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THE HOMOTOPY GROUPS OF A SPECTRUM WHOSE BP_*-HOMOLOGY IS $v_2^{-1} BP_* / (2, v_1^\infty) [t_1] \otimes \Lambda (t_2)$

KEIKO MASAMOTO, TSUYOSHI MATSUHISA and KATSUMI SHIMOMURA

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1. Introduction

In [5], Mahowald gave some examples of ring spectra obtained as Thom spectra. One of them is X_2 in [5], which is a Thom spectrum associated to ω: $\Omega S^2 \to BO$, where ω is a mapping corresponding to the generator of $\pi_1(BO)$. Let BP denote the Brown-Peterson spectrum at the prime 2. Then the spectrum X_2 is also characterized by the BP_*-homology $BP_*(X_2) = BP_*/(2)[t_1]$ as a sub-comodule algera of $BP_*(BP)/(2) = BP_*/(2)[t_1, t_2, \cdots]$, where $BP_* = \boldsymbol{Z}_{(2)}[v_1, v_2, \cdots]$ over Hazewinkel's generators v_i (*cf.* [14]).

Relating to X_2, consider a spectrum X constructed as follows: Let C be a cofiber of the Bousfield localization map $X_2 \to L_1 X_2$ with respect to the Johnson-Wilson spectrum $E(1)$ with $\pi_*(E(1)) = \boldsymbol{Z}_{(2)}[v_1, v_1^{-1}]$. Then C is an X_2-module spectrum since X_2 is a ring spectrum. Consider the element $h_{20} \in \pi_5(X_2)$. Now the spectrum X is a cofiber of a map $h_{20}: \Sigma^5 C \to C$. By this definition, the BP_*-homology of X is $BP_*(X) = BP/(2, v_1^\infty)[t_1] \otimes \Lambda(t_2)$. Once we determined the homotopy groups $\pi_*(L_2 X_2)$ in [17], the homotopy groups $\pi_*(L_2 X)$ can be obtained from it. Here L_2 denotes the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$ with $\pi_*(E(2)) = \boldsymbol{Z}_{(2)}[v_1, v_2, v_2^{-1}]$ as a subalgebra of $v_2^{-1} BP_*$. But, in this paper, we compute, independently of [17], the homotopy groups $\pi_*(L_2 X)$ of the $E(2)_*$-localized spectrum of X by using the Adams-Novikov spectral sequence. The computation of the E_2-term is done in the same manner as that of [17], using the v_1-Bockstein spectral sequence. Different from the odd prime case, there may involve non-trivial differentials of the Adams-Novikov spectral sequence. On the other hand, different from the case for X_2, this case may support at most one family of non-trivial differentials. In this sense, it is a little easier to determine the homotopy groups of $L_2 X$ than those of $L_2 X_2$. By using the results of [7], we show here that the differentials are all trivial, in a different fashion from that of [17], and have the E_∞-term of the spectral sequence. In order to state the result, consider the integers A_n defined by

$$A_0=1,\ A_{2n+1}=1+2A_{2n} \text{ and } A_{2n+2}=2A_{2n+1}$$

for $n\geq 0$, and use the notations:

$C_\infty\langle x\rangle$ is a $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module isomorphic to
$$\boldsymbol{Z}/2[v_1, v_1^{-1}, v_2, v_2^{-1}]/\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$$
generated by elements $\{x/v_1^j\}_{j>0}$ such that $v_1(x/v_1^j)=x/v_1^{j-1}$.

$C_j\langle x\rangle$ is a cyclic $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module isomorphic to
$$\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]/(v_1^j)$$
generated by an element x/v_1^j.

Theorem. *The E_∞-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2X)$ is a $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module*

$$M_*\otimes\Lambda(\rho).$$

Here, the graded $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-module M_* is given by:

$$\begin{aligned}
M_0=&C_\infty\langle 1\rangle\oplus\bigoplus_{n,t\geq 0}C_{A_n}\langle v_3^{2^n(2t+1)}\rangle,\\
M_1=&\bigoplus_{t\geq 0}(C_1\langle v_3^{2t+1}h_{30}\rangle\oplus C_1\langle v_3^{2t+1}h_{31}\rangle\oplus C_3\langle v_3^{4t+2}h_{30}\rangle)\\
&\oplus\bigoplus_{n>0,t\geq 0}C_{A_n}\langle v_3^{2^n(2t+1)+1}h_{21}\rangle\\
&\oplus\bigoplus_{t,k\geq 0}(C_{A_{2k+1}}\langle v_3^{4^k(4t+2)+b_{k+1}}h_{30}\rangle\oplus C_{A_{2k}}\langle x_3^{4^k(2t+1)+b_{k+1}/2}h_{31}\rangle),\\
M_2=&\bigoplus_{t\geq 0}C_1\langle v_3^{2t+1}h_{30}h_{31}\rangle\\
&\oplus\bigoplus_{t,k\geq 0}(C_{A_{2k+1}}\langle v_3^{4^k(4t+2)+b_{k+1}+1}h_{21}h_{30}\rangle\\
&\oplus C_{A_{2k}}\langle v_3^{4^k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}\rangle) \text{ and}\\
M_n=&0 \text{ for } n>2.
\end{aligned}$$

Furthermore, the generators have the following degrees:

$$|v_3|=14,\ |h_{20}|=5,\ |h_{21}|=11,\ |h_{30}|=13, \text{ and } |h_{31}|=27.$$

In the theorem, an element x has a degree r if $x\in\pi_r(L_2X)$.

