth and $f^*(0) = 6\Theta_0 + 3\Theta_1 + \sum_{i=1}^3 (2\Theta_{1,i,1} + \Theta_{1,i,2} + 2\Theta_{2,i})$, where Θ_0 is a smooth rational surface with $\rho(\Theta_0) = 11$, $\Theta_1 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\Theta_{1,i,1} \simeq \Sigma_2$, $\Theta_{1,i,2} \simeq \Sigma_4$ for $i=1, 2, 3$ and $\Theta_{2,i} \simeq \boldsymbol{P}^2$ for $i=1, 2, 3$. $\Theta_1 \cdot \Theta_0$ is a (-2)-curve on Θ_0 and is a fibre of the first projection $\Theta_1 \to \boldsymbol{P}^1$, $\Theta_{1,i,j} \cdot \Theta_0$ is a (-2)-curve on Θ_0, a fibre of the ruling on $\Theta_{1,i,j}$ for $i=1, 2, 3$, $j=1, 2$, $\Theta_{2,i} \cdot \Theta_0$ is a (-3)-curve on Θ_0 and is a line on $\Theta_{2,i}$ for $i=1, 2, 3$. $\Theta_1 \cap \Theta_{1,i,2} = \emptyset$, $\Theta_1 \cap \Theta_{2,j} = \emptyset$, $\Theta_{1,i,j} \cap \Theta_{2,i'} = \emptyset$ for any i, j, i' (see Figure VI_δ-1).*

VI_δ-2 : *$f^*(0) = 6\Theta_0 + 3\Theta_1$, where Θ_i is a normal rational surface with $\rho(\Theta_i) = 2$ such that* $\mathrm{Sing}\, \Theta_i = 3A_{3,1} + 3A_{3,2}$ *for $i=0,1$ The strict transform of $\Theta_1 \cdot \Theta_0$ is a (-2)-curve on the minimal resolution of Θ_0 and is a (-1)-curve on the minimal resolution of Θ_1. The singular locus of X consists of the points $p_i \in \Theta_1 \cap \Theta_0$ $(i=1, 2, 3)$ and the points $p_i^{(j)} \in \Theta_j \backslash (\Theta_1 \cap \Theta_0)$ $(i=1, 2, 3, j=0, 1)$ and analytic locally around p_i, $(p_i \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{xz=0\})/\boldsymbol{Z}_3(1, 1, 2)$ for $i=1, 2, 3$, where $\{x=0\}$ corresponds to Θ_0 and $\{z=0\}$ corresponds to Θ_1, around $p_i^{(0)}$, $(p_i^{(0)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 2, 2)$, around $p_i^{(1)}$, $(p_i^{(1)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{x=0\})/\boldsymbol{Z}_3(1, 2, 2)$ for $i=1, 2, 3$ (see Figure VI_δ-2).*

XII_α-1 : *X is smooth and*

$$f^*(0) = 12\Theta_0 + \sum_{i=1}^{2} (9\Theta_{1,i,1} + 6\Theta_{1,i,2} + 3\Theta_{1,i,3}) + \sum_{j=1}^{3} (8\Theta_{2,j,1} + 4\Theta_{2,j,2})$$
$$+ 6\Theta_3 + \sum_{1 \le i \le 2, 1 \le j \le 3} (5\Theta_{(i,j,1)} + 2\Theta_{(i,j,2)} + \Theta_{(i,j,3)}) + \sum_{j=1}^{3} \Theta_{(3,j)},$$

where $\Theta_0 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\Theta_{1,i,1} \simeq \Sigma_2$, $\Theta_{1,i,2} \simeq \Sigma_4$, $\Theta_{1,i,3} \simeq \Sigma_6$ for $i=1, 2$, $\Theta_{2,j,1}$ is a *smooth rational surface with* $\rho(\Theta_{2,j,1})=12$, $\Theta_{2,j,2}$ *is a smooth rational surface with* $\rho(\Theta_{2,j,2})=6$ *for* $j=1, 2, 3$, $\Theta_3\simeq\Sigma_2$, $\Theta_{(i,j,1)}\simeq\Sigma_2$, $\Theta_{(i,j,2)}\simeq\Sigma_1$, $\Theta_{(i,j,3)}\simeq\Sigma_2$ *and* $\Theta_{(3,j)}\simeq\Sigma_2$ *for* $i=1, 2$, $j=1, 2, 3$. $\Theta_{1,i,1}\cdot\Theta_0$ $(i=1, 2)$ *and* $\Theta_3\cdot\Theta_0$ *are fibres of the first projection* $\Theta_0\to\boldsymbol{P}^1$ *and is a* (-2)*-curve on* $\Theta_{1,i,1}$ $(i=1, 2)$ (*resp. on* Θ_3), $\Theta_{2,j,1}\cdot\Theta_0$ *is a fibre of the second projection* $\Theta_0\to\boldsymbol{P}^1$ *and is a* (-2)*-curve on* $\Theta_{2,j,1}$ *for* $j=1, 2, 3$. $\Theta_{1,i,1}\cdot\Theta_{1,i,2}$ *is a* ∞*-section on* $\Theta_{1,i,1}$ *and is a* (-4)*-curve on* $\Theta_{1,i,2}$, $\Theta_{1,i,2}\cdot\Theta_{1,i,3}$ *is a* ∞*-section of* $\Theta_{1,i,2}$ *and is a* (-6)*-curve on* $\Theta_{1,i,3}$ *for* $i=1, 2$. $\Theta_{2,j,1}\cdot\Theta_{2,j,2}$ *is a* (-1)*-curve on each component for* $j=1, 2, 3$. $\Theta_{1,i,k}\cdot\Theta_{2,j,1}$ (*resp.* $\Theta_3\cdot\Theta_{2,j,1}$) *is a* (-2)*-curve on* $\Theta_{2,j,1}$ *and is a fibre of the ruling on* $\Theta_{1,i,k}$ (*resp.* Θ_3) *for* $i=1, 2$, $j=1, 2, 3$, $k=1, 2, 3$. $\Theta_{(i,j,1)}\cdot\Theta_{(i,j,2)}$ *is a* (-2)*-curve on* $\Theta_{(i,j,1)}$ *and is a fibre of the ruling on* $\Theta_{(i,j,2)}$, $\Theta_{(i,j,2)}\cdot\Theta_{(i,j,3)}$ *is a* (-2)*-curve on* $\Theta_{(i,j,3)}$ *and is a fibre of the ruling on* $\Theta_{(i,j,2)}$, $\Theta_{(i,j,1)}\cdot\Theta_{2,j,1}$ *is a* (-4)*-curve on* $\Theta_{2,j,1}$ *and is a* ∞*-section on* $\Theta_{(i,j,1)}$. $\Theta_{(i,j,1)}\cdot\Theta_{2,j,2}$ *is a* (-2)*-curve on* $\Theta_{2,j,2}$ *and is a fibre of the ruling on* $\Theta_{(i,j,1)}$, $\Theta_{(i,j,2)}\cdot\Theta_{2,j,2}$ *is a* (-1)*-curve on* $\Theta_{2,j,2}$ *and is a* (-1)*-curve on* $\Theta_{(i,j,2)}$, $\Theta_{(i,j,3)}\cdot\Theta_{2,j,2}$ *is a* (-2)*-curve on* $\Theta_{2,j,2}$ *and is a fibre of tce ruling on* $\Theta_{(i,j,3)}$ *for* $i=1, 2$, $J=1, 2, 3$. $\Theta_3\cdot\Theta_{2,j,1}$ *is a* (-2)*-curve on*$\Theta_{2,j,1}$ *and is a fibre of the ruling on* Θ_3. $\Theta_{2,j,1}\cdot\Theta_{(3,j)}$ *is a* (-2)*-curve on* $\Theta_{2,j,1}$ *and is a fibre of the ruling on* $\Theta_{(3,j)}\cdot\Theta_{2,j,2}\cdot\Theta_{(3,j)}$ *is* (-2)*-curve on* $\Theta_{(3,j)}$ *and is a* 0*-curve on* $\Theta_{2,j,2}$ *for* $j=1, 2, 3$. $\Theta_0\cap\Theta_{1,i,k}=\emptyset$ *for* $i=1, 2$, $k=2, 3$. $\Theta_0\cap\Theta_{2,j,2}=\emptyset$ *for* $j=1, 2, 3$. $\Theta_0\cap\Theta_{(i,j,k)}=\emptyset$ *for* $i=1, 2$, j, $k=1, 2, 3$. $\Theta_0\cap\Theta_{(3,j)}=\emptyset$ *for* $j=1, 2, 3$. $\Theta_{1,1,k}\cap\Theta_{1,2,k'}=\emptyset$ *if* $k\neq k'$. $\Theta_{1,i,1}\cap\Theta_{1,i,3}=\emptyset$ *for* $i=1, 2$. $\Theta_{2,j,2}\cap\Theta_{1,i,6}=\emptyset$ *for* $i=1, 2$, j, $k=1, 2, 3$. $\Theta_3\cap\Theta_{1,i,k}=\emptyset$ *for* $i=1, 2$, $k=1, 2, 3$. $\Theta_3\cap\Theta_{2,j,2}=\emptyset$ *for* $j=1, 2, 3$. $\Theta_{1,i,k}\cap\Theta_{(i',j',l)}=\emptyset$ *for any* i, k, i', j', l. $\Theta_{1,i,k}\cap\Theta_{(3,j)}=\emptyset$ *for any* i, k, j. $\Theta_{2,j,k}\cap\Theta_{(i,j',l)}=\emptyset$ *if* $j\neq j'$. $\Theta_{2,j,k}\cap\Theta_{(3,j')}=\emptyset$ *if* $j\neq j'$. $\Theta_{(i,j,k)}\cap\Theta_{(i',j',k')}=\emptyset$ *if* $(i, j)\neq(i', j')$ *or* $(k, k')=(1, 3)$. $\Theta_{(i,j,k)}\cap\Theta_{(3,j')}=\emptyset$ *for any* i, j, j'. $\Theta_3\cap\Theta_{(3,j)}=\emptyset$ *for* $j=1, 2, 3$ (*see Figure* XII_α-1).

XII_α-2 :

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2}(9\Theta_{1,i,1}+6\Theta_{1,i,2}+3\Theta_{1,i,3})+\sum_{j=1}^{3}4\Theta_{2,j}+6\Theta_3+\sum_{1\leq i\leq 2,1\leq j\leq 3}\Theta_{(i,j)}.$$

where $\Theta_0\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$, $\Theta_{1,i,1}\simeq\Sigma_1,\Theta_{1,i,2}\simeq\Sigma_2$, $\Theta_{1,i,3}\simeq\Sigma_3$ for $i=1, 2$, $\Theta_{2,j}$ is a normal rational surface with $\rho(\Theta_{2,j})=10$ which has only one singular point p_j of type $A_{2,1}$ for $j=1, 2, 3$, $\Theta_3\simeq\Sigma_1$, $\Theta_{(i,j)}\simeq\boldsymbol{P}^2$. Sing $X=\{p_j\,;\ 1\leq j\leq 3\}$ *and analytic locally around* p_j, $(p_j\in X, \Theta)\simeq(0\in\boldsymbol{C}^3, \{z=0\})\boldsymbol{Z}_2(1, 1, 1)$ *for* $i=1, 2, 3$. $\Theta_{1,i,1}\cdot\Theta_0(i=1, 2)$ *and* $\Theta_3\cdot\Theta_0$ *is a fibre of the first projection* $\Theta_0\to\boldsymbol{P}^1$, $\Theta_{2,j}\cdot\Theta_0$ *is a fibre of the second projection* $\Theta_0\to\boldsymbol{P}^1$ *for* $j=1, 2, 3$. $\Theta_{1,i,1}\cdot\Theta_{1,i,2}$ *is a* ∞*-section on* $\Theta_{1,i,1}$ *and is a* (-2)*-curve on* $\Theta_{1,i,2}$, $\Theta_{1,i,2}\cdot\Theta_{1,i,3}$ *is a* ∞*-section on* $\Theta_{1,i,2}$ *and is a* (-3)*-curve on* $\Theta_{1,i,3}$ *for* $i=1,2$. $\Theta_{1,i,6}\cdot\Theta_{2,j}$ (*resp.* $\Theta_3\cdot\Theta_{2,j}$) *is a* (-2)*-curve on* $\Theta_{2,j}$ *and is a fibre of the ruling on* $\Theta_{1,i,k}$ (*resp.* Θ_3) *for* $i=1, 2$, $j=1, 2, 3$, $k=1, 2, 3$. $\Theta_{(i,j)}\cdot\Theta_{2,j}$ *is a* (-4)*-curve on* $\Theta_{2,j}$ *and* *is a line on* $\Theta_{(i,j)}$ *for* $i=1, 2, j=1, 2, 3$. $\Theta_0 \cap \Theta_{1,i,k}=\emptyset$ *for* $i=1, 2, k=2, 3$. $\Theta_0 \cap \Theta_{(i,j,k)}=\emptyset$ *for* $i=1, 2, j, k=1, 2, 3$. $\Theta_0 \cap \Theta_{(i,j)}=\emptyset$ *for* $i=1, 2, j=1, 2, 3$. $\Theta_{1,1,k} \cap \Theta_{1,2,k'}=\emptyset$ *if* $k \neq k'$. $\Theta_{1,i,1} \cap \Theta_{1,i,3}=\emptyset$ *for* $i=1, 2$. $\Theta_3 \cap \Theta_{1,i,k}=\emptyset$ *for* $i=1, 2, k=1, 2, 3$. $\Theta_{1,i,k} \cap \Theta_{(i',j')}=\emptyset$ *for any* i, k, i', j'. $\Theta_{2,j} \cap \Theta_{(i,j')}=\emptyset$ *if* $j \neq j'$. $\Theta_{(i,j)} \cap \Theta_{(i',j')}=\emptyset$ *if* $(i, j) \neq (i', j')$. $\Theta_3 \cap \Theta_{(i,j)}=\emptyset$ *for* $i=1, 2, j=1, 2, 3$ (*see Figure* XII_α-2).

XII_α-3 :

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2} 3\Theta_{1,i}+\sum_{j=1}^{3}(8\Theta_{2,j,1}+4\Theta_{2,j,2})+6\Theta_3+\sum_{j=1}^{3} 2\Theta_{3,j},$$

where $\Theta_0 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\Theta_{1,i}$ *is a normal rational surface with* $\rho(\Theta_{1,i})=8$ *which has three singular points* $\{p_j^{(i)}\ ;\ 1 \leq j \leq 3\}$ *of type* $A_{3,1}$ *for* $i=1, 2$, $\Theta_{2,j,1} \simeq \Sigma_1$, $\Theta_{2,j,2} \simeq \Sigma_2$ *for* $j=1, 2, 3$, Θ_3 *is a smooth rational surface with* $\rho(\Theta_3)=11$, $\Theta_{3,j} \simeq \boldsymbol{P}^2$ *for* $j=1, 2, 3$. $\mathrm{Sing}X=\{p_j^{(i)}\ ;\ 1 \leq i \leq 2, 1 \leq j \leq 3\}$ *and analytic locally around* $p_j^{(i)}$, $(p_j^{(i)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 2)$ *for* $i=1, 2, j=1, 2, 3$. $\Theta_{1,i} \cdot \Theta_0 (i=1, 2)$ (*resp.* $\Theta_3 \cdot \Theta_0$) *is a fibre of the first projection* $\Theta_0 \to \boldsymbol{P}^1$ *and is a* (-2)*-curve on* $\Theta_{1,i}$ (*resp.* Θ_3). $\Theta_{2,j} \cdot \Theta_0$ *is a fibre of the second projection* $\Theta_0 \to \boldsymbol{P}^1$ *and is a* (-1)*-curve on* $\Theta_{2,j}$ *for* $j=1, 2, 3$. $\Theta_{2,j,1} \cdot \Theta_{2,j,2}$ *is a* ∞*-section on* $\Theta_{2,j,1}$ *and is a* (-2)*-curve on* $\Theta_{2,j,2}$ *for* $j=1, 2, 3$. $\Theta_{1,i} \cdot \Theta_{2,j,k}$ (*resp.* $\Theta_3 \cdot \Theta_{2,j,k}$) *is a* (-2)*-curve on* $\Theta_{1,i}$ (*resp.* Θ_3) *and is a fibre of the ruling on* $\Theta_{2,j,k}$ *for* $i=1, 2, j=1, 2, 3, k=1, 2$. $\Theta_{3,j} \cdot \Theta_3$ *is a* (-3)*-curve on* Θ_3 *and is a line on* $\Theta_{3,j}$ *for* $j=1, 2, 3$. $\Theta_0 \cap \Theta_{2,j,2}=\emptyset$ *for* $j=1, 2, 3$. $\Theta_0 \cap \Theta_{3,j}=\emptyset$ *for* $J=1, 2, 3$. $\Theta_{1,1} \cap \Theta_{1,2}=\emptyset$. $\Theta_3 \cap \Theta_{1,i}=\emptyset$ *for* $i=1, 2$. $\Theta_{1,i} \cap \Theta_{3,j}=\emptyset$ *for any* i, j. $\Theta_{2,j,k} \cap \Theta_{3,j'}=\emptyset$ *for any* j, j', k. $\Theta_{3,j} \cap \Theta_{3,j'}=\emptyset$ *if* $j \neq j'$. (*see Figure* XII_α-3).

