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INFINITE MARKOV PARTICLE SYSTEMS WITH SINGULAR IMMIGRATION ; MARTINGALE PROBLEMS AND LIMIT THEOREMS

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1. Introduction

In the present paper we are mainly concerned with infinite Markov particle systems $(X_t, \boldsymbol{P}_\mu)$ with singular immigration associated with absorbing Brownian motion $(w^0(t), P_x^0)$ in a half space $H=\boldsymbol{R}^{d-1}\times(0, \infty)$, starting from $\mu\in\mathcal{M}^I$. Here $\mathcal{M}^I=\mathcal{M}^I(H)$ is the spece of all σ-finite counting measures $\mu\in\sum_n\delta_{x_n}$ on H. It is constructed out of infinitely many independent absorbing Brownian particles starting from points of the support of μ and another independent particles which immigrate uniformly from boundary at random time and move according to the excursion law Q^0. The immigration part is obtained as the limit by putting the starting points of independent absorbing Brownian particles which immigrate in H at random times, close to the boundary with infinite mass. From this construction the generator $\mathcal{L}$ of this process should be expressed as the sum of no immigration part $\mathcal{L}^0$ and immigration part $\mathcal{L}^I$. That is, for some suitable functional $F(\mu)$ of integer-valued discrete measures μ,

$$\mathcal{L}^0F(\mu)=\frac{1}{2}\sum_{k=1}^{d}\langle\mu, \mathcal{D}_k^2F(\mu\,;\,\cdot)\rangle,$$

$$\mathcal{L}^IF(\mu)=\frac{1}{2}\langle\tilde{m}, \mathcal{D}_dF(\mu\,;\,\cdot)|_{x_d=0+}\rangle,$$

where $\tilde{m}=d\tilde{x}$ is the Lebesgue measure on $\boldsymbol{R}^{d-1}$, $\mathcal{D}_k$ is a kind of differential operator defined as

$$\mathcal{D}_kG(\mu\,;\,x)=\lim_{h\to 0}\frac{1}{h}[G(\mu+\delta_{x_k(h)}-\delta_x\,;\,x_k(h))-G(\mu\,;\,x)]$$

with $x_k(h)=(x_1, \cdots, x_k+h, x_{k+1}, \cdots, x_d)$, and $\mathcal{D}_k^2=\mathcal{D}_k\circ\mathcal{D}_k$ for $k=1, 2, \cdots, d$. Note that if $G(\mu\,;\,x)=F(\mu)$, then

$$\mathcal{D}_kF(\mu\,;\,x)=\lim_{h\to 0}\frac{1}{h}[F(\mu+\delta_{x_k(h)}-\delta_x)-F(\mu)].$$

We first formulate this operator and give the martingale characterization for $(X_t, \boldsymbol{P}_\mu)$. To treat this process as a diffusion, we introduce a subspace $\bar{\mathscr{M}}^I_p$ $(p>d)$ of $\mathscr{M}^I$ with weak topology so that X_t can be continuous in it. Moreover we investigate Hölder continuity of the sample paths. It is very interesting that the exponent of Hölder continuity changes from 1/4 to 1/2 according to the starting point μ.

By using this process we can construct an *equilibrium process with immigration* $(X_t, \boldsymbol{P})$, which is a stationary Markov particle system with immigration associated with $(w^0(t), P^0_x)$ in H and the Lebesgue measure $m(dx)=dx$ on H. This process is also constructed by using a *Kuznetsov measure* Q_m, which is a stationary σ-finite Markov measure with the same transition law as the absorbing Brownian motion in H. This measure is defined as the integration of the time-shift of excursion law Q^0 with respect to the time in $\boldsymbol{R}^1$.

On the other hand for Brownian motion $(w(t), P_x)$ in $\boldsymbol{R}^d$, F. Spitzer [16] showed that, in case of $d=3$, if B is a compact set with a positive capacity $C(B)$, then

$$P_m(0<T_B<t)=tC(B)+\sqrt{t}\,4(2\pi)^{-3/2}C(B)^2+o(\sqrt{t})$$

as $t\to\infty$, where $P_m=\int dxP_x$ and T_B is the first hitting time for B (note that, in case of $d=4$ the second term is $\log t\ C(B)^2/(4\pi)^2$, which is given by R.K. Getoor [3], and in case of $d\geq 5$ it is $\int P_x(0<T_B<\infty)^2dx<\infty$, which is also given in [16]). This result can be applied to an *equilibrium process* $(X_t, \boldsymbol{P})$ introduced by Shiga and Takahashi [17], which is a stationary Markov particle system associated with the Lebesgue measure $m(dx)=dx$ on $\boldsymbol{R}^d$. Let $X_t(N^B_t)$ denote the number of paticles of the euilibrium process hitting B during the time interval $(0, t)$, where $N^B_t=\{0<T_B<t\}$. By the subadditive ergodic theorem [10], it holds that $X(N^B_t)/t\to C(B)$ $\boldsymbol{P}$-a.s. and in $L^1(\boldsymbol{P})$ as $t\to\infty$. Furtheremore

$$\frac{X(N^B_t)-tC(B)}{\sqrt{t}}\to\begin{cases}N(M^B, C(B)) & \text{if } d=3,\\ N(0, C(B)) & \text{if } d\geq 4\end{cases}$$

in law as $t\to\infty$, where $M^B=4C(B)^2/(2\pi)^{3/2}$ and $N(u, v)$ is the Gaussian distribution with mean u and variance v.

So secondly we consider a hitting rate for Brownian excursions and give the asymptotic behavior under Q_m. For a fixed compact set B in H, let

$$\sigma_B(w)=\inf\{t>0: w(t)\in B\}.$$

If the capacity $C^0(B)$ associated with $(w^0(t), P^0_x, m)$ is positive, then

$$Q_m(0\leq\sigma_B<t)=tC^0(B)+f(t)$$

with $f(t)=\sqrt{2t/\pi}+a(B)+o(1)$ $(d=1)$, $\Phi(B)\log t+o(\log t)$ $(d=2)$, $O(1)$ $(d\geq 3)$

as $t\to\infty$, where $a(B)$, $\Phi(B)$ are certain constants. By applying the result to the equilibrium process with immigration, we derive limit theorems for it (see §3.1).

Now we define infinite Markov particle systems $(X_t, \boldsymbol{P}_\mu)$ with immigration starting from μ associated with absorbing Brownian motion in a half spece H.

Fix an extra point $\Delta\notin H$ and set $H_\Delta=H\cup\{\Delta\}$. Let W^0 be the set of all mappings w^0 from $[0, \infty)$ to H_Δ such that $w^0:[0, \zeta(w^0))\to H$ is continuous and $w^0(t)=\Delta$ if $t\geq\zeta(w^0)$ for a certain constant $\zeta(w^0)>0$. Let $(w^0(t), P_x^0)$ be absorbing Brownian motion starting from x in H with the transition semi-group (P_t^0). We use the same symbol P_x^0 and P_t^0 for the probability and the semi-group, for it is convenient and there is no possibility of confusion. Let W^I be the set of all mappings w^I from $[0, \infty)$ to H_Δ such that $w^I:(\alpha(w^I), \beta(w^I))\to H$ is continuous and $w^I(t)=\Delta$ if $t\notin(\alpha(w^I), \beta(w^I))$ for a certain non empty open interval $(\alpha(w^I), \beta(w^I))$. Set $W=W^0\cup W^I$. For $r>0$, $u>0$, let

$$\nu_r(u)=\frac{u}{\sqrt{2\pi r^3}}\exp\left[-\frac{u^2}{2r}\right]\left(=\frac{1}{2}\frac{\partial}{\partial v}p_r^0(u, v)\Big|_{v=0}\right),$$

where $p_r^0(u, v)$ is the transition density of absorbing Brownian motion on $(0, \infty)$, and

$$\nu_r(dx)=\nu_r(x_d)dx \qquad \text{on } H.$$

Then $\nu_r P_s^0=\nu_{r+s}$ holds for $r>0$, $s\geq 0$, thus $\nu=(\nu_r)_{r>0}$ is an *entrance law for* (P_t^0). Moreover we define the following σ-finite measure on W:

$$Q^0=\lim_{r\downarrow 0}\int_H \nu_r(dx)P_x^0$$

Then Q^0 is supported on $\{\alpha=0\}$ (see [7]) and governs excursions starting uniformly from boundary of H. Q^0 is called the *Brownian excursion law*. Let the following be given:

Ω is the space of σ-finite measures $\omega=\sum_n\delta_{w_n}$, $w_n\in W$,
$X_t(\omega)=\omega(t)|_H$ for $\omega\in\Omega$,
$\mathcal{F}=\sigma(X_s:s<\infty)$, $\mathcal{F}_t=\sigma(X_s:s\leq t)$.

We define a probability measure $\boldsymbol{P}_\mu$ for $\mu\in\mathcal{M}^I$ as follows: For $w=\sum_n\delta_{w_n}\in\Omega$, let ω^0 (resp. ω^I) denote $\sum_{n:w_n\in W^0}\delta_{w_n}$ (resp. $\sum_{n:w_n\in W^I}\delta_{w_n}$), and set $\Omega^0=\{\omega^0\in\Omega:\omega\in\Omega\}$ and $\Omega^I=\{\omega^I\in\Omega:\omega\in\Omega\}$,

$X_t^0(\omega)=\omega^0(t)|_H$: the no immigration part,
$X_t^I(\omega)=\omega^I(t)|_H$: the immigration part.

Hence $X_t=X_t^0+X_t^I$ and we can identify

$$\Omega\equiv\Omega^0\oplus\Omega^I\cong\Omega^0\otimes\Omega^I.$$

So we set

$$\boldsymbol{P}_\mu = \boldsymbol{P}_\mu^0 \otimes \boldsymbol{P}^I,$$

where

$$\boldsymbol{P}_\mu^0 = \bigotimes_{n\geq 1} P_{x_n}^0 \qquad \text{if } \mu = \sum_{n\geq 1} \delta_{x_n},\ x_n \in H$$

and

$$\boldsymbol{P}^I \text{ is the } \int_0^\infty \theta_{-s}(Q^0)ds\text{-Poisson measure,}$$

i.e., the distribution of the Poisson random measure with intensity $\int_0^\infty \theta_{-s}(Q^0)ds$. Then it satisfies that

$$\boldsymbol{E}_\mu[\exp(-\langle X_{t+s}, f\rangle)|\mathcal{F}_s] = \exp\left[-\langle X_s, V_t f\rangle - \int_0^t \langle \nu_r, 1-e^{-f}\rangle dr\right]$$

for positive measurable functions f on H, where

$$V_t f(x) = -\log E_x^0[\exp(-f(w^0(t))] = -\log\{1 - P_t^0(1-e^{-f})(x)\}.$$

Under $\boldsymbol{P}_\mu$, $\{X_t\}$ is an $\mathcal{M}^I$-valued Markov process starting from μ with immigration from boundary. Each particles dies when it reaches boundary of H, and infinitely many particles are born from the boundary. Thus $(X_t, \boldsymbol{P}_\mu)$ is called the *infinite Markov particle system with singular immigration associated with* $(w^0(t), P_x^0, Q^0)$. Note that X_t^I can be also defined as the following:

$$X_t^I = \int_0^t \int_W \delta_{w(t-s)} N^0(dsdw)\bigg|_H,$$

where

$$N^0(\omega\,;\ dsdw) = \#\{n : \alpha(w_n) \in ds,\ w_n(\cdot - \alpha(w_n)) \in dw\} \qquad \text{if } \omega = \sum_{n\geq 1} \delta_{w_n}.$$

Then $N^0(dsdw)$ is a Poisson random measure with intensity $dsQ^0(dw)$. Let $m(dx) = dx$ be the Lebesgue measure on H and Π_m be the m-Poisson measure on $\mathcal{M}^I$, i.e., the distribution of the Poisson random measure with intensity m. Define

$$\boldsymbol{P} = \int \Pi_m(d\mu)\boldsymbol{P}_\mu.$$

Then $(X_t, \boldsymbol{P})$ is a stationary Markov process such that

$$\begin{aligned}\boldsymbol{E}[\exp\{-\langle X_t, f\rangle\}] &= \exp\left[-\langle m, 1-e^{-V_t f}\rangle - \int_0^t \langle \nu_r, 1-e^{-f}\rangle\right] \\ &= \exp[-\langle m, -e^{-f}\rangle]\end{aligned}$$

This process is also defined by using a stationary σ-finite measure Q_m. We extend W to the set of all maps $w : \boldsymbol{R} \rightarrow H_\Delta$ such that there is a nonempty open interval $(\alpha(w), \beta(w))$ on which w is H-valued and continuous, with $w(t) = \Delta$ if $t \leq \alpha(w)$ or $t \geq \beta(w)$, and a constant map $[\Delta]$, i.e., $[\Delta](t) = \Delta$ for all t. We use the same notation W for this set.

$$Q_m=\int_R \theta_{-s}(Q^0)ds.$$

Then $(w(t), Q_m)$ is a sationary Markov process with the same transition probability (P_t^0) as absorbing Brownian motion. Q_m is called the *Kuznetsov measure associated with* $(w^0(t), P_x^0, m)$. If we restrict the time interval to $[0, \infty)$, then $\boldsymbol{P}$ can be also defined as the Q_m-Poisson measure on Ω. $(X_t, \boldsymbol{P})$ is called the *equilibrium process with immigration associated with* $(w^0(t), P_x^0, m)$ or $(w(t), Q_m)$.

In §2 we give a martingale characterization for $(X_t, \boldsymbol{P}_\mu)$ (cf. [8], [11]) and consider Hölder continuity of sample paths.

In §3 we give asymptotic behavior of hitting rates for Brownian excursions and their applications to the equilibrium process with immigration. We first investigate asymptotic behavior of $Q_m(0\leq \sigma_B < t)$ as $t\to\infty$ and limit theorems for the equilibrium process with immigration associated with $(w(t), Q_m)$. We also consider some asymptotic behaviors corresponding to P_m^0 and Q^0 (See §3.3). Moreover since the excursion law Q^0 governs excursions starting uniformly from the boundary of H, we also discuss the non-uniform case and give some results (see §3.4).

A notion of Kuznetsov measures is introduced by S.E. Kuznetsov [12] and recently studied by many authors in relation with capacity theory, e.g., Fitzsimmons and Maisonneuve [2], Getoor [5], and Getoor Steffens [7]. They treat the Kuznetsuv measures in more general situation, thus we give definitions of infinite Markov particle systems with immigration and equilibrium processes with immigration in the general situation (see §2.2). We can also consider the same probrem as Spitzer for the general Kuznetsov measure and give the first term of the asymptotic for it (see §3.2).

2. Martingale Problems and Hölder Continuity

In this section we consider the infinite Markov particle system $(X_t, \boldsymbol{P}_\mu)$ with immigration starting at a counting measure μ, associated with absorbing Brownian motion $(w^0(t), P_x^0)$ in the half space H and the Lebesgue measure $m(dx)=dx$ on H. We give martingale caracterization of $(X_t, \boldsymbol{P}_\mu)$ and investigate the Hölder continuity of sample paths.

2.1. Results

First we define some spaces of functions and measures. Fix $p>d$. Set

$$g_p(x)=(1+|x|^2)^{-p/2} \text{ and } g_{p,0}(x)=g_p(x)h_0(x_d),$$

where $h_0(v)$ is a fixed C^∞-function in $v>0$ satisfying the following properties:

(a) h_0 is non-decreasing and $0<h_0\leq 1$,

(b) $h_0(v)=v$ for $v\in(0, 1/2]$ and $h_0=1$ on $[2, \infty)$.

Then $g_{p,0}(x)$ satisfies that for $x=(\tilde{x}, x_d)\in \boldsymbol{R}^{d-1}\times(0, \infty)$, $g_{p,0}(\tilde{x}, 0+)=\partial_d^2 g_{p,0}(\tilde{x}, 0+)=0$ and $\partial_d g_{p,0}(\tilde{x}, 0+)=g_p(\tilde{x}, 0+)$, where $\partial_d=\partial/\partial x_d$.

Let $C=C(H)$ be the space of all continuous functions in H.

$f\in C_p=C_p(H) \rightleftarrows f\in C$, $\|f/g_p\|_\infty<\infty$.
$f\in C_{p,0}=C_{p,0}(H) \rightleftarrows f\in C$, $\|f/g_{p,0}\|_\infty<\infty$.

The following function space D_p is stable under P_t^0 (see §2.3) :

$f\in D_p=D_p(H) \rightleftarrows f\in C^2$, f, $\partial_d^2 f\in C_{p,0}$ and other partial derivatives are in C_p.

We denote $C_c=C_c(H)$ the space of all continuous functions with compact support and $C_c^\infty=C_c^\infty(H)$ the space of all C^∞-functions with compact support.

Let $\mathcal{M}^I=\mathcal{M}^I(H)$ be the space of all counting measures on H.

$\mu\in\mathcal{M}_p^I=\mathcal{M}_p^I(H) \rightleftarrows \mu\in\mathcal{M}^I$, $\langle\mu, g_p\rangle<\infty$.
$\mu\in\overline{\mathcal{M}}_p^I=\overline{\mathcal{M}}_p^I(H) \rightleftarrows \mu\in\mathcal{M}^I$, $\langle\mu, g_{p,0}\rangle<\infty$.

Then $\mathcal{M}_p^I\subset\overline{\mathcal{M}}_p^I\subset\mathcal{M}^I$. The topology of $\overline{\mathcal{M}}_p^I$ is defined by the weak topology with respect to $C_{p,0}$, i.e.,

$$\mu_n\to\mu \text{ in } \overline{\mathcal{M}}_p^I \rightleftarrows \langle\mu_n, f\rangle\to\langle\mu, f\rangle \text{ for all } f\in C_{p,0} \text{ as } n\to\infty.$$

This is equivalent to that $\langle\mu_n, f\rangle\to\langle\mu, f\rangle$ for all $f\in C_c$ and $f=g_{p,0}$ as $n\to\infty$. Then $\overline{\mathcal{M}}_p^I$ is Polish, i.e., metrizable, complete and separable.