This paper is organized as follows: In the next section, we recall some facts known about the v_1-Bockstein spectral sequence. In §3, we define elements x_n, which will play the main role in the computation of the Bockstein spectral sequence. We compute E_2-terms of the Adams-Novikov spectral sequence computing the homotopy groups $\pi_*(L_2X)$ in §4, by using the tools given in the previous sections. In section 5, we prepare some lemmas to compute the Adams-Novikov differentials in the last section using the results of [7].

2. The Bockstein spectral sequence

Let (A, Γ) denote a Hopf algebroid with Γ A-flat. Then it is known (*cf*. [14, Ch. A1]) that the category of Γ-comodules has enough injectives and so we can define the Ext groups as a cohomology of an injective resolution. Furthermore it is given by a cohomology of the cobar resolution. So we can define $\mathrm{Ext}_\Gamma^n(A, M) = H^n(\Omega_\Gamma^* M)$ for a Γ-comodule M, where $\Omega_\Gamma^* M$ is a cobar complex (*cf*. [14]). The cobar complex $\Omega_\Gamma^* M$ is a defferential graded module with

$$\Omega_\Gamma^s M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (s \text{ copies of } \Gamma),$$

and the differentials $d_r : \Omega_\Gamma^r M \to \Omega_\Gamma^{r+1} M$ defined inductively by

$$d_0(m) = \psi(m) - m \otimes 1 \text{ and } d_r(x \otimes y) = d_s(x) \otimes y + (-1)^s x \otimes d_t(y)$$

for $x \in \Omega_\Gamma^s M$ and $y \in \Omega_\Gamma^t A$. Here $\psi : M \to M \otimes_A \Gamma$ denotes the comodule structure map of M. In the following, every comodule is induced from A and so we use η_R for ψ.

Suppose that $A = \boldsymbol{Z}_{(2)}[v_1, v_2, \cdots]$ and $\Gamma = A[t_1, t_2, \cdots]$. Consider a Hopf algebroid $\Phi = A[t_1] \otimes \Lambda(t_2)$ and a coalgebroid $\Sigma = \Gamma \square_\Phi A$ over A. Then $\Sigma = A[t_2^2, t_3, \cdots]$ and we have the change of rings theorem :

Lemma 2.1. *For a comodule A, there is an isomorphism*

$$\mathrm{Ext}_\Gamma^*(A, M \otimes_A \Phi) \cong \mathrm{Ext}_\Sigma^*(A, M).$$

Proof. Consider a relative injective Γ-resolution of $M \otimes_A \Phi$:

$$M \otimes_A \Phi \longrightarrow I_0 \otimes_A \Gamma \longrightarrow I_1 \otimes_A \Gamma \longrightarrow \cdots,$$

which is split as A-modules. Then apply the cotensor product $- \square_\Phi A$ and we obtain a relative injective Σ-resolution of M :

$$M \longrightarrow I_0 \otimes_A \Sigma \longrightarrow I_1 \otimes_A \Sigma \longrightarrow \cdots,$$

since $\Sigma = \Gamma \square_\Phi A$. Thus the both Ext groups are obtained from the same complex $I_0 \to I_1 \to \cdots$. q.e.d.

In this paper, we will compute $\mathrm{Ext}_\Gamma^*(A, v_2^{-1}A/(2, v_1^\infty) \otimes_A \Phi)$. By virtue of this lemma, we will work in the category of Σ-comodules. In order to compute the Ext groups $\mathrm{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1^\infty))$, we adopt the v_1-Bockstein spectral sequence with E_1-term

$$\mathrm{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1)).$$

To compute the E_1-term we recall [7] the structure

(2.2) $$\mathrm{Ext}_\Gamma^*(A, v_2^{-1}A/(2, v_1)[t_1]) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This is shown by using the change of rings theorems

$$\begin{aligned}\mathrm{Ext}_\Gamma^*(A, v_2^{-1}A/(2, v_1)[t_1]) &= \mathrm{Ext}_{K(2)_*K(2)}^*(K(2)_*, K(2)_*[t_1]) \\ &= \mathrm{Ext}_{S(2,2)}^*(\boldsymbol{Z}/2, \boldsymbol{Z}/2) \otimes_{K(2)_*} K(2)_*[v_3],\end{aligned}$$

in which $K(2)_* = \boldsymbol{Z}/2[v_2, v_2^{-1}]$, $K(2)_*K(2) = K(2)_* \otimes_A \Gamma \otimes_A K(2)_*$ and $S(2,2) =$

$\boldsymbol{Z}/2[t_2, t_3, \cdots]/(t_i^4 - t_i : i>1)$. Note here that the action of A on $K(2)_*$ is given by sending v_i to 0 for $i \neq 2$ and v_2 to v_2, and $(K(2)_*, K(2)_*K(2))$ becomes a Hopf algebroid induced from (A, Γ). The second equation follows from the $K(2)_*K(2)$-comodule structure $K(2)_*[t_1] = K(2)_*[t_1]/(v_2 t_1^4 + v_2^2 t_1) \otimes_{K(2)_*} K(2)_*[v_3]$ which is obtained from Landweber's formula $\eta_R(v_3) \equiv v_3 + v_2 t_1^4 + v_2^2 t_1 \bmod (2, v_1)$.