XII_α-4 :

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2} 3\Theta_{1,i}+\sum_{j=1}^{3} 4\Theta_{2,j}+6\Theta_3,$$

where $\Theta_0 \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, $\Theta_{1,i} \simeq \Sigma_1$ *for* $i=1, 2$, $\Theta_{2,j}$ *is a normal rational surface with* $\rho(\Theta_{2,j})=5$ *which has two singular points* $\{p_i^{(j)}\ ;\ 1 \leq i \leq 2\}$ *of type* $A_{4,3}$ *and one singular point* $p_3^{(j)}$ *of type* $A_{2,1}$ *for* $j=1, 2, 3$, $\Theta_3 \simeq \Sigma_1$. $\mathrm{Sing}\, X=\{p_i^{(j)}\ ;\ 1 \leq i \leq 3, 1 \leq j \leq 3\}$ *and analytic locally,* $(p_i^{(j)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_4(1, 3, 1)$ *for* $i=1, 2, j=1, 2, 3$ *and* $(p_3^{(j)} \in X, \Theta) \simeq (0 \in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_2(1, 1, 1)$ *for* $j=1, 2, 3$. $\Theta_{1,i} \cdot \Theta_0 (i=1, 2)$ (*resp.* $\Theta_3 \cdot \Theta_0$) *is a fibre of the first projection* $\Theta_0 \to \boldsymbol{P}^1$ *and is a* (-1)*-curve on* $\Theta_{1,i}$ (*resp.* Θ_3). $\Theta_{2,j} \cdot \Theta_0$ *is a fibre of the second projection* $\Theta_0 \to \boldsymbol{P}^1$ *and is a* (-1)*-curve on* $\Theta_{2,j}$ *for* $j=1, 2, 3$. $\Theta_{1,i} \cdot \Theta_{2,j}$ *is a* (-4)*-curve on* $\Theta_{2,j}$ *and is a fibre of the ruling on* $\Theta_{1,i}$ *for* $i=1, 2, j=1, 2, 3$. $\Theta_3 \cdot \Theta_{2,j}$ *is a* (-2)*-curve on* $\Theta_{2,j}$ *and is a fibre of the ruling on* Θ_3 *for* $j=1, 2, 3$. $\Theta_{1,1} \cap \Theta_{1,2}=\emptyset$. $\Theta_3 \cap \Theta_{1,i}=\emptyset$ *for* $i=1, 2$. (*see Figure* XII_α-4).

XII_β-1 : X *is smooth and*

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2}(8\Theta_{1,i,1}+4\Theta_{1,i,2})+\sum_{j=1}^{3}6\Theta_{2,j}+\sum_{(i,j)\in S}2\Theta_{(i,j)}$$
$$+\sum_{j=1}^{2}(3\Theta_{(1,j)}+\Theta_{(3,j,1)}+2\Theta_{(3,j,2)}+3\Theta_{(3,j,3)}+7\Theta_{(3,5,1)}),$$

where $\mathcal{S}:=\{(2,1),(2,2),(1,3),(2,3),(3,3)\}$, Θ_0 *is a smooth rational surface with* $\rho(\Theta_0)=6$, $\Theta_{1,1,1}\simeq\Sigma_2$, $\Theta_{1,1,2}\simeq\Sigma_4$, $\Theta_{1,2,1}\simeq\Sigma_1$, $\Theta_{1,2,2}\simeq\Sigma_3$, $\Theta_{2,j}$ *is a smooth rational surface with* $\rho(\Theta_{2,j})=8$ *for* $j=1, 2$, $\Theta_{2,3}$ *is a smooth rational surface with* $\rho(\Theta_{2,3})=11$, $\Theta_{(1,j)}\simeq\Sigma_2$ *for* $j=1, 2$, $\Theta_{(i,j)}\simeq\boldsymbol{P}^2$ *for* $(i,j)\in\mathcal{S}$, $\Theta_{(3,j,1)}\simeq\Sigma_4$, $\Theta_{(3,j,2)}\simeq\Sigma_2$, $\Theta_{(3,j,3)}\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$ *and* $\Theta_{(3,j,4)}$ *is a smooth rational surface with* $\rho(\Theta_{(3,j,4)})=4$. $\Theta_0\cdot\Theta_{1,1,1}$ *is a* 0-*curve on* Θ_0 *and is a* (-2)-*curve on* $\Theta_{1,1,1}\cdot\Theta_0\cdot\Theta_{1,2,1}$ *is a* (-1)-*curve on* Θ_0 *and is a* (-1)-*curve on* $\Theta_{1,2,1}$. $\Theta_0\cdot\Theta_{2,j}$ *is a* (-1)-*curve on* Θ_0 *and is a* (-1)-*curve on* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_0\cdot\Theta_{2,3}$ *is a* 0-*curve on* Θ_0 *and is a* (-2)-*curve on* $\Theta_{2,3}$. $\Theta_0\cdot\Theta_{(1,j)}$ *is a* (-2)-*curve on* Θ_0 *and is a fiber of the ruling on* $\Theta_{(1,j)}$ *for* $j=1, 2$. $\Theta_0\cdot\Theta_{(3,j,4)}$ *is a* (-2)-*curve on* Θ_0 *and is a* 0-*curve on* $\Theta_{(3,j,4)}$ *for* $j=1, 2$. $\Theta_{1,1,1}\cdot\Theta_{1,1,2}$ *is a* ∞-*section on* $\Theta_{1,1,1}$ *and is a* (-4)-*curve on* $\Theta_{1,1,2}$. $\Theta_{1,2,1}\cdot\Theta_{1,2,2}$ *is a* ∞-*section on* $\Theta_{1,2,1}$ *and is a* (-3)-*curve on* $\Theta_{1,2,2}$. $\Theta_{1,1,k}\cdot\Theta_{2,j}$ *is a* (-2) *curve on* $\Theta_{2,j}$ *and is a fibre of the ruling on* $\Theta_{1,1,k}$ *for* $j, k=1, 2$. $\Theta_{1,1,k}\cdot\Theta_{2,3}$ *are two disjoint* (-2) *curves on* $\Theta_{2,j}$ *and are two fibres of the ruling on* $\Theta_{1,1,k}$ *for* $k=1, 2$. $\Theta_{1,2,k}\cdot\Theta_{2,3}$ *is a* (-2) *curve on* $\Theta_{2,3}$ *and is a fibre of the ruling on* $\Theta_{1,2,k}$ *for* $k=1, 2$. $\Theta_{2,j}\cdot\Theta_{(1,j)}$ *is a* (-2)-*curve on* $\Theta_{(1,j)}$ *and is a* 0-*curve on* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_{2,j}\cdot\Theta_{(i,j)}$ *is a* (-3)-*curve on* $\Theta_{2,j}$ *and is a line on* $\Theta_{(i,j)}$ *for* $(i,j)\in\mathcal{S}$. $\Theta_{2,j}\cdot\Theta_{(3,j,k)}$ *is a* (-2)-*curve on* $\Theta_{2,j}$ *and is a fibre of the ruling on* $\Theta_{(3,j,k)}$ *for* $j, k=1, 2$. $\Theta_{2,j}\cdot\Theta_{(3,j,3)}$ *is a* (-1)-*curve on* $\Theta_{2,j}$ *and is a fibre of the first projection* $\Theta_{(3,j,3)}\to\boldsymbol{P}^1$ *for* $j=1, 2$. $\Theta_{2,j}\cdot\Theta_{(3,j,4)}$ *is a* (-3)-*curve on* $\Theta_{2,j}$ *and is a* 1-*curve on* $\Theta_{(3,j,4)}$ *for* $j=1, 2$. $\Theta_{(33,j,1)}\cdot\Theta_{(3,j,2)}$ *is a* (-4)-*curve on* $\Theta_{(3,j,1)}$ *and is a* ∞-*section on* $\Theta_{(3,j,2)}$, $\Theta_{(3,j,2)}\cdot\Theta_{(3,j,3)}$ *is a* (-2)-*curve on* $\Theta_{(3,j,2)}$ *and is a fibre of the second projection* $\Theta_{(3,j,3)}\to\boldsymbol{P}^1$, $\Theta_{(3,j,4)}$ *is a fiber of the second projection* $\Theta_{(3,j,3)}\to\boldsymbol{P}^1$ *and is a* (-2)-*curve on* $\Theta_{(3,j,4)}$ *for* $j=1, 2$. $\Theta_{1,2,k}\cdot\Theta_{(3,j,4)}$ *is a* (-2) *curve on* $\Theta_{(3,j,4)}$ *and is a fibre of the ruling on* $\Theta_{1,2,k}$ *for* $j, k=1, 2$. $\Theta_0\cap\Theta_{1,i,2}=\emptyset$ *for* $i=1, 2$. $\Theta_0\cap\Theta_{(3,j,k)}=\emptyset$ *for* $j=1, 2$, $k=1, 2, 3$. $\Theta_{1,1,k}\cap\Theta_{1,2,k'}=\emptyset$ *for any* k, k'. $\Theta_{2,j}\cap\Theta_{2,j'}=\emptyset$ *if* $j\neq j'$. $\Theta_{1,2,k}\cap\Theta_{2,j}=\emptyset$ *for* $j, k=1, 2$. $\Theta_{1,1,k}\cap\Theta_{(3,j,k')}=\emptyset$ *for any* k, k'. $\Theta_{1,2,k}\cap\Theta_{(3,j,k')}=\emptyset$ *except if* $k'=4$. $\Theta_{(i,j)}\cap(\Theta\backslash\Theta_{2,j})=\emptyset$ *for* $(i,j)\in\mathcal{S}$. $\Theta_{(1,j)}\cap(\Theta\backslash(\Theta_{2,j}\cup\Theta_0))=\emptyset$ *for* $j=1, 2$ (*see Figure* XII_β-1).

XII_β-2:

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2}(8\Theta_{1,i,1}+4\Theta_{1,i,2})+\sum_{j=1}^{3}6\Theta_{2,j}+\sum_{(i,j)\in\mathcal{S}}2\Theta_{(i,j)}+\sum_{j=1}^{2}\Theta_{(3,j)}$$

where $\mathcal{S}:=\{(2,1),(2,2),(1,3),(2,3),(3,3)\}$, Θ_0 *is a normal rational surface with* $\rho(\Theta_0)=2$ *which has four singular points* $\{p_1^{(j)}, q_1^{(j)};\ j=1,2\}$ *of type* $A_{2,1}$,

$\Theta_{1,1,1}\simeq\sum_2$, $\Theta_{1,1,2}\simeq\sum_4$, $\Theta_{1,2,1}$ is a normal rational surface with $\rho(\Theta_{1,2,1})=2$ which has four singular points $\{q_l^{(j)}\,;\ l=1, 2, j=1, 2\}$ of type $A_{2,1}$, $\Theta_{1,2,2}$ is a normal rational surface with $\rho(\Theta_{1,2,2})=2$ which has four singular points $\{q_l^{(j)}\,;\ l=2, 3, j=1, 2\}$ of type $A_{2,1}$, $\Theta_{2,j}$ is a normal rational surface with $\rho(\Theta_{2,j})=8$ which has five sdngular points $\{p_k^{(j)}, q_l^{(j)}\,;\ 1\leq k\leq 2, 1\leq l\leq 3\}$ of type $A_{2,1}$ for $j=1, 2$, $\Theta_{2,3}$ is a smooth rational surface with $\rho(\Theta_{2,3})=11$, $\Theta_{(i,j)}\simeq \boldsymbol{P}^2$ for any i, j. Sing $X=\{p_k^{(j)}, q_l^{(j)}\,;\ 1\leq k\leq 2,\ 1\leq l\leq 3,\ 1\leq j\leq 2\}$ *and analytic locally, $(p_1^{(j)}, q_3^{(j)}\in X, \Theta)=(\boldsymbol{C}^3, \{xy=0\})/\boldsymbol{Z}_2(1, 1, 1)$, $(p_2^{(j)}\in X, \Theta)=(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_2(1, 1, 1)$ and $(q_k^{(j)}\in X, \Theta)=(\boldsymbol{C}^3, \{xyz=0\})/\boldsymbol{Z}_2(1, 1, 1)$ for $k=1, 2$, $j=1, 2$. $\Theta_0\cdot\Theta_{1,1,1}$ is a 0-curve on Θ_0 and is a (-2)-curve on $\Theta_{1,1,1}$. The strict transform of $\Theta_0\cdot\Theta_{1,2,1}$ is a (-1)-curve on the minimal resolution of Θ_0 and is a (-2)-curve on on the minimal resolution of $\Theta_{1,2,1}$. The strict transform of $\Theta_0\cdot\Theta_{2,j}$ is a (-1)-curve on the minimal resolution of Θ_0 and is a (-2)-curve on the minimal resolution of $\Theta_{2,j}$ for $j=1, 2$. $\Theta_0\cdot\Theta_{2,3}$ is a 0-curve on Θ_0 and is a (-2)-curve on $\Theta_{2,3}$. $\Theta_{1,1,1}\cdot\Theta_{1,1,2}$ is a ∞-section on $\Theta_{1,1,1}$ and is a (-4)-curve on $\Theta_{1,1,2}$. The strict transform of $\Theta_{1,2,1}\cdot\Theta_{1,2,2}$ is a 0-curve on the minimal resolution of $\Theta_{1,2,1}$ and is a (-3)-curve on the minimal resolution of $\Theta_{1,2,2}$. $\Theta_{1,1,k}\cdot\Theta_{2,j}$ is a (-2) curve on $\Theta_{2,j}$ and is a fibre of the ruling on $\Theta_{1,1,k}$ for j, $k=1, 2$. $\Theta_{1,1,k}\cdot\Theta_{2,3}$ are two disjoint (-2) curves on $\Theta_{2,j}$ and are two fibres of the ruling on $\Theta_{1,1,k}$ for $k=1, 2$. The strict transform of $\Theta_{1,2,k}\cdot\Theta_{2,j}$ is a (-2) curve on the minimal resolution of $\Theta_{2,j}$ and is a (-1)-curve on the minimal resolution of $\Theta_{1,2,k}$ for $k=1, 2$, $j=1, 2$. $\Theta_{1,2,k}\cdot\Theta_{2,3}$ is a (-2) curve on $\Theta_{2,3}$ and is a fibre of the ruling on $\Theta_{1,2,k}$ for $k=1, 2$. $\Theta_{2,j}\cdot\Theta_{(i,j)}$ is a (-3)-curve on $\Theta_{2,j}$ and is a line on $\Theta_{(i,j)}\in\mathcal{S}$. $\Theta_{2,j}\cdot\Theta_{(3,j)}$ is a (-6)-curve on $\Theta_{2,j}$ and is a line on $\Theta_{(3,j)}$ for $j=1, 2$. $\Theta_0\cap\Theta_{1,i,2}=\emptyset$ for $i=1, 2$. $\Theta_{1,1,k}\cap\Theta_{1,2,k'}=\emptyset$ for any k, k'. $\Theta_{2,j}\cap\Theta_{2,j'}=\emptyset$ if $j\neq j'$. $\Theta_{(i,j)}\cap(\Theta\backslash\Theta_{2,j})=\emptyset$ for any i, j (see Figure XII_β-2).*

XII_β-3 :