Now fix $p>d$. Set

$$\begin{aligned}\boldsymbol{D}_p=\{F(\mu)=\Phi(\langle\mu, f_1\rangle, \cdots, \langle\mu, f_n\rangle) : \Phi\in C^2(\boldsymbol{R}^n),\ |\Phi^{(i)}(x)|\le C(1+|x|)^k,\\ i=0, 1, 2 \text{ for some } C>0,\ k\ge0, \text{ and } f_j\in D_p,\ j=1, 2, \cdots, n\},\end{aligned}$$

and for $F(\mu)=\Phi(\langle\mu, f_1\rangle, \cdots, \langle\mu, f_n\rangle)\in\boldsymbol{D}_p$, $\mu\in\mathcal{M}_p^I$, we define an operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}F(\mu)=&\sum_{i=1}^n\partial_i\Phi(\langle\mu, f_1\rangle, \cdots, \langle\mu, f_n\rangle)\left\{\langle\mu, Af_i\rangle+\frac{1}{2}\langle\tilde{m}, \partial_d f_i(\cdot, 0+)\rangle\right\}\\ &+\frac{1}{2}\sum_{i,j=1}^n\partial_{ij}^2\Phi(\langle\mu, f_1\rangle, \cdots, \langle\mu, f_n\rangle)\langle\mu, \Gamma(f_i, f_j)\rangle,\end{aligned}$$

where $\partial_i=\partial/\partial x_i$, $\tilde{m}=d\tilde{x}$ is the $(d-1)$-dimensional Lebesgue measure, A is the generator of (P_t^0) and $\Gamma(f, g)=Afg-fAg-gAf$, i.e., $Af=\frac{1}{2}\Delta f$ and $\Gamma(f, g)=\nabla f\cdot\nabla g$ for f, $g\in D_p$.

Moreover set $\Omega_p=C([0, \infty)\to\overline{\mathcal{M}}_p^I)$, which is a Polish space.

Theorem 1. *Fix $p>d$ and $\mu\in\overline{\mathcal{M}}_p^I$. Then under $\boldsymbol{P}_\mu$, $\{X_t\}$ is an $\overline{\mathcal{M}}_p^I$-valued*

Markov process having a continuous sample paths relative to the topology in $\overline{\mathcal{M}}_p^I$ *and satisfies that*

(i) $\boldsymbol{P}_\mu(X_0=\mu)=1$,

(ii) *for each* $F\in\boldsymbol{D}_p$, $t>0$ *and* $k\geq 1$, $F(X_t)$, $\int_0^t|\mathcal{L}F(X_s)|ds\in L^k(\boldsymbol{P}_\mu)$,

(iii) *for each* $F\in\boldsymbol{D}_p$, $M_t^F\equiv F(X_t)-F(X_0)-\int_0^t\mathcal{L}F(X_s)ds$ *is a* $\boldsymbol{P}_\mu$-*martingale. Moreover if a probability measure* $\widetilde{\boldsymbol{P}}_\mu$ *on* Ω_p *satisfies* (i) *and if* M_t^F *is a local* $\widetilde{\boldsymbol{P}}_\mu$-*martingale for any* $F(\mu)=\langle\mu, f\rangle$ *and* $\langle\mu, f\rangle^2$ *with* $f\in D_p$, *then* $\widetilde{\boldsymbol{P}}_\mu=\boldsymbol{P}_\mu$ *on* Ω_p.

REMARK 1. (i) When $\mu\in\overline{\mathcal{M}}_p^I-\mathcal{M}_p^I$, $\mathcal{L}F(\mu)$ may not be well-defined because of $\langle\mu, \Gamma(f_i, f_i)\rangle$ being possibly infinite. But for all $t>0$, $\int_0^t\mathcal{L}F(X_s)ds$ is well-defined $\boldsymbol{P}_\mu$-a.s..

(ii) We can see that $\langle X_t, g_p\rangle$, $\int_0^t\langle X_s, g_p\rangle ds\in L^1(\boldsymbol{P}_\mu)$ for all $t>0$ (see §2.3). Hence

$$\boldsymbol{P}_\mu(\langle X_t, g_p\rangle<\infty \text{ for countable numbers of } t)=1$$

and

$$\boldsymbol{P}_\mu(\langle X_t, g_p\rangle<\infty \text{ for } dt\text{-a.a.}t)=1.$$

However we can show that for any time interval (a, b); $0\leq a<b$,

$$\boldsymbol{P}_\mu(\langle X_t, g_p\rangle<\infty \text{ for some } t\in(a, b))=1.$$

We shall prove this result in §2.4.

Theorem 2. *Fix* $p>d$. *Set* $\Gamma f=\Gamma(f, f)=|\nabla f|^2$ *for* $f\in D_p$.

(i) *If* $\mu\in\overline{\mathcal{M}}_p^I$, *then* X_t *is locally* $(1/4-\epsilon)$-*Hölder continuous in* $t\in[0, \infty)$ *for all* $0<\epsilon<1/4$ *under* $\boldsymbol{P}_\mu$, *i.e., for each* $\gamma=1/4-\epsilon$, $f\in D_p$ *and* $T>0$, *there is a constant* $C=C(\mu, f, T, \gamma)>0$ *such that*

$$\boldsymbol{P}_\mu\left(\omega:\sup_{0<|t-s|\leq\eta(\omega),\, s,\, t\in[0,\, T]}\frac{|\langle X_t(\omega)-X_s(\omega), f\rangle|}{|t-s|^\gamma}\leq C\right)=1,$$

where $\eta(\omega)$ *is an a.s. positive random variable.*

(ii) *If* $\mu\in\mathcal{M}_p^I$, *then* X_t *is locally* $(1/2-\epsilon)$-*Hölder continuous in* $t\in[0, \infty)$ *for all* $0<\epsilon<1/2$ *under* $\boldsymbol{P}_\mu$.

(iii) *Even if* $\mu\in\overline{\mathcal{M}}_p^I-\mathcal{M}_p^I$, X_t *is locally* $(1/2-\epsilon)$-*Hölder continuous in* $t\in(0, \infty)$ *for all* $0<\epsilon<1/2$ *under* $\boldsymbol{P}_\mu$.

(iv) *If* $\mu\in\overline{\mathcal{M}}_p^I$, *then for* $f\in D_p$ *such that* $\langle\mu, \Gamma f\rangle>0$, $\langle X_t, f\rangle$ *is not* $1/2$-*Hölder continuous at* $t=0$. *Furthermore if* $f\in D_p$ *such that* $K_f\equiv\{\Gamma f=0\}$ *is a compact subset in* H, *and if* $\mu\in\overline{\mathcal{M}}_p^I$ *satisfies one of the following conditions*:

$$\begin{cases} d=1 \Rightarrow \mu(\{x \geq a\})=\infty & \textit{for some } a>0, \\ d \geq 2 \Rightarrow \langle \mu, \Gamma f \rangle > 0, & \textit{i.e.}, \mu(K_f^c) \geq 1, \end{cases}$$

then $\langle X_t, f \rangle$, $t \geq 0$, *is nowhere* $1/2$*-Hölder continuous under* $\boldsymbol{P}_\mu$, *i.e.*,

$$\{\omega : |\langle X_{t+h}(\omega) - X_t(\omega), f \rangle| \leq C(\omega)\sqrt{h} \quad \textit{for some } t \in [0, \infty) \textit{ and for all } h \in [0, u(\omega)]\}$$

is a $\boldsymbol{P}_\mu$*-null set, where* $C(\omega)$ *and* $u(\omega)$ *are a.s. positive random variables.*

When $\mu \in \overline{\mathcal{M}}_p^I - \mathcal{M}_p^I$, it seems to be difficult to completely determine the exponent of Hölder continuity at $t=0$. However we can give some sufficient conditions for them. Set $S_a = \{|x_i| \leq a,\ i=1, \cdots, d\} \cap H$.

Theorem 3. *Let* $\mu \in \overline{\mathcal{M}}_p^I - \mathcal{M}_p^I$.

(i) *Suppose that for some* $T>0$, $1 \leq \theta < 2$,

$$\langle \mu|_{S_1}, P_t^0 1 \rangle \asymp \int_{S_1} \frac{dx}{x_d^\theta} P_t^0 1(x) \qquad \textit{for all } 0 \leq t \leq T, \tag{2.1}$$

where $g(t) \asymp h(t)$, $0 < t \leq T$ *means* $Ch(t) \leq g(t) \leq C'h(t)$, $0 < t \leq T$ *for some positive constants* C, C' *depending only on* T, *then* X_t *is* $((3-\theta)/4 - \epsilon)$*-Hölder continuous at* $t=0$ *for all* $0 < \epsilon < (3-\theta)/4$, *moreover for each* $f \in D_p$ *such that* $\Gamma f(\tilde{0}, 0+) > 0$, $\langle X_t, f \rangle$ *is not* $(3-\theta)/4$*-Hölder continuous at* $t=0$ *under* $\boldsymbol{P}_\mu$.

(ii) *Suppose that for some* $T>0$ *and* $1 < \theta < 2$, $\eta > 0$ *or* $\theta = 2$, $\eta > 1$,

$$\langle \mu|_{S_{1/e}}, P_t^0 1 \rangle \asymp \int_{S_{1/e}} \frac{dx}{x_d^\theta (\log 1/x_d)^\eta} P_t^0 1(x) \qquad \textit{for all } 0 \leq t \leq T, \tag{2.2}$$

then X_t *is* $(3-\theta)/4$*-Hölder continuous at* $t=0$, *moreover for each* $f \in D_p$ *such that* $\Gamma f(\tilde{0}, 0+) > 0$, $\langle X_t, f \rangle$ *is not* $((3-\theta)/4 + \epsilon)$*-Hölder continuous at* $t=0$ *for all* $\epsilon > 0$ *under* $\boldsymbol{P}_\mu$.

REMARK 2. In the later half of (ii) we can show the following:

$$\limsup_{t \downarrow 0} \frac{|\langle X_t, f \rangle|}{t^{(3-\theta)/4} (\log 1/t)^{-\eta/2}} = \infty \qquad \boldsymbol{P}_\mu\text{-a.s. if } 1 < \theta < 2,\ \eta > 0,$$

$$\limsup_{t \downarrow 0} \frac{|\langle X_t, f \rangle|}{t^{1/4} (\log 1/t)^{-(\eta-1)/2}} = \infty \qquad \boldsymbol{P}_\mu\text{-a.s. if } \theta = 2,\ \eta > 1.$$

Here we give some examples of $\mu = \sum_{n \geq 2} \delta_{x_n} \in \overline{\mathcal{M}}_p^I - \mathcal{M}_p^I$ which satisfy the conditions (2.1), (2.2): For simplicity let $d=1$.

(a) $x_n = r^n$ $(0 < r < 1) \Rightarrow \theta = 1$ in (2.1),

(b) $x_n = n^{-p}$ $(p > 1) \Rightarrow \theta = 1 + 1/p$ in (2.1),

(c) $x_n = n^{-p} (\log n)^{-q}$ $(p > 1,\ q > 0$ or $p = 2 < q) \Rightarrow \theta = 1 + 1/p$, $\eta = q/p$ in (2.2).

Now we can datermine the exponent of Hölder continuity of sample paths of the equilibrium process associated with absorbing Brownian motion in H, since under $\boldsymbol{P}$, X_t has an innitial distribution $\Pi_m(d\mu)$ and $\langle m, g_p\rangle<\infty$ for all $p>d$. Set $D_{d+}=\bigcup_{p>d}D_p$, $\overline{\mathcal{M}}^I_{d+}=\bigcap_{p>d}\overline{\mathcal{M}}^I_p$ and $\Omega_{d+}=\bigcap_{p>d}\Omega_p=C([0, \infty)\to\overline{\mathcal{M}}^I_{d+})$. Then we have the following:

Corollary 1. *$(X_t, \boldsymbol{P})$ is an $\overline{\mathcal{M}}^I_{d+}$-valued stationary Markov process having continuous sample paths relative to the topology in $\overline{\mathcal{M}}^I_{d+}$ with initial distribution $\Pi_m(\mu)$. Furthermore X_t, $t\geq 0$ is locally $(1/2-\epsilon)$-Höder continuous for all $0<\epsilon<1/2$, and for each $f\in D_{d+}$ such that $K_f=\{\Gamma f=0\}$ is compact in H, $\langle X_t, f\rangle$, $t\geq 0$ is nowhere $1/2$-Hölder continuous.*

2.2. Kuznetsov Measures and Infinite Markov Particle Systems with Immigration

In this subsection we give a general theory of Kuznetsov measures according to [2], [5] and [7], and define the infinite Markov particle systems with immigration induced by Kuznetsov measures. We also discuss the Markov property of equilibrium processes with immigration.

Let S be a Lusin space, i.e., a Borel measurable subset of a separable metric space, and fix a point $\Delta\notin S$. Set $S_\Delta=S\cup\{\Delta\}$. Let $(W^0, \mathcal{F}^0, \mathcal{F}^0_t, w^0(t), P^0_x)$, $t\geq 0$, $x\in S$, be a Borel right process with transition semi-group $(P^0_t)_{t\geq 0}$. That is, $(P^0_t)_{t\geq 0}$ is a Borel right semi-group, W^0 is the set of all right-continuous paths w^0: $[0, \infty)\to S_\Delta$, $\theta_t w^0(s)=w^0(t+s)$, $\mathcal{F}^0=\sigma(w^0(t): t\geq 0)$, $\mathcal{F}^0_t=\sigma(w^0(s): 0\leq s\leq t)$ and $\zeta(w^0)=\inf\{t: w^0(t)=\Delta\}$ is the lifetime of w^0. Set $P^0_\Delta=\delta_{[\Delta]}$.

Let W be the of all maps $w: \boldsymbol{R}\to S_\Delta$ such that there is a nonempty open interval $(\alpha(w), \beta(w))$ on which w is S-valued and right-continuous, with $w(t)=\Delta$ if $t\leq\alpha(w)$ or $t\geq\beta(w)$, and a constant map $[\Delta]$, i.e., $[\Delta](t)=\Delta$ for all t. Set $\mathcal{G}^0=\sigma(w(t): t\in\boldsymbol{R})$ and $\mathcal{G}^0_t=\sigma(w(s): -\infty<s\leq t)$. Moreover let γ_t, $t\in\boldsymbol{R}$, be mappings from W to W^0 defined by

$$\gamma_t w(s)=\begin{cases} w(t+s) & \text{for } s\geq 0 \text{ if } t>\alpha(w), \\ \Delta & \text{for } s\geq 0 \text{ if } t\leq\alpha(w). \end{cases}$$

Then the following hold: $\gamma_t=\gamma_0\circ\theta_t$ for $t\in\boldsymbol{R}$. If $\alpha<t<\beta$, then $\zeta\circ\gamma_t=\beta\circ\theta_t$. Moreover γ_t is $\mathcal{G}^0_{s+t}|\mathcal{F}^0_s$-measurable for $s\geq 0$, $t\in\boldsymbol{R}$.

A family of σ-finite measures $\xi=(\xi_t)_{t\in\boldsymbol{R}}$ on S is called an *entrance rule* if $\xi_s P^0_{t-s}\leq\xi_t$ for $s<t$ and $\xi_s P^0_{t-s}\uparrow\xi_t$ as $s\uparrow t$. Moreover $\xi=(\xi_t)_{t\in\boldsymbol{R}}$ is called an *entrance law* if $\xi=(\xi_t)_{t\in\boldsymbol{R}}$ is an entrance rule and satisfies that $\xi_s P^0_{t-s}=\xi_t$ for $s<t$.

The following result by Kuznestov [12] is well-known (see [2], [5] and [7]): For an entrance rule $\xi=(\xi_t)$ there exists a unique σ-finite measure $Q_\xi(dw)$ on W not charging $[\Delta]$ such that for $-\infty<t_1<t_2<\cdots<t_n<+\infty$

$$Q_\xi(\alpha<t_1,\ w(t_1)\in dy_1,\ w(t_2)\in dy_2,\ \cdots,\ w(t_n)\in dy_n,\ t_n<\beta)$$
$$=\xi_{t_1}(dy_1)P^0_{t_2-t_1}(y_1,\ dy_2)\cdots P^0_{t_n-t_{n-1}}(y_{n-1},\ dy_n)$$

In particular $(w(t),\ \mathcal{G}^0_t,\ Q_\xi)$ is Markov in the sense that if $t\in\boldsymbol{R}$ and $V\in\mathcal{F}^0$, then

$$Q_\xi(V\circ\gamma_t|\mathcal{G}^0_t)=P^0_{w(t)}(V) \qquad \text{on } \{\alpha<t\}.$$

DEFINITION 1. Q_ξ is called the *Kuznetsov measure associated with* $(w^0(t),\ P^0_x,\ \xi)$.

If $\xi_t=m$ a σ-finite measure on S for all $t\in\boldsymbol{R}$ then we write Q_m for Q_ξ, and then Q_m is stationary, i.e., $Q_m[f(w(t))]=m(f)$. We denote the class of excessive (resp. invariant, purely excessive) measures by $\boldsymbol{Exc}$ (resp. $\boldsymbol{Inv}$, $\boldsymbol{Pur}$). That is, for a σ-finite measure m

$m\in\boldsymbol{Exc}$ if $mP^0_t\leq m$ for all $t\geq 0$,
$m\in\boldsymbol{Inv}$ if $mP^0_t=m$ for all $t\geq 0$,
$m\in\boldsymbol{Pur}$ if $m\in\boldsymbol{Exc}$ and $\lim_{t\to\infty}mP^0_t=0$.

