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ASYMPTOTIC BEHAVIOR OF RADIAL SOLUTIONS TO AN ELLIPTIC-PARABOLIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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1. Introduction

Chemical reactions that take place in a bounded domain are often described by some reaction-diffusion systems with linear boundary conditions. Such kinds of reaction-diffution systems have been investigated by many reseachers, e.g., Rothe [16], Feng [4], Hoshino-Yamada [5], and others (see also Ruan [17, Theorem 5. 1]). On the other hand interfacial reactions, i.e., chemical reactions that take place on the interface between two phases (as oil and water), are often described by systems of diffusion equations with coupled, nonlinear boundary conditions. Also some important interfacial reactions in chemical engineering are described by elliptic-parabolic systems with coupled, nonlinear boundary conditions. Unfortunately it is difficult to deal with coupled, nonlinear boundary conditions by standard techniques. In fact, not so many fundamental theories are known concerning parabolic systems with non-monotonous, coupled, nonlinear boundary conditions. Recently, surmounting these difficulties, several mathematicians have investigated some systems of 1-dimensional diffusion equations with nonlinear boundary conditions that are related to interfacial reactions (see Yamada-Yotsutani [19], Shinomiya [18], Nagasawa [15], Iida-Yamada-Yotsutani [7], [8], [9], Iida-Yamada-Yanagida-Yotsutani [11], Iida-Ninomiya [6]; see also [17] and the references therein). As for elliptic-parabolic systems related to interfacial reactions, however, there seems to have been no investigations except Yotsutani [21], in which the existence and uniqueness of solutions are shown. The present paper is a first trial to construct a fundamental theory on asymptotic behavior of solutions to such an elliptic-parabolic system with coupled, nonlinear boundary conditions.

Let r_0, r_1 be given numbers with $0<r_0<r_1<1$, and put

$$\begin{aligned}
&\Omega_0=\{x\in\boldsymbol{R}^2;\ |x|<r_0\}, && \Gamma_0=\{x\in\boldsymbol{R}^2;\ |x|=r_0\},\\
&\Omega_*=\{x\in\boldsymbol{R}^2;\ r_0<|x|<r_1\}, && \Gamma_1=\{x\in\boldsymbol{R}^2;\ |x|=r_1\},\\
&\Omega_1=\{x\in\boldsymbol{R}^2;\ r_1<|x|<1\}, && \Gamma_2=\{x\in\boldsymbol{R}^2;\ |x|=1\},
\end{aligned}$$
$$\Omega=\Omega_0\cup\Omega_*\cup\Omega_1$$

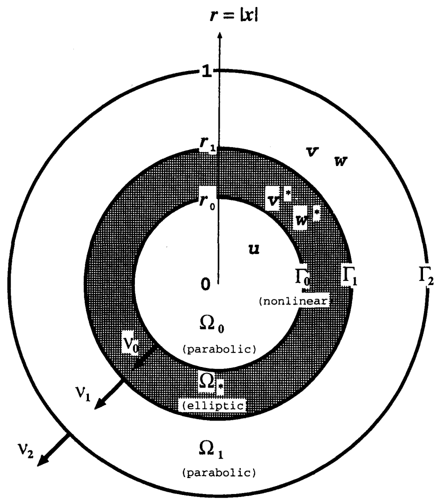

Fig. 1 Domain Ω

(see Fig. 1). We will consider an elliptic-parabolic system in a cylindrical domain $\Omega\times(0, \infty)$:

$$\begin{cases} \Delta_x u - a(|x|)u_z = 0, & (x, z)\in\Omega_0\times(0, \infty), \\ \Delta_x v^* = 0,\ \Delta_x w^* = 0, & (x, z)\in\Omega_*\times(0, \infty), \\ \Delta_x v - b(|x|)v_z = 0,\ \Delta_x w - c(|x|)w_z = 0, & (x, z)\in\Omega_1\times(0, \infty), \end{cases}$$

where Δ_x denotes Laplacian with respect to $x=(x_1, x_2)$. The coefficient $a(|x|)$ is positive in Ω_0 and vanishes on Γ_0; the coefficients $b(|x|)$, $c(|x|)$ are positive in Ω_1 and vanishes on $\Gamma_1\cup\Gamma_2$. The unknown functions u, v^*, w^*, v and w are related to one another by the following boundary conditions. Let ν_j be the outward normal unit vector on a circle Γ_j and $\partial/\partial\nu_j$ the derivative in the direction of ν_j ($j=0, 1, 2$). On $\Gamma_0\times(0, \infty)$ we impose

$$-\frac{\partial u}{\partial\nu_0}=R(u, v^*, w^*),\ \frac{\partial v^*}{\partial\nu_0}=m_0R(u, v^*, w^*),\ \frac{\partial w^*}{\partial\nu_0}=-n_0R(u, v^*, w^*),$$

where $R(u, v^*, w^*)$ is a nonlinear function and m_0, n_0 are positive constants. On $\Gamma_1\times(0, \infty)$ we impose

$$\begin{aligned} &v^*=v, && w^*=w, \\ &\frac{\partial v^*}{\partial\nu_1}=m_1\frac{\partial v}{\partial\nu_1}, && \frac{\partial w^*}{\partial\nu_1}=n_1\frac{\partial w}{\partial\nu_1}, \end{aligned}$$

where m_1 and n_1 are positive constants. On $\Gamma_2\times(0, \infty)$ we impose homogeneous Neumann conditions:

$$\frac{\partial v}{\partial \nu_2}=0, \ \frac{\partial w}{\partial \nu_2}=0.$$

At $z=0$ we impose

$$u(x, 0)=u_0(|x|)\geq 0, \qquad x\in\Omega_0,$$
$$v(x, 0)=v_0(|x|)\geq 0, \ w(x, 0)=w_0(|x|)\geq 0, \qquad x\in\Omega_1,$$

where u_0, v_0 and w_0 are given radially symmetric functions.

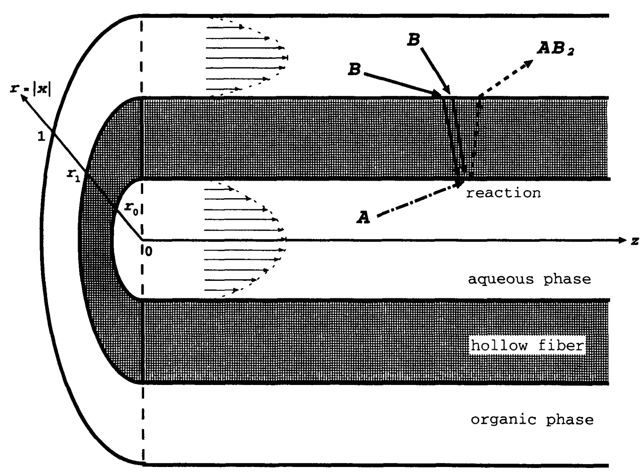

Fig. 2 Chemical situation

This boundary value problem was proposed by Yoshizuka-Kondo-Nakashio [20] as a chemical model. It describes some interfacial reactions that take place in a membrane extractor using a hollow fiber (see Fig. 2). In their model, u denotes the concentration of the metallic reactant A in the aqueous solution flowing through $\Omega_0\times(0, \infty)$; v (resp. w) denotes that of the organic reactant B (resp. the product AB_2) in the organic solution flowing through $\Omega_1\times(0, \infty)$; v^* (resp. w^*) denotes that of B (resp. AB_2) permeating through the hollow fiber $\Omega_*\times(0, \infty)$. Since the hollow fiber is hydrophobic, the interface between the aqueous and organic phases in $\Omega\times(0, \infty)$ is the inner surface of the fiber ($\Gamma_0\times(0, \infty)$). Thus a chemical reaction such as $A+2B\rightleftarrows AB_2$ takes place only on the interface $\Gamma_0\times(0, \infty)$. In order to see the essential effect of the interfacial reaction on the extraction of A by B from the aqueous phase, they considered a simple situation: the reaction has attained a stationary state; the streams through $\Omega_0\times(0, \infty)$ and $\Omega_1\times(0, \infty)$ have become laminar flow. In this situation they derived the boundary value problem stated above. In their model $a(r)$, $b(r)$, $c(r)$ and $R(u, v, w)$ are given by

$$
(1.1)\quad \begin{cases} a(r)=a_0(r_0^2-r^2), \\ b(r)=b_0\left\{1-r^2-(1-r_1^2)\dfrac{\log r}{\log r_1}\right\}, \\ c(r)=c_0\left\{1-r^2-(1-r_1^2)\dfrac{\log r}{\log r_1}\right\}, \\ R(u,\ v,\ w)=\dfrac{uv^2-w}{(s_1+s_2v)^2}, \end{cases}
$$

where a_0, b_0, c_0, s_1, s_2 are positive constants and $r=|x|$ (cf. [20]).

If we restrict our attention to solutions that are radially symmetric with respect to x, then the boundary value problem is reduced to the following system on $[0, 1]\times[0, \infty)$ for $u(r, z)$ $(0\le r\le r_0)$, $v^*(r, z)$ and $w^*(r, z)$ $(r_0\le r\le r_1)$, $v(r, z)$ and $w(r, z)$ $(r_1\le r\le 1)$:

$$
(\mathrm{EP})\quad \begin{cases} a(r)u_z=u_{rr}+\dfrac{1}{r}u_r, & (r,\ z)\in(0,\ r_0)\times(0,\ \infty), \\ v^*_{rr}+\dfrac{1}{r}v^*_r=w^*_{rr}+\dfrac{1}{r}w^*_r=0, & (r,\ z)\in(r_0,\ r_1)\times(0,\ \infty), \\ b(r)v_z=v_{rr}+\dfrac{1}{r}v_r,\ c(r)w_z=w_{rr}+\dfrac{1}{r}w_r, & (r,\ z)\in(r_1,\ 1)\times(0,\ \infty); \\ & \\ u_r(0,\ z)=0, & z\in(0,\ \infty), \\ -u_r(r_0,\ z)=\dfrac{1}{m_0}v^*_r(r_0,\ z)=-\dfrac{1}{n_0}w^*_r(r_0,\ z) & \\ \qquad =R(u(r_0,\ z),\ v^*(r_0,\ z),\ w^*(r_0,\ z)), & z\in(0,\ \infty), \\ v^*(r_1,\ z)=v(r_1,\ z),\ w^*(r_1,\ z)=w(r_1,\ z), & z\in(0,\ \infty), \\ v^*_r(r_1,\ z)=m_1v_r(r_1,\ z),\ w^*_r(r_1,\ z)=n_1w_r(r_1,\ z), & z\in(0,\ \infty), \\ v_r(1,\ z)=w_r(1,\ z)=0, & z\in(0,\ \infty); \\ & \\ u(r,\ 0)=u_0(r)\ge 0, & r\in(0,\ r_0), \\ v(r,\ 0)=v_0(r)\ge 0,\ w(r,\ 0)=w_0(r)\ge 0, & r\in(r,\ 1). \end{cases}
$$

As to this system, Yotsutani [21] has shown the existence and uniqueness of a nonnegative global solution in the framework of Sobolev spaces. Moreover it can be shown that the solution is actually of class C^∞ up to the boundary by developing the method of [21]. The detail will be announced in Iida-Yamada-Yotsutani [10].

The aim of the present paper is to investigate the asymptotic behavior of the solution as $z\to\infty$. From chemical viewpoint, it is expected that the reaction approaches a chemical equilibrium as z increases. We will show that the solution to (EP) actually converges to an equilibrium as $z\to\infty$ and will give the rates of the convergence.

The nonlinear parts of our boundary conditions are essentially in the same form as those of the boundary conditions that are treated in the recent works ([19], [7], [11]) by a group including the author. However, our system (EP) is quite different from theirs in the following sense : in their 3-component parabolic system for u, v and w, u is explicitly associated with v and w by nonlinear boundary conditions ; in our 5-component elliptic-parabolic system for u, v^*, w^*, v and w, u is only implicitly related to v and w through v^* and w^* (see Fig. 1). Therefore we must solve the elliptic equations for v^* and w^* with nonlinearly coupled boundary conditions in order to understand the interaction among u, v and w. This fact makes our analysis more complicated than theirs. Fortunately the equations for v^* and w^* can be solved explicitly. Hence (EP) is reduced to a parabolic system for u, v and w with nonlinear boundary conditions (see §3). Nevertheless, this system is not easy to analyze. The main difficulties come from the following facts : the nonlinear terms in the reduced boundary conditions are much more complicated than those of the original boundary conditions ; comparison principle dose not hold ; the principal eigenvalue of the linearized operator at an equilibrium is always zero. To overcome such difficulties we employ some devices such as to make use of the fact that $R(u, v, w)$ is "component-wise monotonous" in respective components (see (R. 2) of §2), to introduce a Lyapunov function fitting in with the nonlinearity peculiar to chemical reactions, to construct infinite number of energy inequalities systematically, and to take advantage of "mass-conservation" law (see Proposition 2.2).

The organization of this paper is as follows. In the next section we state our main results with the assumptions for $a(r)$, $b(r)$, $c(r)$ and $R(u, v, w)$. In §3 we reduce the elliptic-parabolic system (EP) to a parabolic system (P). This reduction is the basis of the whole argument in the subsequent sections. In §4 we give fundamental lemmas that are useful throughout the paper. In §5 we give a Lyapunov function together with several energy functionals and derive differential inequalities for them. Those inequalities imply the uniform convergence on $\overline{\Omega}$ of the solution to an equilibrium. We devote §6 to the spectral analysis for the linearized operator. Applying the results of §6, we seek the optimal rate of the uniform convergence of the solution in §7. Moreover, in §§8 and 9 we give the optimal rates of decay for all derivatives of the solution by constructing infinite number of energy inequalities.

NOTATION

Let i be one of the subscripts 0, $*$ and 1. Throughout the paper we denote by $C_r^k(\overline{\Omega}_i)$ (resp. $L_r^p(\Omega_i)$, $\cdots$, etc.) the subspace of radially symmetric functions that belong to $C^k(\overline{\Omega}_i)$ (resp. $L^p(\Omega_i)$, $\cdots$, etc.). We also use some weighted L^2-spaces. For a nonnegative function $\omega \in L_r^1(\Omega_i)$, $L_r^2(\Omega_i\,;\,\omega)$ is the Hilbert space of radially symmetric functions ϕ on Ω_i satisfying

$$\int_{\underline{r}_i}^{\overline{r}_i} \phi(r)^2 \omega(r) r dr < \infty,$$

where

$$(\underline{r}_0, \overline{r}_0)=(0, r_0), (\underline{r}_*, \overline{r}_*)=(r_0, r_1), (\underline{r}_1, \overline{r}_1)=(r_1, 1).$$

We use the following symbols to represent various norms of a radially symmetric function ϕ on Ω_i:

$$\|\phi\|_p=\|\phi\|_{p,\Omega_i}: =\left\{\int_{\underline{r}_i}^{\overline{r}_i}|\phi(r)|^p r dr\right\}^{1/p} \qquad (1\leq p<\infty),$$

$$\|\phi\|_\infty=\|\phi\|_{\infty,\Omega_i}: =\operatorname*{ess\,sup}_{\underline{r}_i<r<\overline{r}_i}|\phi(r)|,$$

$$\|\phi\|_{2;\,\omega}=\|\phi\|_{2,\Omega_i;\,\omega}: =\left\{\int_{\underline{r}_i}^{\overline{r}_i}\phi(r)^2\omega(r) r dr\right\}^{1/2}.$$

When we use high order differential operators, we sometimes abbreviate them as

$$D_z^k=\left(\frac{\partial}{\partial z}\right)^k, \quad D_r^k=\left(\frac{\partial}{\partial r}\right)^k,$$

where k is a positive integer.

2. Main results

Throughout the paper we assume the following conditions on $a(r)$, $b(r)$, $c(r)$ and $R(u, v, w)$:

$$\text{(A. 1)} \quad \begin{cases} a(|x|)\in C^\infty(\overline{\Omega}_0), \ a(r)>0 \text{ on } [0, r_0), \\ b(|x|)\in C^\infty(\overline{\Omega}_1), \ b(r)>0 \text{ on } (r_1, 1), \\ c(|x|)\in C^\infty(\overline{\Omega}_1), \ c(r)>0 \text{ on } (r_1, 1). \end{cases}$$

(A. 2) There exists a constant $d_0>0$ such that

$$\lim_{r\to r_0-0}\frac{a(r)}{(r_0-r)^{d_0}}>0, \ \lim_{r\to r_1+0}\frac{b(r)}{(r-r_1)^{d_0}}>0, \ \lim_{r\to r_1+0}\frac{c(r)}{(r-r_1)^{d_0}}>0.$$

(R. 1) There exist an open subset $\mathcal{O}$ of $\boldsymbol{R}^3$ and a positive function $S(u, v, w)\in C^\infty(\mathcal{O})$ such that

$$\mathcal{O}\supset[-\delta_S, \delta_S]^3\cup[0, \infty)^3,$$

$$R(u, v, w)=\frac{u^l v^m-w^n}{S(u, v, w)} \quad \text{on } \mathcal{O},$$

where δ_S is a positive number and l, m, n are positive integers.

(R. 2) $\dfrac{\partial R}{\partial v}(u, v, w)\geq 0$, $\dfrac{\partial R}{\partial w}(u, v, w)\leq 0$ for every $(u, v, w)\in[0, \infty)^3$.

(R. 3) There exists a positive constant C_R such that

$$\max\{-u^{2p-1}R(u, v, w), -v^{2p-1}R(u, v, w), w^{2p-1}R(u, v, w)\} \leq C_R(u^{2p}+v^{2p}+w^{2p})$$

for all $(u, v, w) \in [0, \infty)^3$ and $p \in [1, \infty)$.

Clearly the functions $a(r)$, $b(r)$, $c(r)$ and $R(u, v, w)$ given by (1.1) satisfy these conditions. For the boundary data (u_0, v_0, w_0) we put

$$M_{uw} := \int_0^{r_0} u_0 a r dr + \frac{n_1}{n_0} \int_{r_1}^1 w_0 c r dr,$$
$$M_{vw} := \frac{m_1}{m_0} \int_{r_1}^1 v_0 b r dr + \frac{n_1}{n_0} \int_{r_1}^1 w_0 c r dr.$$

For the subsequent arguments, we summarize fundamental facts. The following two propositions are essentially obtained in [21] (see also [10]).