Lemma 2.3. *The E_1-term is given by*

$$\mathrm{Ext}_\Sigma^*(A, v_2^{-1}A/(2, v_1)) = K(2)_*[v_3] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho),$$

where $K(2)_* = \boldsymbol{Z}/(2)[v_2, v_2^{-1}]$ and h_{21}, h_{30}, h_{31} and ρ are the homology classes represented by t_2^2, t_3, t_3^2 and $v_2^5 t_4 + t_4^2$ in the cobar complex, respectively.

Proof. Let H^*M for a Γ-comodule M denote the Ext group $\mathrm{Ext}_\Gamma^*(A, M)$, and E_* and D_* be Γ-comodules

$$E_* = v_2^{-1}A/(2, v_1)[t_1] \otimes \Lambda(t_2) \text{ and } D_* = v_2^{-1}A/(2, v_1)[t_1].$$

Then the short exact sequence $0 \to D_* \subset E_* \to \Sigma^{-6} D_* \to 0$ of Γ-comodules yields the long exact sequence

$$\cdots \longrightarrow H^{s,t}D_* \longrightarrow H^{s,t}E_* \longrightarrow H^{s,t-6}D_* \xrightarrow{\delta} H^{s+1,t}D_* \longrightarrow \cdots$$

with $\delta(x) = h_{20}x$. By (2.2),

$$H^*D_* = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

This shows that $h_{20} : H^s D_* \to H^{s+1} D_*$ is a monomorphism and we have the lemma. q.e.d.

3. The elements x_n

In this section we will define elements x_n such that

$$x_n \equiv v_3^{2^n} \bmod (2, v_1) \text{ and } d_0(x_n) \equiv v_1^{e_n} g_n,$$

in which g_n repesents a generator of $\mathrm{Ext}_\Sigma^1(A, v_2^{-1}A/(2, v_1))$ and e_n to be taken as greate as possible. These elements play a central role in the Bockstein spectral sequence.

Hereafter we use the following abbreviation:

$\mathrm{Ext}^*(N) = \mathrm{Ext}_\Sigma^*(A, N)$ for a comodule N,
$M(j) = v_2^{-1}A/(2, v_1^j)$ and $M = \varinjlim_j M(j) = v_2^{-1}A/(2, v_1^\infty)$.

Then note that

$$BP_*(L_2X) = M \otimes_A \Phi \text{ and } \mathrm{Ext}^*(M) = \mathrm{Ext}_\Gamma^*(A, BP_*(L_2X)).$$

In $v_2^{-1}BP_*/(2)$, we define elements x_n, which will be used to define elements of $\mathrm{Ext}^*(M)$. From here on, we compute everything with setting $v_2=1$ for the sake of simplicity. We also write

$$x \equiv y \bmod (v_1^j)$$

for $x, y \in \Omega_\Sigma^* M$ if $x=y$ in the cobar complex $\Omega_\Sigma^* M(j)$.

We first introduce elements c_{3i} $(i=0, 1)$ and $\tilde{c}_{31}$ in $\Sigma = A[t_2^2, t_3, \cdots]$ defined by

$$\begin{aligned} v_1^2 c_{30} &= d_0(v_4^2 + v_1^2 v_5) + t_2^8 + t_2^2, \\ v_1 c_{31} &= d_0(v_4) + t_2^4 \text{ and} \\ \tilde{c}_{31} &= c_{31} + v_1(v_3^2 c_{31} + v_3 t_2^2). \end{aligned} \tag{3.1}$$

Lemma 3.2. *The cochains c_{30} and c_{31} are cocycles of the cobar complex $\Omega_\Sigma^1 M(j)$ for any $j>0$. Furthermore,*

$$c_{30} \equiv t_3 + v_3 t_2^8 \bmod (v_1) \quad \textit{and} \quad c_{31} \equiv t_3^2 + v_1 v_3 t_2^2 \bmod (v_1^4).$$

Proof. Since $d_1 d_0 = 0$, $d_1(t_2)=0$ and $d_0(v_1)=0$, the first part of the lemma follows immediately from the definition, since the multiplication by v_1 on $\Omega_\Sigma^1 M(j)$ is monomorphic. The latter half is shown by the direct computation using

$$\begin{aligned} \eta_R(v_1^2) &= v_1^2, \ \eta_R(v_4) \equiv v_4 + v_2 t_2^4 + v_1 t_3^2 + v_1^2 v_3 t_2^2 \bmod (v_1^5), \\ \eta_R(v_4^2) &\equiv v_4^2 + v_2^2 t_2^8 + v_2^8 t_2^2 + v_1^2 t_3^4 + v_1^4 v_3^2 t_2^4 \bmod (v_1^{10}), \text{ and} \\ \eta_R(v_5) &\equiv v_5 + v_3 t_2^8 + v_2 t_3^4 + v_2^8 t_3 \bmod (v_1) \end{aligned} \tag{3.3}$$

in Σ, noticing that $d_0(x) = \eta_R(x) - x$. In fact, $d_0(v_4^2 + v_1^2 v_5) \equiv t_2^8 + t_2^2 + v_1^2 t_3 + v_1^2 v_3 t_2^8 \bmod (v_1^3)$, by setting $v_2=1$, which gives c_{30}. For c_{31}, follows from $\eta_R(v_4)$.

q.e.d.