$m\in\boldsymbol{Exc}$ may be deconmposed uniquely as $m=m_i+m_p$, where $m_i\in\boldsymbol{Inv}$ and $m_p\in\boldsymbol{Pur}$. If $m\in\boldsymbol{Pur}$, then there exists a unique entrance law $\nu=(\nu_t)_{t>0}$ such that

$$m=\int_0^\infty \nu_t dt.$$

From now on we fix $m\in\boldsymbol{Exc}$ (may be infinite) and also m_i, m_p, $\nu=(\nu_t)_{t>0}$, which are uniquely determined by m as above. Then there are Kuznetsov measures

$$Q_m,\ Q_{m_i},\ Q_{m_p},\ \text{and } Q^0\equiv Q_\nu \qquad \text{on } W,$$

which satisfy that

$$Q_m=Q_{m_i}+Q_{m_p}$$

and

$$Q_{m_p}=\int_{\boldsymbol{R}}\theta_{-s}(Q^0)ds=\int_{\boldsymbol{R}}\theta_s(Q^0)ds.$$

In [7] it is shown that $Q^0(\alpha\neq 0)=0$.

Now we consider the infinite Markov particle systems induced by Kuznetsov measures. We restrict the time interval to $[0,\ \infty)$ and the following are defined by the same way as in case of absorbing Brownian motion (see §1): $\mu=\sum_{n\geq 1}\delta_{x_n}\in\mathcal{M}^I=\mathcal{M}^I(S)$, $\omega=\sum_n\delta_{w_n}\in\Omega$ $(w_n\in W)$, $X_t(\omega)=\omega(t)|_S$, $\mathcal{F}$, $\mathcal{F}_t$, $\boldsymbol{P}_\mu\equiv\boldsymbol{P}^0_\mu\otimes\boldsymbol{P}^I$, $\Pi_m(d\mu)$ and $\boldsymbol{P}\equiv\int\Pi_m(d\mu)\boldsymbol{P}_\mu$. Then $(X_t,\ \boldsymbol{P}_\mu)$ is an $\mathcal{M}^I$-valued stationary Markov process with a stationary measure Π_m.

DEFINITION 2. (i) $(X_t, \boldsymbol{P}_\mu)$ is called the *infinite Markov particle system associated with* $(w^0(t), P_x^0)$. In particular if $Q^0 \neq 0$, then $(X_t, \boldsymbol{P}_\mu)$ is called the *infinite Markov particle system with immigration associated with* $(w^0(t), P_x^0, Q^0)$.

(ii) $(X_t, \boldsymbol{P})$ is called the *equilibrium process associated with* $(w^0(t), P_x^0, m)$ or $(w(t), Q_m)$. In particular if $m \in \boldsymbol{Exc} \backslash \boldsymbol{Inv}$, then $(X_t, \boldsymbol{P})$ is called the *equilibrium process with immigration associated with* $(w^0(t), P_x^0, m)$ or $(w(t), Q_m)$.

REMARK 3. (i) $\boldsymbol{P}$ is also defined as the Q_m-Poisson measure Π_{Q_m}, i.e, $\boldsymbol{P} = \Pi_{Q_m}$ in the sense of finite-dimensional distributions.

(ii) When $m \in \boldsymbol{Inv}$, we can identify $Q_m = P_m^0 \left(= \int m(dx) P_x^0 \right)$, so the definition of equilibrium processes is the same as in [17] by T. Shiga and Y. Takahashi.

The Markov property of $\boldsymbol{P}_\mu$ and the identification of $\boldsymbol{P} = \Pi_{Q_m}$ can be proved similarly to [17].

2.3. Proofs

Before proceeding the proofs we give two fundamental properties of the transition semi-group (P_t^0) of absorbing Brownian motion in H. Note that P_t^0 is given by $P_t^0 f(x) = \int P_t^0(x, dy) f(y)$ with $P_t^0(x, dy) = p_t^0(x, y) dy$ and

$$p_t^0(x, y) = p_t(x_1, y_1) \cdots p_t(x_{d-1}, y_{d-1}) p_t^0(x_d, y_d),$$

where for $u, v \in \boldsymbol{R}$,

$$p_t(u, v) = \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{(v-u)^2}{2t} \right]$$

and for $u, v > 0$,

$$\begin{aligned} p_t^0(u, v) &= p_t(u, v) - p_t(u, -v) \\ &= \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left[-\frac{(v-u)^2}{2t} \right] - \exp\left[-\frac{(v+u)^2}{2t} \right] \right\} \\ &= \int_{-u}^{u} dz \frac{v+z}{\sqrt{2\pi t^3}} \exp\left[-\frac{(v+z)^2}{2t} \right] \end{aligned}$$

The following first claim is easily obtained from the proof of Proposition 2.3 in [8] : Let $p > d$. If $f \in C_p$, $f \geq 0$ and $\lim_{|x| \to \infty} |x|^p f(x) = l$ exists, then

$$\lim_{|x| \to \infty} \sup_{t \geq 0} |x|^p P_t^0 f(x) = l. \tag{2.3}$$

The second is that if $f \in C^2$ and $f(\tilde{x}, 0+) = 0$, then

$$P_t^0 f(x) = x_d \partial_d (P_t^0 f)(\tilde{x}, 0+) + \frac{1}{2} x_d^2 P_t^0 (\partial_d^2 f)(\tilde{x}, \theta x_d)$$

for some $\theta\in(0, 1)$. This follows from that $P_t^0 f(\cdot, 0+)=0$ if $t>0$ and $f\in L^1(H, dx)$, and that $\partial_a^2(P_t^0 f)=P_t^0(\partial_a^2 f)$ if $f\in C^2$; $f(\cdot, 0+)=0$. Thus we also see that D_p is stable under P_t^0, i.e., $P_t^0(D_p)\subset D_p$.

From these fact we get the following:

$$\sup_{t\geq 0}\|g_p^{-1}P_t^0 g_p\|_\infty<\infty, \tag{2.4}$$

$$\sup_{t\geq 0}\|g_{p,0}^{-1}P_t^0 g_{p,0}\|_\infty<\infty \tag{2.5}$$

and for all $s>0$, $P_s^0 g_p\in C_{p,0}$, hence

$$\sup_{s\leq t<\infty}\|g_{p,0}^{-1}P_t^0 g_p\|_\infty<\infty. \tag{2.6}$$

Proof of Theorem 1.

(i) is trivial. The integrabilities in (ii) are reduced to the following: For each j, $k\geq 0$ and $t>0$,

$$\boldsymbol{E}_\mu[\langle X_t, g_p\rangle^k],\ \boldsymbol{E}_\mu\left[\left(\int_0^t \langle X_u, g_{p,0}\rangle^j \langle X_u, g_p\rangle du\right)^k\right]<\infty. \tag{2.7}$$

Lemma 1. *Let $p>d$. Fix $t>0$, $\epsilon>0$. Then, for each $k\geq 0$,*

$$\sup_{0\leq a\leq\epsilon}\left\|g_{p,0}^{-1}\frac{\partial^k}{\partial a^k}V_t(ag_p)\right\|_\infty<\infty.$$

Proof. This is deduced to (2.6) by computing $(\partial^k/\partial a^k)V_t(ag_p)$. ■

From this lemma it is easy to see that if $\mu\in\bar{\mathscr{M}}_p^I$, then the Laplace transform

$$\begin{aligned}\varphi_t(a\,;\,f)&\equiv\boldsymbol{E}_\mu[\exp(-a\langle X_t, f\rangle)]\\ &=\exp\left[-\langle\mu, V_t(af)\rangle-\int_0^t\langle\nu_r, 1-e^{-af}\rangle dr\right]\end{aligned}$$

is C^∞ at $a=0+$ if $t>0$ and $f\in D_p$. Hence the first claim follows. For instance, we have for f, $g\in D_p$,

$$\begin{aligned}\boldsymbol{E}_\mu\langle X_t, f\rangle&=\langle\mu, P_t^0 f\rangle+\int_0^t\langle\nu_r, f\rangle dr,\\ \boldsymbol{E}_\mu[\langle X_t, f\rangle\langle X_t, g\rangle]&=\boldsymbol{E}_\mu\langle X_t, f\rangle\boldsymbol{E}_\mu\langle X_t, g\rangle\\ &\quad+\langle\mu, P_t^0(fg)-P_t^0 fP_t^0 g\rangle+\int_0^t\langle\nu_r, fg\rangle dr.\end{aligned}$$

More generally we can give the explicit formula of $\boldsymbol{E}_\mu[\langle X_t, f_1\rangle\cdots\langle X_t, f_n\rangle]$. But it's too tedious to describe it. By using the formula one can easily see the following:

Proposition 1. *For all $0\leq t_1\leq\cdots\leq t_n$ and $f_i\in D_p$; $f_i\geq 0$, $i=1, 2, \cdots, n$,*

$$\begin{aligned}\boldsymbol{E}_\mu[\langle X_{t_1}, f_1\rangle\cdots\langle X_{t_n}, f_n\rangle] &\le \prod_{i=1}^{n}\langle \mu, P^0_{t_i} f_i\rangle + C_1^{(n)}\sum_{i=1}^{n}\prod_{j\ne i}\langle \mu, P^0_{t_j} f_j\rangle \\ &+ C_2^{(n)}\sum_{i_1\ne i_2}\prod_{j\ne i_1, i_2}\langle \mu, P^0_{t_j} f_j\rangle + \cdots + C_{n-1}^{(n)}\sum_{j=1}^{n}\langle \mu, P^0_{t_j} f_j\rangle + C_n^{(n)},\end{aligned}$$

where $C_k^{(n)}$, $k=1, \cdots, n$ *are positive constants, independent of* $\{t_i\}$.

Proof. In case of $t_1=\cdots=t_n=t$ it is easily obtained. Then by using the Markov property we can get the general case, of course by induction on n. ■

To show the second claim in (2.7), we need the following lemma:

Lemma 2. *Let* $\mu\in\overline{\mathcal{M}}_p^I$. *For each* $T>0$, *there exists a constant* $C_T>0$ *such that*

$$\langle \mu, P_t^0 g_p\rangle \le C_T/\sqrt{t} \qquad (0<t\le T).$$

Proof. For simplicity we only consider the case of $d=1$. Suppose that $\langle \mu, g_{p,0}\rangle<\infty$. It is clear that for all $\epsilon>0$, $\sup_t\langle \mu, 1_{(\epsilon,\infty)}P_t^0 g_p\rangle<\infty$ by (2.3) and it is easy to see that

$$\sup_{t\ge 0,\ 0<x\le\epsilon}\frac{\sqrt{t}}{x}P_t^0 g_p(x)\le\sqrt{\frac{2}{\pi}}.$$

The general cases $d\ge 2$ can be proved by a similar way. ■

Hence by (2.5) and Proposition 1 we have (2.7). Thus (ii) in Theorem 1 is proved. Moreover it is easily seen that

Lemma 3. *Let* $\mu\in\overline{\mathcal{M}}_p^I$. *Fix* $T>0$. *For each* $k\ge 1$,

$$\boldsymbol{E}_\mu\left[\left(\int_0^t\langle X_u, g_p\rangle du\right)^k\right]\le C_T t^{k/2} \qquad (0\le t\le T).$$

This lemma is used to show the Hölder continuity of X_t later.

REMARK 4. When $\mu\in\overline{\mathcal{M}}_p^I-\mathcal{M}_p^I$, it holds that

$$\boldsymbol{E}_\mu\left[\int_0^t\langle X_u, g_p\rangle^k du\right]\begin{cases}<\infty & \text{if } k=1,\\ =\infty & \text{if } k\ge 2.\end{cases}$$

On the other hand if $\mu\in\mathcal{M}_p^I$, then for each $T>0$ and $k\ge 1$, $\sup_{0\le t\le T}\langle \mu, P_t^0 g_p\rangle$ is finite because of (2.4). Thus $C_T=\sup_{0\le t\le T}\boldsymbol{E}_\mu[\langle X_t, g_p\rangle^k]$ is finite and it holds that

$$\begin{aligned}\boldsymbol{E}_\mu\left[\left(\int_0^t\langle X_u, g_p\rangle du\right)^k\right]&\le t^{k-1}\int_0^t\boldsymbol{E}_\mu[\langle X_u, g_p\rangle^k]du\\ &\le C_T t^k \qquad (0\le t\le T).\end{aligned}$$

Next we show the martingale property (iii) in Theorem 1.

Lemma 4.

(i) $\sup_{t\geq 0}\left\|g_p^{-1}\frac{\partial}{\partial t}V_t f\right\|_\infty<\infty$ *for each* $f\in D_p$,

(ii) $\sup_{0<r\leq T}\langle \nu_r, g_{p,0}\rangle<\infty$ *for each* $T>0$.

Proof. Since for $h=1-e^{-f}$ and $\Gamma f=\Gamma(f, f)=|\nabla f|^2$,

$$\frac{\partial}{\partial t}V_t f=\frac{P_t^0 Ah}{1-P_t^0 h}\left(=\frac{AP_t^0 h}{1-P_t^0 h}=AV_t f-\frac{1}{2}\Gamma V_t f\right)$$

and $\|1-P_t^0 h\|^{-1}\leq \exp[\|f\|_\infty]<\infty$, (i) is deduced to (2.4). (ii) follows from $\sup_{r>0}\int_0^1 y\nu_r(y)dy=1/2$ and $\int_1^\infty y^{-p}\nu_r(y)dy\leq\int_1^\infty \nu_r(y)dy\leq 1/\sqrt{2\pi r}$. ■

Let $f\in D_p$. Since

$$\lim_{t\downarrow 0}\langle \nu_r, 1-e^{-f}\rangle=\frac{1}{2}\langle\tilde{m}, \partial_d f(\cdot, 0+)\rangle,$$

if $\mu\in\mathcal{M}_p^I$, then

$$\begin{aligned}\mathcal{L}e^{-\langle\cdot, f\rangle}(\mu)&=-\left\{\langle\mu, Af-\frac{1}{2}\Gamma f\rangle+\frac{1}{2}\langle\tilde{m}, \partial_d f(\cdot, 0+)\rangle\right\}e^{-\langle\mu, f\rangle}\\&=\lim_{t\downarrow 0}\frac{1}{t}\{\boldsymbol{E}_\mu[e^{-\langle X_t, f\rangle}]-e^{-\langle\mu, f\rangle}\}.\end{aligned}$$

When $\mu\in\overline{\mathcal{M}}_p^I$, $\mathcal{L}e^{-\langle\cdot, f\rangle}(\mu)$ may not be well-defined. However it holds that $\boldsymbol{P}_\mu(X_u\in\mathcal{M}_p^I$ for du-a.a. $u>0)=1$ as mentioned in Remark 1. Hence by Lemma 4, nothing that

$$\lim_{r\downarrow 0}\langle \nu_r, f^n\rangle=\begin{cases}\frac{1}{2}\langle\tilde{m}, \partial_d f(\cdot, 0+)\rangle & (n=1),\\ 0 & (n\geq 2),\end{cases}$$

if $\mu\in\overline{\mathcal{M}}_p^I$ and $F\in\boldsymbol{D}_p$, then we can get

$$\frac{\partial}{\partial u}\boldsymbol{E}_\mu[F(X_u)|X_s]=\boldsymbol{E}_\mu[\mathcal{L}F(X_u)|X_s]\qquad\text{for } du\text{-a.a.}u>s,\qquad \boldsymbol{P}_\mu\text{-a.s.}$$

Therefore from this equation and by using the Markov property we can see that for each $F\in\boldsymbol{D}_p$ and $0\leq s<t$,

$$\boldsymbol{E}_\mu\left[F(X_t)-F(X_s)-\int_s^t\mathcal{L}F(X_u)du\middle|\mathcal{F}_s\right]=0.$$

That is, (iii) in Theorem 1 holds. In particular, we have

Lemma 5. *For each $f \in D_p$,*

$$M_t(f) = \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \mathscr{L}\langle \cdot, f\rangle(X_u)du$$
$$= \langle X_t, f\rangle - \langle X_0, f\rangle - \int_0^t \langle X_u, Af\rangle du - \frac{t}{2}\langle \tilde{m}, \partial_d f(\cdot, 0+)\rangle$$

is a square integrable $\boldsymbol{P}_\mu$-martingale with quadratic variation

$$\langle\!\langle M(f)\rangle\!\rangle_t = \int_0^t \langle X_u, \Gamma f\rangle du.$$

Now we prove that X_t is continuous relative to the topology in $\overline{\mathscr{M}}_p^I$. We use an approximation by finite particle systems. Set $H_n = \{x \in H : |x| \leq n, x_d \geq 1/n\}$ for $n \geq 1$. Then $\mu(H_n) < \infty$ whenever $\mu \in \overline{\mathscr{M}}_p^I$. Fix $T > 0$ and for each $n \geq 1$, we set

$$W_n = \{w : 0 \leq \alpha(w) < T,\ \beta(w) - \alpha(w) \geq 1/n,\ |w(\alpha(w)+)| \leq n\},$$
$$Q_m^{(n)} = Q_m|_{W_n} \quad \text{and} \quad Q_n^0 = Q^0(\cdot\ ;\ \beta > 1/n)$$

Then $Q_m^{(n)} = \int_0^T \theta_{-s}(Q_n^0)ds$ and $Q_m^{(n)}(W) = T\nu_{1/n}(\{|x| \leq n\}) < \infty$ (note that $Q^0(\alpha \neq 0) = 0$). Now for each $n \geq 1$, we define

$$X^{(n)}(\omega) = \sum_{k\,:\,w_k(0) \in H_n} \delta_{w_k} + \sum_{k\,:\,w_k \in W_n} \delta_{w_k} \qquad \text{if } \omega = \textstyle\sum_k \delta_{w_k}.$$

Then under $\boldsymbol{P}_\mu$, $X_t^{(n)}$ is a finite Markov particle system, that is, $\boldsymbol{P}_\mu(X^{(n)}(W) < \infty) = 1$ and

$$\boldsymbol{E}_\mu[\exp(-\langle X_{t+s}, f\rangle)|\mathscr{F}_s] = \exp\Big[-\langle X_s^{(n)}, V_t f\rangle - \int_0^t \langle Q_n^0, 1 - e^{-f_r}\rangle dr\Big].$$

Clearly $X_t^{(n)}$ is continuous relative to the topology in $\overline{\mathscr{M}}_p^I$ and $X_t^{(n)}(\omega)$ converges to $X_t(\omega)$ as $n \to \infty$ for every ω and $t \geq 0$. Fix $f \in D_p$ and let $M_t^{(n)} = M_t^{(n)}(f)$, $M_t = M_t(f)$ be the martingale part of $X_t^{(n)}$, X_t respectively defined as in Lemma 5. By Doob's maximal inequality and Lebesgue's dominated convergence theorem we see that

$$\boldsymbol{E}_\mu[\sup_{t \leq T}|M_t^{(n)} - M_t^{(m)}|^2] \leq 4\boldsymbol{E}_\mu[|M_T^{(n)} - M_T^{(m)}|^2]$$
$$\to 0 \quad (n, m \to \infty).$$

Hence there is a continuous process $\tilde{M}_t = \tilde{M}_t(f)$ such that $\lim_{k\to\infty}\sup_{t \leq T}|M_t^{(n_k)} - \tilde{M}_t| = 0$, $\boldsymbol{P}_\mu$-a.s. for some suitable subsequence $\{M_t^{(n_k)}\}_{k=1}^\infty$. This implies $\boldsymbol{P}_\mu(M_t = \tilde{M}_t$ for all $t \leq T) = 1$, that is, M_t is continuous. This implies the continuity of X_t in $\overline{\mathscr{M}}_p^I$.