Proposition 2.1. *In addition to* (A. 1), (A. 2), (R. 1), (R. 2) *and* (R. 3), *assume that* (u_0, v_0, w_0) *satisfies*

$$\begin{array}{ll} u_0 \in L^\infty(0, r_0), & u_0 \geq 0 \text{ in } (0, r_0), \\ v_0,\ w_0 \in L^\infty(r_1, 1),\ v_0 \geq 0, & w_0 \geq 0 \text{ in } (r_1, 1). \end{array}$$

Then the boundary value problem (EP) *has a unique nonnegative solution* $(u, v^*, w^*, v, w) \in C^\infty([0, r_0] \times (0, \infty)) \times C^\infty([r_0, r_1] \times (0, \infty))^2 \times C^\infty([r_1, 1] \times (0, \infty))^2$ *satisfying the boundary condition at* $z=0$ *in* L^2*-sense, i.e.*,

$$\lim_{z \to 0} \{\|u(\cdot, z) - u_0\|_{2,\Omega_0} + \|v(\cdot, z) - v_0\|_{2,\Omega_1} + \|w(\cdot, z) - w_0\|_{2,\Omega_1}\} = 0.$$

Moreover the solution is bounded uniformly with respect to z *and its positivity is determined by that of* M_{uw}, M_{vw} *in the following way*:

$$\begin{cases} u>0 \text{ on } [0, r_0]\times(0, \infty),\ v^*>0,\ w^*>0 \text{ on } [r_0, r_1]\times(0, \infty), \\ \qquad v>0,\ w>0 \text{ on } [r_1, 1]\times(0, \infty) \text{ if } M_{uw}>0 \text{ and } M_{vw}>0; \\ u>0 \text{ on } [0, r_0]\times(0, \infty),\ v^* \equiv w^* \equiv 0 \text{ on } [r_0, r_1]\times(0, \infty), \\ \qquad v \equiv w \equiv 0 \text{ on } [r_1, 1]\times(0, \infty) \text{ if } M_{uw}>0 \text{ and } M_{vw}=0; \\ u \equiv 0 \text{ on } [0, r_0]\times(0, \infty),\ v^*>0,\ w^* \equiv 0 \text{ on } [r_0, r_1]\times(0, \infty), \\ \qquad v>0,\ w \equiv 0 \text{ on } [r_1, 1]\times(0, \infty) \text{ if } M_{uw}=0 \text{ and } M_{vw}>0; \\ u \equiv 0 \text{ on } [0, r_0]\times(0, \infty),\ v^* \equiv w^* \equiv 0 \text{ on } [r_0, r_1]\times(0, \infty), \\ \qquad v \equiv w \equiv 0 \text{ on } [r_1, 1]\times(0, \infty) \text{ if } M_{uw}=0 \text{ and } M_{vw}=0. \end{cases}$$

The solution satisfies the law of "mass-conservation." Precisely the following holds.

Proposition 2.2. *Let* (u, v^*, w^*, v, w) *be a solution to* (EP). *Then* (u, v, w) *satisfies*

$$\text{(M)}\qquad \begin{cases} \displaystyle\int_0^{r_0} u(r, z)a(r)rdr + \frac{n_1}{n_0}\int_{r_1}^1 w(r, z)c(r)rdr = M_{uw}, \\ \displaystyle\frac{m_1}{m_0}\int_{r_1}^1 v(r, z)b(r)rdr + \frac{n_1}{n_0}\int_{r_1}^1 w(r, z)c(r)rdr = M_{vw} \end{cases}$$

for $z \in [0, \infty)$.

Consider an "equilibrium" for (EP), i.e., a solution that is independent of z. It is easy to see that an equilibrium for (EP) should be a set $(u_\infty, v_\infty, w_\infty, v_\infty, w_\infty)$ of constant functions if it exists. As for the equilibrium to which the solution of (EP) converges as $z \to \infty$, we should take Proposition 2.2 and the nonnegativity of the solution into consideration. Then the constants u_∞, v_∞ and w_∞ should satisfy

$$(\mathrm{EP}_\infty)\qquad \begin{cases} u_\infty \geq 0,\ v_\infty \geq 0,\ w_\infty \geq 0, \\ R(u_\infty, v_\infty, w_\infty) = 0, \\ \|a\|_{1,\Omega_0} u_\infty + \dfrac{n_1}{n_0}\|c\|_{1,\Omega_1} w_\infty = M_{uw}, \\ \dfrac{m_1}{m_0}\|b\|_{1,\Omega_1} v_\infty + \dfrac{n_1}{n_0}\|c\|_{1,\Omega_1} w_\infty = M_{vw}. \end{cases}$$

We can see the following by Theorem 2 in [7].

Proposition 2.3. *Suppose that* (A. 1) *and* (R. 1) *hold. Then, for each pair* (M_{uw}, M_{vw}) *of nonnegative numbers, there exists a unique solution* $(u_\infty, v_\infty, w_\infty)$ *to* (EP_∞). *More precisely*,

$$\begin{cases} u_\infty > 0,\ v_\infty > 0 \text{ and } w_\infty > 0 & \text{if } M_{uw} > 0 \text{ and } M_{vw} > 0; \\ u_\infty = \dfrac{1}{\|a\|_{1,\Omega_0}}\displaystyle\int_0^{r_0} u_0 a r dr > 0,\ v_\infty = w_\infty = 0 & \text{if } M_{uw} > 0 \text{ and } M_{vw} = 0; \\ v_\infty = \dfrac{1}{\|b\|_{1,\Omega_1}}\displaystyle\int_{r_1}^1 v_0 b r dr > 0,\ u_\infty = w_\infty = 0 & \text{if } M_{uw} = 0 \text{ and } M_{vw} > 0; \\ u_\infty = v_\infty = w_\infty = 0 & \text{if } M_{uw} = 0 \text{ and } M_{vw} = 0. \end{cases}$$

Now let us consider the asymptotic behavior of solutions as $z \to \infty$. When $M_{uw} = 0$ or $M_{vw} = 0$, we can easily obtain the asymptotic behavior from Proposition 2. 1. For instance, we briefly explain the case $M_{uw} = 0 < M_{vw}$. Since $u \equiv 0$, $w^* \equiv 0$ and $w \equiv 0$, v^* and v satisfies

$$\begin{cases} v^*_{rr}+\dfrac{1}{r}v^*_r=0, & (r,\ z)\in(r_0,\ r_1)\times(0,\ \infty),\\ b(r)v_z=v_{rr}+\dfrac{1}{r}v_r, & (r,\ z)\in(r_1,\ 1)\times(0,\ \infty);\\ & \\ v^*_r(r_0,\ z)=0, & z\in(0,\ \infty),\\ v^*(r_1,\ z)=v(r_1,\ z),\ v^*_r(r_1,\ z)=m_1v_r(r_1,\ z), & z\in(0,\ \infty),\\ v_r(1,\ z)=0, & z\in(0,\ \infty);\\ & \\ v(r,\ 0)=v_0(r)\geq 0, & r\in(r_1,\ 1). \end{cases}$$

Hence

$$v^*(r,\ z)\equiv v(r_1,\ z)\quad \text{for } (r,\ z)\in[r_0,\ r_1]\times(0,\ \infty),$$

and v satisfies a linear diffusion equation with homogeneous Neumann boundary condition. Then the standard Energy Method leads us to

$$\left\|v(\cdot,\ z)-\frac{1}{\|b\|_{1,\Omega_1}}\int_{r_1}^{1}v_0brdr\right\|_{\infty,\Omega_1}=O(\exp(-\lambda_+z))\quad \text{as } z\to\infty,$$

where λ_+ is the least positive eigenvalue for

$$\begin{cases} -v_{rr}-\dfrac{1}{r}v_r=\lambda b(r)v & \text{in } (r_1,\ 1),\\ v_r(r_1)=v_r(1)=0. \end{cases}$$

For this reason, here and hereafter we will consider only the case $M_{uw}>0$ and $M_{vw}>0$.

Theorem A. *In addition to* (A. 1), (A. 2), (R. 1), (R. 2) *and* (R. 3), *assume that* $M_{uw}>0$ *and* $M_{vw}>0$ *hold. Let* $(u,\ v^*,\ w^*,\ v,\ w)$ *be the solution to* (EP) *and* $(u_\infty,\ v_\infty,\ w_\infty)$ *the solution to* (EP$_\infty$). *Then*

$$\left.\begin{array}{ll} u(\cdot,\ z)\longrightarrow u_\infty & \textit{uniformly on } \Omega_0\\ v^*(\cdot,\ z)\longrightarrow v_\infty,\ w^*(\cdot,\ z)\longrightarrow w_\infty & \textit{uniformly on } \Omega_*\\ v(\cdot,\ z)\longrightarrow v_\infty,\quad w(\cdot,\ z)\longrightarrow w_\infty & \textit{uniformly on } \Omega_1 \end{array}\right\}\quad \textit{as } z\longrightarrow\infty.$$

To investigate the behavior of solutions near the equilibrium we will study the linearization of (EP) at $(u_\infty,\ v_\infty,\ w_\infty,\ v_\infty,\ w_\infty)$ from spectral analysis. For $(u,\ v,\ w)\in\boldsymbol{R}^3$ we set

$$R^L(u,\ v,\ w)=R^\infty_u u+R^\infty_v v+R^\infty_w w \tag{2.1}$$

with

$$R^\infty_u=\frac{\partial R}{\partial u}(u_\infty,\ v_\infty,\ w_\infty),\ R^\infty_v=\frac{\partial R}{\partial v}(u_\infty,\ v_\infty,\ w_\infty),\ R^\infty_w=\frac{\partial R}{\partial w}(u_\infty,\ v_\infty,\ w_\infty).$$

We introduce an eigenvalue problem associated with the linearization of (EP) at $(u_\infty, v_\infty, w_\infty, v_\infty, w_\infty)$. It is a linear boundary value problem for a set (u, v^*, w^*, v, w) of radially symmetric functions u on $\overline{\Omega}_0$, v^* and w^* on $\overline{\Omega}_*$, v and w on $\overline{\Omega}_1$ with a parameter λ:

$$\text{(EV)}\quad \begin{cases} -u_{rr}-\dfrac{1}{r}u_r=\lambda a(r)u & \text{in } (0, r_0), \\ -v^*_{rr}-\dfrac{1}{r}v^*_r=-w^*_{rr}-\dfrac{1}{r}w^*_r=0 & \text{in } (r_0, r_1), \\ -v_{rr}-\dfrac{1}{r}v_r=\lambda b(r)v,\ -w_{rr}-\dfrac{1}{r}w_r=\lambda c(r)w & \text{in } (r_1, 1)\,; \\ u_r(0)=0, & \\ -u_r(r_0)=\dfrac{1}{m_0}v^*_r(r_0)=-\dfrac{1}{n_0}w^*_r(r_0)=R^L(u(r_0), v^*(r_0), w^*(r_0)), & \\ v^*(r_1)=v(r_1),\ w^*(r_1)=w(r_1), & \\ v^*_r(r_1)=m_1v_r(r_1),\ w^*_r(r_1)=n_1w_r(r_1), & \\ v_r(1)=w_r(1)=0. & \end{cases}$$

We say that a number λ is an eigenvalue for (EV) when there exists a set $(u, v^*, w^*, v, w)\neq(0, 0, 0, 0, 0)$ of radially symmetric functions satisfying (EV). We will prove the following proposition in §6.

Proposition 2.4. *In addition to* (A. 1) *and* (R. 1), *assume that* $M_{uw}>0$ *and* $M_{vw}>0$ *hold. Then there exist countably many eigenvalues for* (EV). *They are all nonnegative, and the set of them has no accumulation points.*

As will be seen in §6, an eigenvalue zero for (EV) always appears. This fact seems to make our analysis complicated. But we can get rid of this difficulty by virtue of Proposition 2.2. Indeed, in an appropriate Hilbert space, the solution moves in the direction normal to the eigenspace corresponding to the eigenvalue zero (see §§6, 7). Thus the least positive eigenvalue for (EV) plays an important role in the local behavior of solutions near the equilibrium.

Theorem B. *Under the same assumption as that of* Theorem **A**, *the solution* (u, v^*, w^*, v, w) *to* (EP) *satisfies*

$$\begin{aligned} &\|u(\cdot, z)-u_\infty\|_{\infty,\Omega_0}+\|v^*(\cdot, z)-v_\infty\|_{\infty,\Omega_*}+\|w^*(\cdot, z)-w_\infty\|_{\infty,\Omega_*} \\ &+\|v(\cdot, z)-v_\infty\|_{\infty,\Omega_1}+\|w(\cdot, z)-w_\infty\|_{\infty,\Omega_1}=O(\exp(-\lambda_+z)) \qquad \text{as } z\longrightarrow\infty, \end{aligned}$$

where λ_+ *is the least positive eigenvalue for* (EV).

Moreover the solution converges in a much stronger sense than in Theorems **A** and **B**.

Theorem C. *Under the same assumption as that of* Theorem **A**, *the derivatives of the solution to* (EP) *decay like*

$$\|D_r^i D_z^j u(\cdot,\ z)\|_{\infty,\Omega_0}+\|D_r^i D_z^j v^*(\cdot,\ z)\|_{\infty,\Omega_*}+\|D_r^i D_z^j w^*(\cdot,\ z)\|_{\infty,\Omega_*}$$
$$+\|D_r^i D_z^j v(\cdot,\ z)\|_{\infty,\Omega_1}+\|D_r^i D_z^j w(\cdot,\ z)\|_{\infty,\Omega_1}=O(\exp(-\lambda_+ z)) \qquad \text{as } z\longrightarrow\infty,$$

where λ_+ *is the least positive eigenvalue for* (EV) *and* i, j *are arbitrary nonnegative integers with* $(i,\ j)\neq(0,\ 0)$.

REMARK. It also holds that

$$\|D_1^i D_2^j D_z^k(u-u_\infty)\|_{\infty,\Omega_0}=O(\exp(-\lambda_+ z)) \qquad \text{as } z\longrightarrow\infty \tag{2.2}$$
$$(i\geq 0,\ j\geq 0,\ k\geq 0),$$

where

$$x=(x_1,\ x_2)\in\Omega_0\,;\quad D_p=\frac{\partial}{\partial x_p}\quad (p=1,\ 2).$$

In fact, as shown in §9, it holds that

$$|[D_z^k(u-u_\infty)]|_i=O(\exp(-\lambda_+ z)) \qquad \text{as } z\longrightarrow\infty \quad (i\geq 1,\ k\geq 0),$$

where $|[\cdot]|_i$ denotes the usual norm in a Sobolev space $H^i(\Omega_0)$. Needless to say, Theorems **B** and **C** imply the corresponding result for v^* (w^*, v or w) to (2.2).

In what follows, the symbols C; C_0, C_1, $\cdots$; $C_{0,0}$, $C_{0,1}$, $\cdots$, $C_{1,0}$, $C_{1,1}$, $\cdots$, etc. denote positive constants that are independent of z unless otherwise stated. For simplicity, we sometimes denote several different constants by one of them if there is no confusion.

3. Reduction to a parabolic system

In this section we only impose (R. 1) and (R. 2).

Lemma 3.1. *Let* d_v *and* d_w *be given positive numbers. For any* $(u,\ v,\ w)\in[0,\ \infty)^3$ *there exists a unique pair* $(v^*,\ w^*)\in[0,\ \infty)^2$ *such that*

$$\begin{cases} v=v^*+d_vR(u,\ v^*,\ w^*), \\ w=w^*-d_wR(u,\ v^*,\ w^*). \end{cases}$$

Moreover the implicit functions $v^*=\beta(u,\ v,\ w)$ *and* $w^*=\gamma(u,\ v,\ w)$ *defined by this relation are of class* $C^\infty([0,\ \infty)^3)$.

Proof. Let u, v and w be any nonnegative numbers. The given relation is equivalent to

$$\begin{cases} \dfrac{v}{d_v}+\dfrac{w}{d_w}=\dfrac{v^*}{d_v}+\dfrac{w^*}{d_w}, \\ v^*-v+d_vR(u,\ v^*,\ w^*)=0. \end{cases}$$

For this reason, we eliminate w^* and consider the following equation for v^*:

$$F(v^*):=v^*-v+d_vR\Big(u,\ v^*,\ w+\frac{d_w}{d_v}(v-v^*)\Big)=0.$$

It is easy to see from (R. 1) and (R. 2) that

$$\begin{cases} F(0)=-v+d_vR\Big(u,\ 0,\ w+\dfrac{d_w}{d_v}v\Big)\leq 0, \\ F\Big(v+\dfrac{d_v}{d_w}w\Big)=\dfrac{d_v}{d_w}w+d_vR\Big(u,\ v+\dfrac{d_v}{d_w}w,\ 0\Big)\geq 0, \\ F'(\xi)=1+d_v\dfrac{\partial R}{\partial v}\Big(u,\ \xi,\ w+\dfrac{d_w}{d_v}(v-\xi)\Big)-d_w\dfrac{\partial R}{\partial w}\Big(u,\ \xi,\ w+\dfrac{d_w}{d_v}(v-\xi)\Big) \\ \qquad \geq 1 \qquad \text{for } \xi\in\Big[0,\ v+\dfrac{d_v}{d_w}w\Big]. \end{cases}$$

Thus the equation $F(v^*)=0$ has a unique solution $v^*\in[0,\ v+d_vw/d_w]$. Consequently there exists a unique pair $(v^*,\ w^*)\in[0,\ \infty)^2$ satisfying the given relation. The regularity of β and γ is shown by Implicit Function Theorem. $\square$

Let us introduce a function $J(u,\ v,\ w)$ that plays an essential role in the reduction of (EP) to a parabolic system:

$$\begin{aligned} J(u,\ v,\ w):=1&+d_v\int_0^1\frac{\partial R}{\partial v}(u,\ \theta v+(1-\theta)v^*,\ \theta w+(1-\theta)w^*)d\theta \\ &-d_w\int_0^1\frac{\partial R}{\partial w}(u,\ \theta v+(1-\theta)v^*,\ \theta w+(1-\theta)w^*)d\theta, \end{aligned} \tag{3.1}$$

where d_v, d_w are the constants in Lemma 3.1 and $v^*=\beta(u,\ v,\ w)$, $w^*=\gamma(u,\ v,\ w)$.

Lemma 3.2. *The function $J(u,\ v,\ w)$ satisfies the following for $(u,\ v,\ w)\in[0,\ \infty)^3$:*

(i) $J(u,\ v,\ w)\geq 1$,
(ii) $R(u,\ v^*,\ w^*)=J(u,\ v,\ w)^{-1}R(u,\ v,\ w)$,

where $v^*=\beta(u,\ v,\ w)$ and $w^*=\gamma(u,\ v,\ w)$.