Lemma 3.4. *Put $\varphi_1 = v_1 v_3^2(v_4 + v_4^4)$, and we have*

$$d_0(\varphi_1) \equiv v_1(c_{30}^2 + \tilde{c}_{31}) \bmod (v_1^3)$$

in $v_2^{-1}\Sigma = v_2^{-1}A[t_2^2, t_3, \cdots]$.

Proof. Since $d_0(x) = \eta_R(x) - x$ and η_R is a map of algebras, this is verified by Lemma 3.2 and the following facts on η_R:

$$\begin{gathered} \eta_R(v_1) = v_1, \ \eta_R(v_2) = v_2, \\ \eta_R(v_3^2) \equiv v_3^2 \bmod (v_1^2), \\ \eta_R(v_4) = v_4 + t_2^4 + v_1 c_{31} \text{ and } \eta_R(v_4^4) \equiv v_4^4 + t_2^{16} + t_2^4 \bmod (v_1^4) \end{gathered}$$

in $v_2^{-1}\Sigma$. In fact, by Lemma 3.2, we see that

$$c_{30}^2 + \tilde{c}_{31} \equiv v_3^2 t_2^{16} + v_1 v_3^2 c_{31}.$$

On the other hand, we compute

$$d_0(\varphi_1)\equiv v_1v_3^2d_0(v_4+v_4^4)\equiv v_1v_3^2(v_1c_{31}+t_2^{16}).$$

q.e.d.

Note that $v_2^{-1}\Sigma$ is not a Hopf algebroid and so (3.1) does not imply the above lemma. In fact, $d_0(v_4^2)=d_0(v_4)^2+t_2^2$. This with (3.1) yields the following

Lemma 3.5. *In* $v_2^{-1}\Sigma$,

$$d_0(v_1^6v_5)=v_1^6(c_{31}^2+c_{30}).$$

Lemma 3.6. *There exist elements x_i of $v_2^{-1}A$ with $x_i\equiv v_3^{2^i}$* mod$(2, v_1)$ *such that*

$$\begin{aligned}
&d_0(x_0)=v_1t_2^2,\\
&d_0(x_1)=v_1^3c_{31},\\
&d_0(x_2)=v_1^6c_{30},\\
&d_0(x_{2n+1})\equiv v_1^{1+2a_n}v_3^{2b_n}(v_3^2c_{31}+v_3t_2^2) \bmod(v_1^{2+2a_n}) \text{ and}\\
&d_0(x_{2n+2})\equiv v_1^{a_{n+1}}v_3^{b_{n+1}}c_{30} \bmod(v_1^{1+a_{n+1}})
\end{aligned}$$

for $n>0$. Here the integers a_n and b_n are given by

$$a_0=1 \text{ and } a_n=4a_{n-1}+2\ (n>0)$$
$$b_0=0,\ b_1=0 \text{ and } b_n=4b_{n-1}+4\ (n>1).$$

Proof. Define the elements x_i inductively as follows:

$$(3.7)\qquad \begin{aligned}
&x_0=v_3,\\
&x_1=v_3^2+v_1^2v_4,\\
&x_2=x_1^2+v_1^6v_5,\\
&x_{2n}=x_{2n-1}^2+v_1^{a_n}v_3^{b_n}v_5 \text{ and}\\
&x_{2n+1}=x_{2n}^2+v_1^{2a_n-1}v_3^{2b_n}\varphi_1+v_1^{2a_n-3}v_3^{2b_n}x_1.
\end{aligned}$$

Then the lemma will be proved by induction. The first equation follows immediately from the Landweber formula: $\eta_R(v_3)=v_3+v_1t_2^2$. The second and the third are verified by (3.1). The others are inductively shown by Lemmas 3.4 and 3.5.

q.e.d.

4. The E_2-term

Put $L=v_2^{-1}BP_*/(2, v_1)$ and $M=v_2^{-1}BP_*/(2, v_1^\infty)$. Then we have the short exact sequence

$$0\longrightarrow L\xrightarrow{f} M\xrightarrow{v_1} M\longrightarrow 0,$$

which yields the long exact sequence

$$(4.1)\qquad \begin{aligned}
&0\longrightarrow \mathrm{Ext}^0(L)\xrightarrow{f_*}\mathrm{Ext}^0(M)\xrightarrow{v_1}\mathrm{Ext}^0(M)\xrightarrow{\delta_0}\\
&\cdots\xrightarrow{\delta_{n-1}}\mathrm{Ext}^n(L)\xrightarrow{f_*}\mathrm{Ext}^n(M)\xrightarrow{v_1}\mathrm{Ext}^n(M)\longrightarrow\cdots.
\end{aligned}$$

Here f is a Σ-comodule map given by $f(x)=x/v_1$,

$$\mathrm{Ext}^n(N)=\mathrm{Ext}^n_\Sigma(A, N)$$

for a Σ-comodule N, and note that the Ext group $\mathrm{Ext}^*(L)$ is determined in Lemma 2.3.