Finally we prove the uniqueness of $\boldsymbol{P}_\mu$ on Ω_p for the martingale problem.

Lemma 6. *If $f \in C_c^\infty$, then $V_t f \in C^\infty \cap D_p$ for every $t \geq 0$, $p > d$, and the following hold: For each $t \geq 0$,*

(i) $\dfrac{\partial}{\partial t} V_t f$ *is continuous in t with respect to the norm* $\|\cdot/g_{p,0}\|_\infty$,

(ii) *$\Gamma V_t f$ is continuous in t with respect to the norm $\|\cdot/g_p\|_\infty$.*

Proof. Set $h=1-e^{-f}\in C_c^\infty$. Since $Ah\in C_c^\infty$,

$$\frac{\partial}{\partial t}V_t f=\frac{P_t^0 Ah}{1-P_t^0 h} \quad \text{and} \quad \|1-P_t^0 h\|_\infty\le e^{\|f\|_\infty}<\infty,$$

(i) is reduced to the continuity of $P_t^0 f$ for $f\in C_c^\infty$, i.e., $\|(P_t^0 f-f)/g_{p,0}\|_\infty\to 0$ $(t\to 0)$. To prove this it suffices to show that, for each $f\in C_c$,

$$\sup_{t\ge 0}\|g_{p,0}^{-1}P_t^0 f\|_\infty<\infty,$$

however it is not difficult. (ii) is reduced to the following: For each $f\in C_c^\infty$,

$$\|(P_t^0 f-P_{t_0}^0 f)/g_{p,0}\|_\infty,\ \|(\partial_i P_t^0 f-\partial_i P_{t_0}^0 f)/g_{p,0}\|_\infty\to 0 \quad i=1,2,\cdots,d\ (t\to t_0).$$

The second follows from the continuity of the transition semi-group $(P_t)_{t\ge 0}$ of Brownian motion in $\boldsymbol{R}^d$. Because if we extend f to on $\boldsymbol{R}^d$ by setting $f=0$ on H^c, then

$$\partial_i P_t^0 f(x)=\begin{cases} P_t^0(\partial_i f)(x) & \text{for } i\neq d,\\ P_t(\partial_d f)(x)+P_t(\partial_d \bar{f})(x) & \text{for } i=d,\end{cases}$$

where $\bar{f}(x)=f(\tilde{x},-x_d)$. One can more easily see that $\|(P_t f-f)/g_p\|_\infty\to 0$ $(t\to 0)$ for each $f\in C_c(\boldsymbol{R}^d)$. ∎

Let $\tilde{\boldsymbol{P}}_\mu$ be a probability measure on Ω_p, under which $X_0=\mu$ a.s. and $F(X_t)-F(X_0)-\int_0^t \mathcal{L}F(X_s)ds$ is a local martingale for $F(\mu)=\langle X_t,f\rangle$, $\langle X_t,f\rangle^2$ ($f\in D_p$). We first assume that $\langle X_t,g_{p,0}\rangle$, $\int_0^t\langle X_s,g_p\rangle ds\in L^2(\tilde{\boldsymbol{P}}_\mu)$ for each $t>0$ and that the above local martingale is a martingale. Then by using Lemma 6 carefully and nothing $\langle \nu_{T-s},1-e^{-f}\rangle=\langle \nu_r,V_{T-s}f\rangle_{r=0}$, we see that for each $f\in C_c^\infty$ and $T>0$,

$$\langle X_t,V_{T-t}f\rangle-\langle X_0,V_T f\rangle-\frac{1}{2}\int_0^t\langle X_s,\Gamma V_{T-s}f\rangle ds-\int_0^t\langle \nu_{T-s},1-e^{-f}\rangle ds$$

is a square integrable continuous $\tilde{\boldsymbol{P}}_\mu$-martingale with quadratic variation

$$\int_0^t\langle X_s,\Gamma V_{T-s}f\rangle ds$$

in $0\le t\le T$. Hence by using Ito's formula we see that

$$\exp\left[-\langle X_t,V_{T-t}f\rangle+\int_0^t\langle \nu_{T-r},1-e^{-f}\rangle dr\right]$$

is a $\tilde{\boldsymbol{P}}_\mu$-martingale. Thus by taking $T=t$ we have

$$\tilde{\boldsymbol{E}}_\mu[\exp(-\langle X_t,f\rangle)|\mathcal{F}_s]=\exp\left[-\langle X_s,V_{t-s}f\rangle-\int_0^{t-s}\langle \nu_u,1-e^{-f}\rangle du\right].$$

Therefore $\tilde{\boldsymbol{P}}_\mu=\boldsymbol{P}_\mu$ in the sense of finite-dimensional distributions, hence it holds on Ω_p, because Ω_p is Polish. In case of a local martingale by using a localization method we can also show $\tilde{\boldsymbol{P}}_\mu=\boldsymbol{P}_\mu$ on Ω_p.

Proof of Theorem 2.

We first show that X_t, $t\geq 0$ is locally γ-Hölder continuous under $\boldsymbol{P}_\mu$ in each cases of (i) $\mu\in\overline{\mathcal{M}}_p^I$, $\gamma\in(0, 1/4)$ and (ii) $\mu\in\mathcal{M}_p^I$, $\gamma\in(0, 1/2)$. We apply the following lemma which can be easily proved (cf. Theorem 2.2.8 in [9]):

Lemma 7. *Fix $T>0$. Suppose that for each $f\in D_p$, there are some constants a, b, $C>0$ such that*

$$\boldsymbol{E}_\mu[|\langle X_t-X_s, f\rangle|^a]\leq C(t-s)^{1+b} \qquad \text{for all } 0\leq s<t\leq T.$$

Then X_t, $0\leq t\leq T$ is locally $(b/a-\epsilon)$-Hölder continuous for every $\epsilon\in(0, b/a)$ in the sense of (i) *in Theorem* 2.

Now fix $\mu\in\overline{\mathcal{M}}_p^I$. Then for each $k\geq 1$, $T>0$ and $f\in D_p$ there is a constant $C_T=C_T(f)>0$ such that

$$\boldsymbol{E}_\mu[\langle X_t-X_s, f\rangle^{2k}]\leq C_T(t-s)^{k/2} \tag{2.8}$$

for all $0\leq s<t\leq T$. In fact, for any fixed $f\in D_p$ and $s\geq 0$, $N_t^s(f)=M_{t\vee s}(f)-M_s(f)$ is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation $\langle\!\langle N^s(f)\rangle\!\rangle_t=\langle\!\langle M(f)\rangle\!\rangle_{t\vee s}-\langle\!\langle M(f)\rangle\!\rangle_s$. Hence by the martingale moment inequality and Lemma 3 we have

$$\begin{aligned}\boldsymbol{E}_\mu[\{M_t(f)-M_s(f)\}^{2k}]&\leq C'_T\boldsymbol{E}_\mu[\{\langle\!\langle M(f)\rangle\!\rangle_t-\langle\!\langle M(f)\rangle\!\rangle_s\}^k]\\&=C'_T\boldsymbol{E}_\mu\left[\left(\int_s^t\langle X_u, \Gamma f\rangle du\right)^k\right]\\&\leq C_T(t-s)^{k/2} \qquad \text{for } 0\leq s<t\leq T.\end{aligned}$$

Furthermore by $|Af|\leq Cg_{p,0}$ and Remark 4 we have

$$\boldsymbol{E}_\mu\left[\left(\int_s^t\langle X_u, |Af|\rangle du\right)^{2k}\right]\leq C_T(t-s)^{2k} \qquad \text{for } 0\leq s<t\leq T.$$

Therefore inequality (2.8) holds. Moreover if $\mu\in\mathcal{M}_p^I$, then it can be seen that

$$\boldsymbol{E}_\mu[\langle X_t-X_s, f\rangle^{2k}]\leq C_T(t-s)^k$$

for all $0\leq s<t\leq T$.

Next we show (iii). Let $\mu\in\overline{\mathcal{M}}_p^I-\mathcal{M}_p^I$. Since for each $r>0$, $\boldsymbol{P}_\mu(X_r\in\mathcal{M}_p^I)=1$, it holds that, by (i) in Theorem 2, X_t, $t\geq 0$ is locally $(1/2-\epsilon)$-Hölder continuous for every $\epsilon\in(0, 1/2)$ under $\boldsymbol{P}_{X_r(\omega)}$ for $\boldsymbol{P}_\mu$-a.a.ω. Thus letting $r\downarrow 0$, the claim follows.

Finally we prove (iv). It suffices to consider the martingale part $M_t(f)=\langle X_t,$

$f\rangle-\langle X_0, f\rangle-\int_0^t \mathcal{L}\langle\cdot, f\rangle(X_s)ds$. First let $\mu\in\bar{\mathcal{M}}_p^I$ and fix $f\in D_p$ such that $\langle\mu, \Gamma f\rangle >0$. We also fix $f_0\in C_{p,0}$ such that $0\le f_0\le\Gamma f$ and $\langle\mu, f_0\rangle>0$. By the continuity of $\langle X_t, f_0\rangle$ there exist a.s. positive random variables C, u such that $\langle\!\langle M(f)\rangle\!\rangle_t\ge \int_0^t\langle X_s, f_0\rangle ds\ge Ct$ for all $0\le t<u$, $\boldsymbol{P}_\mu$-a.s. By the time change for martingale, $M_t(f)=B_{\langle\!\langle M(f)\rangle\!\rangle_t}$ holds a.s., where B_t is a standard one-dimensional Brownian motion. Since B_t, $t\ge 0$ is nowhere 1/2-Hölder continuous, $\langle X_t, f\rangle$ is not 1/2-Hölder continuous at $t=0$. Next fix $f\in D_p$ such that $K=K_f\equiv\{\Gamma f=0\}$ is compact in H. We also fix $f_0\in C_{p,0}$ such that $0\le f_0\le \Gamma f$ and $f_0>0$ on K^c. Let $\mu\in\bar{\mathcal{M}}_p^I$ such that $\mu(\{x\ge a\})=\infty$ for some $a>0$ if $d=1$, and that $\mu(K^c)\ge 1$ if $d\ge 2$. Then $\langle\mu, f_0\rangle>0$ for every $d\ge 1$, more strongly we have

Lemma 8. *Under the above conditions, it holds that*

$$I_T(f_0)\equiv\inf_{0\le t\le T}\langle X_t, f_0\rangle>0 \qquad \boldsymbol{P}_\mu\text{-a.s. for each } T>0.$$

Proof. Recall that $X_t=X_t^0+X_t^I$ and $\boldsymbol{P}_\mu=\boldsymbol{P}_\mu^0\otimes\boldsymbol{P}^I$ (see §1). For $a>0$, set $H_a=\{x_d\ge a\}$. Since

$$\boldsymbol{E}^I[\exp(-X_t^I(H_a))]=\exp\Big[\int_0^t\langle\nu_r, 1-e^{-1_{H_a}}\rangle dr\Big]$$

and $\nu_r(H_a)=\infty$ if and only if $d\ge 2$ for each $r>0$, it holds that $\boldsymbol{P}^I(X_t^I(H_a)=\infty)=1$ for every $t>0$, $a>0$ if $d\ge 2$. Moreover if $\mu\in\bar{\mathcal{M}}_p^I$ such that $\mu(H_a)=\infty$ for some $a>0$,

$$\boldsymbol{P}_\mu(X_t(H)=\infty \text{ for all } t\ge 0)=1.$$

Because for each fixed $T>0$, $\boldsymbol{P}_\mu^0(X_T^0(H)=0)=(\otimes_{k\ge1}P_{x_k}^0)(\zeta\le T)=\prod_{k\ge1}P(|B_T|\ge x_{k,d})\le\prod_{k:x_{k,d}\ge a}P(|B_T|\ge a)=0$, where (B_t, P) is a standard one-dimensional Brownian motion and $x_{k,d}$ is d-th coordinate of x_k. By a similar way it can be easily seen that for each integer $n\ge 0$, $\boldsymbol{P}_\mu(X_T^0(H)=n)=0$. Thus $\boldsymbol{P}_\mu(X_t(H)=\infty$ for all $t\le T)\ge\boldsymbol{P}_\mu(X_T^0(H)=\infty)=1$. Since $T>0$ is arbitrary, the above equation holds. Therefore under the given assumption on μ, we have

$$\begin{aligned}\boldsymbol{P}_\mu(X_t(H)=\infty \text{ for all } t>0)&=\lim_{n\to\infty}\boldsymbol{P}_\mu(X_t(H)=\infty \text{ for all } t\ge 1/n)\\&=\lim_{n\to\infty}\boldsymbol{E}_\mu[\boldsymbol{P}_{X_{1/n}}(X_t(H)=\infty \text{ for all } t\ge 0)]\\&=1.\end{aligned}$$

On the other hand $\boldsymbol{P}_\mu(X_t(K)<\infty$ for all $t\ge 0)=1$ by the $\bar{\mathcal{M}}_p^I$-continuity of X_t. Now if $I_T(f_0)=0$, then by the continuity of $\langle X_t, f_0\rangle$ and $\langle\mu, f_0\rangle>0$ we can find a number $t\in(0, T]$ such that $\langle X_t, f_0\rangle=0$, i.e., $X_t(K)=X_t(H)$. These facts imply that $\boldsymbol{P}_\mu(I_T(f_0)=0)=0$. ■

REMARK 5. When $d=1$, the condition on μ in Lemma 8 can not be weakened to the same as in $d\geq 2$. For example if μ is finite or $\mu=\sum_{n\geq 1}\delta_{x_n}$ with $x_n=n^p$ ($p>1$), then $\boldsymbol{P}_\mu(X_t(H)=0)>0$ for each $t>0$, which implies $\boldsymbol{P}_\mu(I_T(f_0)=0)>0$ for every $T>0$.

In virtue of this lemma, we see that $\langle X_t, f\rangle$ is nowhere 1/2-Hölder continuous in $t\in[0, \infty)$.

Proof of Theorem 3.

We shall prove (i). Suppose that $\mu\in\overline{\mathcal{M}}_p^I-\mathcal{M}_p^I$ satisfies the condition (2.1) with $T>0$, $1\leq\theta<2$. Then for each $a>0$, there exists a small $0<T'<1$ such that

$$\begin{aligned}\langle\mu, P_t^0 g_p\rangle &\asymp\langle\mu, P_t^0 1_{S_a}\rangle\\ &\asymp\langle\mu|_{S_1}, P_t^0 1\rangle\\ &\asymp\begin{cases}t^{-(\theta-1)/2} & (1<\theta<2),\\ \log 1/t & (\theta=1),\end{cases}\end{aligned}\tag{2.9}$$

for all $0<t\leq T'$. In fact by (2.3) we see that $\sup_t\langle\mu|_{S_1^c}, P_t^0 g_p\rangle$ and $\sup_t\langle\mu, P_t^0(g_p 1_{S_a^c})\rangle$ are finite for each $a>0$. By (2.1),

$$\begin{aligned}\langle\mu|_{S_1}, P_t^0 g_p\rangle &\asymp\langle\mu|_{S_1}, P_t^0 1\rangle\\ &\asymp\int_{S_1}\frac{dx}{x_d^\theta}P_t^0 1(x)\\ &=\int_0^1\frac{dx_d}{x_d^\theta}\int_0^{x_d}\sqrt{\frac{2}{\pi t}}\exp\left[-\frac{y^2}{2t}\right]dy.\end{aligned}$$

when $1<\theta<2$, the right hand side is equal to

$$\begin{aligned}\frac{1}{\theta-1}\sqrt{\frac{2}{\pi t}}\int_0^1 dy(y^{1-\theta}-1)\exp\left[-\frac{y^2}{2t}\right]&\asymp\int_0^{1/\sqrt{t}}(\sqrt{t}\,v)^{1-\theta}\exp\left[-\frac{v^2}{2}\right]dv\\ &\asymp t^{-(\theta-1)/2}.\end{aligned}$$

When $\theta=1$, the asymptotic $\log(1/t)$ is similarly obtained. Hence by the same way as in the proof of Theorem 2, we can show the $((3-\theta)/4-\epsilon)$-Hölder continuity at $t=0$ for every $\epsilon\in(0, (3-\theta)/4)$.