Proof. Since u, v, w, v^* and w^* are nonnegative, (i) follows from (R. 2) and (3.1). Observe that

$$R(u,\ v,\ w)=R(u,\ v^*,\ w^*)+\int_0^1\frac{d}{d\theta}R(u,\ v^*+\theta(v-v^*),\ w^*+\theta(w-w^*))d\theta$$

$$=R(u, v^*, w^*)+(v-v^*)\int_0^1 \frac{\partial R}{\partial v}(u, \theta v+(1-\theta)v^*, \theta w+(1-\theta)w^*)d\theta$$
$$+(w-w^*)\int_0^1 \frac{\partial R}{\partial w}(u, \theta v+(1-\theta)v^*, \theta w+(1-\theta)w^*)d\theta.$$

The right-hand side equals $J(u, v, w)R(u, v^*, w^*)$ by virtue of Lemma 3.1. Thus we get (ii). □

Here and hereafter we set

$$d_v=m_0r_0\log(r_1/r_0),\ d_w=n_0r_0\log(r_1/r_0). \tag{3.2}$$

Now we are ready to reduce the elliptic-parabolic system (EP) to a parabolic system (P).

Proposition 3.3. *Let (u, v^*, w^*, v, w) be a solution to* (EP). *Then the following relation holds between (v^*, w^*) and (u, v, w):*

$$\begin{cases} v^*(r_0, z)=\beta(u(r_0, z), v(r_1, z), w(r_1, z)), \\ w^*(r_0, z)=\gamma(u(r_0, z), v(r_1, z), w(r_1, z)), \end{cases} \quad z\in(0, \infty). \tag{3.3}$$

Moreover, (v^*, w^*) satisfies

$$\begin{cases} v^*(r, z)=v^*(r_0, z)+m_0r_0R(u(r_0, z), v^*(r_0, z), w^*(r_0, z))\log\frac{r}{r_0}, \\ w^*(r, z)=w^*(r_0, z)-n_0r_0R(u(r_0, z), v^*(r_0, z), w^*(r_0, z))\log\frac{r}{r_0} \end{cases} \tag{3.4}$$

on $[r_0, r_1]\times(0, \infty)$, and (u, v, w) does

$$\text{(P)}\quad \begin{cases} a(r)u_z=u_{rr}+\frac{1}{r}u_r & \textit{for } (r, z)\in(0, r_0)\times(0, \infty), \\ b(r)v_z=v_{rr}+\frac{1}{r}v_r,\ c(r)w_z=w_{rr}+\frac{1}{r}w_r & \textit{for } (r, z)\in(r_1, 1)\times(0, \infty); \\ u_r(0, z)=0, & \\ -u_r(r_0, z)=\frac{m_1r_1}{m_0r_0}v_r(r_1, z)=-\frac{n_1r_1}{n_0r_0}w_r(r_1, z) & \\ \quad =\frac{R(u(r_0, z), v(r_1, z), w(r_1, z))}{J(u(r_0, z), v(r_1, z), w(r_1, z))}, & \\ v_r(1, z)=w_r(1, z)=0 & \textit{for } z\in(0, \infty); \\ u(r, 0)=u_0(r) & \textit{for } r\in(0, r_0), \\ v(r, 0)=v_0(r),\ w(r, 0)=w_0(r) & \textit{for } r\in(r_1, 1). \end{cases}$$

Conversely, let (u, v, w) be a solution to (P), *and let (v^*, w^*) be defined by* (3.3) *and* (3.4). *Then (u, v^*, w^*, v, w) satisfies* (EP).

Proof. Suppose that (u, v^*, w^*, v, w) is a solution to (EP). A radially symmetric solution $v^*(\cdot, z)$ of the 2-dimensional Laplace equation satisfies $(rv_r^*)_r \equiv 0$ for $z \in (0, \infty)$. Hence we have

(3.5) $rv_r^*(r, z) = m_0 r_0 R(u(r_0, z), v^*(r_0, z), w^*(r_0, z)), (r, z) \in [r_0, r_1] \times (0, \infty)$,

which implies the first equality of (3.4). We also see that

$$\frac{m_1 r_1}{m_0 r_0} v_r(r_1, z) = \frac{r_1 v_r^*(r_1, z)}{m_0 r_0} = R(u(r_0, z), v^*(r_0, z), w^*(r_0, z))$$

by (3.5). Similarly we can derive the corresponding results for w^*. Since $v(r_1, z) = v^*(r_1, z)$ and $w(r_1, z) = w^*(r_1, z)$, we obtain (3.3) from (3.2) and (3.4) by virtue of Lemma 3.1. Then Lemma 3.2 leads us to

$$R(u(r_0, z), v^*(r_0, z), w^*(r_0, z)) = \frac{R(u(r_0, z), v(r_1, z), w(r_1, z))}{J(u(r_0, z), v(r_1, z), w(r_1, z))}.$$

Thus (u, v, w) satisfies (P). The converse is easily verified. □

The following lemmas will be useful when we derive several estimates for derivatives of solutions.

Lemma 3.4. *Let k be a positive integer. For a function $(u, v, w) = (u(z), v(z), w(z)) \in C^k([1, \infty); [0, \infty)^3)$ put*

$$v^*(z) := \beta(u(z), v(z), w(z)), \quad w^*(z) := \gamma(u(z), v(z), w(z)).$$

Suppose that $(d/dz)^j u$, $(d/dz)^j v$ and $(d/dz)^j w$ are bounded on $[1, \infty)$ $(0 \le j \le k-1)$. Then it holds that

$$\left|\frac{d^k v^*}{dz^k}\right| + \left|\frac{d^k w^*}{dz^k}\right| \le B_k \sum_{j=1}^{k} \left(\left|\frac{d^j u}{dz^j}\right| + \left|\frac{d^j v}{dz^j}\right| + \left|\frac{d^j w}{dz^j}\right| \right) \qquad on\ [1, \infty).$$

where B_k is a positive constant independent of z.

Proof. We have

$$\frac{dv^*}{dz} = \frac{\partial \beta}{\partial u}\frac{du}{dz} + \frac{\partial \beta}{\partial v}\frac{dv}{dz} + \frac{\partial \beta}{\partial w}\frac{dw}{dz}.$$

Differentiate both the sides $k-1$ times with respect to z by using Leibniz' formula. Then we get

$$\frac{d^k v^*}{dz^k} = \sum_{j=0}^{k-1} \binom{k-1}{j} \left\{ \frac{d^j}{dz^j}\left(\frac{\partial \beta}{\partial u}\right)\frac{d^{k-j} u}{dz^{k-j}} + \frac{d^j}{dz^j}\left(\frac{\partial \beta}{\partial v}\right)\frac{d^{k-j} v}{dz^{k-j}} + \frac{d^j}{dz^j}\left(\frac{\partial \beta}{\partial w}\right)\frac{d^{k-j} w}{dz^{k-j}} \right\}.$$

Since $\beta \in C^\infty([0, \infty)^3)$, the derivatives $(d/dz)^j(\partial\beta/\partial u)$, $(d/dz)^j(\partial\beta/\partial v)$ and $(d/dz)^j(\partial\beta/\partial w)$ $(0 \le j \le k-1)$ are bounded on $[1, \infty)$. Thus

$$\left|\frac{d^k v^*}{dz^k}\right| \leq C_k \sum_{j=1}^{k}\left(\left|\frac{d^j u}{dz^j}\right|+\left|\frac{d^j v}{dz^j}\right|+\left|\frac{d^j w}{dz^j}\right|\right).$$

We can obtain a similar inequality for $w^*(z)$. □

Lemma 3.5. *Let k be a positive integer and let $u(z)$, $v(z)$, $w(z)$, $v^*(z)$, $w^*(z)$ be the functions that satisfy the conditions in* Lemma 3.4. *Then it holds that*

$$\left|\frac{d^k}{dz^k}R(u, v^*, w^*)\right| \leq B_k \sum_{j=1}^{k}\left(\left|\frac{d^j u}{dz^j}\right|+\left|\frac{d^j v}{dz^j}\right|+\left|\frac{d^j w}{dz^j}\right|\right) \qquad \text{on } [1, \infty),$$

where B_k is a positive constant independent of z.

Proof. An application of Leibniz' formula to

$$\frac{d}{dz}R(u, v^*, w^*)=\frac{\partial R}{\partial u}(u, v^*, w^*)\frac{du}{dz}+\frac{\partial R}{\partial v}(u, v^*, w^*)\frac{dv^*}{dz} + \frac{\partial R}{\partial w}(u, v^*, w^*)\frac{dw^*}{dz}$$

leads us to

$$\frac{d^k}{dz^k}R(u, v^*, w^*)=\sum_{j=0}^{k-1}\binom{k-1}{j}\left\{\frac{d^j}{dz^j}\left(\frac{\partial R}{\partial u}(u, v^*, w^*)\right)\frac{d^{k-j}u}{dz^{k-j}} + \frac{d^j}{dz^j}\left(\frac{\partial R}{\partial v}(u, v^*, w^*)\right)\frac{d^{k-j}v^*}{dz^{k-j}} + \frac{d^j}{dz^j}\left(\frac{\partial R}{\partial w}(u, v^*, w^*)\right)\frac{d^{k-j}w^*}{dz^{k-j}}\right\}.$$

It follows from Lemma 3.1 that

$$\frac{v^*}{d_v}+\frac{w^*}{d_w}=\frac{v}{d_v}+\frac{w}{d_w},$$

which implies the boundedness of v^* and w^* on $[1, \infty)$. Thus, with the aid of Lemma 3.4, we see that

$$\frac{d^j}{dz^j}\left(\frac{\partial R}{\partial u}(u, v^*, w^*)\right), \frac{d^j}{dz^j}\left(\frac{\partial R}{\partial v}(u, v^*, w^*)\right), \frac{d^j}{dz^j}\left(\frac{\partial R}{\partial w}(u, v^*, w^*)\right) \qquad (0 \leq j \leq k-1)$$

are bounded on $[1, \infty)$. Consequently we obtain

$$\left|\frac{d^k}{dz^k}R(u, v^*, w^*)\right| \leq C_k \sum_{j=1}^{k}\left(\left|\frac{d^j u}{dz^j}\right|+\left|\frac{d^j v^*}{dz^j}\right|+\left|\frac{d^j w^*}{dz^j}\right|\right), \quad z\in[1, \infty).$$

Applying Lemma 3.4 to the right-hand side of this inequality, we can obtain the conclusion. □

Corollary 3.6. *Let k be an integer with $k\geq 2$ and let $u(z)$, $v(z)$, $w(z)$, $v^*(z)$, $w^*(z)$ be the functions that satisfy the conditions in* Lemma 3.4. *Then it holds that*

$$\left|\frac{d^k}{dz^k}R(u, v^*, w^*)\right|\leq B_k\left(\left|\frac{d^k u}{dz^k}\right|+\left|\frac{d^k v}{dz^k}\right|+\left|\frac{d^k w}{dz^k}\right|+1\right) \qquad on\ [1, \infty),$$

where B_k is a positive constant independent of z.

4. Lemmas

In this section we prepare fundamental lemmas that will be used in the proofs of Theorems later.

Lemma 4.1. *For a positive integer k_0, let $\{p_k(z)\}_{0\leq k\leq k_0}$, $\{\bar{p}_k(z)\}_{1\leq k\leq k_0}$, $\{q_k(z)\}_{0\leq k\leq k_0}$, $\{\bar{q}_k(z)\}_{0\leq k\leq k_0-1}$ be sequences of nonnegative functions of class $C^1[1, \infty)$ and let $s_0(z)$ be a nonnegative function of class $C[1, \infty)$. Suppose that*

$$\begin{cases} \dfrac{dp_0}{dz}+q_0+s_0\leq 0, \\ \dfrac{d\bar{q}_0}{dz}+\bar{p}_1\leq\dfrac{1}{\eta}q_0, \\ \dfrac{dp_1}{dz}+q_1\leq\dfrac{1}{\eta}(\bar{p}_1+q_0), \\ \dfrac{d\bar{q}_{k-1}}{dz}+\bar{p}_k\leq\dfrac{1}{\eta}(\sum_{j=1}^{k-1}\bar{p}_j+\sum_{j=0}^{k-1}q_j) & (k=2, \cdots, k_0), \\ \dfrac{dp_k}{dz}+q_k\leq\dfrac{1}{\eta}(\sum_{j=1}^{k}\bar{p}_j+\sum_{j=0}^{k-1}q_j) & (k=2, \cdots, k_0), \\ p_k\leq\dfrac{1}{\eta}(\sum_{j=1}^{k}\bar{p}_j+\sum_{j=0}^{k}q_j) & (k=1, 2, \cdots, k_0) \end{cases}$$

for $z\in[1, \infty)$, where η is a positive constant. Then

$$\begin{cases} \displaystyle\int_1^\infty s_0 dz<\infty, \\ \displaystyle\lim_{z\to\infty} p_k(z)=0 \quad (k=1, 2, \cdots, k_0). \end{cases}$$

For the proof see [8, Lemma 3.2].

Lemma 4.2. *For a positive integer k_0, let $\{p_k(z)\}_{0\leq k\leq k_0}$ be a sequence of nonnegative functions of class $C^1[\hat{z}, \infty)$ and let $\{q_k(z)\}_{0\leq k\leq k_0}$, $\{\rho_k(z)\}_{0\leq k\leq k_0}$ be sequences of nonnegative functions of class $C[\hat{z}, \infty)$. Suppose that*

$$\begin{cases} \bar{\rho} := \sup\{\rho_k(z)\,;\ 0 \le k \le k_0,\ z \ge \hat{z}\} < 1, \\ \dfrac{dp_k}{dz} + q_k \le \rho_k \displaystyle\sum_{j=0}^{k} q_j, & z \in [\hat{z}, \infty), \\ \lambda p_k \le q_k, & z \in [\hat{z}, \infty) \end{cases}$$

for $k=0, 1, \cdots, k_0$, *where* λ *is a positive constant. Then*

$$p_k(z) = O(\exp(-\tilde{\lambda} z)) \qquad \text{as } z \longrightarrow \infty \quad (k=0, 1, \cdots, k_0).$$

Here $\tilde{\lambda} \in (0, \lambda)$ *is an appropriate constant such that* $\tilde{\lambda}/\lambda$ *depends only on* $\bar{\rho}$. *Moreover, if* $\rho_k(z) \in L^1(\hat{z}, \infty)$ $(k=0, 1, \cdots, k_0)$, *then we can choose* λ *as* $\tilde{\lambda}$.

For the proof see [11, Lemmas 3.2, 3.3].

Lemma 4.3. *Set* $i=0$ *or* $i=1$. *Let a function* $\omega \in L^1_r(\Omega_i)$ *be positive almost everywhere in* Ω_i. *Then, for any* $\epsilon > 0$ *there exists a positive number* K_ϵ *such that*

$$u(r_i)^2 \le \epsilon \|u_r\|^2_{2,\Omega_i} + K_\epsilon \|u\|^2_{2,\Omega_i;\omega},$$
$$\|u\|^2_{2,\Omega_i} \le \epsilon \|u_r\|^2_{2,\Omega_i} + K_\epsilon \|u\|^2_{2,\Omega_i;\omega}$$

for all $u \in H^1_r(\Omega_i)$.

Proof. We will show the inequalities for $i=0$ (the proof for $i=1$ is easier). Fix any number $\delta \in (0, r_0/2)$. For $\rho, \rho' \in (0, r_0]$ we have

$$u(\rho) = u(\rho') + \int_{\rho'}^{\rho} \frac{1}{\sqrt{r}} u_r(r) \sqrt{r}\, dr.$$

Applying Schwarz' inequality to the right-hand side, we get

$$u(\rho)^2 \le 2u(\rho')^2 + 2\left|\log\frac{\rho}{\rho'}\right| \left|\int_{\rho'}^{\rho} u_r^2 r dr\right|.$$

Multiply both the sides by $\omega(\rho')\rho'$ and integrate them with respect to ρ' over $[\rho - \delta, \rho]$ for $\rho \in (r_0/2, r_0]$. Then we get

$$u(\rho)^2 \int_{\rho-\delta}^{\rho} \omega r dr \le 2\|u\|^2_{2,\Omega_0;\omega} + 2\log\frac{\rho}{\rho-\delta} \int_{\rho-\delta}^{\rho} \omega r dr \|u_r\|^2_{2,\Omega_0}.$$

Hence

$$u(\rho)^2 \le \frac{2}{c_\delta}\|u\|^2_{2,\Omega_0;\omega} + 2\log\frac{\rho}{\rho-\delta}\|u_r\|^2_{2,\Omega_0}, \quad \rho \in \left(\frac{r_0}{2}, r_0\right],$$

where

$$c_\delta := \min_{\delta \le \rho \le r_0} \int_{\rho-\delta}^{\rho} \omega r dr > 0.$$

In particular, we get the first inequality in this lemma by putting

$$\rho=r_0 \text{ and } \epsilon=2\log\frac{r_0}{r_0-\delta}.$$

Similarly we can derive

$$u(\rho)^2\leq\frac{2}{C_\delta}\|u\|^2_{2,\Omega_0;\omega}+2\log\frac{\rho+\delta}{\rho}\|u_r\|^2_{2,\Omega_0},\ \rho\in\left(0,\ \frac{r_0}{2}\right].$$

Consequently we have

$$\int_0^{r_0}u^2\rho d\rho\leq\frac{r_0^2}{C_\delta}\|u\|^2_{2,\Omega_0;\omega}+2\|u_r\|^2_{2,\Omega_0}\left\{\int_0^{r_0/2}\rho\log\frac{\rho+\delta}{\rho}d\rho+\int_{r_0/2}^{r_0}\rho\log\frac{\rho}{\rho-\delta}d\rho\right\}.$$

Observing that

$$\begin{aligned}\int_0^{r_0/2}\rho\log\frac{\rho+\delta}{\rho}d\rho&=\int_0^{r_0/2}\rho\int_\rho^{\rho+\delta}\frac{1}{r}drd\rho\\&=\int_0^\delta\frac{1}{r}\int_0^r\rho d\rho dr+\int_\delta^{r_0/2}\frac{1}{r}\int_{r-\delta}^r\rho d\rho dr+\int_{r_0/2}^{r_0/2+\delta}\frac{1}{r}\int_{r-\delta}^{r_0/2}\rho d\rho dr,\end{aligned}$$

we can find a positive constant C satisfying

$$\int_0^{r_0/2}\rho\log\frac{\rho+\delta}{\rho}d\rho+\int_{r_0/2}^{r_0}\rho\log\frac{\rho}{\rho-\delta}d\rho\leq C\delta$$

for $\delta\in(0,\ r_0/2)$. Thus we obtain the second inequality in this lemma by putting $\epsilon=2C\delta$. □

Lemma 4.4. *Set $i=0$ or $i=1$. Let u be a radially symmetric function on Ω_i. Suppose that*

$$u\in L^2_{loc}(\Omega_i),\quad \Delta u=u_{rr}+\frac{1}{r}u_r\in L^2_r(\Omega_i).$$

Then $u\in H^2_r(\Omega_i)\cap C^1_r(\bar{\Omega}_i)$. Moreover, if $i=0$, then

$$u_r(\rho)^2\leq\frac{1}{2}\int_0^\rho|\Delta u|^2rdr,\quad \rho\in[0,\ r_0]; \tag{4.1}$$

if $i=1$ and if $u_r(1)=0$, then

$$u_r(\rho)^2\leq\frac{1-\rho^2}{2\rho^2}\int_\rho^1|\Delta u|^2rdr,\quad \rho\in[r_1,\ 1]. \tag{4.2}$$

Proof. For simplicity we define $\underline{r}_i$ and $\bar{r}_i$ $(i=0,\ 1)$ by

$$(\underline{r}_0,\ \bar{r}_0)=(0,\ r_0),\quad (\underline{r}_1,\ \bar{r}_1)=(r_1,\ 1).$$

Since u and Δu belong to $L^2_{loc}(\Omega_i)$, the regularity theory for elliptic equations leads us to the fact $u(|x|)\in H^2_{loc}(\Omega_i)$, which yields $ru_r(r)\in L^1(\underline{\rho},\ \bar{\rho})$ with $\underline{r}_i<\underline{\rho}<\bar{\rho}<\bar{r}_i$. On the other hand, we have $(ru_r)_r=r\Delta u\in L^1(\underline{r}_i,\ \bar{r}_i)$. Hence

$$\bar{\rho}u_r(\bar{\rho})-\underline{\rho}u_r(\underline{\rho})=\int_{\underline{\rho}}^{\bar{\rho}} r\Delta u dr,\ \underline{r}_i<\underline{\rho}<\bar{\rho}<\bar{r}_i.$$

In view of $\Delta u\in L_r^2(\Omega_i)$, we can derive from this equality that

$$u(r)\in\begin{cases}C^1(0,\ r_0] & \text{if } i=0,\\ C^1[r_1,\ 1] & \text{if } i=1.\end{cases} \tag{4.3}$$

Consider the case $i=1$ under the condition $u_r(1)=0$. Letting $\bar{\rho}$ tend to 1 and using Schwarz' inequality, we get

$$\underline{\rho}^2 u_r(\underline{\rho})^2\le\int_{\underline{\rho}}^1 rdr\int_{\underline{\rho}}^1|\Delta u|^2 rdr=\frac{1-\underline{\rho}^2}{2}\int_{\underline{\rho}}^1|\Delta u|^2 rdr,$$

which implies (4.2).