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2}4\Theta_{1,i}+\sum_{j=1}^{3}6\Theta_{2,j}+\sum_{j=1}^{2}5\Theta_{(2,j)}$$

where Θ_0 is a normal rational surface with $\rho(\Theta_0)=4$ which has two singular points $\{p_1^{(j)}\,;\ j=1, 2\}$ of type $A_{2,1}$, $\Theta_{1,1}$ is a normal rational surface with $\rho(\Theta_{1,1})=6$ which has four singular points $\{q^{(j)}\,;\ 1\leq j\leq 4\}$ of type $A_{2,1}$, $\Theta_{1,2}$ is a normal rational surface with $\rho(\Theta_{1,2})=5$ which has three singular points $\{r_l\,;\ l=1, 2, 3\}$, $r_l\in\Theta_{1,2}$ is of type $A_{4,3}$ for $l=1, 2$ and $r_3\in\Theta_{1,2}$ is of type $A_{2,1}$. $\Theta_{2,j}$ is a normal rational surface with $\rho(\Theta_{2,j})=2$ which has two singular points $\{p_k^{(j)}\,;\ 1\leq k\leq 2\}$ of type $A_{2,1}$ for $j=1, 2$, $\Theta_{2,3}\simeq\sum_1$ and $\Theta_{(2,j)}\simeq\sum_2$ for $j=1, 2$. Sing $X=\{p_k^{(j)}\,;\ 1\leq j\leq 2, 1\leq k\leq 2)\cup\{q^{(j)}\,;\ 1\leq j\leq 4\}\cup\{r_l\,;\ 1\leq l\leq 3\}$ *and analytic locally, $(p_1^{(j)}\in X, \Theta)=(\boldsymbol{C}_3, \{xy=0)\}/\boldsymbol{Z}_2(1, 1, 1)$ and analytic locally around $p_2^{(j)}(1\leq j\leq 2)$, $q^{(j)}(1\leq j\leq 4)$, r_3, (X, Θ) is isomorphic* *to the germ of the origin of* $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_2(1, 1, 1)$ *and* $(r_l \in X, \Theta)=(0 \in \boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_4(3, 1, 1)$ *for* $l=1, 2$. $\Theta_0 \cdot \Theta_{1,1}$ *is a* 0*-curve on* Θ_0 *and is a* (-2)*-curve on* $\Theta_{1,1,1} \cdot \Theta_0 \cdot \Theta_{1,2}$ *is a* (-1)*-curve on* Θ_0 *and* $\Theta_{1,2}$. *The strict transformof* $\Theta_0 \cdot \Theta_{2,j}$ *is a* (-1)*-curve on the minimal resolution of* Θ_0 *and the minimal resolution of* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_0 \cdot \Theta_{2,3}$ *is a* 0*-curve on* Θ_0 *and is a* (-1)*-curve on* $\Theta_{2,3}$. $\Theta_{1,1} \cdot \Theta_{2,j}$ *is a* (-2) *curve on* $\Theta_{1,1}$ *and is a* 0*-curve on* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_{1,1} \cdot \Theta_{2,3}$ *are two disjoint* (-2) *curves on* $\Theta_{1,1}$ *fnd are two* 0*-curves on* $\Theta_{2,3}$. $\Theta_{1,2} \cdot \Theta_{(2,j)}$ *is a* (-4) *curve on* $\Theta_{1,2}$ *and is a* ∞*-section of* $\Theta_{(2,j)}$ *for* $j=1, 2$. $\Theta_{1,2} \cdot \Theta_{2,3}$ *is a* (-2) *curve on* $\Theta_{1,2}$ *and is a fibre of the ruling on* $\Theta_{2,3}$. $\Theta_{2,j} \cdot \Theta_{(2,j)}$ *is a* 0*-curve on* $\Theta_{2,j}$ *and is a* (-2)*-curve on* $\Theta_{(2,j)}$ *for* $j=1, 2$. $\Theta_{1,1} \cap \Theta_{1,2}=\emptyset$. $\Theta_{2,j} \cap \Theta_{2,j'}=\emptyset$ *if* $j \neq j'$. $\Theta_{1,1} \cap \Theta_{(2,j)}=\emptyset$ *for* $j=1, 2$. $\Theta_{(2,j)} \cap (\Theta \setminus (\Theta_0 \cup \Theta_{1,2} \cup \Theta_{2,j}))=\supsetneqq$ *for* $j=1, 2$ (*see Figure* XII_β-3).

XII_β-4 :

$$f^*(0)=12\Theta_0+\sum_{i=1}^{2} 4\Theta_{1,i}+\sum_{j=1}^{3} 6\Theta_{2,j}+\sum_{j=1}^{2} 3\Theta_{(1,j)}$$

where Θ_0 *is a normal rational surface with* $\rho(\Theta_0)=4$ *which has two singular points* $\{q_1^{(j)}; j=1, 2\}$ *of type* $A_{2,1}$, $\Theta_{1,1}$ *is a normal rational surface with* $\rho(\Theta_{1,1})=6$ *which has four singular points* $\{p^{(j)}; 1 \leq j \leq 4\}$ *of type* $A_{2,1}$, $\Theta_{1,2}$ *is a normal rational surface with* $\rho(\Theta_{1,2})=5$ *which has seven singular points* $q_k^{(j)}$ $(1 \leq j \leq 2, 1 \leq k \leq 3)$, $q^{(3)}$, $q_k^{(j)}$ $(1 \leq j \leq 2, 1 \leq k \leq 2)$, $q^{(3)} \in \Theta_{1,2}$ *are of type* $A_{2,1}$ *and* $q_3^{(j)}$ $(1 \leq j \leq 2) \in \Theta_{1,2}$ *are of type* $A_{4,1}$, $\Theta_{2,j}$ *is a normal rational surface with* $\rho(\Theta_{2,j})=2$ *which has two singular points* $\{q_k^{(j)}; 1 \leq k \leq 2\}$ *of type* $A_{2,1}$ *for* $j=1, 2$, $\Theta_{2,3} \simeq \Sigma_1$ *and* $\Theta_{(1,j)} \simeq \Sigma_2$ *for* $j=1, 2$. Sing $X=\{p^{(j)}; 1 \leq j \leq 4\} \cup \{q_k^{(j)}, q^{(3)}; 1 \leq j \leq 2, 1 \leq k \leq 3\}$ *and analytic locally around* $p^{(j)}$ $(1 \leq j \leq 4)$ *and* $q^{(3)}$, (X, Θ) *is isomorphic to the germ of the origin of* $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_2(1, 1, 1)$, $(q_1^{(j)} \in X, \Theta)=C_3, \{xyz=0\})/\boldsymbol{Z}_2(1, 1, 1)$, $(q_2^{(j)} \in X, \Theta)=(\boldsymbol{C}^3, \{xy=0\})/\boldsymbol{Z}_2(1, 1, 1)$ *and* $(q_3^{(j)} \in X, \Theta)=(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_4(1, 1, 3)$ *for* $j=1, 2$. $\Theta_0 \cdot \Theta_{1,1}$ *is a* 0*-curve on* Θ_0 *and is a* (-2)*-curve on* $\Theta_{1,1}$. *The strict transform of* $\Theta_0 \cdot \Theta_{1,2}$ *is a* (-1)*-curve on the minimal resolution of* Θ_0 *and is a* (-2)*-curve on on the minimal resolution of* $\Theta_{1,2}$. *The strict transform of* $\Theta_0 \cdot \Theta_{2,j}$ *is a* (-1)*-curve on the minimal resolution otf* Θ_0 *and on the minimal resolution of* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_0 \cdot \Theta_{2,3}$ *is a* 0*-curve on* Θ_0 *and is a* (-1)*-curve on* $\Theta_{2,3}$. $\Theta_0 \cdot \Theta_{1,j}$ *is a* (-2)*-curve on* Θ_0 *and is a fibre of the ruling on* $\Theta_{1,j}$ *for* $j=1, 2$. $\Theta_{1,1} \cdot \Theta_{2,j}$ *is a* (-2) *curve on* $\Theta_{1,1}$ *and is a* 0*-curve on* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_{1,1} \cdot \Theta_{2,3}$ *consists of two disjoint* (-2) *curves on* $\Theta_{1,1}$ *and is two fibres of the ruling on* $\Theta_{2,3}$. *The strct transform of* $\Theta_{1,2} \cdot \Theta_{2,j}$ *is a* (-2) *curve on the minimal resolution of* $\Theta_{2,2}$ *and is a* (-1)*-curve on the minimal resolution of* $\Theta_{2,j}$ *for* $j=1, 2$. $\Theta_{1,2} \cdot \Theta_{2,3}$ *is a* (-2) *curve on* $\Theta_{1,2}$ *and is a fibre of the ruling on* $\Theta_{2,3}$. $\Theta_{2,j} \cdot \Theta_{(1,j)}$ *is a fibre of the ruling on* $\Theta_{2,j}$ *and is a* (-2)*-curve on* $\Theta_{(1,j)}$ *for* $j=1, 2$. $\Theta_{1,1} \cap \Theta_{1,2}=\emptyset$. $\Theta_{2,j} \cap \Theta_{2,j'}=\emptyset$ *if* $j \neq j'$. $\Theta_{(1,j)} \cap (\Theta \setminus (\Theta_0 \cup \Theta_{2,j}))=\emptyset$ *for*

$j=1, 2$ (*see Figure* XII_β-4).

REMARK 6.1. The above degeneration $f: X \to \mathcal{D}$ is minimal degeneration except the cases XII_α-2, 3, 4, XII_β-2, 3, 4.

REMARK 6.2. Let (S, Δ) be a ν_0-log surface of abelian type and $\pi: \widetilde{S} \to S$ be the global log canonical cover. Let r be the order of Gal $(\widetilde{S}/S)$. Consider the action of Gal$(\widetilde{S}/S)$ to $\widetilde{S} \times \mathcal{D}$ such that $\sigma((p, t))=(\sigma(p), \zeta^w t)$ for any $(p, t) \in \widetilde{S} \times \mathcal{D}$, where σ is a generator of Gal$(\widetilde{S}/S)$ and ζ is a primitive r-th root of unity. For an appropriate $w \in \boldsymbol{N}$, $\hat{f}: \widetilde{S} \times \mathcal{D}/\langle\sigma\rangle \to \mathcal{D}/\langle\sigma\rangle$ is a log minimal degeneration. In this way, we can easily construct examples of degeneration except the cases II_β, II_γ, II_δ, III_β-1, 2, III_γ-1, 2, III_δ $(0>0, s>0)$, IV_β-1, 2, IV_γ $(t_1, t_2) \neq (0, 0), (4, 4)$ and IV_δ.

From the above theorem, we can calculate the Euler number of the special fibre in certain cases.

Corollary 6.1. *The Euler number of the special fibre of the above degeneration is* 0 *in the cases* I, II_α, II_β, II_γ, II_δ, II_ϵ, III_α-1, 2, III_β-1, 2, III_γ-1, 2, IV_α-1, 2, IV_β-1, 2 *and* VI_α-1, 2, 24 *in the cases* III_δ $(t=9, s=0)$, IV_γ, IV_δ $(t_1=t_2=4)$, VI_γ-1, VI_δ-1, 34 *in the case* V-1, 12 *in the case* V-3, 42 *in the case* VI_β-1, 60 *in the case* VII_α-1, 48 *in the case* XII_β-1.

REMARK 6.3. The above numbers do not depend on the choice of minimal models (see [8]).

To prove Theorem 6.1, we prepare the following lemma.

Lemma 6.1. *Let* $\hat{f}: (\hat{X}, \hat{\Theta}) \to \mathcal{D}$ *be a log minimal degeneration such that* $\hat{\Theta}$ *is irreducible and suppose that* $\mathcal{O}_{\hat{\Theta}}(r(K_{\hat{\Theta}}+\mathrm{Diff}_{\hat{\Theta}}(0))) \simeq \mathcal{O}_{\hat{\Theta}}$. *for* $r \in \boldsymbol{N}$. *Then we have* $\mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta})) \simeq \mathcal{O}_{\hat{X}}$ *after shrinking* $\mathcal{D}$ *if necessary.*

Proof. Put $\mathcal{F} := \mathrm{Coker}\{\mathcal{O}_{\hat{X}}(rK_{\hat{X}}) \to \mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta}))\}$ and $U := \{p \in \hat{X}\,;\ \mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta}))$ is Cartier in the neighborhood of $p.\}$. We note that U is an open subset of $\hat{X}$ which has codimension 3. Let $j: U \to \hat{X}$ be the natural embedding.

Claim $R^1 j_*(j^{-1}\mathcal{O}_{\hat{X}}(rK_{\hat{X}}))=0$.

Proof of the Claim. We may assume that $\hat{X}$ is affine. Let $\pi: \widetilde{X} \to \hat{X}$ be the log canonical cover with respect to $K_{\hat{X}}$, π_U be the restriction of π to $\widetilde{U} := \pi^{-1}(U)$ and $\tilde{j}: \widetilde{U} \to \widetilde{X}$ be the natural embedding. We note that $\widetilde{X}$ has only Gorenstein canonical singularities. Since π is finite and $\widetilde{X}$ is Cohen Macaulay, we have $R^1 j_*(j^{-1}\pi_*\mathcal{O}_{\tilde{X}}) = R^1 j_*(\pi_{U*}\mathcal{O}_{\tilde{U}}) = R^1(j \circ \pi_U)_*\mathcal{O}\,\widetilde{U} = R^1(\pi \circ \tilde{j})_*\mathcal{O}_{\tilde{U}} = \pi_* R^1 \tilde{j}_*\mathcal{O}_{\tilde{U}} = 0$, hence $R^1 j_*(j^{-1}\mathcal{O}_{\hat{X}}(iK_{\hat{X}}))=0$ for any $i \in \boldsymbol{Z}$.

Proof of Lemma 6.1 continued. By the above claim, we get $\mathcal{F}=j_*j^{-1}\mathcal{F}=\mathcal{O}_{\hat{\Theta}}$ and the following exact sequence;

$$0\to\mathcal{O}_{\hat{X}}(rK_{\hat{X}})\to\mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta}))\to\mathcal{O}_{\hat{\Theta}}\to 0,$$

The above exact sequence induces the following exact sequence;

$$\hat{f}_*\mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta}))\xrightarrow{\alpha}H^0(\mathcal{O}_{\hat{\Theta}})\xrightarrow{\beta}R^1\hat{f}_*\mathcal{O}_{\hat{X}}(rK_{\hat{X}})\xrightarrow{\gamma}R^1\hat{f}_*\mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta})).$$

Since $K_{\hat{X}}$ is $\hat{f}$-semi-ample, $R^1\hat{f}_*\mathcal{O}_{\hat{X}}(rK_{\hat{X}})$ is torsion free and γ is an isomorphism on $\mathcal{D}^*$. Therefore we have Ker $\gamma=0$, hence β is a zero map and α is surjective. Let θ be a section of $H^0(\mathcal{O}_{\hat{X}}(r(K_{\hat{X}}+\hat{\Theta}))$ such that $\alpha(\theta)=1$. By construction, we have dim(Supp div $\theta\cap$ Supp $\hat{\Theta})=0$, hence Supp div $\theta\cap$ Supp $\hat{\Theta}=\emptyset$ since Supp div θ is $\boldsymbol{Q}$-Cartier. Thus we get the assertion. ■

Proof of Theorem 6.1. Firstly, we note that Sing $\hat{X}\subset$ Supp $\mathrm{Diff}_{\hat{\Theta}}(0)\cup$ Sing $\hat{\Theta}$ by [20], Corollary 3.7. Put $r:=\mathrm{CI}(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ and Let $\pi:\tilde{X}\to\hat{X}$ be the global log canonical cover with respect to $(\hat{X}, \hat{\Theta})$. Since $(\hat{X}, \hat{\Theta})$ is purely log terminal, $(\tilde{X}, \pi^{-1}\hat{\Theta})$ is also purely log terminal by [20], Corollary 2.2 and in fact, canonical, sdnce $K_{\tilde{X}}+\pi^{-1}\hat{\Theta}$ is Cartier. Taking analytic Stein factorization, we have a surjective and connected projective morphism $\tilde{f}:\tilde{X}\to\tilde{\mathcal{D}}$ from $\tilde{X}$ to a complex disk $\tilde{\mathcal{D}}$ sumh that $f\circ\pi=\tau\circ\tilde{f}$, where $\tau:\tilde{\mathcal{D}}\to\mathcal{D}$ is the induced finite morphism. Put $\tilde{\Theta}:=\pi^{-1}\hat{\Theta}$. Taking adjunction from $K_{\tilde{X}}+\tilde{\Theta}=\pi^*(K_{\hat{X}}+\hat{\Theta})$, we have $0\sim K_{\tilde{\Theta}}=\pi^*(K_{\hat{\Theta}}+\mathrm{Diff}_{\hat{\Theta}}(0))$, hence $\pi:\ \tilde{\Theta}\to\hat{\Theta}$ is the global log canonical cover with respect to $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$. Since $\tilde{\Theta}$ is smooth by assumption, $\tilde{X}$ is smooth by [20], Corollary 3.7, hence $\hat{X}$ has only quotient singularities. Since the support of singular fibre of $\tilde{f}$ is an abelian surface, $\tilde{f}^*(\tilde{t})$ is also an abelian surface for $\tilde{t}\in\tilde{\mathcal{D}}^*$, hence $\hat{f}^*(t)$ is an abelian surface or a hyperelliptic surface for $t\in\mathcal{D}^*$. Assume that $\hat{\Theta}$ is rational. From Lemma 6.1, $rK_{\hat{X}_t}\sim 0$ for $t\in\mathcal{D}^*$. In particular, if $r=5$, then $\hat{X}_t$ is an abelian surface for $t\in\mathcal{D}^*$ by the classification of surfaces. Let $\tilde{m}$ be the multiplicity of $\tilde{\Theta}$. Since $f^*(0)=r\hat{\Theta}$ from [5], Lemma 6.1, we have $r\tilde{\Theta}=\pi^*f^*(0)=\tilde{f}^*\tau^*(0)=\deg\tau\tilde{m}\tilde{\Theta}$. Put $l:=\mathrm{Min}\,\{n\in\boldsymbol{N}\,;\ nK_{\hat{X}_t}\sim 0\ (t\in\mathcal{D}^*)\}$. Then we have $\deg\tau=r/l$, hence $\tilde{m}=l$. Annume that $\hat{X}_t$ is an abelian surface for $t\in\mathcal{D}^*$. By the assumption, $\tilde{f}$ is smooth. Let σ be a generator of $\mathrm{Gal}(\tilde{X}/\hat{X})$. Choose the basis ω_1, ω_2 of $H^0(\tilde{\Theta}, \Omega^1_{\tilde{\Theta}})$such that $\sigma^*\omega_i=\zeta^{w_i}\omega_i(i=1, 2)$, where ζ is a primitive r-th root of unity and w_i, $i=1, 2$ is a non-negative integer. Since $\tilde{\Theta}$ is σ-invariant, we can write $\sigma^*\tilde{f}=\zeta^{w_3}\tilde{f}$, where w_3 is a non-negative integer. Noting that $\tilde{\Theta}$ is an abelian surface, for all fixed point $p\in\tilde{\Theta}$ under the action of σ, $\pi(p)\in\hat{X}$ is a quotient singularity of type $(1/r)(w_1, w_2, w_3)$, that is, all singularities of images of fixed points of σ are of the same type if the generic fibre is an abelian surface.