On the other hand the non-Hölder continuity is proved in the following manner: When $\theta=1$, we've already shown the non 1/2-Hölder continuity. So it is enough to consider the case $1<\theta<2$. Fix $f\in D_p$ such that $\Gamma f(\tilde{0}, 0+)>0$ and let $M_t=M_t(f)$ be the martingale part of $\langle X_t, f\rangle$. By the same way as in the proof of Proposition 1 if $0\leq s<u$, then for $f_0=\Gamma f$, we have

$$\boldsymbol{E}_\mu[\langle X_s, f_0\rangle\langle X_u, f_0\rangle]=\boldsymbol{E}_\mu\langle X_u, f_0\rangle+V(s, u)$$

with

$$V(s, u)=\langle\mu, P_s^0(f_0 P_{u-s}^0 f_0)-(P_s^0 f_0)(P_u^0 f_0)\rangle$$

$$+\int_0^s \langle \nu_r,\ f_0 P^0_{u-s} f_0\rangle dr + \boldsymbol{E}_\mu \langle X_s,\ f_0\rangle \int_0^{u-s} \langle \nu_r,\ f_0\rangle dr.$$

Hence for all $0 \leq t \leq T$,

$$\begin{aligned}\boldsymbol{E}_\mu[\ll M \gg_t^2] - (\boldsymbol{E}_\mu \ll M \gg_t)^2 &= 2\iint_{0\leq s<u\leq t} V(s,\ u) ds du \\ &\leq C \iint_{0\leq s<u\leq t} \langle \mu,\ P^0_s(f_0 P^0_{u-s} f_0)\rangle ds du \\ &\leq C' t \int_0^t \langle \mu,\ P^0_s \Gamma f\rangle ds \\ &= C' t \boldsymbol{E}_\mu \ll M \gg_t,\end{aligned}$$

where C, C' are constants, independent of $0 \leq t \leq T$. Moreover by (2.9) and $\Gamma f \geq c$ on S_a for some constants a, $c>0$, we have

$$\begin{aligned}\boldsymbol{E}_\mu \ll M \gg_t &\asymp \int_0^t \langle \mu,\ P^0_s 1_{S_a}\rangle ds \\ &\asymp t^{(3-\theta)/2}.\end{aligned}$$

Therefore for some $C_T>0$,

$$\boldsymbol{E}_\mu\left[\left|\frac{\ll M \gg_t}{\boldsymbol{E}_\mu \ll M \gg_t} - 1\right|^2\right] \leq C_T \frac{t}{\boldsymbol{E}_\mu \ll M \gg_t} \asymp t^{(\theta-1)/2} \quad (\to 0 \text{ as } t \downarrow 0).$$

Thus for a sequence $t_n = 1/2^n$, we have

$$\lim_{n\to\infty} \frac{\ll M \gg_{t_n}}{\boldsymbol{E}_\mu \ll M \gg_{t_n}} = 1 \qquad \boldsymbol{P}_\mu\text{-a.s.}.$$

Hence it is easily seen that

$$\liminf_{t\downarrow 0} \frac{\ll M \gg_t}{\boldsymbol{E}_\mu \ll M \gg_t} > 0 \qquad \boldsymbol{P}_\mu\text{-a.s.}.$$

Now by

$$\frac{|M_t|}{t^{(3-\theta)/4}} = \frac{|B_{\ll M \gg_t}|}{\sqrt{\ll M \gg_t}} \left(\frac{\ll M \gg_t}{t^{(3-\theta)/2}}\right)^{1/2}$$

and the well-known result

$$\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \qquad \boldsymbol{P}_\mu\text{-a.s.},$$

it holds that

$$\limsup_{t\downarrow 0} \frac{|M_t|}{t^{(3-\theta)/4}} = \infty \qquad \boldsymbol{P}_\mu\text{-a.s.}. \tag{2.10}$$

Hence $M_t(f)$ is not $(3-\theta)/4$-Hölder continuous at $t=0$, so is not $\langle X_t,\ f\rangle$.

Next we show (ii). Suppose that $\mu \in \overline{\mathcal{M}}^I_p - \mathcal{M}^I_p$ satisfies the condition (2.2) with

$T>0$, $1<\theta\leq 2$ and $\eta>0$ $(1<\theta<2)$ or $\eta>1$ $(\theta=2)$. Then for $0<a<1$, there exists $0<T'<1$ such that

$$\begin{aligned}(2.11)\qquad \langle\mu, P_t^0 g_p\rangle &\asymp \langle\mu, P_t^0 1_{S_a}\rangle \\ &\asymp \langle\mu|_{S_a}, P_t^0 1\rangle \\ &\asymp \begin{cases} t^{-(\theta-1)/2}(\log 1/t)^{-\eta} & (1<\theta<2,\ \eta>0), \\ t^{-1/2}(\log 1/t)^{-(\eta-1)} & (\theta=2,\ \eta>1),\end{cases}\end{aligned}$$

for all $0<t\leq T'$. In fact in this case for a small $0<a<1$,

$$\langle\mu|_{S_a}, P_t^0 1\rangle \asymp \int_0^a \frac{dx_d}{x_d^\theta(\log 1/x_d)^\eta}\int_0^{x_d}\sqrt{\frac{2}{\pi t}}\exp\left[-\frac{y^2}{2t}\right]dy.$$

Let $p=1/(\theta-1)$, $q=\eta/(\theta-1)$ and set $x=z^{-p}(\log z)^{-p}$. We have $p>1$, $q>0$ or $p=1<q$ and

$$\begin{aligned} dx &= -\frac{p\log z+q}{z^{p+1}(\log z)^{q+1}}dz \\ &\asymp -x^{1+1/p}(\log 1/x)^{q/p}dz,\end{aligned}$$

that is, $x^{-\theta}(\log 1/x)^{-\eta}dx \asymp -dz$. Hence for some $b>1$; $a=b^{-p}(\log b)^{-q}$,

$$\langle\mu|_{S_a}, P_t^0 1\rangle \asymp \int_b^\infty dz\int_0^{z^{-p}(\log z)^{-q}}\sqrt{\frac{2}{\pi t}}\exp\left[-\frac{y^2}{2t}\right]dy.$$

If $\sqrt{t}=z^{-p}(\log z)^{-q}$, then $z=h(t)\asymp t^{-1/(2p)}(\log 1/t)^{-q/p}=t^{-(\theta-1)/2}(\log 1/t)^{-\eta}$. We divide the integral area of z at $h(t)$. Note that $z\in(b, h(t))$ if and only if $z^{-p}(\log z)^{-q}>\sqrt{t}$. Thus it easy to see that

$$\frac{1}{\sqrt{t}}\int_b^{h(t)}dz\int_0^{z^{-p}(\log z)^{-q}}\exp\left[-\frac{y^2}{2t}\right]dy \asymp h(t)$$

and

$$\begin{aligned}\frac{1}{\sqrt{t}}\int_{h(t)}^\infty dz\int_0^{z^{-p}(\log z)^{-q}}\exp\left[-\frac{y^2}{2t}\right]dy &\asymp \frac{1}{\sqrt{t}}\int_{h(t)}^\infty \frac{dz}{z^p(\log z)^q} \\ &\asymp \frac{1}{\sqrt{t}}\int_{\log h(t)}^\infty \frac{1}{v^q}e^{-(p-1)v}dv.\end{aligned}$$

If $p>1$, $q>0$, i.e., $1<\theta<2$, $\eta>0$, then the right hand side is equal to or less than

$$\begin{aligned}\frac{1}{\sqrt{t}(\log h(t))^q}\int_{\log h(t)}^\infty e^{-(p-1)v}dv &\asymp t^{-1/2}(\log h(t))^{-q}h(t)^{-(p-1)} \\ &\asymp h(t)\end{aligned}$$

If $p=1<q$, i.e., $\theta=2$, $\eta=q>1$, then the right hand side is equal to

$$\begin{aligned}\frac{1}{\sqrt{t}}\int_{\log h(t)}^\infty \frac{dv}{v^q} &\asymp t^{-1/2}(\log h(t))^{-(q-1)} \\ &\asymp t^{-1/2}(\log 1/t)^{-(\eta-1)}.\end{aligned}$$

Therefore (2.11) holds. Now nothing that

$$(t^{(3-\theta)/2}(\log 1/t)' \asymp t^{-(\theta-1)/2}(\log 1/t)^{-\eta}$$

and

$$(\sqrt{t}(\log 1/t)^{-(\eta-1)})' \asymp t^{-1/2}(\log 1/t)^{-(\eta-1)},$$

we have the following: For each $0<a<1$, there exists a constant $0<T'<1$ such that

$$\begin{aligned}\int_0^t \langle \mu, P_s^0 g_p\rangle ds &\asymp \int_0^t \langle \mu, P_s^0 1_{S_a}\rangle ds \\ &\asymp \int_0^t \langle \mu|_{S_a}, P_s^0 1\rangle ds \\ &\asymp \begin{cases} t^{(3-\theta)/2}(\log 1/t)^{-\eta} & (1<\theta<2,\ \eta>0), \\ t^{1/2}(\log 1/t)^{-(\eta-1)} & (\theta=2,\ \eta>1). \end{cases}\end{aligned}$$

We shall show that for each $f\in D_p$,

$$\limsup_{t\downarrow 0} \frac{|M_t(f)|}{t^{(3-\theta)/4}}=0, \tag{2.12}$$

which implies the $(3-\theta)/4$-Hölder continuity of X_t at $t=0$. We first consider the case $1<\theta<2$, $\eta>0$. For simplicity of notations we set $M_t=M_t(f)$, $\gamma=(3-\theta)/4$. Fix an integer $N\geq 1$ such that $N_\eta>1$. By a martingale moment inequality we have

$$\begin{aligned}\boldsymbol{P}_\mu(\sup_{s\leq t}|M_s|\geq \epsilon) &\leq \frac{C}{\epsilon^{2N}}\boldsymbol{E}_\mu[\langle\!\langle M\rangle\!\rangle_t^N] \\ &\leq \frac{C'}{\epsilon^{2N}}\frac{t^{2N\gamma}}{(\log 1/t)^{N\eta}}\end{aligned}$$

for all $\epsilon>0$ and small $t>0$, where C, C' are independent of (ϵ, t). For each fixed k, $n\geq 1$, set $t=1/2^n$ and $\epsilon=t^\gamma/k$. Then the right hand side is equal to $C'k^{2N}n^{-N\eta}(\log 2)^{-N\eta}$. This is the general term of a convergent series. Hence by Borel-Cantelli's lemma, for $\boldsymbol{P}_\mu$-a.a. ω, it holds that for each $k\geq 1$, there is an integer-valued random variable $N(\omega)$ such that if $n\geq N(\omega)$, then $\sup_{s\leq 2^{-n}}|M_s(\omega)|\leq 2^{-n\gamma}k^{-1}$, that is,

$$\sup_{s\in(2^{-n-1},2^{-n}]}\frac{|M_s(\omega)|}{s^\gamma}\leq\frac{2^\gamma}{k}.$$

Therefore the equation (2.12) follows. The case of $\theta=2$, $\eta>1$ can be proved by the same way.

We can show the equations in Remark 2 by the same way as the proof of (2.10). These imply the non-Hölder continuity in (ii) of Theorem 3..

Corollary 1 immediately follows from Theorem 1 and Theorem 2.

The proofs are complete.

2.4. Further Remarks

In this subsection we give some more properties of X_t^0 and X_t^I. Also we shall prove the unboundedness of $\langle X_t, g_p\rangle$.

(a) The results of Theorem 1, Theorem 2 (except the later half in (iv)) and Theorem 3 are also valid for the infinite Markov particle system X_t^0 with no immigration. Of course all terms related to the immigration part are deleted. Moreover if we assume $\mu(\{x_d \geq a\})=\infty$ for some $a>0$ and each $d\geq 1$, then the same result as the later half in (iv) of Theorem 2 is also valid, because this guarantee $\boldsymbol{P}_\mu^0(X_t^0(H)=\infty$ for all $t\geq 0)=1$, and hence Lemma 8 holds.

(b) For any (short) time interval, infinitely many particles are born and die, i.e., for all $0\leq s<t$, $\boldsymbol{P}^I(X_\cdot^I(s\leq \alpha<\beta\leq t)=\infty)=1$. In fact, in case of $d=1$. Since

$$Q_m(s\leq\alpha<t)=\int_0^\infty Q^0(s\leq\alpha+u<t)du=(t-s)Q^0(W)=\infty,$$

we see that the number of particles which are born in the time interval $[s, t)$ is infinite, i.e., $\boldsymbol{P}^I X_\cdot^I(s\leq\alpha<t)=\infty)=1$. Moreover

$$Q_m(s\leq\alpha<t<\beta)=\int_s^t Q^0(w(u)\in H)du=\sqrt{\frac{2}{\pi}}(\sqrt{t}-\sqrt{s})<\infty$$

implies that the number of particles which are born after time s and survive until time t is finite. Thus our claim follows. In general case $(d\geq 2)$, restrict on $\{|w(\alpha+)|\leq n\}$.

To prove the unboundedness of $\langle X_t, g_p\rangle$ mentioned in (ii) of Remark 1, it suffices to show the following result: Let $d=1$.

Theorem 4. *For any* $T>0$, $\boldsymbol{P}^I(X_t^I((0, 1))=\infty$ *for some* $t\in(0, T))=1$.

Let $W_0=W\cap\{\alpha=0\}$ be the totality of excursions in $H=(0,\infty)$. $X_t^I((0,1))$ can be expressed as

$$N_t=N^0(D_t) \quad \text{with} \quad D_t=\{(s, w)\in[0,\infty)\times W_0: w(t-s)\in(0,1),\ 0\leq s<t\}.$$

For each fixed $k\geq 1$, we define a smaller process $S_{k,t}$ as follows: Let $a_k=1/2^k$, $t_j^k=j/4^k$ $(j=1, 2, \cdots)$ and

$$\xi^k=N^0(V^k)$$

with

$$V^k=\{(s, w): s\in[0, a_k^2),\ w(a_k^2-s),\ w(2a_k^2-s)\in[a_k, 2a_k),$$

$$w(3a_k^2-s)=\Delta,\ \tau_{2a_k}(w)>\beta(w)\},$$

where $\tau_a(w)$ is the passage time to $a>0$ of w. For each $j\geq 1$, if $t_j^k\leq t<t_{j+1}^k$, then set

$$\xi_t^k=N^0(V_j^k) \quad \text{with} \quad V_j^k=\theta_{-t_{j-1}^k}^k(V^k)$$

(note that ξ_t^k is undefined for $0\leq t<t_1^k$). It holds that $\xi_t^k\overset{(d)}{=}\xi^k$. In particular, if we set $\xi_{t_j}^k=\xi_{t_j^k}^k$, then $\{\xi_{t_j}^k : j=1, 2, \cdots, k=1, 2, \cdots\}$ are independent.

REMARK 6. ξ^k denotes the number of particles which are born during the time interval $[0, a_k^2)$, stay in $[a_k, 2a_k)$ at time points a_k^2, $2a_k^2$ and die during the time interval $(2a_k^2, 3a_k^3]$, and also which never hit $2a_k$.

Now for each $k\geq 1$, set $S_{k,t}=\sum_{n=k}^{\infty}\xi_t^n$. Clearly if $t\geq t_1^k(=a_k^2=1/4^k)$, then $S_{k,t}\leq N_t$. Hence to prove Theorem 4 it is enough to show the following proposition:

Proposition 2. *For each k, $i\geq 1$, $\boldsymbol{P}^I(S_{k,t}=\infty$ for some $t_1^k\leq t<t_{i+1}^k)=1$.*

Proof. We define a random variable $U_k^{*,i}$ for each k, $i\geq 1$ as follows: Set

$$U_k^{1,i}=\max_{1\leq j\leq i}\xi_{t_j}^k$$

and

$$I_1=\{j=1, 2, \cdots, i : \xi_{t_j}^k=U_k^{1,i}\},$$
$$J_1=4^k+[4I_1-3)\cup(4I_1-2)\cup(4I_1-1)\cup 4I_1].$$

Also define

$$U_k^{2,i}=U_k^{1,i}+\max_{j\in J_1}\xi_{t_j}^{k+1}.$$

If we have I_n, J_n, $U_k^{n+1,i}$, then set

$$I_{n+1}=\{j\in J_n : \xi_{t_j}^{k+n}=U_k^{n+1,i}\},$$
$$J_{n+1}=4^{k+n}+[(4I_{n+1}-3)\cup(4I_{n+1}-2)\cup(4I_{n+1}-1)\cup 4I_{n+1}]$$

and

$$U_k^{n+2,i}=U_k^{n+1,i}+\max_{j\in J_{n+1}}\xi_{t_j}^{k+n+1}.$$

So we define

$$U_k^{*,i}=\lim_{n\to\infty}U_k^{n,i}.$$

We can show the following two claims:

Claim 1. $\lambda_k\equiv\boldsymbol{E}^I\xi^k=C/2^k$ for all $k\geq 1$ $(C=\boldsymbol{E}^I\xi^0>0)$.

Claim 2. For each k, $i\geq 1$, $\boldsymbol{P}^I(U_k^{*,i})=\infty=1$.

Obviously Claim 2 implies Proposition 2.