Now we consider the case $i=0$. We will show (4.1) and

$$\lim_{r\to 0} u_r(r)=0. \tag{4.4}$$

Observing $ru_r(r)\in L^1(0,\ r_0)$, we can see that $ru_r(r)$ is abosolutely continuous on $[0,\ r_0]$ and satisfies

$$\begin{aligned}&\lim_{r\to 0} ru_r(r)=\eta,\\ &\rho u_r(\rho)=\eta+\int_0^\rho (ru_r)_r dr=\eta+\int_0^\rho r\Delta u dr,\quad \rho\in(0,\ r_0]\end{aligned} \tag{4.5}$$

for a number η. Suppose that $\eta>0$. Then we have

$$u_r(r)\ge\frac{\eta}{2r},\quad r\in(0,\ \delta)$$

for some $\delta\in(0,\ r_0)$. Since $u(r)\in C^1(0,\ r_0]$, we see that

$$u(\rho)-u(\epsilon)=\int_\epsilon^\rho u_r dr\ge\frac{\eta}{2}\int_\epsilon^\rho\frac{dr}{r},\quad 0<\epsilon<\rho<\delta.$$

The right-hand side of the inequality tends to ∞ as $\epsilon\to 0$. This contradicts the fact $u\in H_{loc}^2(\Omega_0)\subset C(\Omega_0)$. Similarly $\eta<0$ implies contradiction. Therefore (4.5) is rewritten as

$$\rho u_r(\rho)=\int_0^\rho r\Delta u dr,\quad \rho\in(0,\ r_0].$$

By virtue of Schwarz' inequality, we have

$$\rho^2 u_r(\rho)^2=\int_0^\rho rdr\int_0^\rho|\Delta u|^2 rdr=\frac{\rho^2}{2}\int_0^\rho|\Delta u|^2 rdr,\quad \rho\in(0,\ r_0],$$

which implies (4.1) and (4.4).

To conclude the proof, we verify the regularity of u in both the cases $i=0,\ 1$.

It follows from (4.3) and (4.4) that $u\in C^1_r(\bar{\Omega}_i)$. Moreover, the boundary values of $u(|x|)$ on $\partial\Omega_i$ equal a constant or two constants. Thus we can see that $u\in H^2_r(\Omega_i)$ by the regularity theory for elliptic boundary value problems. $\square$

Lemma 4.5. *Let I be a closed interval in $\boldsymbol{R}$ with $I\ni 0$; Ω an open subset of $\boldsymbol{R}^2$ with smooth boundary; $F(\xi)$ a sufficiently smooth function on I with $F(0)=0$; $u(x, z)$ a sufficiently smooth function from $\bar{\Omega}\times[1, \infty)$ to I; k_0 a nonnegative integer.*

(i) *Suppose that $D_z^k u$ $(0\leq k\leq k_0)$ are bounded on $\bar{\Omega}\times[1, \infty)$. Then*

$$\sum_{k=0}^{k_0}|[D_z^k F(u(\cdot, z))]|_{1,\Omega}\leq L\sum_{k=0}^{k_0}|[D_z^k u(\cdot, z)]|_{1,\Omega}, \quad z\in[1,\infty)$$

holds for a positive constant L that is independent of z but dependent on k_0.

(ii) *Let i_0 be an integer with $i_0\geq 2$. Suppose that $|[D_z^k u(\cdot, z)]|_{i_0,\Omega}$ $(0\leq k\leq k_0)$ are bounded on $[1, \infty)$. Then*

$$\sum_{k=0}^{k_0}|[D_z^k F(u(\cdot, z))]|_{i_0,\Omega}\leq N\sum_{k=0}^{k_0}|[D_z^k u(\cdot, z)]|_{i_0,\Omega}, \quad z\in[1,\infty)$$

holds for a positive constant N that is independent of z but dependent on i_0 and k_0.

Here $|[\cdot]|_{i,\Omega}$ denotes the usual norm in a Sobolev space $H^i(\Omega)$.

Proof. According to the convention, we use the following abbreviation for differential operators with respect to $x=(x_1, x_2)$:

$$\alpha=(\alpha_1, \alpha_2), \quad |\alpha|=\alpha_1+\alpha_2,$$
$$D_x^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2},$$

where α_1 and α_2 are nonnegative integers. We also abbreviate $|[\cdot]|_{i,\Omega}$ to $|[\cdot]|_i$.

(i) Seeing that $F(0)=0$, we have

$$\|F(u)\|_2=\left\|\int_0^u F'(\xi)d\xi\right\|_2\leq \sup_{|\xi|\leq\sup|u|}|F'(\xi)|\|u\|_2.$$

Moreover we have

$$D_z^k F(u)=D_z^{k-1}\{F'(u)D_z u\}=\sum_{j=0}^{k-1}\binom{k-1}{j}D_z^j F'(u)D_z^{k-j}u, \qquad 1\leq k\leq k_0;$$

$$D_x^\alpha D_z^k F(u)=D_z^k\{F'(u)D_x^\alpha u\}=\sum_{j=0}^{k}\binom{k}{j}D_z^j F'(u)D_x^\alpha D_z^{k-j}u, \qquad |\alpha|=1,\ 0\leq k\leq k_0.$$

Hence the uniform boundedness of $D_z^k u$ leads us to

$$|[D_z^k F(u)]|_1 \leq C_k \sum_{j=0}^{k} |[D_z^j u]|_1, \quad 0 \leq k \leq k_0.$$

(ii) Since $H^2(\Omega) \subset C(\bar{\Omega})$, $D_z^k u$ $(0 \leq k \leq k_0)$ are bounded on $\bar{\Omega} \times [1, \infty)$. Then the argument in (i) is still valid, so it suffices to show that

$$\|D_x^\alpha D_z^k F(u)\|_2 \leq C_{i_0, k_0} \sum_{j=0}^{k} |[D_z^j u]|_{i_0}, \quad 0 \leq k \leq k_0. \tag{4.6}$$

for all α with $2 \leq |\alpha| \leq i_0$. In the case $|\alpha|=2$, setting $\alpha=\beta+\gamma$ with $|\beta|=|\gamma|=1$, we have

$$D_x^\alpha D_z^k F(u) = D_z^k \{F'(u) D_x^\alpha u + F''(u) D_x^\beta u D_x^\gamma u\},$$

which implies

$$\|D_x^\alpha D_z^k F(u)\|_2 \leq C_k \{\sum_{j=0}^{k} \|D_x^\alpha D_z^j u\|_2 + (\sum_{j=0}^{k} \|D_x^\beta D_z^j u\|_4)(\sum_{j=0}^{k} \|D_x^\gamma D_z^j u\|_4)\}.$$

Recall the fact $H^1(\Omega) \subset L^4(\Omega)$. Then the boundedness of $|[D_z^k u]|_2$ leads us to (4.6) with $|\alpha|=2$. For $i_0 \geq 3$ we will show (4.6) with $3 \leq |\alpha| \leq i_0$ by induction. Let i be an integer with $3 \leq i \leq i_0$, and suppose that (4.6) holds for all α with $|\alpha|=i-1$. In this case we may further assume the same inequality where $F(u)$ is replaced with $F'(u)$. Setting $\alpha=(\alpha_1, \alpha_2)$, $\beta=(\beta_1, \beta_2)$ and $\gamma=(\gamma_1, \gamma_2)$ with

$$|\alpha|=i, \ |\beta|=|\gamma|=1, \ \beta+\gamma \leq \alpha,$$

we have

$$\begin{aligned}
D_x^\alpha D_z^k F(u) &= D_x^\beta [D_z^k D_x^{\alpha-\beta-\gamma} \{F'(u) D_x^\gamma u\}] \\
&= D_x^\beta \Big\{ \sum_{j=0}^{k} \sum_{0 \leq \delta < \alpha-\beta-\gamma} \binom{k}{j} c_{\alpha\beta\gamma\delta} D_x^\delta D_z^j F'(u) D_x^{\alpha-\beta-\delta} D_z^{k-j} u \Big\} \\
&\quad + D_x^\beta \Big\{ \sum_{j=0}^{k} \binom{k}{j} D_x^{\alpha-\beta-\gamma} D_z^j F'(u) D_x^\gamma D_z^{k-j} u \Big\} \\
&= \sum_{j=0}^{k} \sum_{0 \leq \delta < \alpha-\beta-\gamma} \binom{k}{j} c_{\alpha\beta\gamma\delta} D_x^{\beta+\delta} D_z^j F'(u) D_x^{\alpha-\beta-\delta} D_z^{k-j} u \\
&\quad + \sum_{j=0}^{k} \sum_{0 \leq \delta < \alpha-\beta-\gamma} \binom{k}{j} c_{\alpha\beta\gamma\delta} D_x^\delta D_z^j F'(u) D_x^{\alpha-\delta} D_z^{k-j} u \\
&\quad + \sum_{j=0}^{k} \binom{k}{j} D_x^{\alpha-\gamma} D_z^j F'(u) D_x^\gamma D_z^{k-j} u \\
&\quad + \sum_{j=0}^{k} \binom{k}{j} D_x^{\alpha-\beta-\gamma} D_z^j F'(u) D_x^{\beta+\gamma} D_z^{k-j} u.
\end{aligned}$$

Here $c_{\alpha\beta\gamma\delta}$ is a positive integer determined only by α, β, γ and δ. Besides $D_x^{\beta+\delta} D_z^j F'(u)$, $D_x^\delta D_z^j F'(u)$, $D_x^\gamma D_z^{k-j} u$ and $D_x^{\alpha-\beta-\gamma} D_z^j F'(u)$ are bounded on $\bar{\Omega} \times [1, \infty)$, because $H^{i_0}(\Omega) \subset C^{i_0-2}(\bar{\Omega})$ and $|\delta| \leq i_0-3$. Hence

$$\|D_x^\alpha D_z^k F(u)\|_2 \leq C_{i,k} \sum_{j=0}^{k} (|[D_z^j u]|_i + \|D_x^{\alpha-\gamma} D_z^j F'(u)\|_2).$$

Since $|\alpha-\gamma|=i-1$, the assumption in the induction leads us to (4.6) with $|\alpha|=i$. Thus we complete the proof. $\square$

In the same manner we can prove the following lemma.

Lemma 4.6. *Let $\mathcal{K}$ be a closed subset of $\boldsymbol{R}^3$ with $\mathcal{K}\ni(0, 0, 0)$; Ω an open subset of $\boldsymbol{R}^2$ with smooth boundary; $F(\xi, \eta, \zeta)$ a sufficiently smooth function on $\mathcal{K}$ with $F(0, 0\ 0)=0$; $u(x, z)$, $v(x, z)$ and $w(x, z)$ sufficiently smooth functions on $\bar{\Omega}\times[1, \infty)$ with $(u(x, z), v(x, z), w(x, z))\in\mathcal{K}$ on $\bar{\Omega}\times[1, \infty)$; k_0 a non-negative integer.*

(i) *Suppose that $D_z^k u$, $D_z^k v$ and $D_z^k w$ $(0\leq k\leq k_0)$ are bounded on $\bar{\Omega}\times[1, \infty)$. Then*

$$\sum_{k=0}^{k_0}\|[D_z^k F(u(\cdot, z), v(\cdot, z), w(\cdot, z))]\|_{1,\Omega}$$
$$\leq L\sum_{k=0}^{k_0}\{\|[D_z^k u(\cdot, z)]\|_{1,\Omega}+\|[D_z^k v(\cdot, z)]\|_{1,\Omega}+\|[D_z^k w(\cdot, z)]\|_{1,\Omega}\},\ z\in[1, \infty)$$

holds for a positive constant L that is independent of z but dependent on k_0.

(ii) *Let i_0 be an integer with $i_0\geq 2$. Suppose that $\|[D_z^k u(\cdot, z)]\|_{i_0,\Omega}$, $\|[D_z^k v(\cdot, z)]\|_{i_0,\Omega}$ and $\|[D_z^k w(\cdot, z)]\|_{i_0,\Omega}$ $(0\leq k\leq k_0)$ are bounded on $[1, \infty)$. Then*

$$\sum_{k=0}^{k_0}\|[D_z^k F(u(\cdot, z), v(\cdot, z), w(\cdot, z))]\|_{i_0,\Omega}$$
$$\leq N\sum_{k=0}^{k_0}\{\|[D_z^k u(\cdot, z)]\|_{i_0,\Omega}+\|[D_z^k v(\cdot, z)]\|_{i_0,\Omega}+\|[D_z^k w(\cdot, z)]\|_{i_0,\Omega}\},$$
$$z\in[1, \infty)$$

holds for a positive constant N that is independent of z but dependent on i_0 and k_0.

Here $\|[\cdot]\|_{i,\Omega}$ denotes the usual norm in a Sobolev space $H^i(\Omega)$.

Finally we give a lemma that is useful when we derive the uniform boundedness of derivatives of solutions from their L^2-boundedness. Only here we use (A. 2).

Lemma 4.7. *Let $u=u(r, z)$ be a function of class $C^1([1, \infty); C[0, r_0])\cap C([1, \infty); C^2[0, r_0])$ and $v=v(r, z)$, $w=w(r, z)$ functions of class $C^1([1, \infty); C[r_1, 1])\cap C([1, \infty); C^2[r_1, 1])$. In addition to* (A. 1) *and* (A. 2), *assume that (u, v, w) satisfies*

$$
(4.7)\quad \begin{cases}
a(r)u_z = u_{rr} + \dfrac{1}{r}u_r, & (r,\ z)\in(0,\ r_0)\times[1,\ \infty),\\
b(r)v_z = v_{rr} + \dfrac{1}{r}v_r,\ c(r)w_z = w_{rr} + \dfrac{1}{r}w_r, & (r,\ z)\in(r_1,\ 1)\times[1,\ \infty);\\
u_r(0,\ z)=0, & z\in[1,\ \infty),\\
v_r(1,\ z)=w_r(1,\ z)=0, & z\in[1,\ \infty),\\
|u_r(r_0,\ z)|+|v_r(r_1,\ z)|+|w_r(r_1,\ z)| & \\
\quad \le \varkappa\{|u(r_0,\ z)|+|v(r_1,\ z)|+|w(r_1,\ z)|+1\}, & z\in[1,\ \infty)
\end{cases}
$$

and

$$K:=\sup_{z\ge 1}(\|u\|_{2,\Omega_0;a}+\|v\|_{2,\Omega_1;b}+\|w\|_{2,\Omega_1;c})<\infty,$$

where $\varkappa$ is a positive constant independent of z. Then

$$\|u(\cdot,\ z)\|_{\infty,\Omega_0}+\|v(\cdot,\ z)\|_{\infty,\Omega_1}+\|w(\cdot,\ z)\|_{\infty,\Omega_1}\le C_K,\quad z\in[1,\ \infty)$$

holds for a positive constant C_K that is independent of z but dependent on $\|u(\cdot,\ 1)\|_{\infty,\Omega_0}$, $\|v(\cdot,\ 1)\|_{\infty,\Omega_1}$, $\|w(\cdot,\ 1)\|_{\infty,\Omega_1}$, $\varkappa$ and K.

Yamada and Yotsutani showed a similar fact to this lemma in [19, proposition 8.1], where the inequality corresponding to the last one in (4.7) has no constant term in the right-hand side. In our case we can not omit a constant term in the right-hand side because of later necessity. However, we can prove this lemma in a similar manner to their proof. The idea of the proof is based on Alikakos [2, Theorem 3.1], which is an application of an iteration technique due to Moser [12], [13] and [14].