Cases where $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is of type I, II, III_α, III_β, III_γ, IV_α, IV_β or IV_α can be treated in the same way as degeneration of elliptic curves.

Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is of type III_δ in Theorem 5.1. Let $p\in\tilde{X}$ be a fixed point of σ. From the above argument, analytic locally around $\pi(p)$, $(\hat{X}, \hat{\Theta})$ is isomorphic to the germ of the orign of (1) $(\boldsymbol{C}_3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 1)$ or (2) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 2)$. We blow up the singular points of type (1) and we are in the case III_δ in Theorem 6.1.

Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is of type IV_γ or IV_δ in Theoren 5.1. Let $p\in\tilde{X}$ be a fixed point of σ. Then analytic locally around $\pi(p)$, $(\hat{X}, \hat{\Theta})$ is isomorphic to the germ of the origin of (1)$(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_4(1, 2, 1)$ or (2)$(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_4(1, 2, 3)$. By taking a crepant blowing-up, we see that we are in the case IV_γ or IV_δ in Theorem 6.1.

Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is of type V in Theorem 5.1. Let $p\in\tilde{X}$ be a fixed point of σ. Then analytic locally around $\pi(p)$, $(\hat{X}, \hat{\Theta})$ is isomorphic to the germ of the origin of (1)$(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_5(1, 2, 2)$ or(2)$(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_5(1, 2, 3)$. (3)$(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_5(1, 2, 1)$. From the above argument, all of the singularities of $\hat{X}$ is of the same type. In case (1), (resp. (2)), we are in the case V-1 (resp. V-2) in Theorem 6.1. Assume that we are in the case (3). Let : $Y\to\hat{X}$ be the resolution as in the Figure V-3.1. Then we have

$$K_Y=\mu^*K_{\hat{X}}-\sum_{i=1}^{5}(1/5)\tilde{\Theta}_{1,i}+\sum_{i=1}^{5}(2/5)\tilde{\Theta}_{2,i},$$
$$\mu^*\hat{\Theta}=\tilde{\Theta}_0+\sum_{i=1}^{5}(1/5)\tilde{\Theta}_{1,i}+\sum_{i=1}^{5}(3/5)\tilde{\Theta}_{2,i},$$

where $\tilde{\Theta}_0:=\mu_*^{-1}\hat{\Theta}_0$ and $\tilde{\Theta}_{1,i}$, $\tilde{\Theta}_{2,i}$ are μ-exceptional divisors for $1\leq i\leq 5$. Since $K_Y\cdot l_i=-1$, where $l_i\subset\tilde{\Theta}_{2,i}$ be a line, we can see that $\{l_i\,;\,1\leq i\leq 5\}$ generate extremal rays. Let φ_1 : $Y_0:=Y\to Y_1$ be the blow down of all of these rays (see Figure V-3.2) nd put $\Theta_0^{(1)}:=\varphi_{1*}\tilde{\Theta}_0$, $\tilde{\Theta}_{1,i}^{(1)}:=\varphi_{1*}\tilde{\Theta}_{1,i}$. We note that all of the support of extremal rays are contained in $\tilde{\Theta}_0^{(1)}$. Let φ_2 : $Y_1\to Y_2$ be the contraction of an extremal ray. Assume first that $\tilde{\Theta}_0^{(1)}$ is divisorially contracted. By the $\boldsymbol{Q}$-factoriality otf Y_2, $\varphi_2(\tilde{\Theta}_0^{(1)})$ is a curve. We use the same notation for the induced morphism φ_2 : $\tilde{\Theta}_0^{(1)}\to\varphi_2(\tilde{\Theta}_0^{(1)})^\nu\simeq\boldsymbol{P}^1$. For any point $p\in\varphi_2(\tilde{\Theta}_0^{(1)})^\nu$, $\varphi_2^*(p)$ is written as $\varphi_2^*(p)=\sum_j m_jl_j$, where $l_j\simeq\boldsymbol{P}^1$ and $\cup_j l_j$ is a tree of rational curves. Since we have $1=(-K_{Y_1}, \varphi_2^*(p))=\sum_j m_j(-K_{Y_1}, l_j)$ and $2(-K_{Y_1}, l_j)\in\boldsymbol{N}$ for any j, $\varphi_2^*(p)$ is one of the following ; (1) $\varphi_2^*(p)=l$, where $l\simeq\boldsymbol{P}^1$ and $\tilde{\Theta}_0^{(1)}$ is smooth in the neighborhood of l, (2) $\varphi_2^*(p)=l_1+l_2$, where $l_j\simeq\boldsymbol{P}^1$ for $j=1$, 2 and $(l_1, l_2)=1$. $\tilde{\Theta}_0^{(1)}$ has two singular points $q_j(j=1, 2)$ of type $A_{2,1}$ on Supp $\varphi_2^*(p)$ such that $q_j\in l_j\backslash l_1\cap l_2(j=1, 2)$, (3) $\varphi_2^*\,(p)=2l$, where $l\simeq\boldsymbol{P}^1$ and $\tilde{\Theta}_0^{(1)}$ has one singular point q of type $A_{2,1}$ on Supp $\varphi_2^*(p)$ or (4) $\varphi_2^*(p)=l_1+l_2$, where $l_j\simeq\boldsymbol{P}^1$ for $j=1$, 2 and l_1 and $l_2=1$ intersect at one point q. $\tilde{\Theta}_0^{(1)}$ has one singular point q of type $A_{2,1}$ on Supp $\varphi_2^*(p)$. The cases (2) and (3) are excluded by trivial reason. Put $C_{1,i}^{(1)}:=\tilde{\Theta}_{1,i}^{(1)}|_{\Theta_0^{(1)}}$ and let f be a general fibre of φ_2 : $\tilde{\Theta}_0^{(1)}\to\boldsymbol{P}^1$. Since we have $\sum_{i=1}^5(C_{1,i}^{(1)}, f)=5$ and $(C_{1,i}^{(1)}, f)>0$ for any i by the $\boldsymbol{Q}$-factoriality of Y_2, we get $(C_{1,i}^{(1)}, f)=1$ for any i. Let $q\in\tilde{\Theta}_{1,i}^{(1)}$ be any point described as in (4). If $C_{1,i}^{(1)}$ does not pass through q, then we may assume that $(\tilde{\Theta}_{1,i}^{(1)}, l_1)=1$ and $(\tilde{\Theta}_{1,i}^{(1)}, l_2)=0$. Since φ_2 is an extremal contraction, we get a contradiction. Therefore, for any i, $C_{1,i}^{(1)}$ passes through q, but which is absurd. Thus we conclude that φ_2 is a small contraction. Let C be an irreducible curve which is contained in the exceptional locus of φ_2. From [13], we can see that $C\simeq \boldsymbol{P}^1$ and C passes through only one singular point of Y_1. Put $C_{1,i}^{(0)}:=\widetilde{\Theta}_{1,i}^{(0)}|_{\Theta_0^{(0)}}$ and $C_{2,i}^{(0)}:=\widetilde{\Theta}_{2,i}^{(0)}|_{\Theta_0^{(0)}}$ for $1\leq i\leq 5$. Let C' be the strict transform of C on $\Theta_0^{(0)}$. Since we have $(-K_{Y_1}, C)=1/2$, we have

$$5/2=\left(\mu^*\left(\sum_{i=1}^{5} C_{1,i}^{(1)}, C'\right)=\left(\sum_{i=1}^{5} C_{1,i}^{(0)}, C'\right)+(1/2)\left(\sum_{i=1}^{5} C_{2,i}^{(0)}, C'\right).$$

Noting that K_Y and $K_Y+\widetilde{\Theta}_0+(1/5)\sum_{i=1}^5 \widetilde{\Theta}_{1,i}+(3/5)\sum_{i=1}^5 \widetilde{\Theta}_{2,i}$ is relatively numerical equivalent over $\mathscr{D}$, we see that $(Y_2, \widetilde{\Theta}_0^{(2)}+(1/5)\sum_{i=1}^5 \widetilde{\Theta}_{1,i}^{(2)})$ is divisorially log terminal, hence $\widetilde{\Theta}_0^{(2)}$ is normal and $(\widetilde{\Theta}_0^{(2)}, (1/5)\sum_{i=1}^5 C_{1,i}^{(2)})$ is also divisorially log terminal by [20], (3.2.3), where $\widetilde{\Theta}_0^{(2)}:=\varphi_{2*}\widetilde{\Theta}_0^{(1)}$, $\widetilde{\Theta}_{1,i}^{(2)}:=\varphi_{2*}\widetilde{\Theta}_{1,i}^{(2)}$ and $C_{1,i}^{(2)}:=\varphi_{2*}C_{1,i}^{(1)}$. Thus we conclude that $(\sum_{i=1}^5 C_{1,i}^{(0)}, C'))=2$ and $(\sum_{i=1}^5 C_{2,i}^{(0)}, C')=1$. Moreover, since we have $K_{\widetilde{\Theta}_0}\cdot C'=(-(2/5)\sum_{i=1}^5 C_{1,i}^{(0)}-(1/5)\sum_{i=1}^5 C_{2,i}^{(0)}, C')=-1$ and $C'^2<0$, we get $C'^2<0$, we get $C'^2=-1$. We can get the flip of C by blowing-up along C' and contract the exceptional divisor which is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ along the other ruling (see Figure V-3.2 and V-3.3). By the same way as above, we carry out flips four more times and we get the model as in Figure V-3.4. The strict transform of $\widetilde{\Theta}_0$ on this model is isomorphic to a Hirzebruch surface and after contracting this component along fibres of the ruling, we get a minimal model as described in Theorem 6.1 V-3.

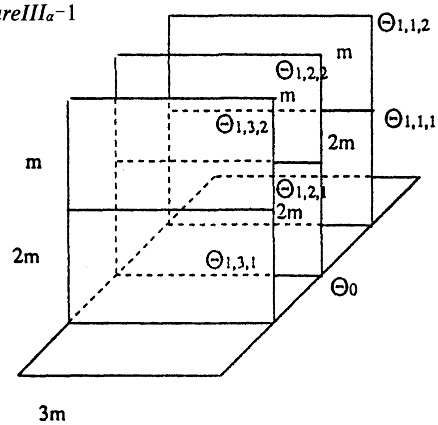
Assume that $(\widehat{\Theta}, \mathrm{Diff}_{\widehat{\Theta}}(0)$ is of type VI_β in Theorem 5.1. Lep $p\in\widetilde{X}$ be a fixed point of σ. We see that analytic locally around $\pi(p)$, $(\widehat{X}, \widehat{\Theta})$ is isomorphic to the germ of the origin of (1) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_6(3, 2, 1)$ or (2) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_6(3, 2, 5)$. We resolve these singularities and calculate the intersection number with the special fibre of the induced fibration and the strict transform of an irreducible component of $\mathrm{Diff}_{\widehat{\Theta}}(0)$ whose coefficient is $1/2$ to see that for all of the fixed points $p\in\widetilde{X}$ of σ, $\pi(p)$ has the same type described as above. Thus we are in the case VI_β-1 or VI_β-1 or VI_β-2 in Theorem 6.1.

Assume that $(\widehat{\Theta}, \mathrm{Diff}_{\widehat{\Theta}}(0))$ is of type VI_γ or VI_δ in Theorem 5.1. Let $p\in\widetilde{X}$ be a fixed point of σ. We can see that analytic locally around $\pi(p)$, $(\widetilde{X}, \widehat{\Theta})$ is isomorphic to the germ of the origin of (1) $(\boldsymbol{C}^3, \{z=0\}/\boldsymbol{Z}_6(2,5,5))$ or (2) $(\boldsymbol{C}^3, \{z=0\}/\boldsymbol{Z}_6(2, 5, 1))$. We resolve these singularities and calculate the intersection number with the special fibre of the induced fibration and the strict transform of an irreducible component of $\mathrm{Diff}_{\widehat{\Theta}}(0)$ whose coefficient is $1/2$ to see that for all of the fixed point $p\in\widetilde{X}$ of σ, $\pi(p)$ has the same type described as above. From the proof of Theorem 5.1, $\widehat{\Theta}$ has a structure of $\boldsymbol{P}^1$-fibration all of whose fibres are irreducible. Take an irreducible reduced curve Γ which is contained in a fibre and passes through the singular points of $\widehat{\Theta}$. We resolve $\widehat{X}$ and calculate the intersection number with the special fibre of the induced fibration and the strict transform of Γ to see that analytic locally around all the other singular points of $\widehat{X}$, $(\widehat{X}, \widehat{\Theta})$ is isomorphic to the germ of the origin of $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 1)$ in the case (1) and $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_3(1, 1, 2)$ in the case (2). Thus we are in the case VI_γ-1, VI_δ-1, VI_γ-2 or VI_δ-2 in Theorem 6.1.

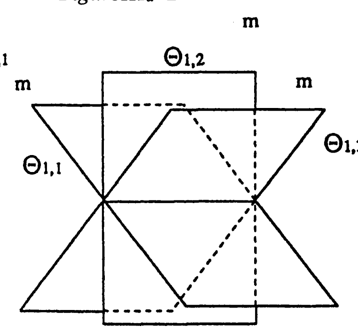
Assume that $(\hat{\Theta}, \mathrm{Diff}_{\hat{\Theta}}(0))$ is of type XII_α in Theorem 5.1. Let $p \in \tilde{X}$ be a fixed point of σ. From the above argument, analytic locally around $\pi(p)$, $(\hat{X}, \hat{\Theta})$ is isomorphic to the germ of the origin of (1) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_{12}(4, 3, 5)$ or (2) $(\boldsymbol{C}^3, \{z=0\})/Z_{12}(4, 3, 1)$. (3) $(C^3, \{z=0\})/\boldsymbol{Z}_{12}(4, 3, 11)$, (4) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_{12}(4, 3, 7)$. In the same way as above, we see that for all of the fixed points $p \in \tilde{X}$ of σ, $\pi(p)$ has the same type described as above and we are in the cases XII_α-1, 2, 3, 4 in Theorem 6.1.