Proof of Claim 1. Let V_s^k denote the s-section of V^k.

$$\begin{aligned}
\lambda_k &= \boldsymbol{E}^I N^0(V^k)=\int_0^{a_k^2} ds Q^0(V_s^k)\\
&=\int_0^{a_k^2} ds\int_{a_k}^{2a_k} Q^0(w(a_k^2-s)\in dy_1\,;\ a_k^2-s<\tau_{2a_k})\\
&\qquad \int_{a_k}^{2a_k} P_{y_1}^0(w^0(a_k^2)\in dy_2\,;\ a_k^2<T_{2a_k})P_{y_2}^0(\zeta\leq a_k^2,\ \zeta<T_{2a_k})\\
&=\int_0^{a_k^2} du\int_{a_k}^{2a_k} dy_1 \lim_{r\downarrow 0}\int_0^{a_k} dx\nu_r(x)p_u^{0,2a_k}(x,\ y_1)\\
&\qquad \int_{a_k}^{2a_k} dy_2 p_{a_k^2}^{0,2a_k}(y_1,\ y_2)P_{y_2}(T_0\leq a_k^2,\ T_0<T_{2a_k})\\
&=\int_{a_k}^{2a_k} dy_1\int_{a_k}^{2a_k} dy_2 p_{a_k^2}^{0,2a_k}(y_1,\ y_2)\\
&\qquad \int_0^{a_k^2} du \lim_{r\downarrow 0}\int_0^{a_k} dx\nu_r(x)p_u^{0,2a_k}(x,y_1)P_{y_2}(T_0\leq a_k^2,\ T_0<T_{2a_k})\\
&=a_k^2\int_1^2 dz_1\int_1^2 dz_2 p_{a_k^2}^{0,2a_k}(a_kz_1,\ a_kz_2)\\
&\qquad \int_0^{a_k^2} du \lim_{r\downarrow 0}\int_0^{a_k} dx\nu_r(x)p_u^{0,2a_k}(x,a_kz_1)P_{a_kz_2}(T_0\leq a_k^2,\ T_0<T_{2a_k}),
\end{aligned}$$

where $p_t^{0,b}(x,\ y)$ denotes the transition density for Brownian motion absorbed at 0 and $b>0$ ($x,\ y>0$). Note that

$$\begin{aligned}
p_t^{0,b}(x,\ y)(&=P_x^0(w^0(t)=y\,;\ t<T_b)=P_x(B_t=y\,;\ t<T_0\wedge T_b))\\
&=\sum_{n=-\infty}^{\infty} p_t^0(x,\ y+2nb)
\end{aligned}$$

and

$$P_x(T_0\in dt\,;\ T_0<T_b)=\frac{dt}{\sqrt{2\pi t^3}}\sum_{n=-\infty}^{\infty}(x+2nb)\exp\left[-\frac{(x+2nb)^2}{2t}\right].$$

By using the scaling properties; for any $a>0$,

$$\nu_{a^2s}(ay)=\frac{1}{a^2}\nu_s(y) \quad\text{and}\quad P_{a^2v}^{0,2a}(ay,\ az)=\frac{1}{a}p_v^{0,2}(y,\ z),$$

we can get

$$\begin{aligned}
\boldsymbol{E}^I\xi^k &= a_k\int_1^2 dz_1\int_1^2 dz_2 p_1^{0,2}(z_1,\ z_2)\int_0^1 dv\lim_{s\downarrow 0}\int_0^1 dy\nu_s(y)p_v^{0,2}(y,\ z_1)P_{z_2}(T_0\leq 1,\ T_0<T_2)\\
&=a_k\boldsymbol{E}^I\xi^0.
\end{aligned}$$

Proof of Claim 2. We shall show that for each k, $i\geq 1$ and $m\geq 0$, $\boldsymbol{P}^I(U_k^{*,i}\leq$

$m)=0$ by mathematical induction.

(1) For each k, $i\geq 1$, $\boldsymbol{P}^{I}(U_{k}^{*,i}=0)=0$.

In fact if $U_{k}^{*,i}=0$, then $\xi_{t_j}^{k+n}=0$ for all $n\geq 0$, $4^{n}\leq j<4^{n}(i+1)$. But the sum of these expectations is given as $i(\lambda_{k}+4\lambda_{k+1}+4^{2}\lambda_{k+2}+\cdots)$ and this is infinity by Claim 1. Hence the probability of this event is 0.

(2) If we assume that $\boldsymbol{P}^{I}U_{k}^{*,i}\leq m-1)=0$ for all k, $i\geq 1$, then

$$\begin{aligned}P^{I}(U_{k}^{*,i}\leq m)&=\sum_{m_k=0}^{m}\boldsymbol{P}^{I}(\xi^{k}=m_{k})^{i}\boldsymbol{P}^{I}(U_{k+1}^{*,4i}\leq m-m_{k})\\&\quad+\sum_{j=1}^{i-1}\binom{i}{j}\sum_{m_k=1}^{m}\boldsymbol{P}^{I}(\xi^{k}=m_{k})^{j}\boldsymbol{P}^{I}(\xi^{k}\leq m_{k}-1)^{i-j}\boldsymbol{P}^{I}(U_{k+1}^{*,4j}\leq m-m_{k})\\&=\boldsymbol{P}^{I}(\xi^{k}=0)^{i}\boldsymbol{P}^{I}(U_{k+1}^{*,4i}\leq m)\\&=\boldsymbol{P}^{I}(\xi^{k}=0)^{i}\boldsymbol{P}^{I}(\xi^{k+1}=0)^{4i}\boldsymbol{P}^{I}(U_{k+2}^{*,4^{2}i}\leq m)\\&\leq\boldsymbol{P}^{I}(\xi^{k}=0)^{i}\boldsymbol{P}^{I}(\xi^{k+1}=0)^{4i}\cdots\boldsymbol{P}^{I}(\xi^{k+n}=0)^{4^{n}i}\to 0\ (n\to\infty)\end{aligned}$$

by Claim 1. This implies $\boldsymbol{P}^{I}(U_{k}^{*,i}\leq m)=0$ for all k, $i\geq 1$.

(3) From the above results (1) and (2) we have Claim 2. ■

3. Asymptotic Behavior of Hitting Rates for Brownian Excursions; Applications to Equilibrium Processes

In this section we consider the Kuznetsov measure $Q_{m}(dw)$ and the canonical process $\{w(t)\}_{t\in R}$ associated with absorbing Brownian motion $(w^{0}(t), P_{x}^{0})$ in the half space H and the Lebesgue measure $m(dx)=dx$ on H. We give the asymptotic behavior of $Q_{m}(0\leq\sigma_{B}<t)$ as $t\to\infty$ for the hitting time $\sigma_{B}(w)=\inf\{t>0: w(t)\in B\}(=\infty$ if $\{\cdot\}=\emptyset)$ of a compact subset B in H. Moreover by applying the result to the equilibrium process with immigration $(X_{t}, \boldsymbol{P})$ associated with $(w(t), Q_{m})$, we also give limit theorems for it.

3.1. Main Results

Let B be a compact subset of H and $\pi_{B}(dx)$ be the capacitary measure of B, that is, π_{B} is supported by ∂B and satisfies that

$$P_{x}^{0}(T_{B}<\infty)=\int g^{0}(x, y)\pi_{B}(dy),$$

with the potential kernel

$$g^{0}(x, y)=\int_{0}^{\infty}p_{t}^{0}(x, y)dt,$$

where $p_t^0(x, y)$ is the transition density of $(w^0(t), P_x^0)$. The *capacity* is defined by

$$C^0(B)=\pi_B(1)=\int_{\partial B}\pi_B(dx).$$

Moreover if we set $\tau_B(w)=\inf\{t\in\boldsymbol{R} : w(t)\in B\}(=\infty$ if $\{\cdot\}=\emptyset)$, then

$$C^0(B)=Q_m(0<\tau_B<1)=Q^0(\tau_B<\infty)=Q^0(\sigma_B<\infty)$$

(see §3.3).

We shall use a symbol $Q^0[\cdot]$ as the integral by the measure Q^0.

Our main result is the following:

Theorem 5. *Let B be a compact subset of H with a positive capacity. Then it holds that*

$$Q_m(0\leq\sigma_B<t)=tC^0(B)+f(t)$$

with

$$f(t)=\begin{cases}\sqrt{\dfrac{2t}{\pi}}-\dfrac{2}{3}b+c+o(1) & \text{if } d=1,\\ \log t\ \Phi(B)+o(\log t) & \text{if } d=2,\\ O(1) & \text{if } d\geq 3\end{cases}$$

as $t\to\infty$, where

$$b=\inf\{x : x\in B\}\quad \textit{and}\quad c=\sup\{x : x\in B\}\qquad \textit{if } d=1$$

and

$$\Phi(B)=\frac{1}{\pi}\int_{\partial B}x_2\pi_B(dx)Q^0[w^{(2)}(\sigma_B)<\infty]=\frac{1}{\pi}\left(\int_{\partial B}x_2\pi_B(dx)\right)^2$$

with $w=(w^{(1)}, w^{(2)})$ if $d=2$.

REMARK 7. If $d=1$, then $g^0(x, y)=2(x\wedge y)$, $\pi_B(dx)=\delta_b(dx)/(2b)$ and $C^0(B)=1/(2b)$.

Let $(\{X_t\}_{t\in\boldsymbol{R}}, \boldsymbol{P})$ be the equilibrium process associated with $(w(t), Q_m)$. Set $N_t^B=\{w\in W : 0\leq\sigma_B(w)<t\}$. Then $X(N_t^B)$ is the number of particles hitting B during time interval $[0, t)$. Now we have the following result:

Theorem 6. *Let B be a compact subset of H with a positive capacity. Then*

$$\frac{X(N_t^B)}{t}\to C^0(B)\qquad \boldsymbol{P}\textit{-a.s. and in } L^1(\boldsymbol{P})$$

as $t\to\infty$. Moreover

$$\frac{X(N_t^B)-tC^0(B)}{\sqrt{t}}\to\begin{cases}N\sqrt{2/\pi},\ C^0(B)) & \text{if } d=1,\\ N(0,\ C^0(B)) & \text{if } d\geq 2\end{cases}$$

in law as $t\to\infty$.

3.2. Kuznetsov Measures and Capacity Theory

In this subsection we consider the asymptotic of $Q_m(0\leq\tau_B<t)$ in more general situation and give the first term of the asymptotic.

The situation is the same as in §2.2. That is, S is a Lusin space, $S_\Delta\equiv S\cup\{\Delta\}(\Delta\notin S)$, $(W^0, \mathcal{F}^0, \mathcal{F}, w^0(t), P_x^0)$, $t\geq 0$, $x\in S$, is a Borel right process with transition semi-group $(P_t^0)_{t\geq 0}$ and let W be the set of all maps $w:\boldsymbol{R}\to S_\Delta$ such that w is S-valued right continuous on some $(\alpha(w), \beta(w))$ and $w(t)=\Delta$ for $t\notin(\alpha(w), \beta(w))$.

We fix $m\in\boldsymbol{Exc}$. It is decomposed uniquely as $m=m_i+m_p\in\boldsymbol{Inv}\oplus\boldsymbol{Pur}$. Moreover there exists a unique entrance law $\nu=(\nu_t)_{t>0}$ such that

$$m_p=\int_0^\infty \nu_t dt.$$

Then there are Kuznetsov measures

$$Q_m,\ Q_{m_i},\ Q_{m_p}\quad\text{and}\quad Q^0=Q_\nu\qquad\text{on } W,$$

which satisfy that

$$Q_m=Q_{m_i}+Q_{m_p},\quad\text{and}\quad Q_{m_p}=\int_{\boldsymbol{R}}\theta_{-s}(Q^0)ds=\int_{\boldsymbol{R}}\theta_s(Q^0)ds.$$

Note that Q_{mi} is supported on $\{\alpha=-\infty\}$ and Q^0 is supported on $\{\alpha=0\}$.

Let B be a Borel subset in S. Recall that $\tau_B(w)=\inf\{t\in\boldsymbol{R}: w(t)\in B\}$ and note that $\tau_B\circ\theta_s=\tau_B-s$ for all $s\in\boldsymbol{R}$. According to [7] B is *co-transient* if $Q_m(\tau_B=-\infty)=0$, and for such B we define the following *co-capacities*:

$$\begin{aligned}\hat{C}(B)&=Q_m(0<\tau_B<1),\\ \hat{C}^i(B)&=Q_{m_i}(0<\tau_B<1),\\ \hat{C}^p(B)&=Q_{m_p}(0<\tau_B<1).\end{aligned}$$

Then

$$\hat{C}(B)=\hat{C}^i(B)+\hat{C}^p(B)\quad\text{and}\quad\hat{C}^p(B)=Q^0(\tau_B<\infty)$$

The last equation is due to

$$\begin{aligned}Q_{m_p}(0<\tau_B<t)&=\int_{\boldsymbol{R}}ds\int_s^{s+t}Q^0(\tau_B\in du)\\ &=\int_{\boldsymbol{R}}Q^0(\tau_B\in du)\int_{u-t}^u ds\end{aligned}$$

$$=tQ^0(\tau_B<\infty).$$

Now recall $\sigma_B(w)=\inf\{t>0: w(t)\in B\}$ and note that under Q^0, $\tau_B=\sigma_B$ and $\tau_B\circ\theta_{-s}=\sigma_B\circ\theta_{-s}=\sigma_B+s$ for $s\geq 0$, but in general $\tau_B\circ\theta_s\neq\sigma_B\circ\theta_s$ for $s>0$. We have the following result:

Theorem 7. *If B is a co-transient set with a positive and finite co-capacity, then*

$$Q_m(0\leq\sigma_B<t)=t\hat{C}(B)+o(t) \qquad \text{as } t\to\infty.$$

Proof. One can easily see that

$$Q_m(0\leq\sigma_B<t)=Q_{m_i}(0\leq\sigma_B<t)+Q_{m_p}(0\leq\sigma_B<t),$$
$$Q_{m_i}(0\leq\sigma_B<t)=P^0_{m_i}(T_B<t)$$

and

$$\begin{aligned}Q_{m_p}(0\leq\sigma_B<t)&=\int_0^\infty Q^0(0\leq\sigma_B\circ\theta_s<t)ds+\int_0^t Q^0(-s\leq\sigma_B<t-s)ds\\&=\int_0^\infty ds\int_S\nu_s(dx)P^0_x(\sigma_B<t)+\int_0^t Q^0(\sigma_B<t-s)ds\\&=P^0_{m_p}(T_B<t)+\int_0^t Q^0(\sigma_B<s)ds.\end{aligned}$$

That is,

$$Q_m(0\leq\sigma_B<t)=P^0_m(T_B<t)+\int_0^t Q^0(\sigma_B<s)ds. \tag{3.1}$$

By Spitzer's formula ([7])

$$\lim_{t\to\infty}\frac{1}{t}P^0_m(T_B<t)=\hat{C}^i(B),$$

and clearly

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Q^0(\sigma_B<s)ds=Q^0(\sigma_B<\infty)=Q^0(\tau_B<\infty)=\hat{C}^p(B).$$

Therefore our claim follows. ■

REMARK 8. According to [7] the capacity of B is defined by $C(B)=Q_m(0<\lambda_B<1)$, where $\lambda_B=\sup\{t\in\boldsymbol{R}: w(t)\in B\}(=-\infty \text{ if } \{\cdot\}=\emptyset)$ is the last exit time of B for $w(t)$. For a Borel set B in S, B is *transient* if $Q_m(\lambda_B=\infty)=0$. If B is both transient and co-transient, then $C(B)=\hat{C}(B)$. Let $L_B(w^0)=\sup\{t>0: w^0(t)\in B\}(=0 \text{ if } \{\cdot\}=\emptyset)$ be the last exit time for w^0. B is strongly transient if $P^0_m(L_B\geq\zeta)=0$, where $\zeta(w^0)$ is the life time of w^0. If (w^0, P^0_x) and $(\hat{w}^0, \hat{P}^0_x)$ are transient standard processes in weak duality relative to an excessive measure m and if B is strongly transient with finite capacity, then there is a measure

$$\pi_B(dx)=\lim_{t\downarrow 0}\frac{1}{t}P_x^0(0<L_B\leq t)m(dx)=\lim_{t\downarrow 0}\frac{1}{t}P_x^0(T_B\leq t,\ T_B\circ\theta_t=\infty)m(dx) \qquad \text{on } S$$

such that $P_x^0(T_B<\infty)m(dx)=\int\hat{G}^0(y,\ dx)\pi_B(dy)$, where $\hat{G}^0(y,\ dx)=\int_0^\infty dt\hat{P}_t^0(y, dx)$ [4]. Moreover $Q_m(\lambda_B\in dt,\ w(\lambda_B-)\in dy)=dt\pi_B(dy)$ and hence $Q_m(0<\lambda_B<1)=\pi_B(1)$. If B is strongly co-transient then $Q_m(\tau_B\in dt,\ w(\tau_B)\in dy)=dt\hat{\pi}_B(dy)$ and $Q_m(0<\tau_B<1)=\hat{\pi}_B(1)$ hold [6].

3.3. Proof of Main Results

Recall that $H=\{x_d>0\}$, $m(dx)=dx$ on H, $(w^0(t),\ P_x^0)$ is absorbing Brownian motion in H, $(P_t^0)_{t\geq 0}$ is its transition semi-group. Q^0 is a Brownian excursion law, Q_m is a Kuznetsov measure associated with $(w^0(t),\ P_x^0,\ m)$, and $\nu_r(dx)=\nu_r(x_d)dx$ with

$$\nu_r(u)=\frac{u}{\sqrt{2\pi r^3}}\exp\left[-\frac{u^2}{2r}\right]$$

for $r>0$, $u>0$. Here we give some useful results for the entrance law:

$$\lim_{r\downarrow 0}\int_\epsilon^\infty\nu_r(dx_d)=0,\qquad 2\lim_{r\downarrow 0}\int_0^\epsilon x_d^n\nu_r(dx_d)=\begin{cases}1 & \text{if } n=1,\\ 0 & \text{if } n\geq 2\end{cases}$$

for all $\epsilon>0$, and

$$\int_H\nu_r(dx)p_t^0(x,\ y)=\int_0^\infty\nu_r(dx_d)p_t^0(x_d,\ y_d)=\nu_{r+t}(y_d),$$

where $\nu_r(dx_d)=\nu_r(x_d)dx_d$ on $(0,\ \infty)$.