5. Proof of Theorem A

For functions $u(r)\in C[0,\ r_0]$ and $v(r),\ w(r)\in C[r_1,\ 1]$ we use the abbreviation $U=(u,\ v,\ w)$ and define energy functionals:

$$E(U):=\frac{1}{2}(\|u\|_{2,\Omega_0}^2+\|v\|_{2,\Omega_1}^2+\|w\|_{2,\Omega_1}^2),$$

$$E(U;\ \omega_0,\ \omega_1,\ \omega_2):=\frac{1}{2}(\|u\|_{2,\Omega_0;\omega_0}^2+\|v\|_{2,\Omega_1;\omega_1}^2+\|w\|_{2,\Omega_1;\omega_2}^2),$$

where

$$\omega_0\in L^1_r(\Omega_0),\qquad \omega_0>0\quad \text{a.e. in } \Omega_0;$$
$$\omega_1,\ \omega_2\in L^1_r(\Omega_1),\qquad \omega_1>0,\ \omega_2>0\quad \text{a.e. in } \Omega_1.$$

Additionally if u, v and w are all positive functions, we use the energy functional

$$\Psi(U) := lu_\infty \int_0^{r_0} \phi\left(\frac{u}{u_\infty}\right) ardr + mv_\infty \frac{m_1}{m_0} \int_{r_1}^1 \phi\left(\frac{v}{v_\infty}\right) brdr$$
$$+ nw_\infty \frac{n_1}{n_0} \int_{r_1}^1 \phi\left(\frac{w}{w_\infty}\right) crdr,$$

where

$$\phi(t) := \int_1^t \log \tau d\tau = t \log t - (t-1) \geq 0 \qquad \text{for } t > 0$$

(note that $u_\infty > 0$, $v_\infty > 0$ and $w_\infty > 0$; cf. Proposition 2.3). It is easy to see that

$$\Psi(U) = 0 \text{ if and only if } U = (u_\infty, v_\infty, w_\infty). \tag{5.1}$$

REMARK. Rothe [16] employed a Lyapunov function that resembles Ψ. He investigated the asymptotic behavior of solutions to a reaction-diffusion system with homogeneous Neumann boundary conditions. His system is a mathematical model that describes chemical reactions not *on the boundary* but *in the domain*.

The following energy estimates play an essential role.

Proposition5.1. *The solution $U = (u(r, z), v(r, z), w(r, z))$ to* (P) *satisfies*

$$\frac{d}{dz}\Psi(U) + \epsilon E(D_r U) + \epsilon\{u(r_0, z)^l v(r_1, z)^m - w(r_1, z)^n\}^2 \leq 0, \tag{5.2}$$
$$\frac{d}{dz}E(D_r U\,;\, f, g, h) + 2E(D_z U\,;\, af, bg, ch) = E(D_r U),$$
$$\frac{d}{dz}E(D_z U\,;\, a, b, c) + E(D_r D_z U) \leq N_1 E(D_z U\,;\, af, bg, ch),$$
$$E(D_z U\,;\, a, b, c) \leq N_1\{E(D_z U\,;\, af, bg, ch) + E(D_r D_z U)\}$$

for $z \in (0, \infty)$. Here ϵ, N_1 are positive constants that are independent of z, and

$$f(r) = \int_r^{r_0} a(t) t \log \frac{r_0}{t} dt, \qquad r \in (0, r_0),$$
$$g(r) = \int_{r_1}^r b(t) t \log \frac{t}{r_1} dt,\ h(r) = \int_{r_1}^r c(t) t \log \frac{t}{r_1} dt, \quad r \in (r_1, 1),$$
$$D_r U = (u_r, v_r, w_r),\ D_z U = (u_z, v_z, w_z).$$

We prove Theorem **A** by using this proposition, whose proof is given later.

Proof of Theorem **A**. We will show that the solution $U = U(z) = (u(\cdot, z), v(\cdot, z), w(\cdot, z))$ to (P) converges to the equilibrium $U_\infty = (u_\infty, v_\infty, w_\infty)$. Combining the inequalities in Proposition 5.1, we can derive

$$\int_1^\infty \{u(r_0, z)^l v(r_1, z)^m - w(r_1, z)^n\}^2 dz < \infty, \tag{5.3}$$

$$\lim_{z\to\infty} E(D_z U;\ a,\ b,\ c)=0 \tag{5.4}$$

(use, for instance, Lemma 4.1 with $k_0=1$). Since u, v and w satisfy diffusion equations, Lemma 4.4 yields

$$\|u_r\|_\infty^2+\|v_r\|_\infty^2+\|w_r\|_\infty^2\leq CE(D_z U;\ a,\ b,\ c).$$

Thus we obtain from (5.4) that

$$\lim_{z\to\infty}\{\|u_r(\cdot,\ z)\|_\infty+\|v_r(\cdot,\ z)\|_\infty+\|w_r(\cdot,\ z)\|_\infty\}=0. \tag{5.5}$$

Therefore the family $\{u(\cdot,\ z)\}_{z\geq1}$ is uniformly bounded and equi-continuous on $[0, r_0]$, so are the families $\{v(\cdot,\ z)\}_{z\geq1}$ and $\{w(\cdot,\ z)\}_{z\geq1}$ on $[r_1,\ 1]$. On account of (5.3), there exists a sequence $\{z_j\}_{j=1,2,3,\cdots}$ such that

$$\lim_{j\to\infty} z_j=\infty \quad\text{and}\quad \lim_{j\to\infty}\{u(r_0,\ z_j)^l v(r_1,\ z_j)^m-w(r_1,\ z_j)^n\}^2=0.$$

By Ascoli-Arzelã's theorem we can choose a subsequence from $\{z_j\}$ (denoting it by $\{z_j\}$ again) such that

$$\lim_{j\to\infty}\{\|u(\cdot,\ z_j)-\overline{u}\|_\infty+\|v(\cdot,\ z_j)-\overline{v}\|_\infty+\|w(\cdot,\ z_j)-\overline{w}\|_\infty\}=0.$$

Here it follows from (5.5) and Proposition 2.2 that $\overline{u}$, $\overline{v}$, $\overline{w}$ are nonegative constant functions satisfying

$$\begin{cases} R(\overline{u},\ \overline{v},\ \overline{w})=0, \\ \|a\|_1\,\overline{u}+\dfrac{n_1}{n_0}\|c\|_1\,\overline{w}=M_{uw}, \\ \dfrac{m_1}{m_0}\|b\|_1\,\overline{v}+\dfrac{n_1}{n_0}\|c\|_1\,\overline{w}=M_{vw}. \end{cases}$$

Recalling Proposition 2.3, we see that $(\overline{u},\ \overline{v},\ \overline{w})=U_\infty$. Hence

$$\lim_{j\to\infty}\{\|u(\cdot,\ z_j)-u_\infty\|_\infty+\|v(\cdot,\ z_j)-v_\infty\|_\infty+\|w(\cdot,\ z_j)-w_\infty\|_\infty\}=0,$$

which implies

$$\lim_{j\to\infty}\Psi(U(z_j))=\Psi(U_\infty)=0.$$

Moreover we see by (5.2) that $\Psi(U(z))$ monotonically decreases. Thus we obtain

$$\lim_{z\to\infty}\Psi(U(z))=0. \tag{5.6}$$

Since $\{U(z)\}_{z\geq1}$ is uniformly bounded and equi-continuous, the facts (5.1) and (5.6) lead us to

$$\lim_{z\to\infty}\{\|u(\cdot,\ z)-u_\infty\|_\infty+\|v(\cdot,\ z)-v_\infty\|_\infty+\|w(\cdot,\ z)-w_\infty\|_\infty\}=0. \tag{5.7}$$

Finally we show the convergence of v^* and w^*. Using Lemma 3.2 and (3.3), we obtain from (5.7) that

$$\lim_{z\to\infty} R(u(r_0, z), v^*(r_0, z), w^*(r_0, z)) = \frac{R(u_\infty, v_\infty, w_\infty)}{J(u_\infty, v_\infty, w_\infty)} = 0, \tag{5.8}$$

which, combined with Lemma 3.1, implies

$$\lim_{z\to\infty} v^*(r_0, z) = v_\infty, \quad \lim_{z\to\infty} w^*(r_0, z) = w_\infty.$$

Letting z tend to ∞ in (3.4), we complete the proof. □

In order to prove Proposition 5.1, we prepare some differential (in)equalities.

Lemma 5.2. *The solution* $U=(u, v, w)$ *to* (P) *satisfies*

$$\begin{aligned}&\frac{d}{dz}\Psi(U) + l\int_0^{r_0} \frac{u_r^2}{u} r dr + m\frac{m_1}{m_0}\int_{r_1}^1 \frac{v_r^2}{v} r dr + n\frac{n_1}{n_0}\int_{r_1}^1 \frac{w_r^2}{w} r dr \\ &+ r_0\{\log u(r_0, z)^l v(r_1, z)^m - \log w(r_1, z)^n\} \frac{R(u(r_0, z), v(r_1, z), w(r_1, z))}{J(u(r_0, z), v(r_1, z), w(r_1, z))} \\ &= 0.\end{aligned}$$

We can prove this lemma in the same way as the calculation of $d\Psi(u, u^\infty)/dz$ in [7, p. 493].

Lemma 5.3. *The solution* (u, v, w) *to* (P) *satisfies*

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dz}\|u_r\|_{2;f}^2 + \|u_z\|_{2;af}^2 = \dfrac{1}{2}\|u_r\|_2^2, \\ \dfrac{1}{2}\dfrac{d}{dz}\|v_r\|_{2;g}^2 + \|v_z\|_{2;bg}^2 = \dfrac{1}{2}\|v_r\|_2^2, \\ \dfrac{1}{2}\dfrac{d}{dz}\|w_r\|_{2;h}^2 + \|w_z\|_{2;ch}^2 = \dfrac{1}{2}\|w_r\|_2^2. \end{cases}$$

Proof. Observe that

$$u_r(0, z) = f(r_0) = 0, \ \lim_{r\to r_0} \frac{f_r(r)}{a(r)} = 0, \ \left(\frac{f_r}{ra}\right)_r = \frac{1}{r}.$$

Moreover rf_r/a is bounded near $r=0$. Accordingly we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dz}\|u_r\|_{2;f}^2 &= \int_0^{r_0} u_{zr} r u_r f dr \\ &= -\int_0^{r_0} u_z (r u_r)_r f dr - \int_0^{r_0} u_z r u_r f_r dr \\ &= -\int_0^{r_0} u_z r a u_z f dr - \int_0^{r_0} r a u_z r u_r \frac{f_r}{ra} dr\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^{r_0} u_z^2 afr dr - \int_0^{r_0} (ru_r)_r ru_r \frac{f_r}{ra} dr \\
&= -\|u_z\|_{2;af}^2 - \frac{1}{2}\int_0^{r_0} \{(ru_r)^2\}_r \frac{f_r}{ra} dr \\
&= -\|u_z\|_{2;af}^2 + \frac{1}{2}\int_0^{r_0} (ru_r)^2 \frac{1}{r} dr \\
&= -\|u_z\|_{2;af}^2 + \frac{1}{2}\|u_r\|_2^2.
\end{aligned}
$$

Similarly the facts

$$
\begin{aligned}
&v_r(1, z) = g(r_1) = 0, \ \lim_{r\to r_1} \frac{g_r(r)}{b(r)} = 0, \ \left(\frac{g_r}{rb}\right)_r = \frac{1}{r}, \\
&w_r(1, z) = h(r_1) = 0, \ \lim_{r\to r_1} \frac{h_r(r)}{c(r)} = 0, \ \left(\frac{h_r}{rc}\right)_r = \frac{1}{r}
\end{aligned}
$$

and the boundedness of rg_r/b, rh_r/c near $r=1$ lead us to the equalities for v and w. □

Lemma 5.4. *The solution U to* (P) *satisfies*

$$
\frac{d}{dz}E(D_zU\,;\ a,\ b,\ c) + E(D_rD_zU) \le N_1 E(D_zU\,;\ af,\ bg,\ ch).
$$

Proof. Differentiate equations for u in (P) with respect to z. Thereby we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dz}\|u_z\|_{2;a}^2 = \int_0^{r_0} u_z r a u_{zz} dr = \int_0^{r_0} u_z (r u_{zr})_r dr \\
&= -u_z(r_0, z) r_0 \frac{d}{dz} R(u(r_0, z),\ v^*(r_0, z),\ w^*(r_0, z)) - \|u_{zr}\|_2^2
\end{aligned}
$$

(in the last equality we have used (ii) of Lemma 3.2). After similar calculations for v and w, we see that

$$
\begin{aligned}
&\frac{d}{dz}E(D_zU\,;\ a,\ b,\ c) + 2E(D_rD_zU) \\
&= r_0\left\{-u_z(r_0, z) - \frac{m_0}{m_1} v_z(r_1, z) + \frac{n_0}{n_1} w_z(r_1, z)\right\} \\
&\quad \frac{d}{dz} R(u(r_0, z),\ v^*(r_0, z),\ w^*(r_0, z)).
\end{aligned}
$$

In view of Lemma 3.5, the right-hand side is bounded from above by

$$
C\{u_z(r_0, z)^2 + v_z(r_1, z)^2 + w_z(r_1, z)^2\}
$$

on $(0, \infty)$. Thus we arrive at the conclusion by using Lemma 4.3. □

Proof of Proposition 5.1. The last inequality immediately follows from Lemma 4.3, so it suffices to show (5.2). Observe that

$$(\log x-\log y)(x-y)\geq \varkappa(x-y)^2 \qquad \text{for } x,\ y\in(0,\ x_0],$$

where $\varkappa=1/x_0$. In particular, by choosing

$$x_0=\max\{\sup_{z>0} u(r_0,\ z)^l\sup_{z>0} v(r_1,\ z)^m,\ \sup_{z>0} w(r_1,\ z)^n\}(<\infty),$$

we obtain that

$$\begin{aligned}&\{\log u(r_0,\ z)^l v(r_1,\ z)^m-\log w(r_1,\ z)^n\}\frac{R(u(r_0,\ z),\ v(r_1,\ z),\ w(r_1,\ z))}{J(u(r_0,\ z),\ v(r_1,\ z),\ w(r_1,\ z))}\\ &\geq\frac{\varkappa\{u(r_0,\ z)^l v(r_1,\ z)^m-w(r_1,\ z)^n\}^2}{S(u(r_0,\ z),\ v(r_1,\ z),\ w(r_1,\ z))J(u(r_0,\ z),\ v(r_1,\ z),\ w(r_1,\ z))}\end{aligned}$$

for $z\in(0,\ \infty)$. Here we have used the positivity of S and J. Consequently we can derive (5.2) from Lemma 5.2, because U is positive and uniformly bounded. □

6. Eigenvalue problem

In this section we will show that the eigenvalues for (EV) are nonnegative and will characterize the least positive eigenvalue by a quadratic form.

Lemma 6.1. *In addition to* (A. 1) *and* (R. 1), *assume that* $M_{uw}>0$ *and* $M_{vw}>0$ *hold. Then*

$$\begin{cases} R_u^\infty=\dfrac{\partial R}{\partial u}(u_\infty,\ v_\infty,\ w_\infty)>0,\\ R_v^\infty=\dfrac{\partial R}{\partial v}(u_\infty,\ v_\infty,\ w_\infty)>0,\\ R_w^\infty=\dfrac{\partial R}{\partial w}(u_\infty,\ v_\infty,\ w_\infty)<0.\end{cases}$$

Proof. By Proposition 2.3 we have

$$u_\infty>0,\ v_\infty>0,\ w_\infty>0.$$

Combined with $R(u_\infty,\ v_\infty,\ w_\infty)=0$, the identity

$$\frac{\partial R}{\partial u}(u_\infty,\ v_\infty,\ w_\infty)=\frac{l u_\infty^{l-1}v_\infty^m}{S(u_\infty,\ v_\infty,\ w_\infty)}-R(u_\infty,\ v_\infty,\ w_\infty)\frac{\dfrac{\partial S}{\partial u}(u_\infty,\ v_\infty,\ w_\infty)}{S(u_\infty,\ v_\infty,\ w_\infty)}$$

implies $R_u^\infty>0$. Similarly we get $R_v^\infty>0$ and $R_w^\infty<0$. □

In the following lemmas d_v and d_w denote the positive constants defined by (3. 2).

Lemma 6.2. *For any* $(u, v, w)\in \boldsymbol{R}^3$ *there exists a unique pair* $(v^*, w^*)\in \boldsymbol{R}^2$ *such that*

$$\begin{cases} v=v^*+d_vR^L(u, v^*, w^*), \\ w=w^*-d_wR^L(u, v^*, w^*). \end{cases}$$

Proof. It suffices to solve the linear equation

$$\begin{pmatrix} 1+d_vR_v^\infty & d_vR_w^\infty \\ -d_wR_v^\infty & 1-d_wR_w^\infty \end{pmatrix}\begin{pmatrix} v^* \\ w^* \end{pmatrix}=\begin{pmatrix} v-d_vR_u^\infty u \\ w+d_wR_u^\infty u \end{pmatrix}$$

(recall the definition (2.1) of R^L). We see by Lemma 6.1 that

$$\begin{vmatrix} 1+d_vR_v^\infty & d_vR_w^\infty \\ -d_wR_v^\infty & 1-d_wR_w^\infty \end{vmatrix}=1+d_vR_v^\infty-d_wR_w^\infty>1\,;$$

so that the above linear equation is uniquely solved. □

We denote v^*, w^* corresponding to u, v, w in this lemma by

$$v^*=\beta^L(u, v, w),\ w^*=\gamma^L(u, v, w).$$

For simplicity we put

$$J_\infty:\ =J(u_\infty, v_\infty, w_\infty).$$

Lemma 6.3. *It holds that*

(i) $J_\infty=1+d_vR_v^\infty-d_wR_w^\infty>1$,

(ii) $R^L(u, v^*, w^*)=J_\infty^{-1}R^L(u, v, w)$ *for* $(u, v, w)\in \boldsymbol{R}^3$,

where $v^*=\beta^L(u, v, w)$ *and* $w^*=\gamma^L(u, v, w)$.

Proof. Since $R(u_\infty, v_\infty, w_\infty)=0$, we can see by Lemma 3.1 that

$$v_\infty=\beta(u_\infty, v_\infty, w_\infty),\ w_\infty=\gamma(u_\infty, v_\infty, w_\infty).$$

Thus the definition (3.1) of J implies the equality in (i). The inequality in (i) is derived from Lemma 6.1.