We here introduce some notations:

$$K(2)_*=\boldsymbol{Z}/2[v_2, v_2^{-1}],\ K=K(2)_*[v_1]=\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}].$$

For an element $x\in\mathrm{Ext}^*(L)$,

$C_n\langle x\rangle$ denotes a cyclic K-module isomorphic to $K/(v_1^n)$ generated by $\{x/v_1^n+z/v_1^{n-1}\}\in\mathrm{Ext}^*(M)$ for some $z\in\Omega^*_\Sigma v_2^{-1}BP_*/(2)$.

$C_\infty\langle x\rangle$ denotes a K-module isomorphic to $v_1^{-1}K/K$ with basis $\{x/v_1^j+z/v_1^{j-1}\}_{j>0}\subset\mathrm{Ext}^*(M)$ for some $z\in\Omega^*_\Sigma v_2^{-1}BP_*/(2)$.

Note that these $C_*\langle x\rangle$ are sub-K-module of $\mathrm{Ext}^*(M)$.

We compute $\mathrm{Ext}^*(M)=\mathrm{Ext}^*_\Sigma(A, v_2^{-1}A/(2, v_1^\infty))$ from $\mathrm{Ext}^*(L)=\mathrm{Ext}^*_\Sigma(A, v_2^{-1}A/(2, v_1))$ by using the following

Lemma 4.2. ([8, Remark 3.11]) *Let* $\{x_\lambda\}_{\lambda\in\Lambda}$ *be a set of generators of* $K(2)_*$*-module* $\mathrm{Ext}^i(L)$, *and* $\{\xi_\lambda\}_{\lambda\in\Lambda_0}$ *and* $\{\xi_{\lambda,j}\}_{\lambda\in\Lambda_1}$ *subsets of* $\mathrm{Ext}^i(M)$ *such that* $\Lambda=\Lambda_0\coprod\Lambda_1$,

1) *there exists a positive integer* $a(\lambda)$ *for each* $\lambda\in\Lambda_0$ *such that*

$$v_1^{a(\lambda)-1}\xi_\lambda=f_*(x_\lambda)\ \textit{and}$$
$$\delta_i(\xi_\lambda)\neq 0,$$

2) $\xi_{\lambda,1}=f_*(x_\lambda)$, $v_1\xi_{\lambda,j}=\xi_{\lambda,j-1}$ *and* $\delta_i(\xi_{\lambda,j})=0$ *for* $\lambda\in\Lambda_1$.

Suppose that the set $\{\delta_i(\xi_\lambda)\}_{\lambda\in\Lambda_0}$ *is linearly independent over* $K(2)_*$. *Then* $\mathrm{Ext}^i(M)=\bigoplus_{\lambda\in\Lambda_0}C_{a(\lambda)}\langle x_\lambda\rangle\oplus\bigoplus_{\lambda\in\Lambda_1}C_\infty\langle x_\lambda\rangle$.

In this section, we will use Lemma 4.2 to compute $\mathrm{Ext}^*(M)$, which is the E_2-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2X)$. Let ρ denote the homology class of $\mathrm{Ext}^1(L)$ given in Lemma 2.3.

Lemma 4.3. *There exist elements* $\rho_i\in\Omega^1_\Sigma v_2^{-1}A/(2)$ *such that*

$$\rho_i\equiv\rho \bmod(2, v_1)$$

up to homology and

$$d_1(\rho_i)\equiv 0 \bmod(2, v_1^{2^i}).$$

Proof. In [9], Moreira constructed an element $u\in\Omega_{\Sigma}^{1}L$ such that

$$\begin{aligned} d_0(u) &= (\tilde{\rho}+\zeta)+(\tilde{\rho}+\zeta)^2 \\ &= (\tilde{\rho}+t_2^2)+\tilde{\rho}^2+t_2^2+t_2^4 \end{aligned}$$

in the cobar complex $\Omega_{\Sigma}^{2}L$. Here ζ is represented by a cochain $t_2+t_2^2$ in $\Omega_{\Gamma}^{1}L$, and $\tilde{\rho}$ denotes a cocycle which represents the cohomology class ρ. Since t_2^4 is homologous to 0, so is $\tilde{\rho}$ to $\tilde{\rho}^2$. Hence define $\rho_i=\tilde{\rho}^{2^i}$ and we have the lemma. q.e.d.

For each j, there is an integer i such that ρ_i/v_1^j is a cocycle. In this case, we write

$$x\rho/v_1^j=x\rho_i/v_1^j.$$

Such an abbreviation would not cause any confusion.

The main lemma of the last section implies

Lemma 4.4. *For the connecting homomorphism δ_0 in* (4.1),

$$\begin{aligned} \delta_0(v_3^{2t+1}/v_1) &= v_3^{2t}h_{21}, \\ \delta_0(v_3^{4t+2}/v_1^3) &= v_3^{4t}h_{31}, \\ \delta_0(v_3^{8t+4}/v_1^6) &= v_3^{8t}h_{30}, \\ \delta_0(v_3^{4^n(4t+2)}/v_1^{1+2a_n}) &= v_3^{4^{n+1}t+2b_n}(v_3^2h_{31}+v_3h_{21}) \text{ and} \\ \delta_0(v_3^{4^{n+1}(2t+1)}/v_1^{a_{n+1}}) &= v_3^{2\cdot 4^{n+1}t+b_{n+1}}h_{30} \end{aligned}$$

for $t\geq 0$, $n>0$.