Note that the facts in Remark 8 and that w^0 is symmetric relative to m, i.e., $w^0=\hat{w}^0$. For each compact subset B of H, it is strongly transient, it holds that

$$C^0(B)=\pi_B(1)=Q_m(0<\tau_B<1)=Q^0(\tau_B<\infty)=Q^0(\sigma_B<\infty)$$

and this is finite (cf. Theorem 6.5.3 in [14]). We can show $\pi_B(1)=Q^0(\sigma_B<\infty)$ directly. In fact, since $\int_H p_t^0(x,\ y)\nu_r(dx)=\nu_{r+t}(y_d)$ and $\int_0^\infty\nu_r(y_d)dr=1$, we have $Q^0(\sigma_B<\infty)=\lim_{r\downarrow 0}\int_H\nu_r(dx)P_x^0(T_B<\infty)=\lim_{r\downarrow 0}\int_{\partial B}\pi_B(dy)\int_H g^0(x,\ \ y)\nu_r(dx)=\pi_B(1)$. Moreover we see that

$$\begin{aligned}\pi_B(dy)&=\hat{\pi}_B(dy)=Q_m[w(\tau_B)\in dy:0<\tau_B<1]\\ &=\int_0^\infty Q^0[w(\tau_B)\in dy:s<\tau_B<s+1]ds+\int_0^1 Q^0[w(\tau_B)\in dy:\tau_B<1-s]ds\end{aligned}$$

$$=Q^0[w(\tau_B)\in dy : \tau_B<\infty].$$

By a simple computation one can get

$$\int_0^t Q^0(\sigma_B<s)ds=tQ^0(\sigma_B<\infty)-Q^0[\sigma_B\wedge t : \sigma_B<\infty].$$

Thus by the equation (3.1), nothing that $m\in \boldsymbol{Pur}$, we have

$$Q_m(0\leq\sigma_B<t)=tC^0(B)+P_m^0(T_B<t)-Q^0[\sigma_B\wedge t : \sigma_B<\infty]. \tag{3.2}$$

Now Theorem 5 can be proved as follows: First we consider the one-dimensional case. In this case our claim is immediately obtained by the following proposition:

Proposition 3. *Let $d=1$ and B be a non-empty compact subset of $H=(0,\infty)$. Set $b=\inf B$ and $c=\sup B$. Then*

$$P_m^0(T_B<t)=\sqrt{\frac{2t}{\pi}}-\frac{b}{2}+c+o(1)$$

as $t\to\infty$, and

$$Q^0[\sigma_B : \sigma_B<\infty]=\frac{b}{6}.$$

Proof. It is easy to see that

$$\int_0^\infty P_x^0(T_B<t)dx=\int_0^b P_x^0(T_b<t)dx+\int_b^c P_x^0(T_B<t)dx+\int_c^\infty P_x^0(T_c<t)dx.$$

When $t\to\infty$, the first term of the right hand side goes to

$$\int_0^b P_x^0(T_b<\infty)dx=\int_0^b \frac{x}{b}dx=\frac{b}{2}$$

and the second term goes to

$$\int_b^c P_x^0(T_B<\infty)dx=c-b.$$

The last term is equal to

$$\int_0^\infty P_x(T_0<t)dx=\int_0^\infty 2P_0(B_t>x)dx=\sqrt{\frac{2t}{\pi}},$$

where (B_t, P_x) is Brownian motion in $\boldsymbol{R}$. Hence the first claim follows. To prove the second claim we need some well-known results (cf. [9]). If $0\leq x\leq b$, then

$$E_x^0[e^{-\lambda T_b} : T_b<\infty]=E_x[e^{-\lambda T_b} : T_b<T_0]=\frac{\sinh(x\sqrt{2\lambda})}{\sinh(b\sqrt{2\lambda})}$$

and

$$P_x^0(T_b<\infty)=P_x(T_b<T_0)=x/b.$$

Thus for all $0<\epsilon\leq b$ it holds that

$$\begin{aligned}\lim_{u\downarrow 0}\int_0^\epsilon \nu_u(dx)E_x^0[e^{-\lambda T_b}:T_b<\infty]&=\frac{\sqrt{\lambda}}{\sqrt{2}\sinh(b\sqrt{2\lambda})}\\&=\left(b\int_{-1}^1 e^{b\sqrt{2\lambda}v}dv\right)^{-1}\end{aligned}$$

and

$$\lim_{u\downarrow 0}\int_0^\epsilon \nu_u(dx)P_x^0(T_b<\infty)=1/(2b).$$

Now since $\lambda^{-1}(1-e^{-\lambda\sigma_B})=\lambda^{-2}\int_0^{\sigma_B}e^{-\lambda u}dv\nearrow\sigma_B$ as $\lambda\downarrow 0$ we have

$$\begin{aligned}Q^0[\sigma_B:\sigma_B<\infty]&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}Q^0[1-e^{-\lambda\sigma_B}:\sigma_B<\infty]\\&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\lim_{u\downarrow 0}\int_0^b\nu_u(dx)E_x^0[1-e^{-\lambda T_b}:T_b<\infty]\\&=\lim_{\lambda\downarrow 0}\frac{1}{\lambda}\left\{\frac{1}{2b}-\left(b\int_{-1}^1 e^{b\sqrt{2\lambda}v}dv\right)^{-1}\right\}\\&=\frac{b}{6}.\end{aligned}$$

Therefore our claim follows. ■

We proceed with the proof of Theorem 5 in the higher dimensional case ($d\geq 2$). In this case we use a technique which is a kind of approximation by the capacitary measure. We further decompose the equation (3.2) to

$$\begin{aligned}&Q_m(0\leq\sigma_B<t)-tC^0(B)\\&\quad=P_m^0(T_B<t)-tQ^0(t\leq\sigma_B<\infty)-Q^0[\sigma_B:\sigma_B<t].\end{aligned}$$

Moreover

$$P_m^0(T_B<t)=\int_H dx\int_{\partial B}\pi_B(dy)\int_0^t p_s^0(x,y)ds+P_m^0(T_B<t,\ T_B\circ\theta_t<\infty),$$

$$Q^0(t\leq\sigma_B<\infty)=Q^0(\sigma_B\circ\theta_t<\infty)-Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)$$

and

$$Q^0[\sigma_B:\sigma_B<t]=\lim_{u\downarrow 0}\int_H\nu_u(dx)\int_{\partial B}\pi_B(dy)\int_0^t sp_s^0(x,y)ds+g(t)$$

with

$$g(t)=Q^0[\sigma_B : \sigma_B<t]-\lim_{u\downarrow 0}\int_H \nu_u(dx)\int_{\partial B}\pi_B(dy)\int_0^t sp_s^0(x, y)ds.$$

The second equality is shown as follows:

$$\begin{aligned}
&P_m^0(T_B<t)-\int_H dx\int_{\partial B}\pi_B(dy)\int_0^t p_s^0(x, y)ds\\
&=\int_H dx\left\{P_x^0(T_B<t)-\int_{\partial B}\pi_B(dy)\int_0^t p_s^0(x, y)ds\right\}\\
&=\int_H dx\left\{\int_{\partial B}\pi_B(dy)\int_t^\infty p_s^0(x, y)ds-P_x^0(t\leq T_B<\infty)\right\}\\
&=\int_H dx\{P_x^0(T_B\circ\theta_t<\infty)-P_x^0(t\leq T_B<\infty)\}\\
&=P_m^0(T_B<t,\ T_B\circ\theta_t<\infty).
\end{aligned}$$

Furthermore we see that

$$\begin{aligned}
&\int_H dx\int_{\partial B}\pi_B(dy)\int_0^t p_s^0(x, y)ds\\
&=\int_{\partial B}\pi_B(dy)\int_0^t ds\int_0^\infty dr\int_H \nu_r(dx)p_s^0(x, y)\\
&=\int_{\partial B}\pi_B(dy)\int_0^t ds\int_s^\infty \nu_u(y_d)du\\
&=\int_{\partial B}\pi_B(dy)\left\{t\int_t^\infty \nu_u(y_d)du+\int_0^t u\nu_u(y_d)du\right\},
\end{aligned}$$

$$\begin{aligned}
Q^0(\sigma_B\circ\theta_t<\infty)&=\int_H \nu_t(dx)P_x^0(T_B<\infty)\\
&=\int_H \nu_t(dx)\int_{\partial B}\pi_B(dy)\int_0^\infty p_s^0(x, y)ds\\
&=\int_{\partial B}\pi_B(dy)\int_0^\infty ds\int_0^\infty \nu_t(dx_d)p_s^0(x_d, y_d)\\
&=\int_{\partial B}\pi_B(dy)\int_t^\infty \nu_u(y_d)du\\
&\left(=\sqrt{\frac{2}{\pi t}}\int_{\partial B}y_d\pi_B(dy)+O\left(\frac{1}{\sqrt{t^3}}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{u\downarrow 0}\int_H \nu_u(dx)\int_{\partial B}\pi_B(dy)\int_0^t sp_s^0(x, y)ds\\
&=\int_{\partial B}\pi_B(dy)\int_0^t s\nu_s(y_d)ds\\
&\left(=\sqrt{\frac{2t}{\pi}}\int_{\partial B}y_d\pi_B(dy)-\int_{\partial B}y_d^2\pi_B(dy)+o(1)\right)
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&Q_m(0\leq\sigma_B<t)-tC^0(B)\\
&=P_m^0(T_B<t,\ T_B\circ\theta_t<\infty)+tQ^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)-g(t).
\end{aligned}$$

Now we can prove that

$$P^0_m(T_B<t,\ T_B\circ\theta_t<\infty)=o(1), \tag{3.3}$$

$$tQ^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)=\begin{cases}\Phi(B)+o(1) & \text{if } d=2,\\ O(1/\sqrt{t}) & \text{if } d\geq 3\end{cases} \tag{3.4}$$

and

$$g(t)=\begin{cases}-\log t\ \Phi(B)+o(\log t) & \text{if } d=2,\\ O(1) & \text{if } d\geq 3\end{cases} \tag{3.5}$$

as $t\to\infty$. Hence our claim follows.

To prove the above equations (3.3), (3.4) and (3.5) we need several lemmas.

Lemma 9. *For each $h>0$, $C_h\equiv\int_H y_d P^0_y(T_B<h)dy$ is finite and it holds that*

$$P^0_m(t\leq T_B<t+h)\leq C_h\sqrt{\frac{2}{\pi t}},\qquad Q^0(t\leq\sigma_B<t+h)\leq C_h\frac{1}{\sqrt{2\pi t^3}}$$

for all $t>0$.

Proof. Since $w^0(t)$ is bounded on a finite time interval, for each $t>0$, there is a bounded set K_t in H such that for all $x\in B$, $P^0_x(w^0(s)\in K_t$ for all $s\in[0,\ t])>1/2$. Hence for each $x\in H$, $P^0_x(w^0(t)\in K_t)\geq P^0_x(w^0(t)\in K_t,\ T_B<t)\geq E^0_x[P^0_{w^0(T_B)}(w^0(t-s)\in K_t)|_{s=T_B}:\ T_B<t]\geq P^0_x(T_B<t)/2$, that is, $P^0_x(T_B<t)\leq 2P^0_x(w^0(t)\in K_t)$. Therefore

$$\begin{aligned}C_h&\leq 2\int_H x_d P^0_x(w^0(h)\in K_h)dx\\&=2\int_H dx x_d\int_{K_h}p^0_h(x,\ y)dy\\&=2\int_{K_h}dy\int_0^\infty x_d p^0_h(x_d,\ y_d)dx_d\\&=2\int_{K_h}y_d dy<\infty\end{aligned}$$

and

$$\begin{aligned}P^0_m(t\leq T_B<t+h)&\leq\int_H P^0_x(T_B\circ\theta_t<h)dx\\&=\int_H dx\int_H dy p^0_t(x,\ y)P^0_y(T_B<h)\\&=\int_H dy P^0_y(T_B<h)\int_0^\infty dx_d p^0_t(x_d,\ y_d)\\&=\int_H dy P^0_y(T_B<h)\int_{-y_d}^{y_d}\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{z^2}{2t}\right]dz\end{aligned}$$

$$\leq\sqrt{\frac{2}{\pi t}}\int_H y_d P_y^0(T_B<h)dy.$$

On the other hand

$$\begin{aligned}Q^0(t\leq\sigma_B<t+h)&\leq Q^0(\sigma_B\circ\theta_t<h)\\&=\int_H \nu_t(dx)P_x^0(T_B<h)\\&\leq\frac{1}{\sqrt{2\pi t^3}}\int_H x_d P_x^0(T_B<h)dx.\end{aligned}$$ ■

REMARK 9. From this lemma we see that $P_m^0(T_B<t)=O(\sqrt{t})$ as $t\to\infty$.

Lemma 10.

$$\int_t^\infty p_s^0(x, y)ds\begin{cases}\leq\dfrac{4x_d y_d}{(2\pi)^{d/2}dt^{d/2}} & \textit{for all } t>0,\\ =\dfrac{4x_d y_d}{(2\pi)^{d/2}dt^{d/2}}+O\Big(\dfrac{1}{t^{d/2+1}}\Big) & \textit{as } t\to\infty,\end{cases}$$

where the $O(t^{-d/2-1})$-constant is bounded whenever (x, y) is bounded.

Proof. Since

$$\int_t^\infty p_s^0(x, y)ds=\int_t^\infty ds(2\pi s)^{-d/2}\exp\Big[-\frac{|\tilde{x}-\tilde{y}|^2}{2s}\Big]\int_{(x_d-y_d)^2/(2s)}^{(x_d+y_d)^2/(2s)}e^{-z}dz,$$

where $\tilde{x}=(x_1, \cdots, x_{d-1})\in\boldsymbol{R}^{d-1}$, our first claim immediately follows. Moreover by using Taylor's formula

$$\exp\Big[-\frac{|\tilde{x}-\tilde{y}|^2}{2s}-z\Big]=1-\Big(\frac{|\tilde{x}-\tilde{y}|^2}{2s}+z\Big)e^{-\theta},$$

where $\theta>0$ depends on (s, z, x, y), we see that

$$\begin{aligned}0&\leq\frac{4x_d y_d}{(2\pi)^{d/2}dt^{d/2}}-\int_t^\infty p_s^0(x, y)ds\\&\leq\int_t^\infty ds(2\pi s)^{-d/2}\int_{(x_d-y_d)^2/(2s)}^{(x_d+y_d)^2/(2s)}\Big(\frac{|\tilde{x}-\tilde{y}|^2}{2s}+z\Big)dz\\&=t^{-d/2-1}\frac{2x_d y_d}{(2\pi)^{d/2}(d+2)}(|x-y|^2+2x_d y_d).\end{aligned}$$ ■

Using these lemmas we can prove the following:

Lemma 11. *Let $d\geq2$.*

$$P_m^0(T_B<t,\ T_B\circ\theta_t<\infty)=\begin{cases}O\Big(\dfrac{\log t}{\sqrt{t}}\Big) & \text{if } d=2,\\ O(1/\sqrt{t}) & \text{if } d\geq 3\end{cases}$$

and

$$Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)=\begin{cases}\Phi(B)/t+o(1/t) & \text{if } d=2,\\ O(1/\sqrt{t^3}) & \text{if } d\geq 3\end{cases}$$

as $t\to\infty$.

Proof. We first prove the two-dimensional case.

$$\begin{aligned}
&P_m^0(T_B<t,\ T_B\circ\theta_t<\infty)\\
&\quad\leq\int_H dxP_x^0(T_B<[t]-1,\ T_B\circ\theta_t<\infty)+\int_H dxP_x^0([t]-1\leq T_B<t)\\
&\quad=\int_H dx\int_H P_x^0(w^0(t)\in dy:T_B<[t]-1)P_y^0(T_B<\infty)+O(1/\sqrt{t})\quad\text{by Lemma 9}\\
&\quad=\int_{\partial B}\pi_B(dz)\int_H dx\int_{v\in\partial B}\int_0^{[t]-1}d_sP_x^0(w^0(T_B)\in dv:T_B\leq s)\int_H P_{t-s}^0(v,\ dy)g^0(y,\ z)\\
&\qquad+O(1/\sqrt{t})\\
&\quad=\int_{\partial B}\pi_B(dz)\int_H dx\int_{v\in\partial B}\int_0^{[t]-1}d_sP_x^0(w^0(T_B)\in dv:T_B\leq s)\int_{t-s}^{\infty}p_u^0(v,\ z)du\\
&\qquad+O(1/\sqrt{t}).
\end{aligned}$$

By Lemma 10 the first term is equal or less than

$$\begin{aligned}
&\frac{1}{\pi}\int_{\partial B}\pi_B(dz)z_2\int_H dx\int_{v\in\partial B}v^2\int_0^{[t]-1}d_sP_x^0(w^0(T_B)\in dv:T_B\leq s)\frac{1}{t-s}\\
&\quad\leq\frac{c^2}{\pi}C^0(B)\int_H dx\int_0^{[t]-1}\frac{P_x^0(T_B\in ds)}{t-s}\quad(c=\sup\{x_2>0:x\in\partial B\})\\
&\quad=\frac{c^2}{\pi}C^0(B)\int_H dx\Big\{\frac{P_x^0(T_B\leq t)}{t}+\int_0^{[t]-1}\Big(\frac{1}{t-s}-\frac{1}{t}\Big)P_x^0(T_B\in ds)\Big\}\\
&\quad=\frac{c^2}{\pi}C^0(B)\Big\{\frac{P_m^0(T_B\leq t)}{t}+\int_0^{[t]-1}\frac{s}{(t-s)t}P_m^0(T_B\in ds)\Big\}.
\end{aligned}$$

Now by using Lemma 9 the first term and the second term are equal to $O(1/\sqrt{t})$ and $O(\log t/\sqrt{t})$ as $t\to\infty$, respectively. In fact,

$$\begin{aligned}
\int_0^{[t]-1}\frac{s}{t-s}P_m^0(T_B\in ds)&\leq\sum_{k=1}^{[t]-1}\frac{k}{t-k}P_m^0(k-1\leq T_B<k)\\
&=O\Big(\sum_{k=1}^{[t]-1}\frac{k}{t-k}\cdot\frac{1}{\sqrt{k}}\Big)\\
&=O\Big(\int_1^{t/2}\frac{ds}{\sqrt{s}}+\sqrt{t}\int_1^{t/2}\frac{ds}{s}\Big)\\
&=O(\sqrt{t}\log t).
\end{aligned}$$

Therefore

$$P_m^0(T_B<t,\ T_B\circ\theta_t<\infty)=O(\log t/\sqrt{t})\qquad \text{if } d=2.$$