Recalling (2.1), we see that

$$\begin{aligned} R^L(u, v, w)&=R^L(u, v^*, w^*) \\ &\quad+\int_0^1 \frac{d}{d\theta}R^L(u, v^*+\theta(v-v^*), w^*+\theta(w-w^*))d\theta \\ &=R^L(u, v^*, w^*)+(v-v^*)R_v^\infty+(w-w^*)R_w^\infty. \end{aligned}$$

The right-hand side is equal to $J_\infty R^L(u, v^*, w^*)$ by virtue of Lemma 6.2. Consequently we obtain (ii). □

Using Lemmas 6.2 and 6.3, we can reduce (EV) to an eigenvalue problem for u, v and w in the same way as the proof of Proposition 3.3:

Proposition 6.4. *Suppose that a set (u, v^*, w^*, v, w) of smooth and radially symmetric functions u on Ω_0, v^* and w^* on Ω_*, v and w on Ω_1 satisfies* (EV) *for a parameter λ. Then (u, v, w) satisfies*

$$(\mathrm{EV})' \quad \begin{cases} -u_{rr}-\dfrac{1}{r}u_r=\lambda a(r)u & \text{in } (0, r_0), \\ -v_{rr}-\dfrac{1}{r}v_r=\lambda b(r)v, \ -w_{rr}-\dfrac{1}{r}w_r=\lambda c(r)w & \text{in } (r_1, 1); \\ u_r(0)=0, \\ -u_r(r_0)=\dfrac{m_1r_1}{m_0r_0}v_r(r_1)=-\dfrac{n_1r_1}{n_0r_0}w_r(r_1)=J_\infty^{-1}R^L(u(r_0), v(r_1), w(r_1)), \\ v_r(1)=w_r(1)=0 \end{cases}$$

for the same λ. Moreover (v^*, w^*) is represented by (u, v, w) as

$$(6.1) \quad \begin{cases} v^*(r_0)=\beta^L(u(r_0), v(r_1), w(r_1)), \ w^*(r_0)=\gamma^L(u(r_0), v(r_1), w(r_1)); \\ v^*(r)=v^*(r_0)+m_0r_0R^L(u(r_0), v^*(r_0), w^*(r_0))\log\dfrac{r}{r_0}, \\ w^*(r)=w^*(r_0)-n_0r_0R^L(u(r_0), v^*(r_0), w^*(r_0))\log\dfrac{r}{r_0} \qquad \text{on } [r_0, r_1]. \end{cases}$$

Conversely, suppose that a set (u, v, w) of smooth and radially symmetric functions u on Ω_0, v and w on Ω_1 satisfies (EV)$'$ *for a parameter λ. Then (u, v^*, w^*, v, w), where v^* and w^* are defined by* (6.1), *satisfies* (EV) *for the same λ.*

We will formulate (EV)$'$ as an eigenvalue problem for a linear operator in $L^2_r(\Omega_0\,;\, a)\times L^2_r(\Omega_1\,;\, b)\times L^2_r(\Omega_1\,;\, c)$. Define a linear operator $\mathcal{L}$ with its definition domain $D(\mathcal{L})$ by

$$\begin{aligned} D(\mathcal{L}):=\{\mathcal{U}=(u, v, w)\in L^2_r(\Omega_0\,;\, a)\times L^2_r(\Omega_1\,;\, b)\times L^2_r(\Omega_1\,;\, c)\,; \\ \frac{(ru_r)_r}{a(r)r}\in L^2_r(\Omega_0\,;\, a), \frac{(rv_r)_r}{b(r)r}\in L^2_r(\Omega_1\,;\, b), \frac{(rw_r)_r}{c(r)r}\in L^2_r(\Omega_1\,;\, c), \\ u_r(0)=v_r(1)=w_r(1)=0, \quad -u_r(r_0)=\frac{m_1r_1}{m_0r_0}v_r(r_1) \\ =-\frac{n_1r_1}{n_0r_0}w_r(r_1)=J_\infty^{-1}R^L(u(r_0), v(r_1), w(r_1))\}, \\ \mathcal{L}\mathcal{U}:=-\left(\frac{(ru_r)_r}{a(r)r}, \frac{(rv_r)_r}{b(r)r}, \frac{(rw_r)_r}{c(r)r}\right) \quad \text{for } \mathcal{U}=(u, v, w)\in D(\mathcal{L}). \end{aligned}$$

By Lemma 4.4 we can see that $D(\mathcal{L})$ is well defined and that

$$D(\mathcal{L})\subset(H^2_r(\Omega_0)\cap C^1_r(\bar{\Omega}_0))\times(H^2_r(\Omega_1)\cap C^1_r(\bar{\Omega}_1))^2.$$

Clearly λ is an eigenvalue of $\mathcal{L}$ if and only if there exists a $(u, v, w)\neq(0, 0, 0)$ satisfying (EV)$'$ for λ. Thus the set of the eigenvalues for (EV) coincides with that of $\mathcal{L}$. Moreover the linearization of (P) at $(u_\infty, v_\infty, w_\infty)$ can be represented by the abstract form

$$\frac{d\mathcal{U}}{dz}+\mathcal{L}\mathcal{U}=0 \qquad \text{in } L^2_r(\Omega_0\,;\ a)\times L^2_r(\Omega_1\,;\ b)\times L^2_r(\Omega_1\,;\ c).$$

Consequently it is important for us to analyze the spectrum of $\mathcal{L}$.

A remarkable point among properties of $\mathcal{L}$ is the fact that $\mathcal{L}$ is self-adjoint in $L^2_r(\Omega_0\,;\ a)\times L^2_r(\Omega_1\,;\ b)\times L^2_r(\Omega_1\,;\ c)$ equipped with an appropriate inner product. We will show this fact.

Let us construct a symmetric bilinear from associated with $\mathcal{L}$. For $(u, v, w)\in D(\mathcal{L})$ and $(u', v', w')\in C^1_r(\bar{\Omega}_0)\times C^1_r(\bar{\Omega}_1)^2$ we have

$$\begin{aligned}\int_0^{r_0}\left\{-\frac{(ru_r)_r}{ar}\right\}u'ardr&=-\int_0^{r_0}(ru_r)_ru'dr=-[ru_ru']_0^{r_0}+\int_0^{r_0}ru_ru'_rdr\\&=\int_0^{r_0}u_ru'_rrdr+\frac{r_0}{J_\infty}R^L(u(r_0),\ v(r_1),\ w(r_1))u'(r_0),\\ \int_{r_1}^1\left\{-\frac{(rv_r)_r}{br}\right\}v'brdr&=\int_{r_1}^1v_rv'_rrdr+\frac{m_0r_0}{m_1J_\infty}R^L(u(r_0),\ v(r_1),\ w(r_1))v'(r_1),\\ \int_{r_1}^1\left\{-\frac{(rw_r)_r}{cr}\right\}v'crdr&=\int_{r_1}^1w_rw'_rrdr-\frac{n_0r_0}{n_1J_\infty}R^L(u(r_0),\ v(r_1),\ w(r_1))w'(r_1).\end{aligned}$$

Recall the definition (2.1) of R^L. Then these three identities lead us to

$$\begin{aligned}&R_u^\infty\int_0^{r_0}\left\{-\frac{(ru_r)_r}{ar}\right\}u'ardr+\frac{m_1}{m_0}R_v^\infty\int_{r_1}^1\left\{-\frac{(rv_r)_r}{br}\right\}v'brdr\\&+\frac{n_1}{n_0}(-R_w^\infty)\int_{r_1}^1\left\{-\frac{(rw_r)_r}{cr}\right\}w'crdr\\&=R_u^\infty\int_0^{r_0}u_ru'_rrdr+\frac{m_1}{m_0}R_v^\infty\int_{r_1}^1v_rv'_rrdr+\frac{n_1}{n_0}(-R_w^\infty)\int_{r_1}^1w_rw'_rrdr\\&\quad+\frac{r_0}{J_\infty}R^L(u(r_0),\ v(r_1),\ w(r_1))R^L(u'(r_0),\ v'(r_1),\ w'(r_1)).\end{aligned}\tag{6.2}$$

Seeing both the sides, we introduce the following symmetric bilinear forms:

$$\langle\mathcal{U},\ \mathcal{U}'\rangle:=R_u^\infty\int_0^{r_0}uu'ardr+\frac{m_1}{m_0}R_v^\infty\int_{r_1}^1vv'brdr+\frac{n_1}{n_0}(-R_w^\infty)\int_{r_1}^1ww'crdr$$
$$\text{for } \mathcal{U}=(u,\ v,\ w),\ \mathcal{U}'=(u',\ v',\ w')\in L^2_r(\Omega_0\,;\ a)\times L^2_r(\Omega_1\,;\ b)\times L^2_r(\Omega_1\,;\ c);$$
$$\begin{aligned}Q(\mathcal{U},\ \mathcal{U}'):=&R_u^\infty\int_0^{r_0}u_ru'_rrdr+\frac{m_1}{m_0}R_v^\infty\int_{r_1}^1v_rv'_rrdr+\frac{n_1}{n_0}(-R_w^\infty)\int_{r_1}^1w_rw'_rrdr\\&+\frac{r_0}{J_\infty}R^L(u(r_0),\ v(r_1),\ w(r_1))R^L(u'(r_0),\ v'(r_1),\ w'(r_1))\end{aligned}$$
$$\text{for } \mathcal{U}=(u,\ v,\ w),\ \mathcal{U}'=(u',\ v',\ w')\in H^1_r(\Omega_0)\times H^1_r(\Omega_1)^2.$$

In view of Lemma 6.1, we can define a Hilbert space $\mathcal{H}$ by $L^2_r(\Omega_0\,;\ a)\times L^2_r(\Omega_1\,;\ b)$

$\times L^2_r(\Omega_1; c)$ equipped with an inner product $\langle\cdot,\cdot\rangle$. We denote by $|||\cdot|||$ the norm in $\mathcal{H}$ and abbreviate $Q(\mathcal{U},\mathcal{U})$ to $Q(\mathcal{U})$, i.e.,

$$|||\mathcal{U}|||^2=R^\infty_u\|u\|^2_{2,\Omega_0;a}+\frac{m_1}{m_0}R^\infty_v\|v\|^2_{2,\Omega_1;b}+\frac{n_1}{n_0}(-R^\infty_w)\|w\|^2_{2,\Omega_1;c} \tag{6.3}$$

for $\mathcal{U}=(u, v, w)\in\mathcal{H}$ and

$$\begin{aligned}Q(\mathcal{U})&=R^\infty_u\|u_r\|^2_{2,\Omega_0}+\frac{m_1}{m_0}R^\infty_v\|v_r\|^2_{2,\Omega_1}+\frac{n_1}{n_0}(-R^\infty_w)\|w_r\|^2_{2,\Omega_1}\\&\quad+\frac{r_0}{J_\infty}R^L(u(r_0), v(r_1), w(r_1))^2\\&\geq 0\end{aligned} \tag{6.4}$$

for $\mathcal{U}=(u, v, w)\in H^1_r(\Omega_0)\times H^1_r(\Omega_1)^2$. Clearly the identity (6.2) also holds for $(u', v', w')\in(H^1_r(\Omega_0)\cap C_r(\Omega_0))\times H^1_r(\Omega_1)^2$. Thus we have obtained the following lemma.

Lemma 6.5. *The identity*

$$\langle\mathcal{L}\mathcal{U},\mathcal{U}'\rangle=Q(\mathcal{U},\mathcal{U}')$$

holds for $\mathcal{U}\in D(\mathcal{L})$, $\mathcal{U}'\in(H^1_r(\Omega_0)\cap C_r(\Omega_0))\times H^1_r(\Omega_1)^2$.

Lemma 6.6. *The operator* $\mathcal{L}$ *is self-adjoint and positive semidefinite in* $\mathcal{H}$. *It has a compact resolvent in* $\mathcal{H}$.

Proof. Fix an arbitrary element $\mathcal{F}$ of $\mathcal{H}$. We see by Lemma 4.3 that $\langle\cdot,\cdot\rangle+Q(\cdot,\cdot)$ is an inner product equivalent to the usual inner product in $H^1_r(\Omega_0)\times H^1_r(\Omega_1)^2$. Hence Riesz' theorem leads us to the fact: there exists a unique element $\mathcal{U}\in H^1_r(\Omega_0)\times H^1_r(\Omega_1)^2$ such that

$$\langle\mathcal{U},\mathcal{U}'\rangle+Q(\mathcal{U},\mathcal{U}')=\langle\mathcal{F},\mathcal{U}'\rangle \qquad \text{for all } \mathcal{U}'\in H^1_r(\Omega_0)\times H^1_r(\Omega_1)^2.$$

By virtue of Lemma 4.4, we can show $\mathcal{U}\in D(\mathcal{L})$ in the standard manner for elliptic boundary value problems. Thus, with the aid of Lemma 6.5, we get

$$\mathcal{U}+\mathcal{L}\mathcal{U}=\mathcal{F},$$

i.e., the range of $I+\mathcal{L}$ coincides with $\mathcal{H}$. On the other hand Lemma 6.5 yields

$$\left.\begin{aligned}\langle\mathcal{L}\mathcal{U},\mathcal{U}'\rangle&=\langle\mathcal{U},\mathcal{L}\mathcal{U}'\rangle,\\ \langle\mathcal{L}\mathcal{U},\mathcal{U}\rangle&=Q(\mathcal{U})\geq 0\end{aligned}\right\}\qquad \text{for } \mathcal{U},\mathcal{U}'\in D(\mathcal{L}).$$

Consequently $\mathcal{L}$ is self-adjoint and positive semidefinite. We can derive the compactness of resolvents for $\mathcal{L}$ by a standard argument about elliptic differential operators in a bounded domain. $\square$

By virtue of this lemma, the spectrum of $\mathscr{L}$ consists of countably many nonnegative eigenvalues and has no accumulation points. We can see that dim Ker $\mathscr{L}=2$ by the following lemma.

Lemma 6.7. *Zero is an eigenvalue of $\mathscr{L}$ and the corresponding eigenspace consists of constant functions. More precisely,*

$$\operatorname{Ker}\mathscr{L}=\{s\Phi_1+t\Phi_2\,;\ s,\ t\in\boldsymbol{R}\}$$

with

$$\Phi_1=\left(\frac{1}{R_u^\infty},\ 0,\ -\frac{1}{R_w^\infty}\right),\ \Phi_2=\left(0,\ \frac{1}{R_v^\infty},\ -\frac{1}{R_w^\infty}\right)\in\boldsymbol{R}^3.$$

Proof. We have only to see that $\mathcal{U}=(u,\ v,\ w)\in\operatorname{Ker}\mathscr{L}$ if and only if $\mathcal{U}$ is a constant satisfying $R^L(u,\ v,\ w)=0$. □

On the basis of Lemmas 6.6 and 6.7 we denote the eigenvalues of $\mathscr{L}$ by $\{\lambda_j\}_{j=1,2,3,\cdots}$ with

$$(0=)\lambda_1=\lambda_2<\lambda_3\leq\lambda_4\leq\cdots.$$

Since $\mathscr{L}$ is self-adjoint, we have

$$\lambda_3=\min\left\{\frac{\langle\mathscr{L}\mathcal{U},\ \mathcal{U}\rangle}{|||\mathcal{U}|||^2};\ \mathcal{U}\in D(\mathscr{L})\backslash 0,\ \mathcal{U}\perp\operatorname{Ker}\mathscr{L}\text{ in }\mathscr{H}\right\}. \tag{6.5}$$

Seeing that

$$\left.\begin{aligned}\langle\mathcal{U},\ \Phi_1\rangle&=\int_0^{r_0}uardr+\frac{n_1}{n_0}\int_{r_1}^1 wcrdr\\ \langle\mathcal{U},\ \Phi_2\rangle&=\frac{m_1}{m_0}\int_{r_1}^1 vbrdr+\frac{n_1}{n_0}\int_{r_1}^1 wcrdr\end{aligned}\right\}\quad\text{for }\mathcal{U}=(u,\ v,\ w)\in\mathscr{H},$$

we can easily obtain the following.

Lemma 6.8. *For $\mathcal{U}=(u,\ v,\ w)\in\mathscr{H}$ the following holds: $\mathcal{U}\perp\operatorname{Ker}\mathscr{L}$ in $\mathscr{H}$ if and only if $(u,\ v,\ w)$ satisfies*

$$\int_0^{r_0}uardr+\frac{n_1}{n_0}\int_{r_1}^1 wcrdr=\frac{m_1}{m_0}\int_{r_1}^1 vbrdr+\frac{n_1}{n_0}\int_{r_1}^1 wcrdr=0. \tag{M$_0$}$$

Now we are ready to characterize λ_+ by $Q(\cdot)$ and $|||\cdot|||$.

Proposition 6.9. *The eigenvalues for* (EV) *are all nonnegative and the set of them has no accumulation points. The least positive eigenvalue λ_+ is represented as*

$$\lambda_{+}=\inf\left\{\frac{Q(\mathcal{U})}{|||\mathcal{U}|||^{2}};\ \mathcal{U}\in(H_{r}^{1}(\Omega_{0})\times H_{r}^{1}(\Omega_{1})^{2})\backslash 0,\ \mathcal{U}\ \textit{satisfies}\ (\mathrm{M}_{0})\right\}.$$

Proof. Since the set of the eigenvalues for (EV) coincides with that of $\mathcal{L}$, it suffices to show the latter part. Observe that we can choose $\langle\cdot,\ \cdot\rangle+Q(\cdot,\ \cdot)$ as an inner product in $H_{r}^{1}(\Omega_{0})\times H_{r}^{1}(\Omega_{1})^{2}$. Then, by a standard argument about self-adjoint operators, we can derive

$$\begin{aligned}
&\inf\left\{\frac{Q(\mathcal{U})}{|||\mathcal{U}|||^{2}};\ \mathcal{U}\in(H_{r}^{1}(\Omega_{0})\times H_{r}^{1}(\Omega_{1})^{2})\backslash 0,\ \mathcal{U}\ \text{satisfies}\ (\mathrm{M}_{0})\right\}\\
&\quad=\inf\left\{\frac{Q(\mathcal{U})}{|||\mathcal{U}|||^{2}};\ \mathcal{U}\in(H_{r}^{1}(\Omega_{0})\times H_{r}^{1}(\Omega_{1})^{2})\backslash 0,\ \mathcal{U}\perp\operatorname{Ker}\mathcal{L}\ \text{in}\ \mathcal{H}\right\}\\
&\quad=\inf\left\{\frac{Q(\mathcal{U})}{|||\mathcal{U}|||^{2}};\ \mathcal{U}\in D(\mathcal{L})\backslash 0,\ \mathcal{U}\perp\operatorname{Ker}\mathcal{L}\ \text{in}\ \mathcal{H}\right\}\\
&\quad=\lambda_{3}=\lambda_{+}
\end{aligned}$$

(see, e.g., Courant-Hilbert [3]). □

7. Proof of Theorem B

As an application of the preceding section, we get the positivity of $Q(\cdot)$.

Lemma 7.1. *If $\mathcal{U}=(u,\ v,\ w)\in H_{r}^{1}(\Omega_{0})\times H_{r}^{1}(\Omega_{1})^{2}$ satisfies* (M_{0}), *then*

$$\lambda_{+}|||\mathcal{U}|||^{2}\leq Q(\mathcal{U}), \tag{7.1}$$

$$u(r_{0})^{2}+v(r_{1})^{2}+w(r_{1})^{2}\leq KQ(\mathcal{U}), \tag{7.2}$$

where K is a positive constant independent of $\mathcal{U}$.