Assume that $(\hat{\Theta}, \mathrm{Dfff}_{\hat{\Theta}}(0))$ is of type XII_β in Theorem 5.1. Let $p \in \tilde{X}$ be a fixed point of σ. From the above argument, analytic locally around $\pi(p)$, $(\hat{X}, \hat{\Theta})$ is isomorphic to the germ of the origin of (1) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_{12}(2, 3, 7)$ or (2) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_{12}(2, 3, 1)$. (3) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_{12}(2, 3, 5)$, (4) $(\boldsymbol{C}^3, \{z=0\})/\boldsymbol{Z}_{12}(2, 3, 11)$. In the same way as above, we see that for all of the fixed point $p \in \tilde{X}$ of σ, $\pi(p)$ has the same type described as above and we are in the cases XII_β-1, 2, 3, 4 in Theorem 6.1. ■

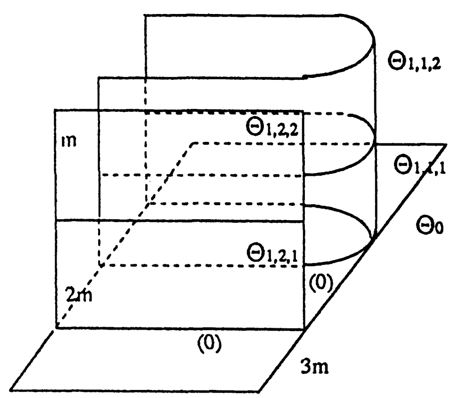
FigureIII$_\alpha$-1



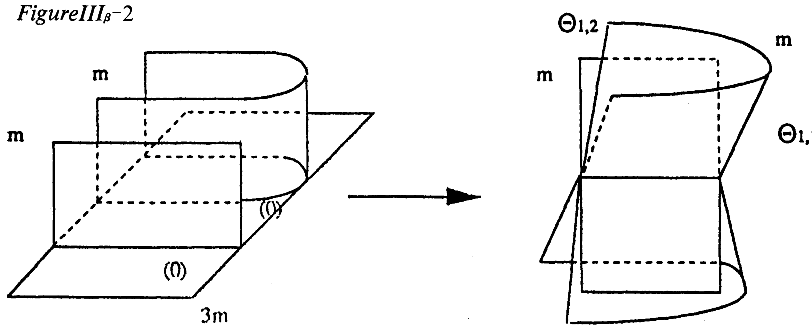
FigureIII$_\alpha$-2



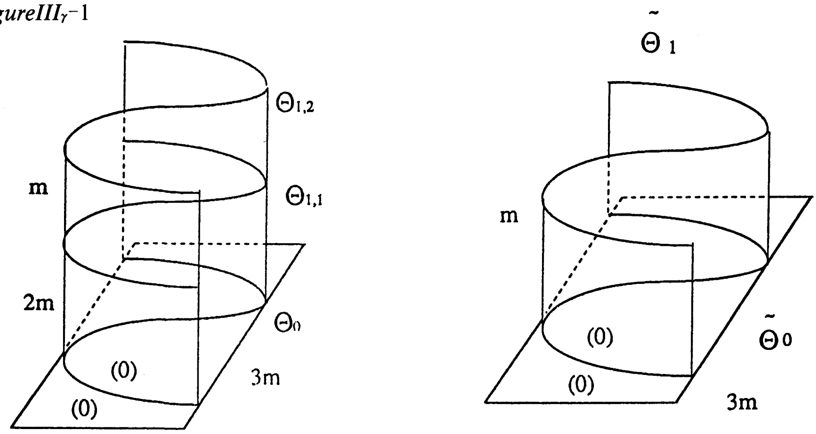
FigureIII$_\beta$-1



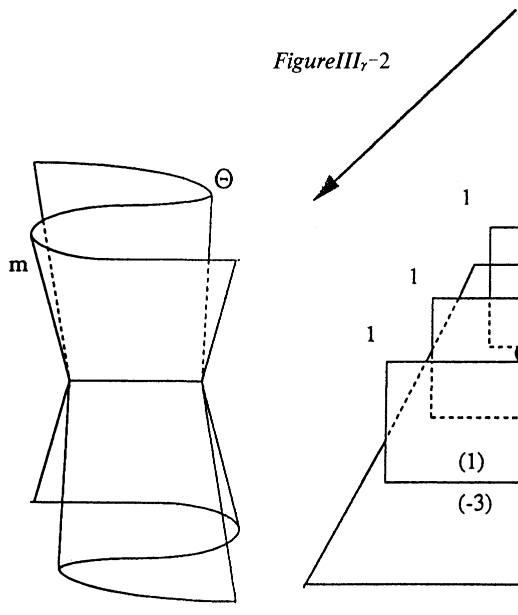
FigureIII$_\beta$-2



FigureIII$_\gamma$-1



FigureIII$_\gamma$-2



FigureIIIδ

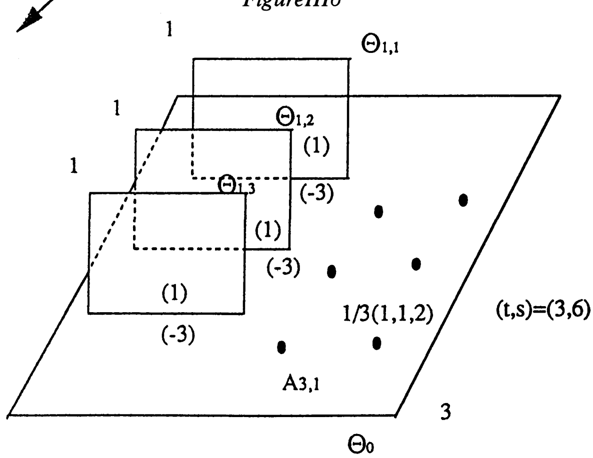


Figure IV_{α}-1

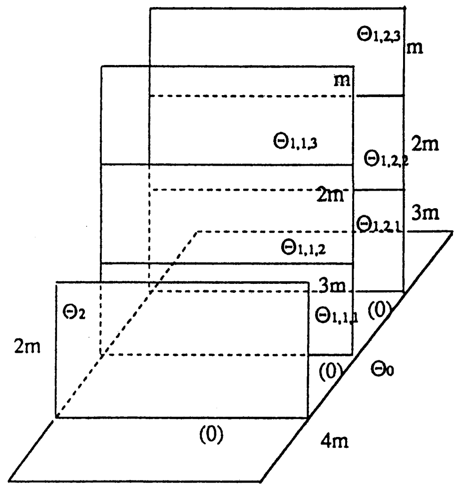
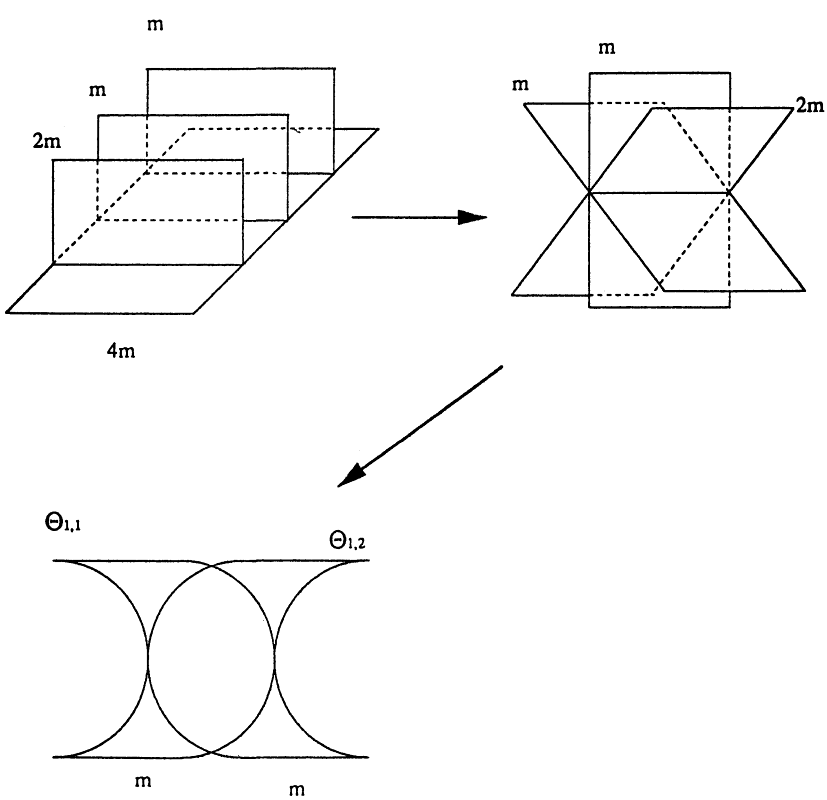
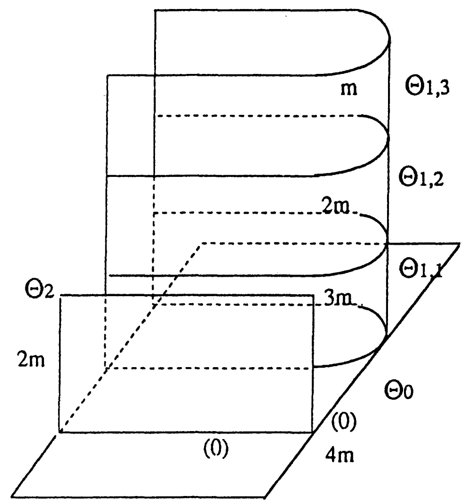