Here v_3^s/v_1^j denotes a cocycle of the cobar complex whose leading term is v_3^s/v_1^j. Therefore, we obtain the lemma by setting $v_3^{2^n s}/v_1^j=x_n^s/v_1^j$ from Lemma 3.6. Now apply Lemma 4.2 to obtain

Proposition 4.5. *The* Ext *group* $\mathrm{Ext}^0(M)$ *is a direct sum of* $C_\infty\langle 1\rangle$ *and* $C_{A_n}\langle v_3^{2^n(2t+1)}\rangle$ *for $n\geq 0$ and $t\geq 0$. Here $A_{2n}=a_n$ and $A_{2n+1}=1+2a_n$.*

These give us the cokernel of δ_0:

Corollary 4.6. *The cokernel of* $\delta_0\colon \mathrm{Ext}^0(M)\to\mathrm{Ext}^1(L)$ *is a $K(2)_*$-free module generated by*

$$v_3^{2t+1}h_{21},\ v_3^{u'}h_{30},\ v_3^{u}h_{31} \text{ and } v_3^{t}\rho$$

for $t\geq 0$, $u\notin T$ and $u'\notin 2T$. Here T is a subset of the natural numbers $\boldsymbol{N}$:

$$T=\{n:\ 4|n \text{ or } 4^{i+1}|(n-2b_i-2) \text{ for some } i>0\},$$

for $b_i=4(4^{i-1}-1)/3$.

Lemma 4.7. *The complement* $U=N-T$ *is given as*

$$U=\{n:\ 2\nmid n \ \text{or}\ n=2\cdot 4^k t+6\cdot 4^{k-1}+2(4^{k-1}-1)/3 \\ \text{for some } k>0 \text{ and } t\geq 0\}$$

For the computation of δ_1, we introduce other elements:

Lemma 4.8. *Consider an element* $\varphi=v_5+v_3v_4^2$. *Then there exist elements* H_{21} *and* H_{32} *in* Σ *such that*

$$d_0(\varphi)=H_{32}+t_3+H_{21},\ \ d_1(H_{21})=0=d_1(H_{32}),$$
$$H_{21}\equiv t_2^2 \quad \text{and} \quad H_{32}\equiv t_3^4 \mod(v_1)$$

in the cobar complex $\Omega_\Sigma^1 v_2^{-1}A/(2)$.

Proof. *For an element* $\psi=v_3^2+v_1^7v_3$, *we compute* $d_0(\psi)=v_1^2t_2^4$ *by* $\eta_R(v_3)=v_3+v_1t_2^2+v_1^4t_2$ *in* $BP_*[t_2, t_3, \cdots]$. *Now put*

$$H_{32}=t_3^4+v_1^2\psi t_2^4.$$

Then, the formula $\Delta(t_3^4)=t_3^4\otimes 1+1\otimes t_3^4+v_1^4t_2^4\otimes t_2^4$ yields

$$d_1(H_{32})=0 \quad \text{and} \quad H_{32}\equiv t_3^4 \mod(v_1).$$

Furthermore, we compute

$$d_0(\varphi)\equiv t_3^4+t_3+v_3t_2^2 \mod(v_1),$$

and so

$$d_0(\varphi)\equiv H_{32}+t_3+v_3t_2^2 \mod(v_1).$$

Put, then,

$$H_{21}=d_0(\varphi)+H_{32}+t_3$$

and we have

$$d_1(H_{21})=0 \quad \text{and} \quad H_{21}\equiv v_3t_2^2 \mod(v_1).$$

q.e.d.

Lemma 4.9. *For the connecting homomorphism* $\delta_1:\mathrm{Ext}^1(M)\longrightarrow \mathrm{Ext}^2(L)$, *we have*

$$\delta_1(v_3^{4t+3}h_{21}/v_1^3)=v_3^{4t+1}h_{21}h_{31},$$
$$\delta_1(v_3^{8t+5}h_{21}/v_1^6)=v_3^{8t+1}h_{21}h_{30},$$
$$\delta_1(v_3^{4^n(4t+2)+1}h_{21}/v_1^{1+2a_n})=v_3^{4^{n+1}t+2b_n+1}h_{21}(v_3^2h_{31}+v_3h_{21}),$$
$$\delta_1(v_3^{4^{n+1}(2t+1)+1}h_{21}/v_1^{a_{n+1}})=v_3^{2\cdot 4^{n+1}t+b_{n+1}+1}h_{21}h_{30}$$
$$\delta_1(v_3^{2t+1}h_{30}/v_1)=v_3^{2t}h_{21}h_{30},$$

$$\delta_1(v_3^{4t+2}h_{30}/v_1^3)=v_3^{4t}h_{30}h_{31},$$
$$\delta_1(v_3^{4^k(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k})=v_3^{4^k(4t+2)-2}h_{30}(h_{31}+v_3^{-1}h_{21}),$$
$$\delta_1(v_3^{2t+1}h_{31}/v_1)=v_3^{2t}h_{21}h_{31} \quad and$$
$$\delta_1(v_3^{4^k(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k})=v_3^{4^k(2t+1)-2}h_{30}(h_{31}+v_3^{-1}h_{21}).$$