Moreover

$$\begin{aligned}
&Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)\\
&\quad =Q^0(\sigma_B<[t]-1,\ \sigma_B\circ\theta_t<\infty)+Q^0([t]-1\leq\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)\\
&\quad =\lim_{r\downarrow 0}\int_H\nu_r(dx)\int_{\partial B}\pi_B(dz)\int_{v\in\partial B}\int_0^{[t]-1}d_sP_x^0(w^0(T_B)\in dv:T_B\leq s)\int_{t-s}^{\infty}p_u^0(v,z)du\\
&\qquad +O(1/\sqrt{t^3}).
\end{aligned}$$

Thus

$$\begin{aligned}
&Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)-\frac{1}{\pi t}\int_{\partial B}z_2\pi_B(dz)\int_{\partial B}v_2Q^0(w^0(\sigma_B)\in dv:\sigma_B\leq[t]-1)\\
&\quad =\lim_{r\downarrow 0}\int_H\nu_r(dx)\int_{\partial B}\pi_B(dz)\int_{v\in\partial B}\int_0^{[t]-1}d_sP_x^0(w^0(T_B)\in dv:T_B\leq s)\\
&\qquad \times\left(\int_{t-s}^{\infty}p_u^0(v,z)du-\frac{z_2v_2}{\pi t}\right)+O\left(\frac{1}{\sqrt{t^3}}\right).
\end{aligned}$$

Since by Lemma 10

$$\int_{t-s}^{\infty}p_u^0(v,z)du-\frac{z_2v_2}{\pi t}=\frac{1}{\pi}z_2v_2\left(\frac{1}{t-s}-\frac{1}{t}\right)+O\left(\frac{1}{(t-s)^2}\right)$$

as $t\to\infty$. Hence we have

$$\begin{aligned}
&|Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)-\Phi(B)/t|\\
&\quad\leq\frac{c^2}{\pi}C^0(B)\lim_{u\downarrow 0}\int_H\nu_u(dx)\int_0^{[t]-1}\left(\frac{1}{t-s}-\frac{1}{t}\right)P_x^0(T_B\in ds)\\
&\qquad +C\lim_{u\downarrow 0}\int_H\nu_u(dx)\int_0^{[t]-1}(t-s)^{-2}P_x^0(T_B\in ds)+o\left(\frac{1}{t}\right),
\end{aligned}$$

where C is a finite constant. By using Lemma 9 we can also see that the first term is equal to $O(\log t/\sqrt{t^3})$ and the second term is to $O(1/\sqrt{t^3})$ as $t\to\infty$. Thus

$$Q^0(\sigma_B<t,\ \sigma_B\circ\theta<\infty)=\Phi(B)/t+o(1/t)\qquad \text{if } d=2.$$

In the higher dimensional cases ($d\geq 3$) by a similar way our claim immediately follows. In fact, by using Lemma 10 it is easy to see that

$$\int_0^{[t]-1}((t-s)^{-d/2}-t^{-d/2})P_m^0(T_B\in ds)=O\left(\frac{1}{\sqrt{t}}\right)$$

and

$$\lim_{u\downarrow 0}\int_H\nu_u(dx)\int_0^{[t]-1}((t-s)^{-d/2}-t^{-d/2})P_x^0(T_B\in ds)=O\left(\frac{1}{\sqrt{t^3}}\right)$$

as $t \to \infty$. From this we can get

$$P_m^0(T_B < t,\ T_B \circ \theta_t < \infty) = O(1/\sqrt{t})$$

and

$$Q^0(\sigma_B < t,\ \sigma_B \circ \theta_t < \infty) = O(1/\sqrt{t^3})$$

as $t \to \infty$. ■

From this lemma the equations (3.3) and (3.4) are obtained.
Note that in the above computations we also have the following result:

Theorem 8. *Let B be a compact subset of H with a positive capacity. Then*

$$P_m^0(T_B < t) = \sqrt{\frac{2t}{\pi}} \int_H 2x_d \pi_B(dx) + \begin{cases} -\dfrac{b}{2} + c + o(1) & \text{if } d=1, \\ -\displaystyle\int_H x_d^2 \pi_B(dx) + o(1) & \text{if } d \geq 2 \end{cases}$$

and

$$Q^0(t \leq \sigma_B < \infty) + \begin{cases} o(1/t) & \text{if } d=1, \\ \sqrt{\dfrac{2}{\pi t}} \displaystyle\int x_d \pi_B(dx) - \begin{cases} \Phi(B)/t + o(1/t) & \text{if } d=2, \\ O(1/\sqrt{t^3}) & \text{if } d \geq 3 \end{cases} \end{cases}$$

as $t \to \infty$.

Note that if $d=1$, then $\int_H 2x_d \pi_B(dx) = 1$ by Remark 7, and that $Q^0(\sigma_B < t) \to C^0(B)$ as $t \to \infty$. And if $d \geq 2$, then

$$P_m^0(T_B < t) = \int \pi_B(dx) \int_0^t ds \int_s^\infty \nu_u(x_d) du + P_m^0(T_B < t,\ T_B \circ \theta_t < \infty),$$
$$Q^0(t \leq \sigma_B < \infty) = \int \pi_B(dx) \int_t^\infty \nu_u(x_d) du - Q^0(\sigma_B < t,\ \sigma_B \circ \theta_t < \infty).$$

We prove the equation (3.5). It is enough to show that for all $h > 0$

$$g(t+h) - g(t) = \begin{cases} -\dfrac{h}{t} \Phi(B) + o\left(\dfrac{h}{t}\right) & \text{if } d=2, \\ O\left(\dfrac{h}{\sqrt{t^3}}\right) & \text{if d} \geq 3 \end{cases}$$

as $t \to \infty$. This can be proved by a similar way to the proof of Lemma 11. Since

$$P_x^0(T_B < t) - \int_{\partial B} \pi_B(dy) \int_0^t p_s^0(x, y) ds = P_x^0(T_B < t,\ T_B \circ \theta_t < \infty),$$

we have

$$
\begin{aligned}
&g(t+h)-g(t)\\
&=\lim_{u\downarrow 0}\int_H \nu_u(dx)\Big\{E_x^0[T_B : t\le T_B<t+h]-\int_{\partial B}\pi_B(dy)\int_t^{t+h} sp_s^0(x,y)ds\Big\}\\
&\le\lim_{u\downarrow 0}\int_H \nu_u(dx)\Big\{(t+h)P_x^0(t\le T_B<t+h)-t\int_{\partial B}\pi_B(dy)\int_t^{t+h} p_s^0(x,y)ds\Big\}\\
&=t\lim_{u\downarrow 0}\int_H \nu_u(dx)\Big\{P_x^0(t\le T_B<t+h)-\int_{\partial B}\pi_B(dy)\int_t^{t+h} p_s^0(x,y)ds\Big\}\\
&\quad +h\lim_{u\downarrow 0}\int_H \nu_u(dx)P_x^0(t\le T_B<t+h)\\
&=t\lim_{u\downarrow 0}\int_H \nu_u(dx)\{P_x^0(T_B<t+h,\ T_B\circ\theta_{t+h}<\infty)-P_x^0(T_B<t,\ T_B\circ\theta_t<\infty)\}\\
&\quad +hQ^0(t\le\sigma_B<t+h)\\
&=t\{Q^0(\sigma_B<t+h,\ \sigma_B\circ\theta_{t+h}<\infty)-Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)\}\\
&\quad +hQ^0(t\le\sigma_B<t+h)\\
&=\begin{cases} -\dfrac{h}{t}\Phi(B)+o\Big(\dfrac{h}{t}\Big) & \text{if } d=2,\\[2mm] O\Big(\dfrac{h}{\sqrt{t^3}}\Big) & \text{if } d\ge 3\end{cases}
\end{aligned}
$$

as $t\to\infty$ by Lemma 9 and Lemma 11. A lower estimate is given by the same way:

$$
\begin{aligned}
&g(t+h)-g(t)\\
&\quad\ge t\{Q^0(\sigma_B<t+h,\ \sigma_B\circ\theta_{t+h}<\infty)-Q^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)\}\\
&\quad\quad -h\{Q^0(\sigma_B\circ\theta_t<\infty)-Q^0(\sigma_B\circ\theta_{t+h}<\infty)\}\\
&\quad=\begin{cases} -\dfrac{h}{t}\Phi(B)+o\Big(\dfrac{h}{t}\Big) & \text{if } d=2,\\[2mm] O\Big(\dfrac{h}{\sqrt{t^3}}\Big) & \text{if } d\ge 3\end{cases}
\end{aligned}
$$

as $t\to\infty$. Therefore the equation (3.5) follows.

Finally we prove Theorem 6. Recall $N_t^B=\{0\le\sigma_B<t\}$. The first claim follows by the subadditive ergodic theorem [10] and $\boldsymbol{E}[X(N_t^B)]=Q_m(N_t^B)$. For the second claim it is enough to show that if

$$Q_m(N_t^B)=tC_1+\sqrt{t}\,C_2+o(\sqrt{t}\,)$$

as $t\to\infty$, where $C_1>0$ and $C_2\ge 0$, then

$$\boldsymbol{E}\exp\Big[iz\frac{X(N_t^B)-tC_1}{\sqrt{t}}\Big]\to\exp\Big[iC_2z-\frac{C_1}{2}z^2\Big]$$

as $t\to\infty$. However this can be easily obtained by using the following formula:

$$\boldsymbol{E}\exp[iz\langle X,F\rangle]=\exp\Big\{\int_W(1-e^{izF(w)})Q_m(dw)\Big\}$$

for all L^1-functions $F(w)$ on (W, Q_m). In fact, as $t\to\infty$,

$$-\int_W\left(1-\exp\left[\frac{iz}{\sqrt{t}}1_{N_t^B}\right]\right)Q_m(dw)-iz\sqrt{t}\,C_1\to izC_2-\frac{z^2}{2}C_1.$$

3.4. Further Results

The Brownian excursion law Q^0 can be also defined by the following: For $x=(\tilde{x}, x_d)\in \boldsymbol{R}^{d-1}\times(0, \infty)=H$,

$$Q_{\tilde{x}}^0=\lim_{r\downarrow 0}\int_0^\infty \nu_r(dx_d)P_x^0 \quad\text{and}\quad Q^0=\int_{\boldsymbol{R}^{d-1}}d\tilde{x}\,Q_{\tilde{x}}^0.$$

So in this final subsection we change the Lebesgue measure $d\tilde{x}$ on $\boldsymbol{R}^{d-1}\cong\partial H$, the boundary of H, to an arbitrary σ-finite measure $\mu(d\tilde{x})$ and consider the same problem as above. Let $p_r^{(d-1)}(\tilde{y}, \tilde{x})$ denote the transition density of the $(d-1)$-dimensional Brownian motion. Define

$$Q_\mu^0=\int_{\boldsymbol{R}^{d-1}}\mu(d\tilde{x})Q_{\tilde{x}}^0=\lim_{r\downarrow 0}\int_H\nu_r^\mu(dx)P_x^0,$$

where $\nu_r^\mu(dx)=\nu_r^\mu(x)dx$ with

$$\nu_r^\mu(x)=\int_{\boldsymbol{R}^{d-1}}p_r^{(d-1)}(\tilde{y}, \tilde{x})\mu(d\tilde{y})\nu_r(x_d).$$

Then $\nu^\mu=(\nu_r^\mu)_{r>0}$ is an entrance law for $(P_t^0)_{t\geq 0}$, i.e., $\nu_r^\mu P_t^0=\nu_{r+t}^\mu$. Moreover set

$$m^\mu(dx)=\int_0^\infty dr\nu_r^\mu(dx) \quad\text{and}\quad Q_{m^\mu}=\int_{-\infty}^\infty\theta_{-s}(Q_\mu^0)ds,$$

Note that $m^\mu(dx)=m^\mu(x)dx$ with

$$m^\mu(x)=\int_0^\infty\nu_r^\mu(x)dr=\frac{\Gamma(d/2)}{(2\pi)^{d/2}}x_d\int_{\boldsymbol{R}^{d-1}}\left(\frac{2}{|\tilde{x}-\tilde{y}|^2+x_d^2}\right)^{d/2}\mu(d\tilde{y}),$$

where Γ is gamma function. Then $m^\mu\in\boldsymbol{Pur}$ and Q_{m^μ} is the Kuznetsov measure associated with $(w^0(t), P_x^0, m^\mu)$. For this measure m^μ, we have the hollowing: The transition density $p_t^m(x, y)=p_t^0(x, y)/m^\mu(x)$, the capacity measure $\pi_B^m(dx)=m^\mu(x)\pi_B(dx)$, the capacity $C^m(B)=\pi^m(1)$, the co-capacitary measure $\hat{\pi}_B^m(dx)=Q_{m^\mu}(w(\tau_B)\in dx: 0<\tau_B<1)=Q_\mu^0(w(\sigma_B)\in dx: \sigma_B<\infty)$ and the co-capacity $\hat{C}^m(dx)=Q_{m^\mu}(0<\tau_B<1)=Q_\mu^0(\sigma_B<\infty)$. In this case w^0 is not symmetric relative to m^μ, thus in general $C^m(B)\neq\hat{C}^m(B)$.

Now fix a compact subset B of H with positive co-capacity. By the same computation as in §3.3, we can get

$$\begin{aligned}&Q_{m^\mu}(0\leq\sigma_B<t)-t\hat{C}^m(B)\\&=P_{m^\mu}^0(T_B<t)-tQ_\mu^0(t\leq\sigma_B<\infty)-Q_\mu^0[\sigma_B:\sigma_B<t]\\&=P_{m^\mu}^0(T_B<t,\ T_B\circ\theta_t<\infty)+tQ_\mu^0(\sigma_B\circ\theta_t<\infty)-g_\mu(t),\end{aligned}$$

where

$$g_\mu(t)=Q_\mu^0[\sigma_B:\sigma_B<t]-\lim_{r\downarrow 0}\int_H \nu_r^\mu(dx)\int_{\partial B}\pi_B(dy)\int_0^t p_s^0(x,\,y)ds.$$

We only consider the case that $\mu(d\tilde{x})=\delta_{x^{(k-1)}(0)}(dx^{(k-1)})dx^{(d-k)}$ for a certain $k=1,\,2,\,\cdots,\,d$ and a certain fixed point $x^{(k-1)}(0)\in \boldsymbol{R}^{k-1}$. Then we have the following:

$$\nu_r^\mu(x)=\frac{x_d}{(2\pi)^{k/2}r^{1+k/2}}\exp\left[-\frac{|x^{(k-1)}-x^{(k-1)}(0)|^2+x_d^2}{2r}\right]$$

and

$$m^\mu(x)=\frac{\Gamma(k/2)}{\pi^{k/2}}\frac{x_d}{\{|x^{(k-1)}-x^{(k-1)}(0)|^2+x_d^2\}^{k/2}}.$$

Note that if $k=1$, then $\nu_r^\mu=\nu_r$ and $m^\mu(x)=1$. So it suffices to consider the case of $2\leq k\leq d$. In this case it can be seen that

$$P_{m^\mu}^0(T_B<t,\ T_B\circ\theta_t<\infty)=\begin{cases}O\left(\dfrac{\log t}{t}\right) & (k=d=2),\\ O(1/t) & (k=2,\ d\geq 3),\\ O(1/t^{k/2}) & (3\leq k\leq d)\end{cases}$$

and

$$Q_\mu^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)=\begin{cases}\Phi_\mu(B)/t+o(1/t) & (d=2),\\ O(1/\sqrt{t^3}) & (d=3),\\ O(1/t^2) & (d\geq 4).\end{cases}$$

More exactly when $d\geq 4$, we have

$$Q_\mu^0(\sigma_B<t,\ \sigma_B\circ\theta_t<\infty)=\begin{cases}O(1/t^{1+k/2}) & (k\leq d-3),\\ O(1/t^{d/2}) & (k=d-2,\ d-1,\ d).\end{cases}$$

Furthermore

$$g_\mu(t)=\begin{cases}-\log t\Phi_\mu(B)+o(\log t) & (d=2),\\ O(1) & (d\geq 3)\end{cases}$$

as $t\to\infty$, where

$$\begin{aligned}\Phi_\mu(B)&=\frac{1}{\pi}\int_{\partial B}x_2\pi_B(dx)Q_\mu^0[w^{(2)}(\sigma_B):\sigma_B<\infty]\\ &=\frac{1}{\pi}\int_{\partial B}x_2 m^\mu(x)^{-1}\pi_B^m(dx)\int_{\partial B}y_2\hat{\pi}_B^m(dy).\end{aligned}$$

Therefore we have the following result: For each $2\leq k\leq d$,

$$Q_{m^\mu}(0 \leq \sigma_B < t) = t\widehat{C}^m(B) + \begin{cases} \log t\Phi_\mu(B) + o(\log t) & (d=2), \\ O(1) & (d \geq 3) \end{cases}$$

and $P^0_{m^\mu}(T_B < t)$ is equal to

$$\begin{cases} \log t \int \frac{x_d}{(2\pi)^{k/2}} \pi_B(dx) + O(1) & (k=2), \\ \frac{\Gamma(k/2-1)}{2\pi^{k/2}} \int x_d \{|x^{(k-1)} - x^{(k-1)}(0)|^2 + x_d^2\}^{1-k/2} \pi_B(dx) + o(1) & (k \geq 3) \end{cases}$$

as $t \to \infty$. Moreover since

$$Q^0_\mu(\sigma_B \circ \theta_t < \infty) = \frac{2t^{-k/2}}{k(2\pi)^{k/2}} \int x_d \pi_B(dx)$$
$$- \frac{2t^{-k/2-1}}{(k+2)(2\pi)^{k/2}} \int x_d \{|x^{(k-1)} - x^{(k-1)}(0)|^2 + x_d^2\} \pi_B(dx) + O(t^{-k/2-2}),$$

we can get the asymptotic behavior of

$$Q^0_\mu(t \leq \sigma_B < \infty) = Q^0_\mu(\sigma_B \circ \theta_t < \infty) - Q^0_\mu(\sigma_B < t,\ \sigma_B \circ \theta_t < \infty).$$

However it is too tedious to describe it.

Of course we also apply the above result to the equilibrium process with immigration and obtain a similar result to Theorem 6.

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