Figure IV_{α}-2



FigureIV$_\beta$-1



FigureIV$_\beta$-2

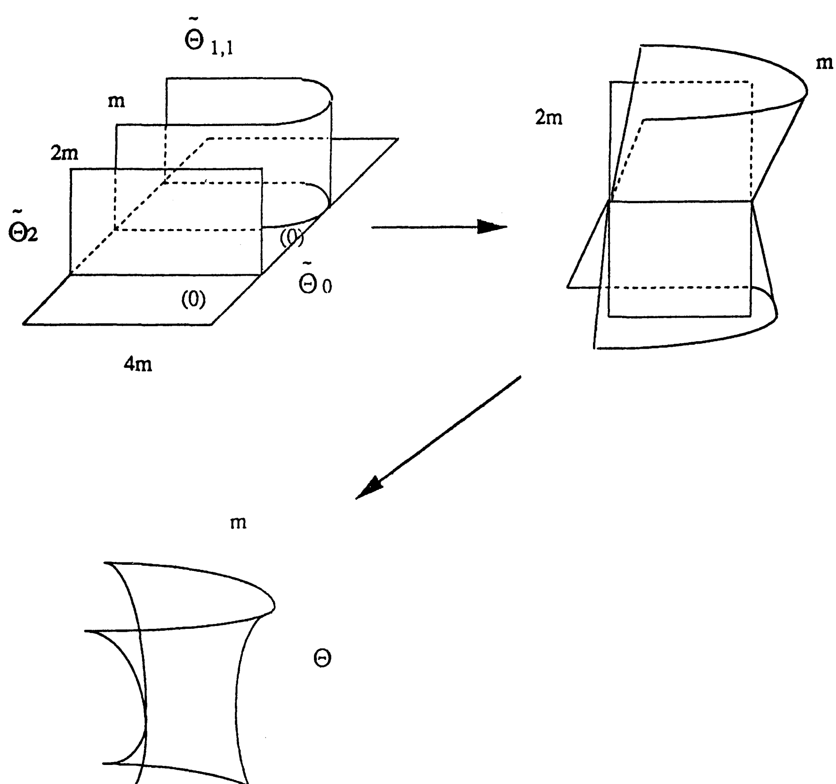


Figure IV₇

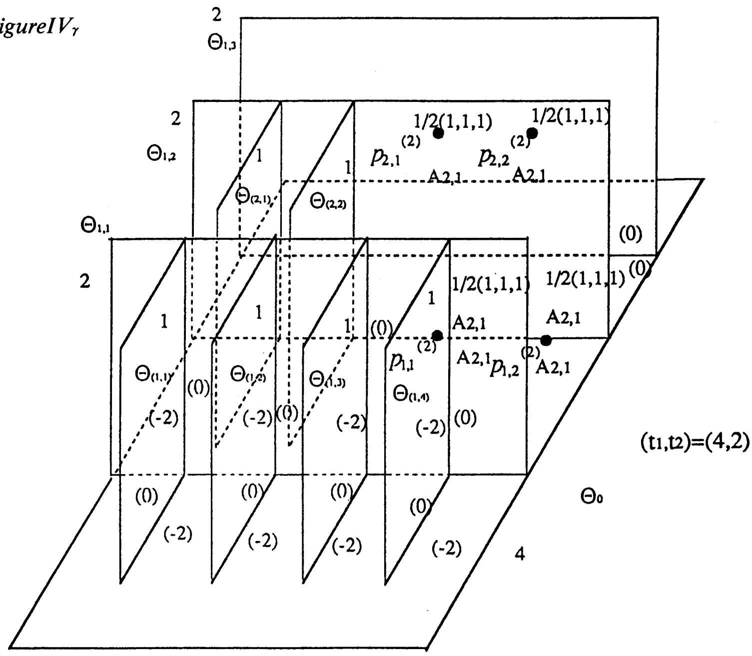


Figure IV₈

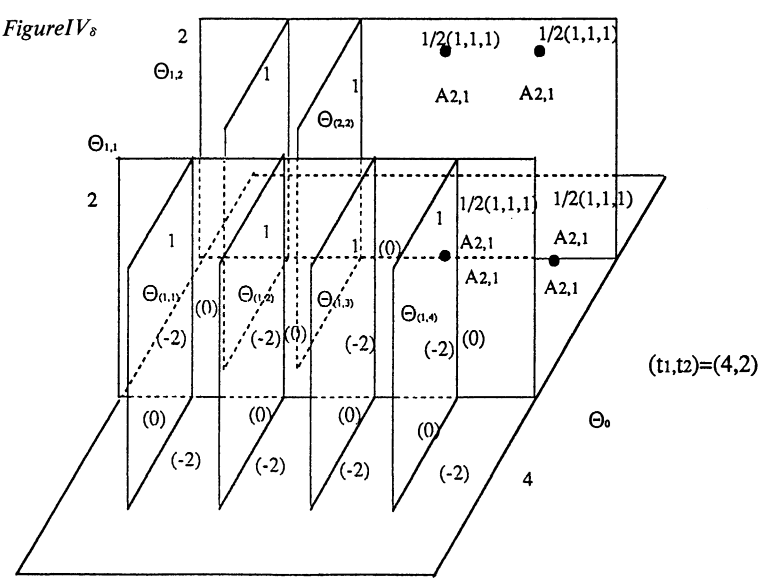


Figure V-1

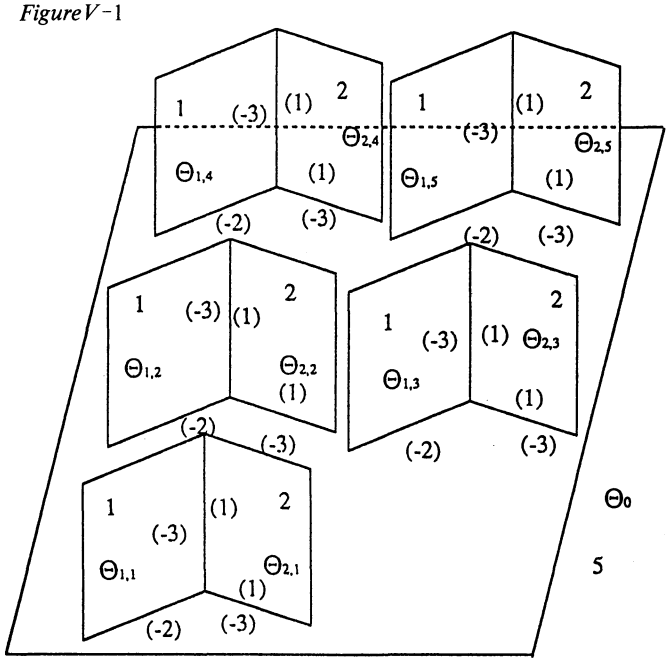


Figure V-2

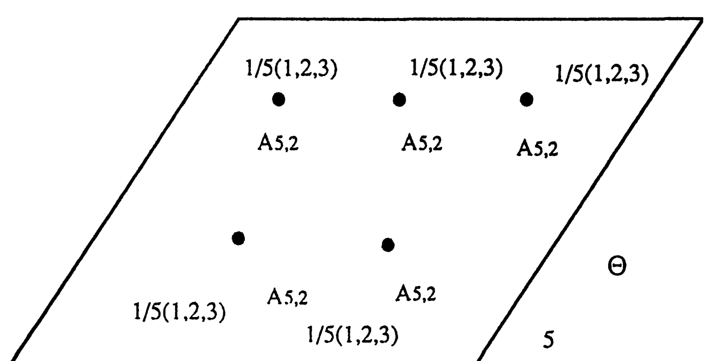


Figure V-3.1

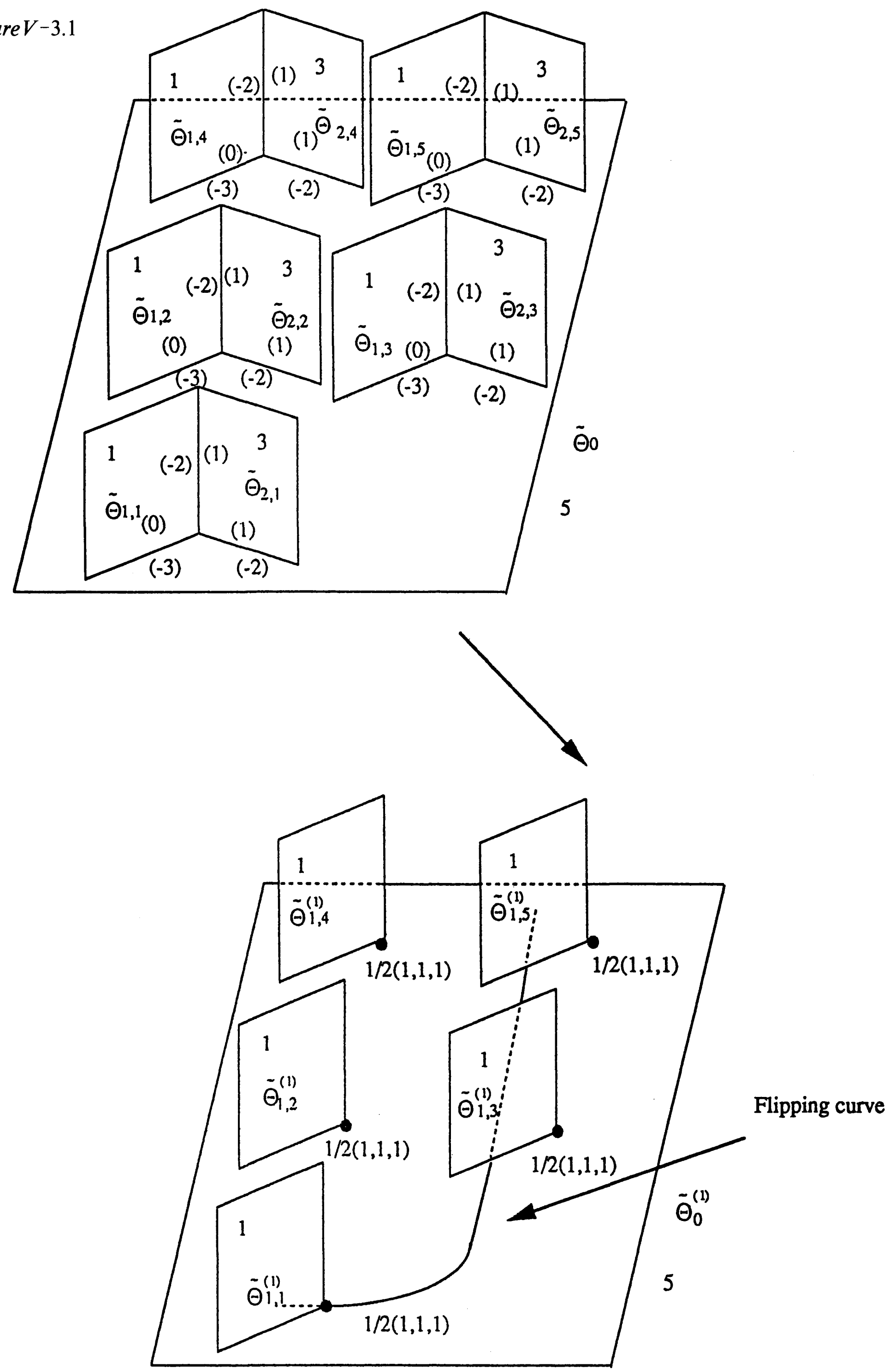
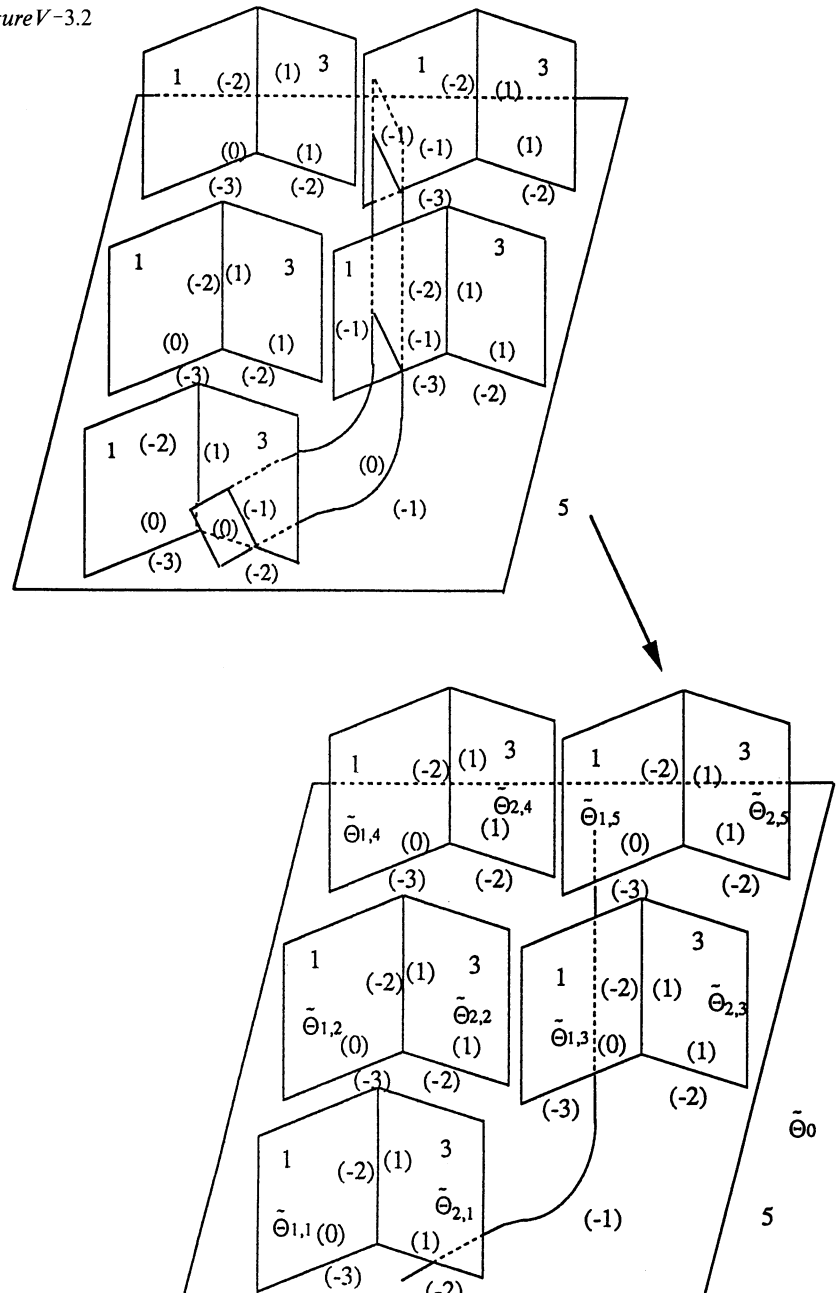
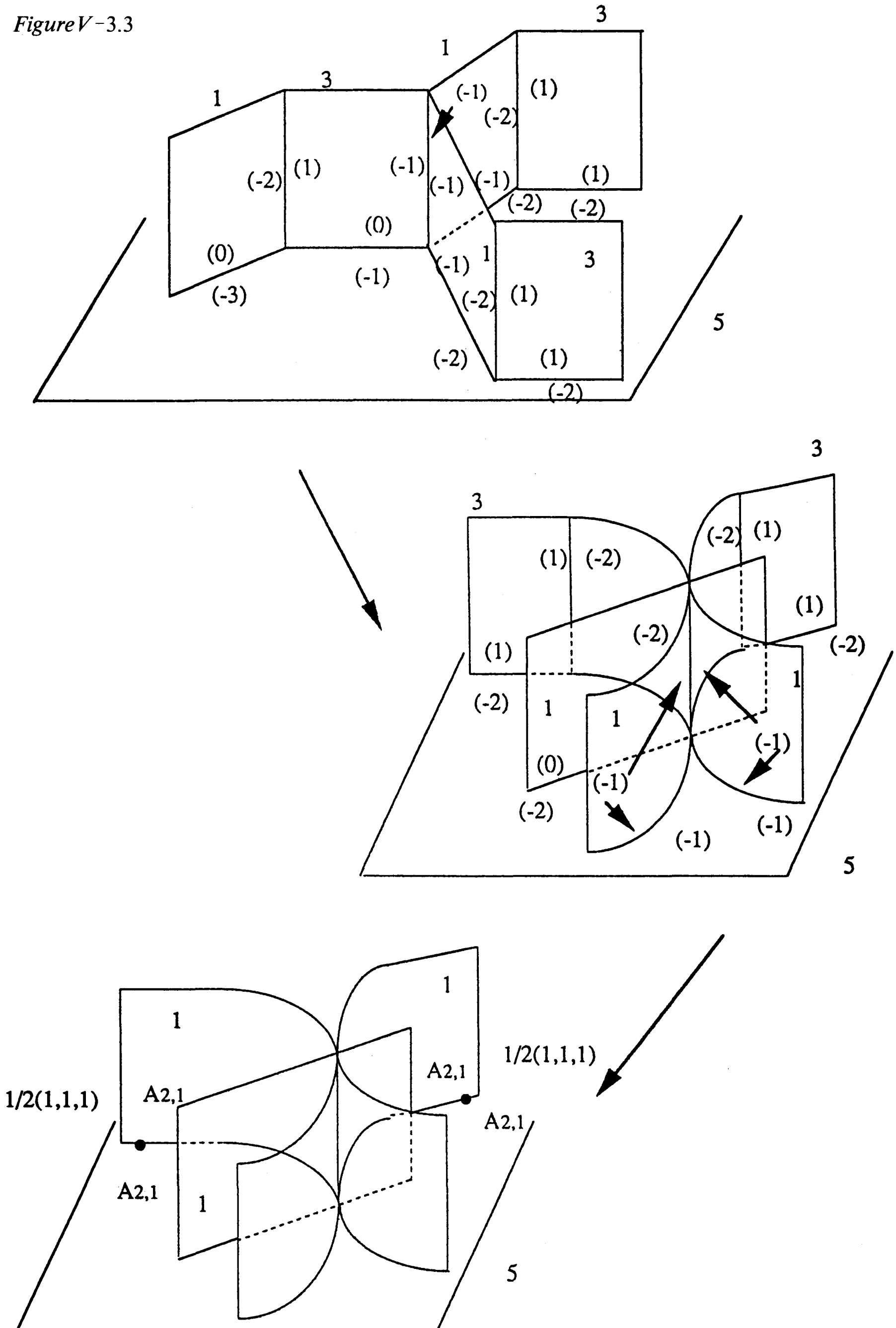

1
(-2)
(1)
3
$\tilde{\Theta}_{1,4}$
(0)
(1)
$\tilde{\Theta}_{2,4}$
(-3)
(-2)
$\tilde{\Theta}_{1,5}$
$\tilde{\Theta}_{2,5}$
$\tilde{\Theta}_{1,2}$
$\tilde{\Theta}_{2,2}$
$\tilde{\Theta}_{1,3}$
$\tilde{\Theta}_{2,3}$
$\tilde{\Theta}_{1,1}$
$\tilde{\Theta}_{2,1}$
$\tilde{\Theta}_0$
5
$\tilde{\Theta}^{(1)}_{1,4}$
$\tilde{\Theta}^{(1)}_{1,5}$
$\tilde{\Theta}^{(1)}_{1,2}$
$\tilde{\Theta}^{(1)}_{1,3}$
$\tilde{\Theta}^{(1)}_{1,1}$
1/2(1,1,1)
Flipping curve
$\tilde{\Theta}^{(1)}_0$

Figure V-3.2



1
3
5
(-2)
(1)
(0)
(-1)
(-3)
(0)
(-1)
5
$\tilde{\Theta}_{1,1}$
$\tilde{\Theta}_{2,1}$
$\tilde{\Theta}_{1,2}$
$\tilde{\Theta}_{2,2}$
$\tilde{\Theta}_{1,3}$
$\tilde{\Theta}_{2,3}$
$\tilde{\Theta}_{1,4}$
$\tilde{\Theta}_{2,4}$
$\tilde{\Theta}_{1,5}$
$\tilde{\Theta}_{2,5}$
$\tilde{\Theta}_0$
(-1)
5

Figure V-3.3


1
3
1
3
(-1)
(1)
(-2)
(-2)
(1)
(-1)
(-1)
(-1)
(1)
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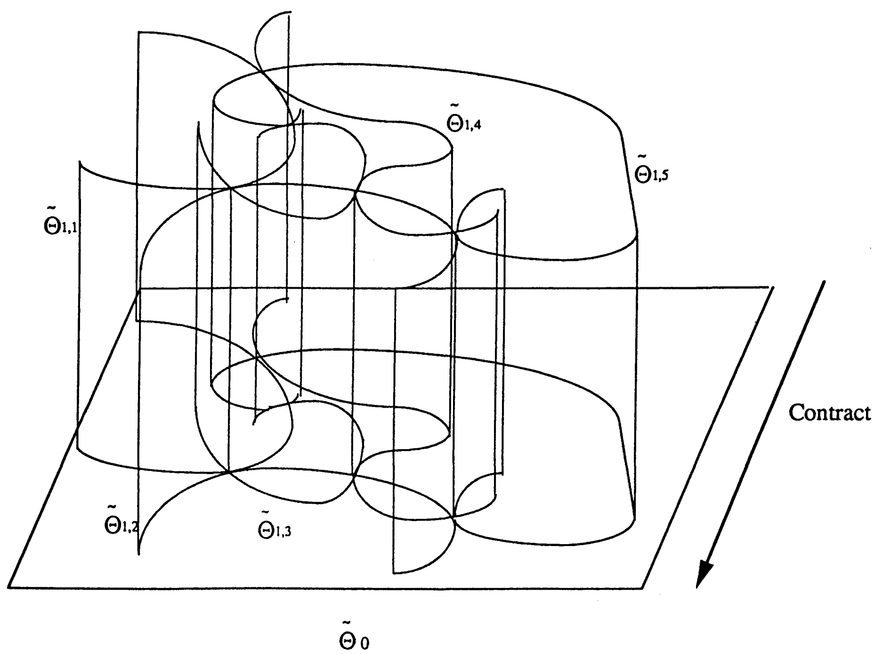
(-2)
(-2)
(0)
(-1)
1
3
(-3)
(-1)
(-2)
(1)
5
(-2)
(1)
(-2)
3
3
(1)
(-2)
(-2)
(1)
(1)
(-2)
(1)
(-2)
(-2)
1
1
1
(-1)
(0)
(-1)
(-2)
(-1)
(-1)
5
1
1
1/2(1,1,1)
A2,1
A2,1
1/2(1,1,1)
A2,1
A2,1
1
5