Proof. The first four equations follow immediately from Lemmas 4.4 and 4.8 with replacing v_3h_{21} by H_{21}. The fifth, sixth and eighth equations follow immediately from Lemmas 3.2 and 3.6. For the other equations, just put

$$v_3^{4^k(4t+2)+b_{k+1}}h_{30}/v_1^{1+2a_k}=v_3^{4^k(4t+2)}d_0(x_{2k+2})/v_1^{1+2a_k+a_{k+1}} \quad \text{and}$$
$$v_3^{4^k(2t+1)+b_{k+1}/2}h_{31}/v_1^{a_k}=v_3^{4^k(2t+1)}d_0(x_{2k+1})/v_1^{a_k+1+2a_k},$$

and we have the result by Lemma 3.6. q.e.d.

Now use Lemma 4.2, and we obtain

Proposition 4.10. $\mathrm{Ext}^1(M)$ *is a direct sum of* $\rho\mathrm{Ext}^0(M)$ *and*

$$e^1(M)=\bigoplus_{t\geq 0}(C_1\langle v_3^{2t+1}h_{30}\rangle\oplus C_1\langle v_3^{2t+1}h_{31}\rangle\oplus C_3\langle v_3^{4t+2}h_{30}\rangle)$$
$$\oplus\bigoplus_{n>0,t\geq 0}C_{A_n}\langle v_3^{2^n(2t+1)+1}h_{21}\rangle$$
$$\oplus\bigoplus_{t,k\geq 0}(C_{1+2a_k}\langle v_3^{4^k(4t+2)+b_{k+1}}h_{30}\rangle\oplus C_{a_k}\langle v_3^{4^k(2t+1)+b_{k+1}/2}h_{31}\rangle).$$

Corollary 4.11. *The cokernel of* δ_1: $\mathrm{Ext}^1(M)\to\mathrm{Ext}^2(L)$ *is a direct sum of* $\rho\mathrm{Coker}\ \delta_0$ *and a* $K(2)_*$*-module generated by*

$$v_3^{2t+1}h_{30}h_{31},\ v_3^{2u+1}h_{21}h_{31}\ \ and\ \ v_3^{2u'+1}h_{21}h_{30}$$

for $t\geq 0$, $2u\notin T$ *and* $u'\notin 2T$.

Lemma 4.12. *For the connecting homomorphism* δ_2: $\mathrm{Ext}^1(M)\to\mathrm{Ext}^2(L)$, *we have*

$$\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1)=v_3^{2t}h_{21}h_{30}h_{31},$$
$$\delta_2(v_3^{4t+3}h_{21}h_{30}/v_1^3)=v_3^{4t+1}h_{21}h_{30}h_{31},$$
$$\delta_2(v_3^{4^k(4t+2)+b_{k+1}+1}h_{21}h_{30}/v_1^{1+2a_k})=v_3^{4^k(4t+2)-1}h_{21}h_{30}h_{31},$$
$$\delta_2(v_3^{4^k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}/v_1^{a_k})=v_3^{4^k(2t+1)-1}h_{21}h_{30}h_{31}.$$

Proof. Note that $\delta_2(v_3^{2t+1}h_{30}h_{31}/v_1)=\delta_0(v_3^{2t+1}/v_1)h_{30}h_{31}$ since $h_{3i}=c_{3i}$'s are cocycles by Lemma 3.2. Now the first equation follows from Lemmas 4.4 and 4.9. For the other equations, use Lemmas 4.8 and 4.9 since $\delta_2(v_3^{2t+1}h_{21}h_{3i}/v_1^j)=\delta_1(v_3^{2t}h_{3i}/v_1^j)v_3h_{21}$ if we use the representative H_{21} for the cohomology class v_3h_{21}.

Again by Lemma 4.2, we obtain q.e.d.

Proposition 4.13. $\mathrm{Ext}^2(M)$ *is a direct sum of* $\rho e^1(M)$ *and*

$$e^2(M)=\oplus_{t,k\geq 0}(C_{1+2a_k}\langle v_3^{4^k(4t+2)+b_{k+1}+1}h_{21}h_{30}\rangle$$
$$\oplus\ C_{a_k}\langle v_3^{4^k(2t+1)+(b_{k+1}/2)+1}h_{21}h_{31}\rangle)\oplus C_1\langle v_3^{2t+1}h_{30}h_{31}\rangle).$$

Corollary 4.14. *The cokernel of* $\delta_2: \mathrm{Ext}^2(M) \to \mathrm{Ext}^3(L)$ *is a* $K(2)_*$-*module* $\rho\mathrm{Coker}\ \delta_1$.

Now the following proposition follows immediately, by the same argument as above.