Proof. Proposition 6.9 implies (7.1). We can show (7.2) by using (7.1) and Lemma 4.3. □

Let $(u,\ v,\ w)$ be the solution to (P). Throughout this section we use the following abbreviations:

$$\begin{cases}
\tilde{u}:=u-u_{\infty},\quad \tilde{v}:=v-v_{\infty},\quad \tilde{w}:=w-w_{\infty},\\
\mathcal{U}(z):=(\tilde{u}(\cdot,\ z),\ \tilde{v}(\cdot,\ z),\ \tilde{w}(\cdot,\ z)),\\
D_{z}\mathcal{U}(z):=(\tilde{u}_{z}(\cdot,\ z),\ \tilde{v}_{z}(\cdot,\ z),\ \tilde{w}_{z}(\cdot,\ z)),\\
\|\mathcal{U}(z)\|_{\infty}:=\|\tilde{u}(\cdot,\ z)\|_{\infty,\Omega_{0}}+\|\tilde{v}(\cdot,\ z)\|_{\infty,\Omega_{1}}+\|\tilde{w}(\cdot,\ z)\|_{\infty,\Omega_{1}}.
\end{cases}$$

Since $(u,\ v,\ w)$ satisfies (M) for all $z\in[0,\ \infty)$, we have

$$
(7.3)\qquad \begin{cases} \displaystyle\int_0^{r_0} \tilde{u}\, ardr + \frac{n_1}{n_0}\int_{r_1}^1 \tilde{w}\, crdr \\ \displaystyle\qquad = \int_0^{r_0} uardr + \frac{n_1}{n_0}\int_{r_1}^1 wcrdr - M_{uw} = 0, \\ \displaystyle\frac{m_1}{m_0}\int_{r_1}^1 \tilde{v}\, brdr + \frac{n_1}{n_0}\int_{r_1}^1 \tilde{w}\, crdr \\ \displaystyle\qquad = \frac{m_1}{m_0}\int_{r_1}^1 vbrdr + \frac{n_1}{n_0}\int_{r_1}^1 wcrdr - M_{vw} = 0. \end{cases}
$$

Hence $\mathcal{U}(z)$ satisfies (M_0) for all $z \in [0, \infty)$. Moreover, by differentiating (7.3) with respect to z, we can see that $D_z\mathcal{U}(z)$ also satisfies (M_0) for all $z \in (0, \infty)$. These facts play an essential role in the proof of Theorem **B**.

Proposition 7.2. *There exists a positive constant N such that*

$$
(7.4)\qquad \begin{cases} \displaystyle\frac{1}{2}\frac{d}{dz}|||\mathcal{U}(z)|||^2 + Q(\mathcal{U}(z)) \\ \qquad \leq NQ(\mathcal{U}(z))\{|\tilde{u}(r_0, z)| + |\tilde{v}(r_1, z)| + |\tilde{w}(r_1, z)|\}, \\ \displaystyle\frac{1}{2}\frac{d}{dz}|||D_z\mathcal{U}(z)|||^2 + Q(D_z\mathcal{U}(z)) \\ \qquad \leq NQ(D_z\mathcal{U}(z))\{|\tilde{u}(r_0, z)| + |\tilde{v}(r_1, z)| + |\tilde{w}(r_1, z)|\} \end{cases}
$$

for $z \in (0, \infty)$.

Proof. Since $\|\mathcal{U}(z)\|_\infty$ is bounded on $(0, \infty)$, it follows from $R(u_\infty, v_\infty, w_\infty) = 0$ that

$$
(7.5)\qquad \frac{R(u(r_0, z), v(r_1, z), w(r_1, z))}{J(u(r_0, z), v(r_1, z), w(r_1, z))} = J_\infty^{-1} R^L(\tilde{u}(r_0, z), \tilde{v}(r_1, z), \tilde{w}(r_1, z)) + \tilde{R}(z),
$$

where $\tilde{R}(z)$ is a function of class $C^\infty(0, \infty)$ satisfying

$$
(7.6)\qquad |\tilde{R}(z)| \leq C\{\tilde{u}(r_0, z)^2 + \tilde{v}(r_1, z)^2 + \tilde{w}(r_1, z)^2\},
$$

$$
(7.7)\qquad \left|\frac{d}{dz}\tilde{R}(z)\right| \leq C\{|\tilde{u}(r_0, z)| + |\tilde{v}(r_1, z)| + |\tilde{w}(r_1, z)|\} \{|\tilde{u}_z(r_0, z)| + |\tilde{v}_z(r_1, z)| + |\tilde{w}_z(r_1, z)|\}
$$

for $z \in (0, \infty)$.

We will show the first inequality of (7.4). Multiplying both the sides of $a(r)u_z = u_{rr} + r^{-1}u_r (= r^{-1}(ru_r)_r)$ by $r\tilde{u}$ and integrating them from 0 to r_0, we have

$$
\frac{1}{2}\frac{d}{dz}\int_0^{r_0} \tilde{u}^2 ardr = \int_0^{r_0} \tilde{u}\, au_z rdr = \int_0^{r_0} \tilde{u}(ru_r)_r dr
$$

$$
= [\tilde{u}\, ru_r]_0^{r_0} - \int_0^{r_0} \tilde{u}_r ru_r dr
$$

$$= -\tilde{u}(r_0, z) r_0 \frac{R(u(r_0, z), v(r_1, z), w(r_1, z))}{J(u(r_0, z), v(r_1, z), w(r_1, z))} - \int_0^{r_0} \tilde{u}_r^2 r dr$$
$$= -\tilde{u}(r_0, z)\frac{r_0}{J_\infty} R^L(\tilde{u}(r_0, z), \tilde{v}(r_1, z), \tilde{w}(r_1, z)) - \tilde{u}(r_0, z) r_0 \tilde{R}(z)$$
$$- \int_0^{r_0} \tilde{u}_r^2 r dr.$$

Accordingly,

$$\frac{1}{2}\frac{d}{dz}\| \tilde{u} \|_{2;a}^2 + \| \tilde{u}_r \|_2^2 + \frac{r_0}{J_\infty} \tilde{u}(r_0, z) R^L(\tilde{u}(r_0, z), \tilde{v}(r_1, z), \tilde{w}(r_1, z))$$
$$= -r_0 \tilde{u}(r_0, z) \tilde{R}(z).$$

Similarly we can derive

$$\frac{1}{2}\frac{d}{dz}\| \tilde{v} \|_{2;b}^2 + \| \tilde{v}_r \|_2^2 + \frac{m_0 r_0}{m_1 J_\infty} \tilde{v}(r_1, z) R^L(\tilde{u}(r_0, z), \tilde{v}(r_1, z), \tilde{w}(r_1, z))$$
$$= -\frac{m_0 r_0}{m_1} \tilde{v}(r_1, z) \tilde{R}(z),$$
$$\frac{1}{2}\frac{d}{dz}\| \tilde{w} \|_{2;c}^2 + \| \tilde{w}_r \|_2^2 - \frac{n_0 r_0}{n_1 J_\infty} \tilde{w}(r_1, z) R^L(\tilde{u}(r_0, z), \tilde{v}(r_1, z), \tilde{w}(r_1, z))$$
$$= \frac{n_0 r_0}{n_1} \tilde{w}(r_1, z) \tilde{R}(z).$$

Summing up these three equalities, we get

$$\frac{1}{2}\frac{d}{dz}|||\mathcal{U}(z)|||^2 + Q(\mathcal{U}(z)) = -r_0 R^L(\tilde{u}(r_0, z), \tilde{v}(r_1, z), \tilde{w}(r_1, z)) \tilde{R}(z)$$

(recall (6.3) and (6.4)). Since $\mathcal{U}(z)$ satisfies (M_0), (7.2) holds with $\mathcal{U} = \mathcal{U}(z)$. Thus, with the aid of (7.6), we obtain the first inequality of (7.4).

After the differentiation of equations in (P) with respect to z, similar calculation yields

$$\frac{1}{2}\frac{d}{dz}|||D_z\mathcal{U}(z)|||^2 + Q(D_z\mathcal{U}(z))$$
$$= -r_0 R^L(\tilde{u}_z(r_0, z), \tilde{v}_z(r_1, z), \tilde{w}_z(r_1, z)) \frac{d}{dz}\tilde{R}(z).$$

In view of (7.7) and (7.2), we obtain the second inequality of (7.4). □

Lemma 7.3. *There exists a number* $\epsilon \in (0, \lambda_+)$ *such that*

$$\|\mathcal{U}(z)\|_\infty = O(\exp(-\lambda_+ z + \epsilon z)) \qquad \text{as } z \longrightarrow \infty.$$

Proof. The inequality (7.1) holds with $\mathcal{U} = \mathcal{U}(z)$ and with $\mathcal{U} = D_z\mathcal{U}(z)$, because $\mathcal{U}(z)$ and $D_z\mathcal{U}(z)$ satisfy (M_0). Then Theorem **A** and (7.4) lead us to

$$\left.\begin{aligned}|||\mathcal{U}(z)||| &= O(\exp(-\lambda_+ z+\epsilon z)),\\ |||D_z\mathcal{U}(z)||| &= O(\exp(-\lambda_+ z+\epsilon z))\end{aligned}\right\} \quad \text{as } z\to\infty$$

with some $\epsilon<0$ (use, for instance, Lemma 4.2 with $k_0=1$). Now it suffices to show that

$$\|\mathcal{U}(z)\|_\infty \le C\{|||\mathcal{U}(z)|||+|||D_z\mathcal{U}(z)|||\}. \tag{7.8}$$

Applying Lemmas 4.3 and 4.4 to the right-hand side of

$$\tilde{u}(r,\ z)=\tilde{u}(r_0,\ z)-\int_r^{r_0}\tilde{u}_r(\rho,\ z)d\rho,$$

we get

$$\|\tilde{u}\|_\infty \le C(\|\tilde{u}\|_{2;a}+\|\Delta\tilde{u}\|_2).$$

Moreover $\tilde{u}$ satisfies a diffusion equation. Thus we have

$$\|\tilde{u}\|_\infty \le C\|\tilde{u}\|_{2;a}+\|\tilde{u}_z\|_{2;a}).$$

We can derive similar inequalities for $\tilde{v}$ and $\tilde{w}$. Consequently we obtain (7.8) and complete the proof. □

Lemma 7.4. $\|\mathcal{U}(z)\|_\infty=O(\exp(-\lambda_+ z))$ *as* $z\longrightarrow\infty$.

Proof. We have only to repeat the argument in the proof of Lemma 7.3, using the fact

$$|\tilde{u}(r_0,\ z)|+|\tilde{v}(r_1,\ z)|+|\tilde{w}(r_1,\ z)|=O(\exp(-\lambda_+ z+\epsilon z)) \quad \text{as } z\to\infty$$

in place of Theorem **A**. □

Proof of Theorem **B**. Using (ii) of Lemma 3.2 and (3.3), we can rewrite (7.5) as

$$R(u(r_0,\ z),\ v^*(r_0,\ z),\ w^*(r_0,\ z))=J_\infty^{-1}R^L(\tilde{u}(r_0,\ z),\ \tilde{v}(r_1,\ z),\ \tilde{w}(r_1,\ z))+\tilde{R}(z),$$

which, combined with (7.6) and Lemma 7.4, implies

$$R(u(r_0,\ z),\ v^*(r_0,\ z),\ w^*(r_0,\ z))=O(\exp(-\lambda_+ z)) \quad \text{as } z\to\infty. \tag{7.9}$$

Hence Lemma 3.1 and (3.3) lead us to

$$\begin{aligned}v^*(r_0,\ z)-v_\infty &= \tilde{v}(r_1,\ z)-d_vR(u(r_0,\ z),\ v^*(r_0,\ z),\ w^*(r_0,\ z))\\ &= O(\exp(-\lambda_+ z)) \qquad \text{as } z\to\infty.\end{aligned}$$

Consequently, using (3.4), we obtain from (7.9) that

$$\|v^*(\cdot,\ z)-v_\infty\|_\infty=O(\exp(-\lambda_+ z)) \quad \text{as } z\to\infty.$$

Similarly we can derive the corresponding result for w^*; so that the proof is completed. □

8. Proof of Theorem C-(i) L^∞-decay of derivatives

In this section we prove that all derivatives of solution converge to 0 uniformly for r as $z\to\infty$. Using it, we will derive the rates of their convergence in the succeeding section. First we give some energy estimates for derivatives of solutions. Here again we use the abbreviation and the energy functionals in §5.

Proposition 8.1. *Let k be an integer with $k\geq 2$. The solution $U=(u, v, w)$ to* (P) *satisfies*

$$\frac{d}{dz}E(D_rD_z^{k-1}U\,;\ f,\ g,\ h)+2E(D_z^kU\,;\ af,\ bg,\ ch)=E(D_rD_z^{k-1}U),$$
$$E(D_z^kU\,;\ a,\ b,\ c)\leq N_k\{E(D_z^kU\,;\ af,\ bg,\ ch)+E(D_rD_z^kU)\}$$

for $z\in(0,\infty)$. Here N_k is a positive constant that is independent of z; f, g and h are the weight functions that are given in Proposition 5.1;

$$D_z^kU=(D_z^ku,\ D_z^kv,\ D_z^kw),\ D_rD_z^kU=(D_rD_z^ku,\ D_rD_z^kv,\ D_rD_z^kw).$$

Proof. The inequality immediately follows from Lemma 4.3. To obtain the equality, differentiate (P) $k-1$ times with respect to z. Then the same calculation for $D_z^{k-1}U$ as the proof of Lemma 5.3 leads us to the equality. □

Proposition 8.2. *Let k be an integer with $k\geq 2$. Suppose that $D_z^ju(r_0,\cdot)$, $D_z^jv(r_1,\cdot)$, $D_z^jw(r_1,\cdot)$ $(0\leq j\leq k-1)$ are bounded on $[1,\infty)$ for the solution $U=(u, v, w)$ to* (P). *Then U satisfies*

$$\begin{aligned}&\frac{d}{dz}E(D_z^kU\,;\ a,\ b,\ c)+E(D_rD_z^kU)\\ &\leq N_k\{\sum_{j=1}^{k}E(D_z^jU\,;\ af,\ bg,\ ch)+\sum_{j=1}^{k-1}E(D_rD_z^jU)\}\end{aligned}\tag{8.1}$$

for $z\in[1,\infty)$. Here N_k is a positive constant that is independent of z.

Proof. After differentiating (P) k times with respect to z, repeat the same argument as the proof of Lemma 5.4. □

Let ϕ be a function of class $C^\infty[r_1, 1]$ satisfying

$$0\le\phi(r)\le 1 \qquad \text{on } [r_1, 1],$$

$$\phi(r)=\begin{cases}1 & \text{if } r_1\le r\le\dfrac{2r_1+1}{3},\\[2mm] 0 & \text{if } \dfrac{r_1+2}{3}\le r\le 1.\end{cases}$$

For the solution (u, v, w) to (P) we use the abbreviation

$$\tilde{u}=u-u_\infty,\ \tilde{v}=v-v_\infty,\ \tilde{w}=w-w_\infty$$

and introduce three functions

$$G_u(r, z):=-\frac{R(u(r, z), v(r_1, z), w(r_1, z))}{J(u(r, z), v(r_1, z), w(r_1, z))}, \qquad (r, z)\in[0, r_0]\times[1, \infty),$$

$$G_v(r, z):=\phi(r)\frac{m_0r_0R(u(r_0, z), v(r, z), w(r_1, z))}{m_1r_1J(u(r_0, z), v(r, z), w(r_1, z))}, \quad (r, z)\in[r_1, 1]\times[1, \infty),$$

$$G_w(r, z):=-\phi(r)\frac{n_0r_0R(u(r_0, z), v(r_1, z), w(r, z))}{n_1r_1J(u(r_0, z), v(r_1, z), w(r, z))}, (r, z)\in[r_1, 1]\times[1, \infty).$$

We give a priori estimates for Sobolev norms of the solution to (P).

Lemma 8.3. *For any nonnegative integer i there exists a positive number L_i such that*

$$\begin{aligned}
&|[D_z^k\tilde{u}]|_{i+2,\Omega_0}\le L_i(\|D_z^k\tilde{u}\|_{2,\Omega_0}+|[D_z^{k+1}\tilde{u}]|_{i,\Omega_0}+|[D_z^kG_u]|_{i+1,\Omega_0}), && z\in[1, \infty),\\
&|[D_z^k\tilde{v}]|_{i+2,\Omega_1}\le L_i(\|D_z^k\tilde{v}\|_{2,\Omega_1}+|[D_z^{k+1}\tilde{v}]|_{i,\Omega_1}+|[D_z^kG_v]|_{i+1,\Omega_1}), && z\in[1, \infty),\\
&|[D_z^k\tilde{w}]|_{i+2,\Omega_1}\le L_i(\|D_z^k\tilde{w}\|_{2,\Omega_1}+|[D_z^{k+1}\tilde{w}]|_{i,\Omega_1}+|[D_z^kG_w]|_{i+1,\Omega_1}), && z\in[1, \infty)
\end{aligned}$$

for $k=0, 1, 2, \cdots$, where $|[\cdot]|_{i,\Omega_j}$ denotes the usual norm in a Sobolev space $H^i(\Omega_j)$ $(j=0, 1)$.

Proof. Since

$$\begin{cases}G_v(|x|, z)=\dfrac{m_0r_0R(u(r_0, z), v(r_1, z), w(r_1, z))}{m_1r_1J(u(r_0, z), v(r_1, z), w(r_1, z))}, & x\in\Gamma_1,\\ G_v(|x|, z)=0, & x\in\Gamma_2,\end{cases}$$

the function $D_z^k\tilde{v}(|x|, z)$ of x satisfies

$$\begin{cases}\Delta D_z^k\tilde{v}=b(|x|)D_z^{k+1}\tilde{v} & \text{in } \Omega_1,\\ \dfrac{\partial}{\partial\nu_1}D_z^k\tilde{v}=D_z^kG_v & \text{on } \Gamma_1,\\ \dfrac{\partial}{\partial\nu_2}D_z^k\tilde{v}=D_z^kG_v & \text{on } \Gamma_2\end{cases}$$

for $z\in[1, \infty)$. Hence we get

$$|[D_z^k\tilde{v}]|_{i+2,\Omega_1}\le C_i(\|D_z^k\tilde{v}\|_{2,\Omega_1}+|[bD_z^{k+1}\tilde{v}]|_{i,\Omega_1}+|[D_z^kG_v]|_{i+1/2,\partial\Omega_1})$$

by virtue of Agmon-Douglis-Nirenberg [1]. This a priori estimate yields the conclusion for $\tilde{v}$. Similarly we obtain the estimates for $\tilde{u}$ and $\tilde{w}$. □

Now we prove that

$$\begin{aligned}(8.2)\qquad &\|D_r^i D_z^j u\|_{\infty,\Omega_0}+\|D_r^i D_z^j v^*\|_{\infty,\Omega_*}+\|D_r^i D_z^j w^*\|_{\infty,\Omega_*}\\ &+\|D_r^i D_z^j v\|_{\infty,\Omega_1}+\|D_r^i D_z^j w\|_{\infty,\Omega_1}\longrightarrow 0 \qquad \text{as } z\to\infty\end{aligned}$$

for all nonnegative integers i, j with $(i, j)\neq(0, 0)$ by using Propositions 8.1, 8.2 and Lemma 8.3.