Figure V-3.4



Case $\tilde{\Theta}_0 \cong P^1 \times P^1$

Figure VI_a-1

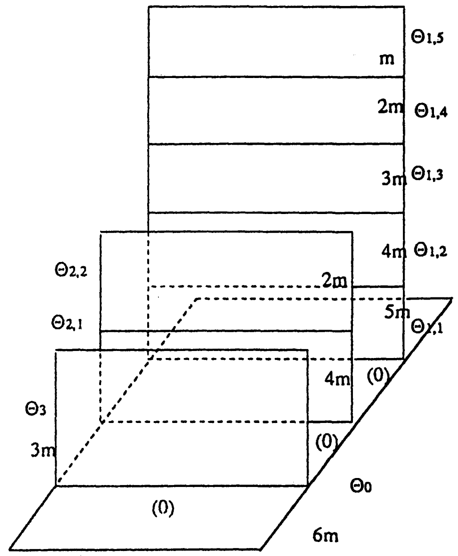


Figure VI_a-2

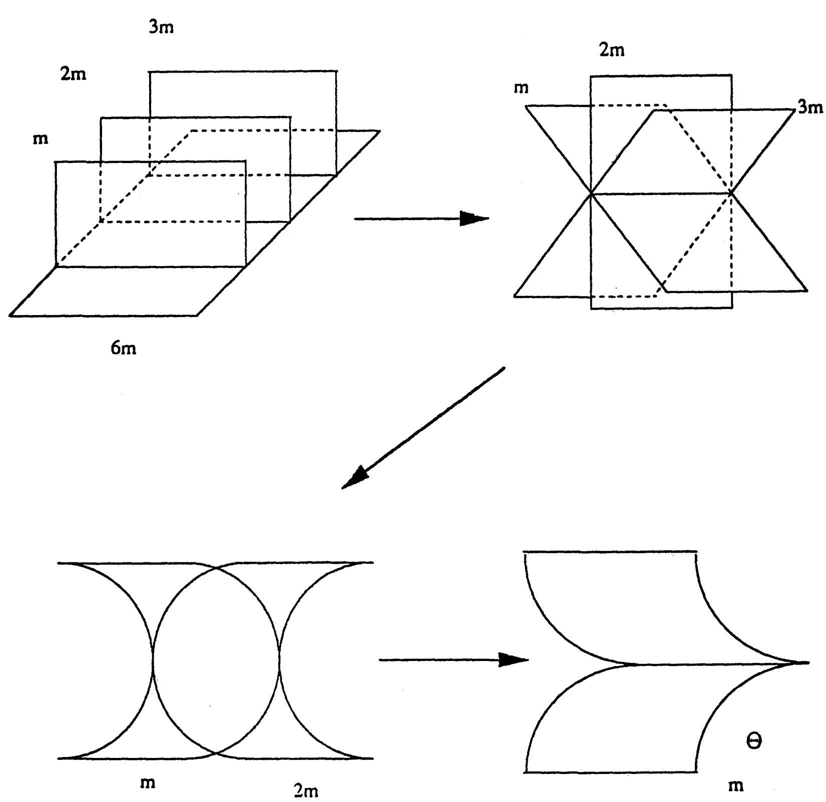


Figure VI_{β}-1

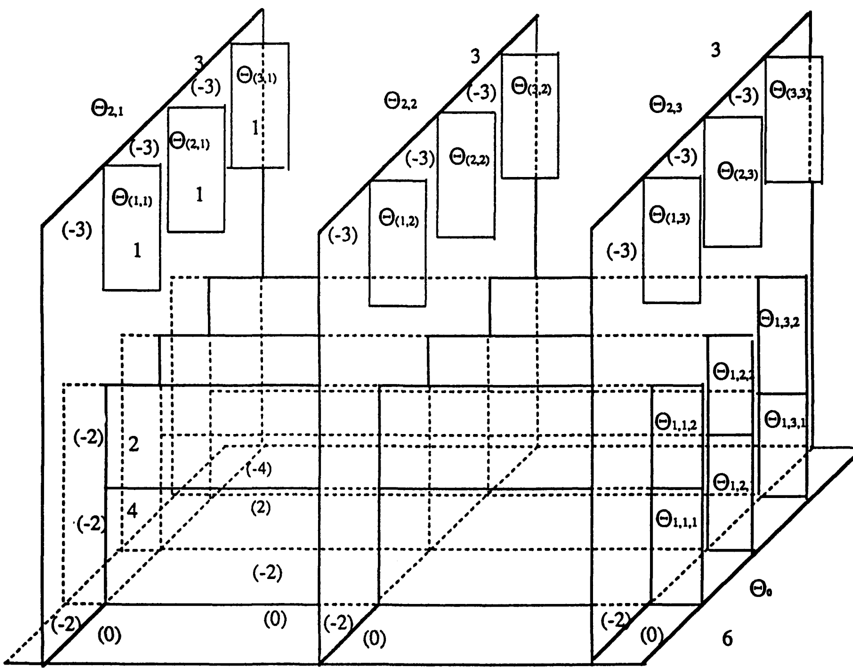


Figure VI_β-2

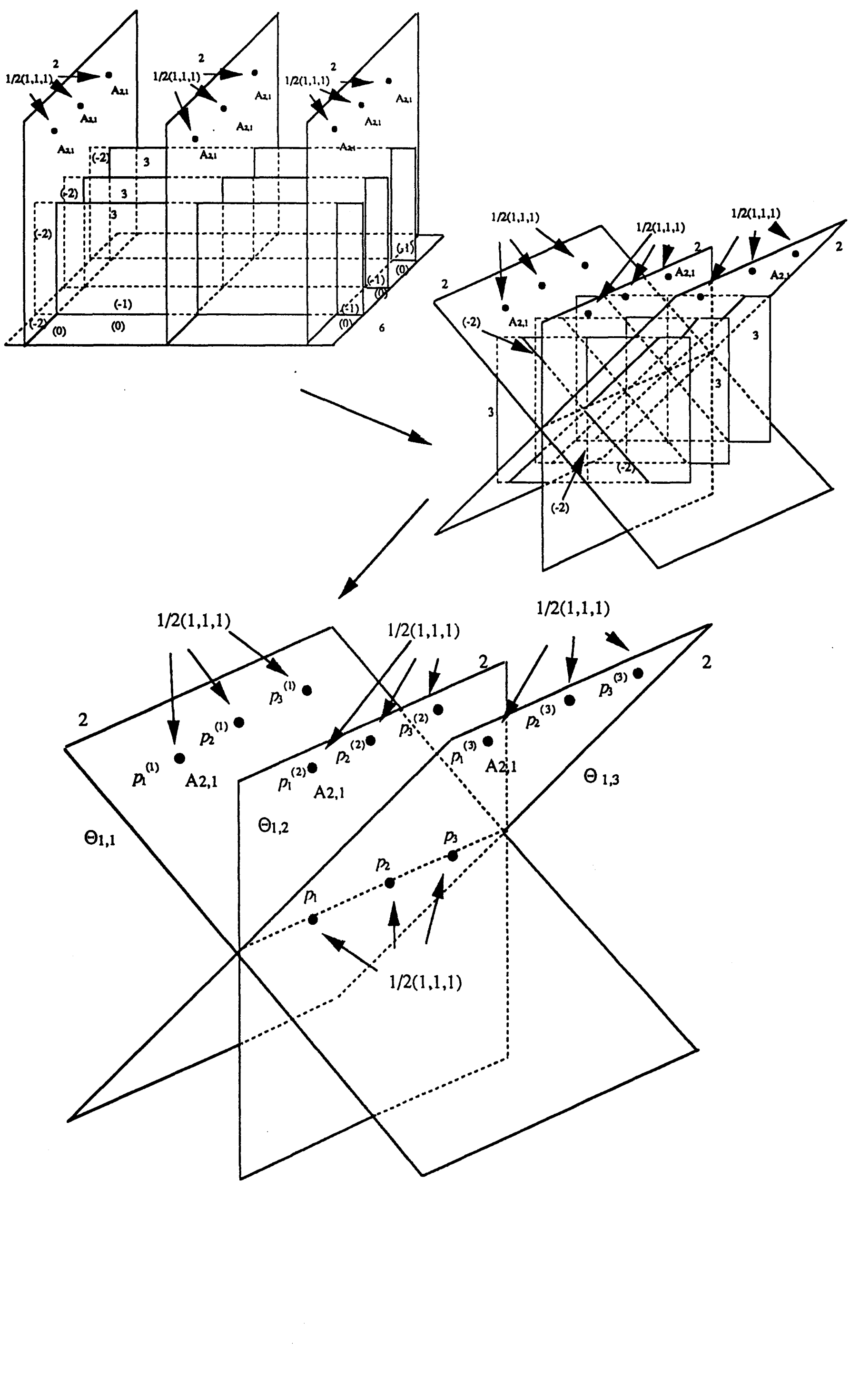


Figure VI_γ-1

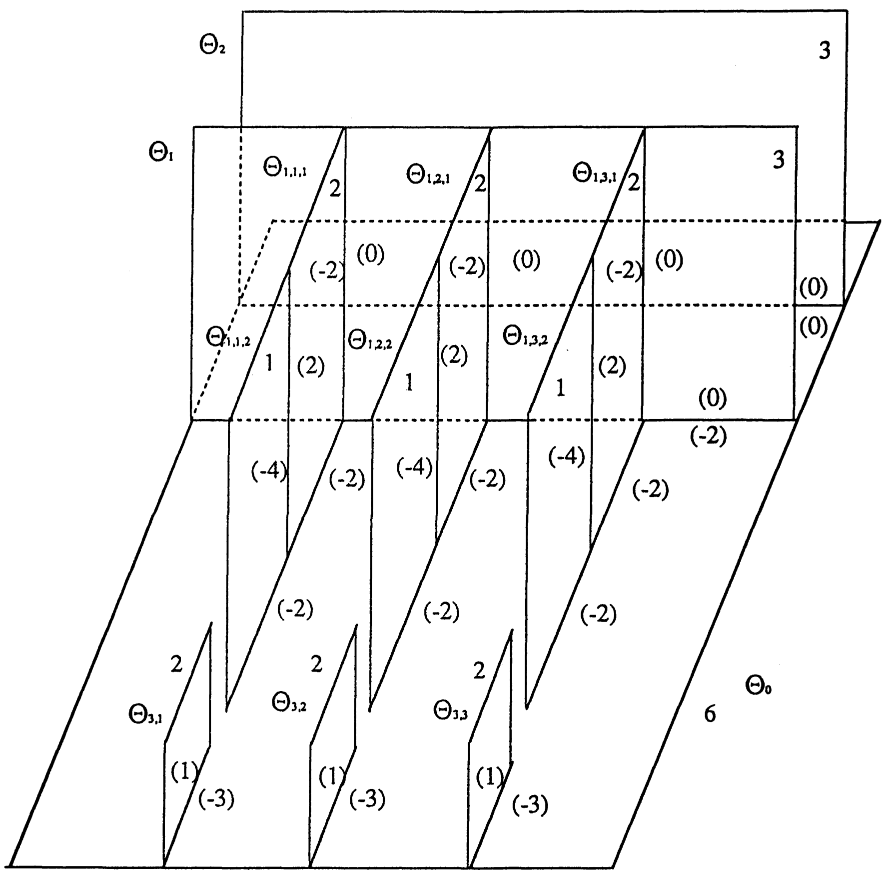


Figure VI_7-2

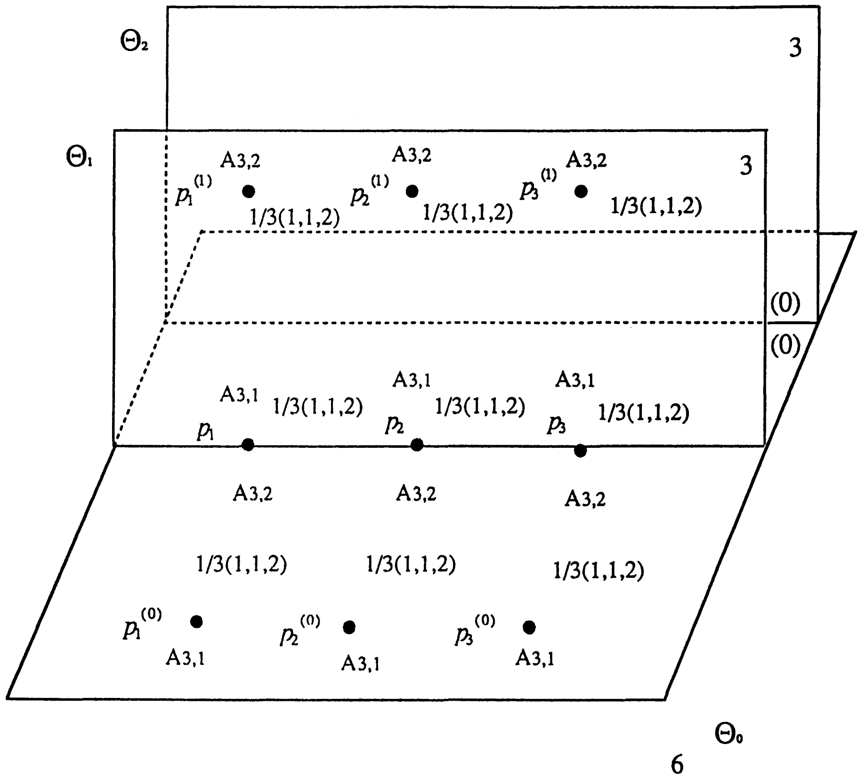

Θ2
3
Θ1
3
A3,2
A3,2
A3,2
1/3(1,1,2)
1/3(1,1,2)
1/3(1,1,2)
(0)
(0)
A3,1
A3,1
A3,1
1/3(1,1,2)
1/3(1,1,2)
1/3(1,1,2)
A3,2
A3,2
A3,2
1/3(1,1,2)
1/3(1,1,2)
1/3(1,1,2)
A3,1
A3,1
A3,1
Θ0
6

Figure VI_{δ}-1

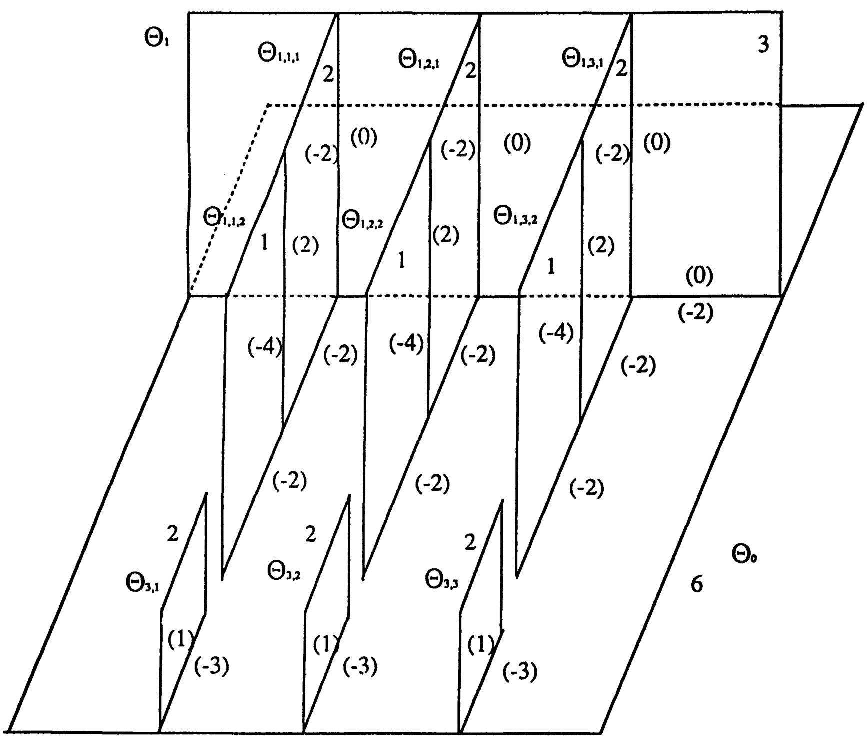
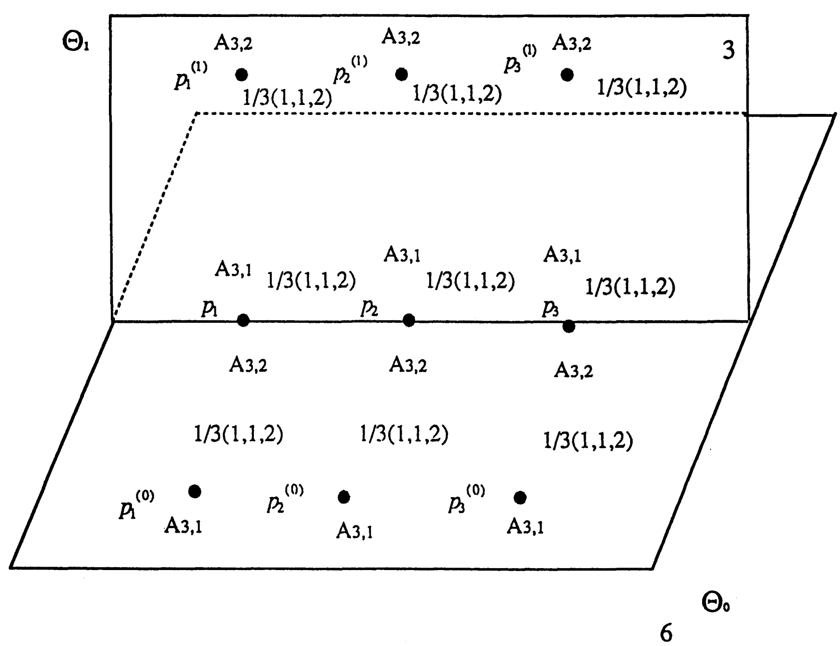
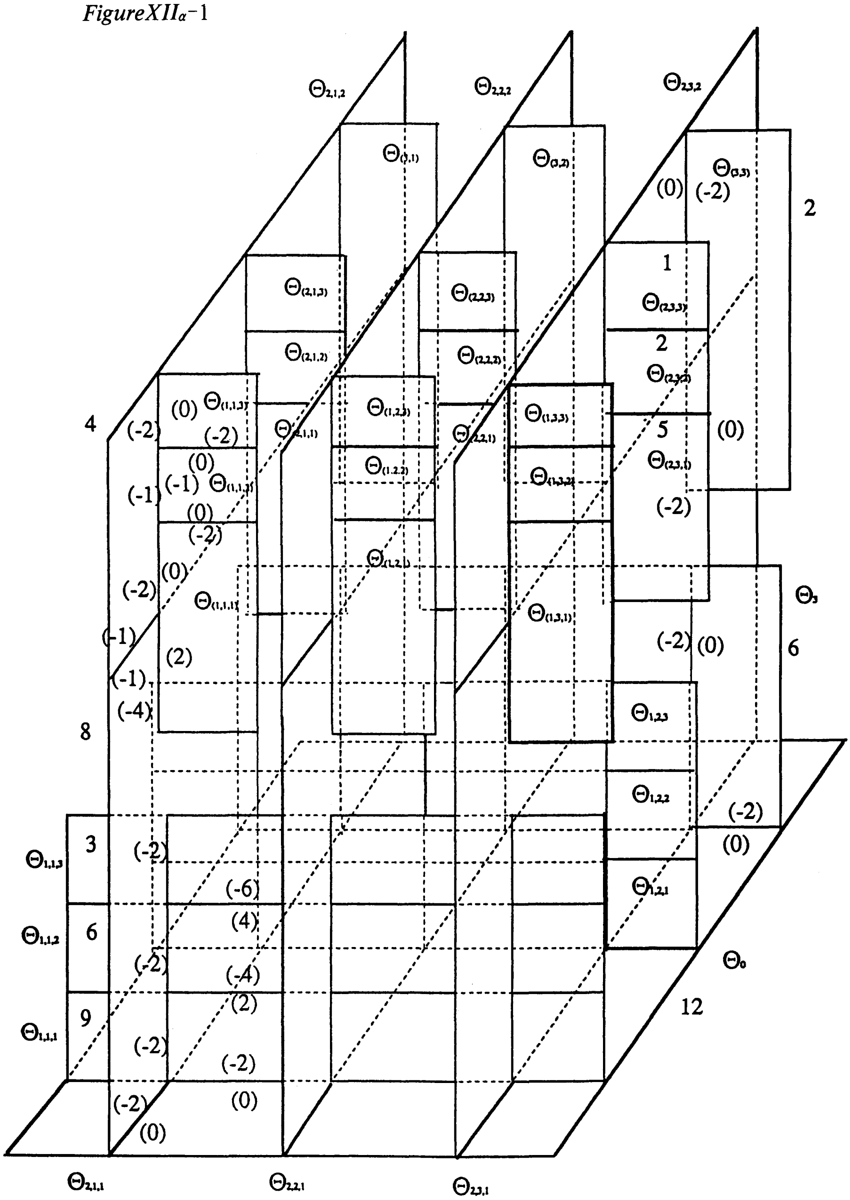