Proof of (8.2). We divide the proof into three steps.
First we will show that

$$\sup_{z\geq 1}\{\|D_z^{k-1}u(\cdot, z)\|_\infty+\|D_z^{k-1}v(\cdot, z)\|_\infty+\|D_z^{k-1}w(\cdot, z)\|_\infty\}<\infty, \tag{8.3}$$

$$\lim_{z\to\infty}(\|D_z^k u(\cdot, z)\|_{2;a}+\|D_z^k v(\cdot, z)\|_{2;b}+\|D_z^k w(\cdot, z)\|_{2;c})=0 \tag{8.4}$$

for $k=1, 2, 3, \cdots$. We have already verified them for $k=1$ (see (5.4)). Let us consider the case $k=2$. By virtue of (8.3) and Lemma 3.5 with $k=1$, we see that $(D_z u, D_z v, D_z w)$ satisfies (4.7). Hence we can derive (8.3) with $k=2$ from (8.4) with $k=1$ by using Lemma 4.7. According to Proposition 8.2, we obtain (8.1) with $k=2$ from (8.3) with $k\leq 2$. Consequently, as an application of Lemma 4.1, we can derive (8.4) with $k=2$ from Propositions 5.1, 8.1 and (8.1) with $k=2$. For $k\geq 3$, repeat this argument with Lemma 3.5 replaced by Corollary 3.6. Then we can inductively prove (8.3) and (8.4) for $k=3, 4, 5, \cdots$.

Next we will show the convergence of Sobolev norms of u, v and w:

$$\lim_{z\to\infty}|[D_z^k \tilde{u}(\cdot, z)]|_i+|[D_z^k \tilde{v}(\cdot, z)]|_i+|[D_z^k \tilde{w}(\cdot, z)]|_i=0 \tag{8.5}$$

for $k=0, 1, 2,\cdots$; $i=1, 2, 3, \cdots$. Here we denote by $|[\cdot]|_i$ the usual norm in a Sobolev space $H^i(\Omega_j)$ ($j=0$ or $j=1$). By Lemma 4.4 we have

$$\|D_r D_z^k u\|_\infty^2\leq\frac{1}{2}\|\Delta D_z^k u\|_2^2=\frac{1}{2}\|aD_z^{k+1}u\|_2^2\leq\frac{1}{2}\|a\|_\infty\|D_z^{k+1}u\|_{2;a}^2,$$

which, together with Lemma 4.3, implies

$$|[D_z^k \tilde{u}]|_1\leq C(\|D_z^k \tilde{u}\|_{2;a}+\|D_z^{k+1} \tilde{u}\|_{2;a}).$$

Similar inequalities hold for v and w. Thus we obtain

$$\begin{aligned}(8.6)\qquad &|[D_z^k \tilde{u}]|_1+|[D_z^k \tilde{v}]|_1+|[D_z^k \tilde{w}]|_1\\ &\leq C(\|D_z^k \tilde{u}\|_{2;a}+\|D_z^k \tilde{v}\|_{2;b}+\|D_z^k \tilde{w}\|_{2;c}\\ &\qquad+\|D_z^{k+1} \tilde{u}\|_{2;a}+\|D_z^{k+1} \tilde{v}\|_{2;b}+\|D_z^{k+1} \tilde{w}\|_{2;c}).\end{aligned}$$

Consequently we get (8.5) with $i=1$ from (8.4) and Theorem **A**. Consider the case

$i=2$. Take

$$\begin{aligned}
&F(\xi, \eta, \zeta)=\frac{R(u_\infty+\xi, v_\infty+\eta, w_\infty+\zeta)}{J(u_\infty+\xi, v_\infty+\eta, w_\infty+\zeta)},\\
&\Omega=\Omega_0,\\
&u(x, z)=\tilde{u}(|x|, z),\ v(x, z)=\tilde{v}(r_1, z),\ w(x, z)=\tilde{w}(r_1, z)
\end{aligned}$$

in Lemma 4.6. Then it follows from (8.3) that

$$\begin{aligned}
|[D_z^k G_u]|_{1,\Omega_0} &\le C_k \sum_{j=0}^{k}\{|[D_z^j \tilde{u}(\cdot, z)]|_{1,\Omega_0}+|[D_z^j \tilde{v}(r_1, z)]|_{1,\Omega_0}+|[D_z^j \tilde{w}(r_1, z)]|_{1,\Omega_0}\}\\
&\le C_k \sum_{j=0}^{k}\{|[D_z^j \tilde{u}(\cdot, z)]|_{1,\Omega_0}+|D_z^j \tilde{v}(r_1, z)|+|D_z^j \tilde{w}(r_1, z)|\}.
\end{aligned}$$

As a result, we see by Lemma 4.3 that

$$|[D_z^k G_u]|_{1,\Omega_0} \le C_k \sum_{j=0}^{k}\{|[D_z^j \tilde{u}(\cdot, z)]|_{1,\Omega_0}+|[D_z^j \tilde{v}(\cdot, z)]|_{1,\Omega_1}+|[D_z^j \tilde{w}(\cdot, z)]|_{1,\Omega_1}\}.$$

Combining this inequality with Lemma 8.3, we get

$$|[D_z^k \tilde{u}]|_2 \le C_{2,k}\Big\{|[D_z^{k+1} \tilde{u}]|_0+\sum_{j=0}^{k}(|[D_z^j \tilde{u}]|_1+|[D_z^j \tilde{v}]|_1+|[D_z^j \tilde{w}]|_1)\Big\}.$$

Similar a priori estimates hold for $\tilde{v}$ and $\tilde{w}$. Thus we obtain

$$\begin{aligned}
&|[D_z^k \tilde{u}]|_2+|[D_z^k \tilde{v}]|_2+|[D_z^k \tilde{w}]|_2\\
&\le C_{2,k}\{|[D_z^{k+1} \tilde{u}]|_0+|[D_z^{k+1} \tilde{v}]|_0+|[D_z^{k+1} \tilde{w}]|_0\\
&\qquad +\sum_{j=0}^{k}(|[D_z^j \tilde{u}]|_1+|[D_z^j \tilde{v}]|_1+|[D_z^j \tilde{w}]|_1)\},
\end{aligned} \tag{8.7}$$

which implies (8.5) with $i=2$. In the case $i=3$, use the boundedness for the H^2-norms of $D_z^k \tilde{u}$, $D_z^k \tilde{v}$, $D_z^k \tilde{w}$ instead of (8.3). Thereby we can similarly derive (8.5) with $i=3$. Repeating this argument, we can inductively derive (8.5) for $i=4, 5, 6, \cdots$. Consequently we obtain by Sobolev's lemma that

$$\lim_{z\to\infty}(\|D_r^i D_z^k u(\cdot, z)\|_\infty+\|D_r^i D_z^k v(\cdot, z)\|_\infty+\|D_r^i D_z^k w(\cdot, z)\|_\infty)=0$$

for all nonnegative integers i, k with $(i, k)\neq(0, 0)$.

Finally we will show that

$$\lim_{z\to\infty}(\|D_r^i D_z^k v^*(\cdot, z)\|_\infty+\|D_r^i D_z^k w^*(\cdot, z)\|_\infty)=0 \tag{8.8}$$

for all nonnegative integers i, k with $(i, k)\neq(0, 0)$. Differentiating (3.4) i times with respect to r, we get

$$\begin{aligned}
&\|D_r^i v^*(\cdot, z)\|_\infty+\|D_r^i w^*(\cdot, z)\|_\infty \le C_i|R(u(r_0, z), v^*(r_0, z), w^*(r_0, z))|,\\
&\qquad\qquad z\in(0, \infty)\,;\ i=1, 2, 3, \cdots.
\end{aligned} \tag{8.9}$$

Accordingly (5.8) yields (8.8) with $i\ge 1$ and $k=0$. Differentiate (3.4) i times with respect to r and k times with respect to z. Then, using Lemmas 3.4 and 3.5, we can derive

$$\begin{aligned}
&\|D_r^i D_z^k v^*(\cdot, z)\|_\infty + \|D_r^i D_z^k w^*(\cdot, z)\|_\infty \\
(8.10)\qquad &\le C_{i,k} \sum_{j=1}^{k} \{|D_z^j u(r_0, z)| + |D_z^j v(r_1, z)| + |D_z^j w(r_1, z)|\}, \\
&\qquad z \in [1, \infty);\ i=0, 1, 2, \cdots;\ k=1, 2, 3, \cdots.
\end{aligned}$$

Consequently we obtain (8.8) with $i \ge 0$ and $k \ge 1$. □

9. Proof of Theorem C-(ii) rates of decay

Let (u, v, w) be the solution to (P). Throughout this section we use the following abbreviation:

$$\begin{cases}
\tilde{u} := u - u_\infty, \quad \tilde{v} := v - v_\infty, \quad \tilde{w} := w - w_\infty, \\
\mathcal{U}(z) := (\tilde{u}(\cdot, z), \tilde{v}(\cdot, z), \tilde{w}(\cdot, z)), \\
D_z^k \mathcal{U}(z) := (D_z^k \tilde{u}(\cdot, z), D_z^k \tilde{v}(\cdot, z), D_z^k \tilde{w}(\cdot, z)), \\
\|D_z^k \mathcal{U}(z)\|_\infty := \|D_z^k \tilde{u}(\cdot, z)\|_{\infty, \Omega_0} + \|D_z^k \tilde{v}(\cdot, z)\|_{\infty, \Omega_1} + \|D_z^k \tilde{w}(\cdot, z)\|_{\infty, \Omega_1},
\end{cases}$$

where $k=1, 2, 3, \cdots$. It is important that $D_z^k \mathcal{U}(z)$ $(k=1, 2, 3, \cdots)$ satisfy (M_0) for all $z \in (0, \infty)$. We can see this fact by differentiating (7.3) k times with respect to z. Consequently the inequalities (7.1) and (7.2) hold with $\mathcal{U} = D_z^k \mathcal{U}(z)$ $(k=1, 2, 3, \cdots)$.

Proposition 9.1. *Let k be a positive integer. There exists a positive constant N_k such that*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dz}|||D_z^k \mathcal{U}(z)|||^2 + Q(D_z^k \mathcal{U}(z)) \\
(9.1)\qquad &\le N_k \{|D_z^k \tilde{u}(r_0, z)| + |D_z^k \tilde{v}(r_1, z)| + |D_z^k \tilde{w}(r_1, z)|\} \sum_{j=0}^{k} Q(D_z^j \mathcal{U}(z))
\end{aligned}$$

for $z \in [1, \infty)$.

Proof. Differentiate (7.5) and each equation of (P) k times with respect to z. Then a similar argument to the proof of Proposition 7.2 leads us to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dz}|||D_z^k \mathcal{U}(z)|||^2 + Q(D_z^k \mathcal{U}(z)) \\
&= -r_0 R^L(D_z^k \tilde{u}(r_0, z), D_z^k \tilde{v}(r_1, z), D_z^k \tilde{w}(r_1, z)) \frac{d^k}{dz^k} \tilde{R}(z).
\end{aligned}$$

Since $\|D_z^j \mathcal{U}(z)\|_\infty$ $(j=0, 1, 2, \cdots)$ are bounded on $[1, \infty)$, we can easily derive

$$\left|\frac{d^k}{dz^k} \tilde{R}(z)\right| \le C_k \sum_{j=0}^{k} \{|D_z^j \tilde{u}(r_0, z)|^2 + |D_z^j \tilde{v}(r_1, z)|^2 + |D_z^j \tilde{w}(r_1, z)|^2\}, \ z \in [1, \infty).$$

Thus we obtain the conclusion with use of (7.2). $\square$

Proof of Theorem C. We have obtained by Theorem A and (8.2) that

$$\lim_{z\to\infty}\{|D_z^k\tilde{u}(r_0, z)|+|D_z^k\tilde{v}(r_1, z)|+|D_z^k\tilde{w}(r_1, z)|\}=0 \quad (k=0, 1, 2, \cdots).$$

By similar arguments to the proofs of Lemmas 7.3 and 7.4, we can derive from (7.1), (7.4) and (9.1) that

$$\|D_z^k\mathcal{U}(z)\|_\infty=O(\exp(-\lambda_+z)) \qquad \text{as } z\to\infty \ (k=1, 2, 3, \cdots)$$

(use, for instance, Lemma 4.2). Accordingly we obtain from (8.6) that

$$\|[D_z^k\tilde{u}]\|_1+\|[D_z^k\tilde{v}]\|_1+\|[D_z^k\tilde{w}]\|_1=O(\exp(-\lambda_+z)) \qquad \text{as } z\to\infty \ (k=0, 1, 2, \cdots)$$

where $\|[\cdot]\|_1$ denotes the usual norm in a Sobolev space $H^1(\Omega_j)$ ($j=0$ or $j=1$). Recalling that $\|[D_z^k\tilde{u}]\|_i$, $\|[D_z^k\tilde{v}]\|_i$ and $\|[D_z^k\tilde{w}]\|_i$ ($i\geq 2$, $k\geq 0$) are bounded for $z\in[1, \infty)$, we can show in the same manner as the proof of (8.7) that

$$\begin{aligned}&\|[D_z^k\tilde{u}]\|_i+\|[D_z^k\tilde{v}]\|_i+\|[D_z^k\tilde{w}]\|_i\\ &\leq C_{i,k}\{\|[D_z^{k+1}\tilde{u}]\|_{i-2}+\|[D_z^{k+1}\tilde{v}]\|_{i-2}+\|[D_z^{k+1}\tilde{w}]\|_{i-2}\\ &\qquad+\sum_{j=0}^{k}\|[D_z^j\tilde{u}]\|_{i-1}+\|[D_z^j\tilde{v}]\|_{i-1}+\|[D_z^j\tilde{w}]\|_{i-1}\}.\end{aligned}$$

for $i\geq 2$ and $k\geq 0$. Thus, by induction with respect to i, we can derive

$$\|[D_z^k\tilde{u}]\|_i+\|[D_z^k\tilde{v}]\|_i+\|[D_z^k\tilde{w}]\|_i=O(\exp(-\lambda_+z)) \qquad \text{as } z\to\infty$$
$$(i=2, 3, 4, \cdots;\ k=0, 1, 2, \cdots).$$

Hence we see by Sobolev's lemma that

$$\|D_r^iD_z^ku\|_\infty+\|D_r^iD_z^kv\|_\infty+\|D_r^iD_z^kw\|_\infty=O(\exp(-\lambda_+z)) \qquad \text{as } z\to\infty$$
$$(i=1, 2, 3, \cdots;\ k=0, 1, 2, \cdots).$$

On the other hand, in view of (8.9) we get from (7.9) that

$$\|D_r^iv^*\|_\infty+\|D_r^iw^*\|_\infty=O(\exp(-\lambda_+z)) \qquad \text{as } z\to\infty \qquad (i=1, 2, 3, \cdots).$$

Moreover, it follows from (8.10) that

$$\|D_r^iD_z^kv^*\|_\infty+\|D_r^iD_z^kw^*\|_\infty=O(\exp(-\lambda_+z)) \qquad \text{as } z\to\infty$$
$$(i=0, 1, 2, \cdots;\ k=1, 2, 3, \cdots).$$

Thus we accomplish the proof. $\square$

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References

[1] S. Agmon, A. Douglis and L. Nirenberg: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions* I, Comm. Pure Appl. Math. **12** (1959), 623-727.

[2] N. D. Alikakos: *An application of invariance principle to reaction-diffusion equations*, J. Differential Equations **33** (1979), 201-225.

[3] R. Courant and D. Hilbert: Methods of Mathematical Physics I, II, Interscience, New York, 1953, 1962.

[4] W. Feng: *Coupled system of reaction-diffusion equations and applications in carrier facilitated diffusion*, Nonlinear Analysis **17** (1991), 285-311.

[5] H. Hoshino and Y. Yamada: *Asymptotic behavior of global solutions for some reaction-diffusion systems*, Nonlinear Analysis **23** (1994), 639-650.

[6] M. Iida and H. Ninomiya: *Asymptotic expansion of solutions to a chemical model with boundary reaction terms*, Funkcial. Ekvac. (to appear).

[7] M. Iida, Y. Yamada and S. Yotsutani: *Asymptotic behavior of solutions for a mathematical model on chemical interfacial reactions*, Osaka J. Math. **29** (1992), 483-495.

[8] ——————: *Convergence of solutions of a chemical interfacial reaction model*, Funkcial. Ekvac. **36** (1993), 311-328.

[9] ——————: *On the convergence rates for solutions of some chemical interfacial reaction problems*, Osaka J. Math. **32** (to appear).

[10] ——————: *On radial solutions for an elliptic-parabolic system related to chemical interfacial reaction problems*, in preparation.

[11] M. Iida, Y. Yamada, E. Yanagida and S. Yotsutani: *Exponential convergence of solutions for a mathematical model on chemical interfacial reactions*, Advances in Math. Sci. and Appl. **3** (1993/94), 335-352.

[12] J. Moser: *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457-468.

[13] ———: *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577-591.

[14] ———: *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101-134.

[15] T. Nagasawa: *The rate of convergence for chemical interfacial reaction models*, Funkcial. Ekvac. **35** (1992), 515-531.

[16] F. Rothe: Global Solutions of Reaction-Diffusion Systems, Lec. Notes in Math. Vol. 1072, Springer-Verlag, Berlin, 1984.

[17] W. Ruan: *Bounded solutions for reaction-diffusion systems with nonlinear boundary conditions*, Nonlinear Analysis **14** (1990), 1051-1077.

[18] Y. Shinomiya: *On the asymptotic stability as $t\to\infty$ of diffusion equations with nonlinear boundary conditions*, Master Thesis, Osaka Univ. (in Japanese)(1990).

[19] Y. Yamada and S. Yotsutani: *A mathematical model on chemical interfacial reactions*, Japan J. Appl. Math. **7** (1990), 369-398.

[20] K. Yoshizuka, K. Kondo and F. Nakashio: *Effect of interfacial reaction on rates of extraction and stripping in membrane extractor using a hollow fiber*, J. Chem. Eng. Japan **19** (1986), 312-318.

[21] S. Yotsutani: *Chemical interfacial reaction models with radial symmetry*, in "Nonlinear Diffusion Equations and Their Equilibrium States, 3" N. G. Lloyd, W. M. Ni, L. A. Peletier and J. Serrin eds., Birkhäuser, Boston, 1992, 561-572.

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