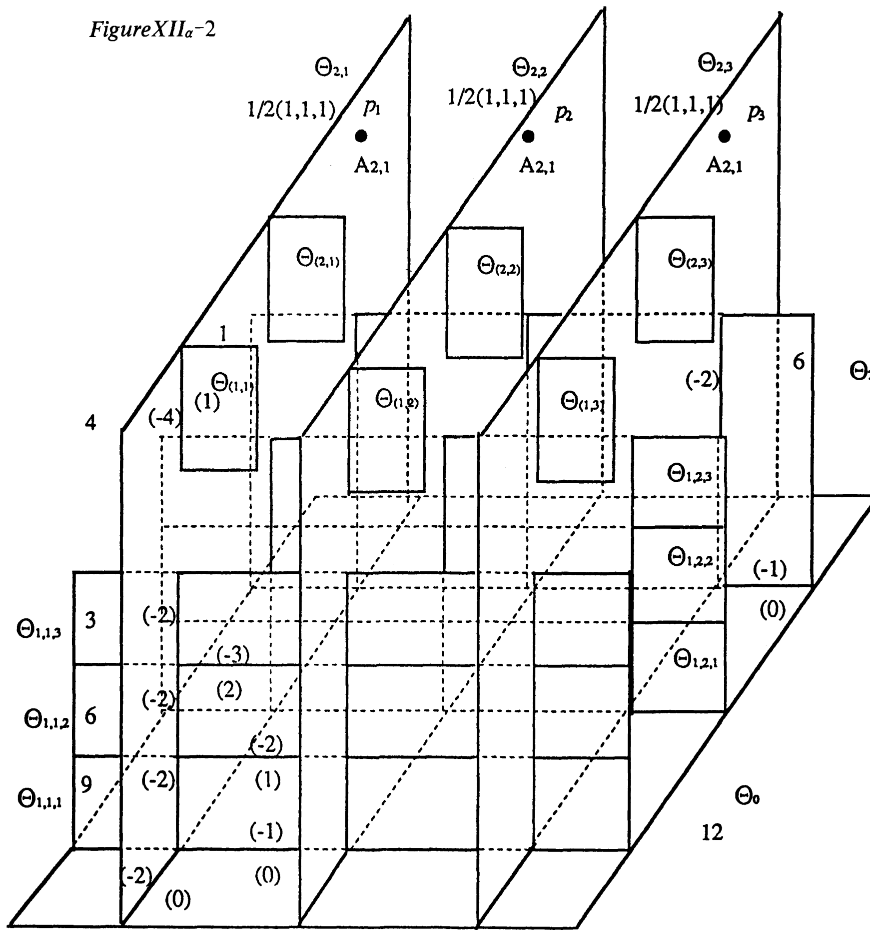
Figure VI_8-2


Θ1
A3,2
p1(1)
1/3(1,1,2)
A3,2
p2(1)
1/3(1,1,2)
p3(1)
A3,2
1/3(1,1,2)
3
A3,1
1/3(1,1,2)
A3,1
1/3(1,1,2)
A3,1
1/3(1,1,2)
p1
p2
p3
A3,2
A3,2
A3,2
1/3(1,1,2)
1/3(1,1,2)
1/3(1,1,2)
p1(0)
A3,1
p2(0)
A3,1
p3(0)
A3,1
Θ0
6

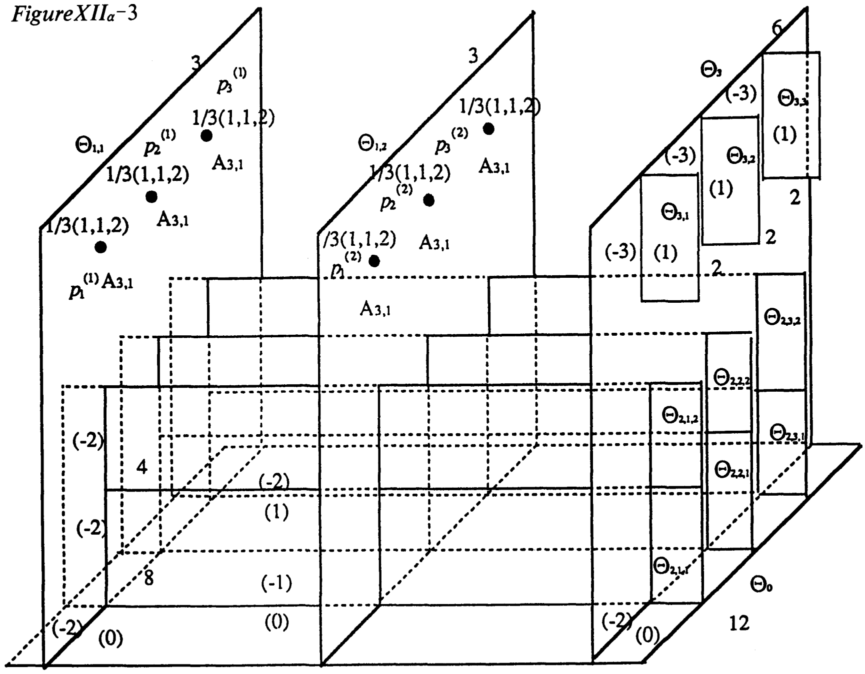
FigureXIIα-1



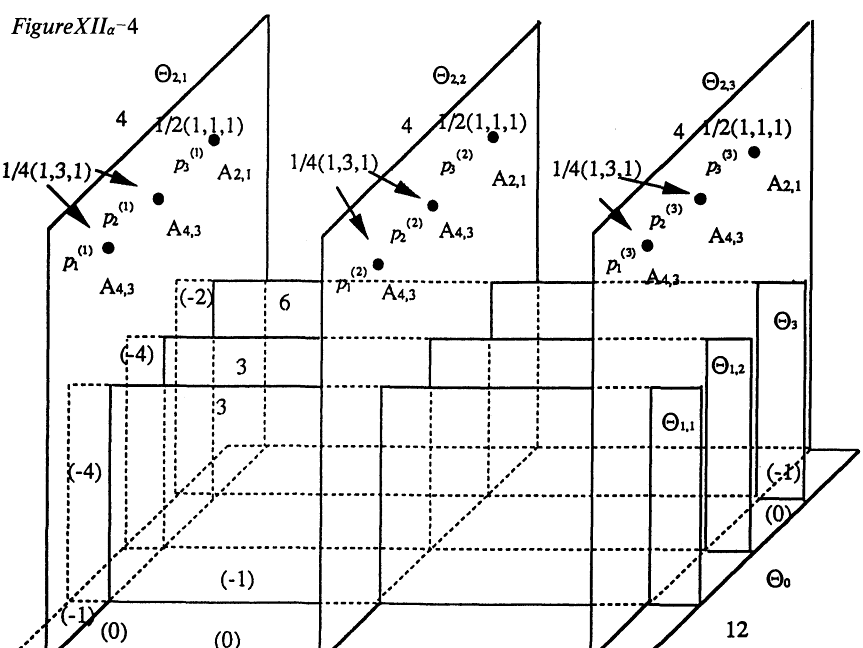
FigureXII$_a$-2



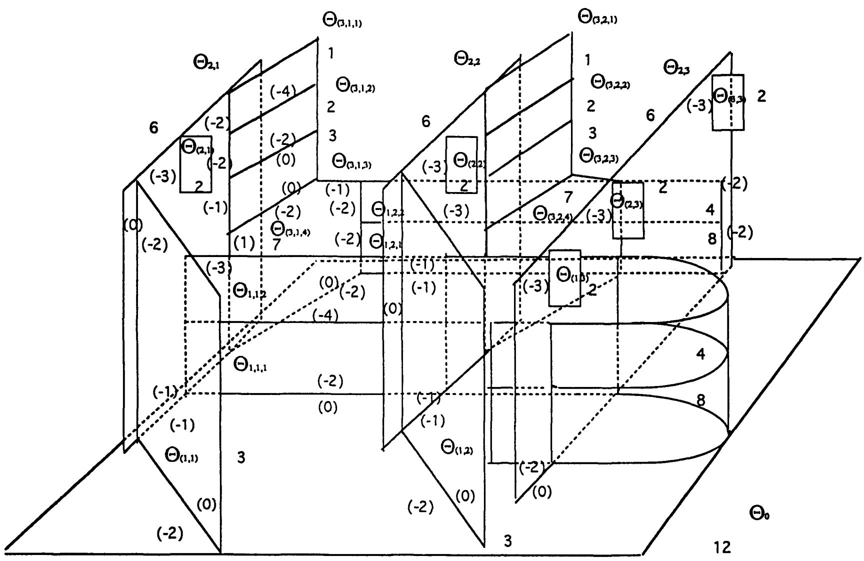
FigureXIIα-3



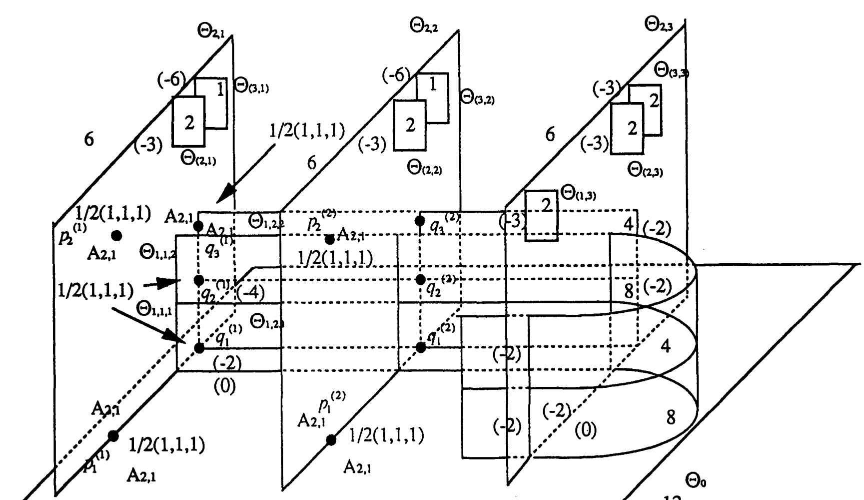
FigureXIIα-4



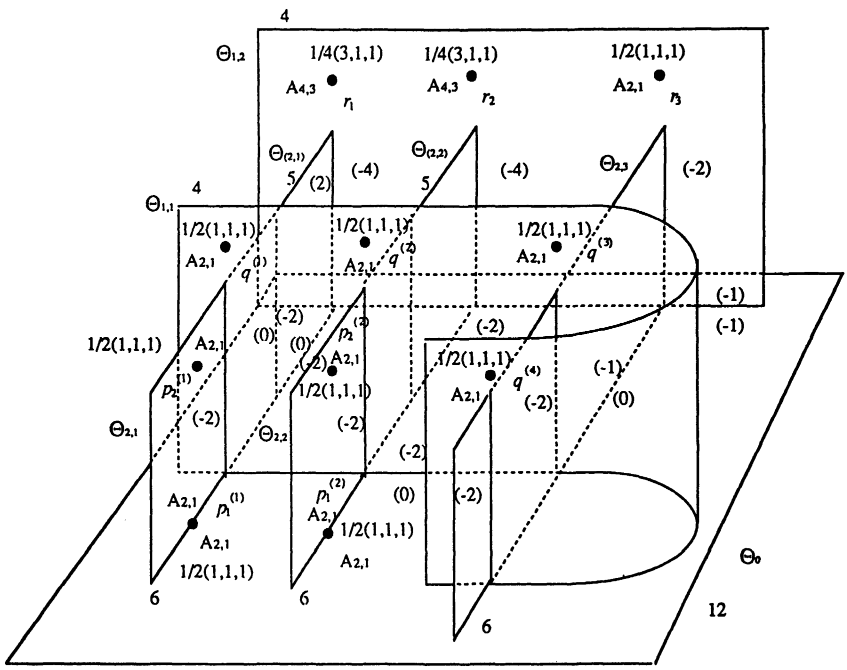
FigureXII$_\beta$-1



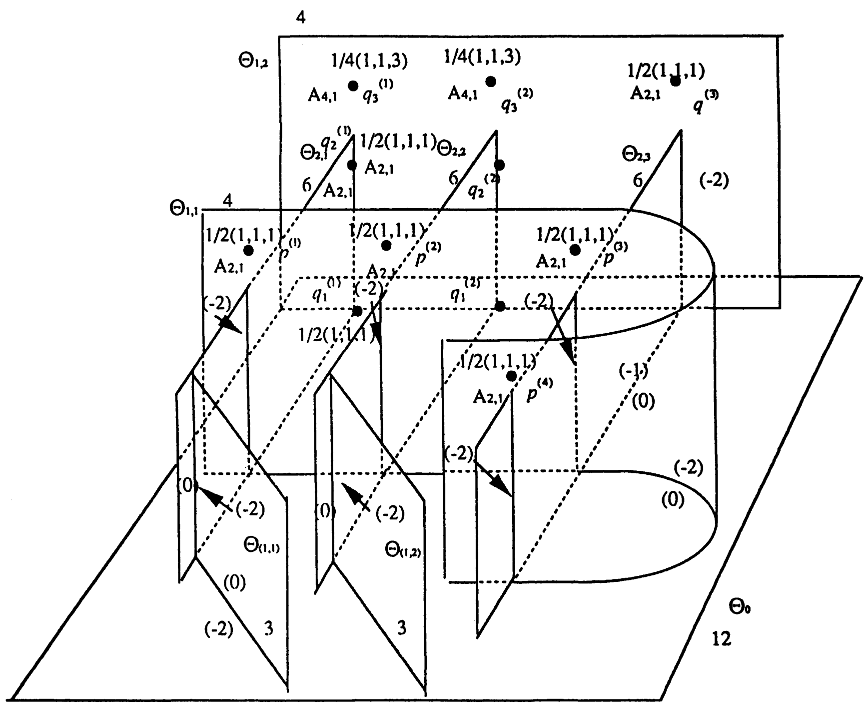
FigureXII$_\beta$-2



FigureXII$_\beta$-3



FigureXII$_\beta$-4



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