Proposition 4.15. *For* $n>3$, $\mathrm{Ext}^n(M)=0$, *and*

$$\mathrm{Ext}^3(M)=\rho e^2(M).$$

5. On the map $j_*: E_2(X) \to E_2(C)$

As is stated in the introduction, C denotes the cofiber of $X_2 \to L_2X_2$. Then it is an X_2-module spectrum and $h_{20} \in \pi_5(X_2)$ induces a map $h_{20}: C \to C$. In fact, it is the composition

$$C=S^0 \wedge C \xrightarrow{h_{20}\wedge C} X_2 \wedge C \xrightarrow{\nu} C,$$

in which ν denotes the X_2-module structure. Then we have a cofiber sequence

$$\Sigma^5 C \xrightarrow{h_{20}} C \xrightarrow{i} X \xrightarrow{j} \Sigma^6 C.$$

Let $E_r^*(Y)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to $\pi_*(L_2 Y)$ for a spectrum Y, and d_r^{AN}, its differentials. Then this gives rise to the exact sequence

$$0 \longrightarrow E_2^{0,t}(C) \xrightarrow{i_*} E_2^{0,t}(X) \xrightarrow{j_*} E_2^{0,t-6}(C) \xrightarrow{\delta} E_2^{1,t}(C) \longrightarrow \cdots.$$

Here $E_2^{s,t}(X)=\mathrm{Ext}^{s,t}(M)$, whose structure is given in the previous section. We further consider a cofiber E of $h_{20}: C \to C$. Then we have a commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C \\
\downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 \\
C & \xrightarrow{h_{20}} & C & \xrightarrow{i} & X & \xrightarrow{j} & \Sigma C, \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
\Sigma D & \xrightarrow{h_{20}} & \Sigma D & \xrightarrow{i} & \Sigma E & \xrightarrow{j} & \Sigma^2 D
\end{array} \tag{5.1}$$

in which rows and columns are cofibrations.

Lemma 5.2. *Let* v_3^t/v_1^A *denote a generator of* $E_2(X)$ *as a* $\boldsymbol{Z}/2[v_1, v_2, v_2^{-1}]$-*module. Then*

$$j_*(v_3^t/v_1^{A-1})=0.$$

Proof. If $t=2^n(2s+1)$ for some n, $s \geq 0$, then v_3^t/v_1^A is a homology class represented by $x_n^{2s+1}/v_1^{A_n}$. For $n=0$, the lemma is trivial. Now suppose that $j_*(x_n^{2s+1}/v_1^{A_n})=0$ for even $n=2m$. Then squaring this, we obtain

$$j_*(x_{n+1}^{2s+1}/v_1^{A_{n+1}})=v_3^w/v_1$$

for some $w \geq 0$. Consider the diagram

$$\begin{array}{ccccc} & & E_2^0(X) & \xrightarrow{j_*} & E_2^0(C) \\ & & \downarrow \delta & & \downarrow \delta \\ E_2^1(D) & \xrightarrow{i_*} & E_2^1(E) & \xrightarrow{j_*} & E_2^1(D) \end{array}$$

induced from (5.1). Since $\delta(x_{n+1}^{2s+1}/v_1^{An+1})$ is in the image of i_* by Lemma 4.4, $\delta(v_3^w/v_1)=0$ in $E_2^1(D)$ by the above diagram, and so $2|w$ since $\delta(v_3^w/v_1)=wv_3^{w-1}h_{21}$ by Landweber's formula $d_0(v_3)=v_1t_2^2+v_1^4t_2$ in $BP_*[t_2, t_3, \cdots]$. Thus we have

$$j_*(x_{n+1}^{2s+1}/v_1^{An+1})=v_3^{2u}/v_1.$$

Square this, and we have

$$j_*(x_{n+2}^{2s+1}/v_1^{An+2})=v_3^{4u}/v_1^2.$$

Notice that $j_*(x)=y$ if $d_0(x)=yt_2$, where $d_0(x)=\eta_R(x)-x$. A direct computation shows us $d_0(v_3^{4u}x_1/v_1^4)=v_3^{4u}t_2/v_1^2$ in the cobar complex Ω_Σ^2M. Thus we have shown inductively that $j_*(v_3^{2^n(2s+1)}/v_1^{An})$ equals to 0 if n is even, and to v_3^{2s}/v_1 for some u if n is odd. q.e.d.

6. The Adams-Novikov differential

Consider the cofiber E of $h_{20}: \Sigma^5D \to D$. Then by [7, Th. 7.1], we immediately obtain the following

Proposition 6.1. *The Adams-Novikov spectral sequence for computing $\pi_*(L_2E)$ collapses from the E_2-term.*

Note that the E_2-term for our X is

$$E_2^*(X)=\mathrm{Ext}_\Gamma^*(A, v_2^{-1}BP_*(X))=\mathrm{Ext}^*(M).$$

Lemma 6.2. *For the Adams-Novikov differential $d_3^{AN}: E_2^0(X) \to E_2^3(X)$, $d_3^{AN}(v_3^t/v_1^A)$ is a sum of the elements of the form $v_3^{2u+1}h_{21}h_{3i}\rho/v_1^k$ for $i=0, 1$ and $k>1$. Here v_3^t/v_1^A is a generator of the $\mathbf{Z}/2[v_1, v_2, v_2^{-1}]$-module M_0.*

Proof. Consider the diagram (5.1). The third column induces the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^3(M) \xrightarrow{v_1} \mathrm{Ext}^3(M) \xrightarrow{\delta_3} \mathrm{Ext}^4(L) \longrightarrow \cdots$$

of the E_2-terms. If the δ_0 image of v_3^t/v_1^A is $x \neq 0$, then $\delta_3(d_3^{AN}(v_3^t/v_1^A))=d_3^{AN}(x)=0$ by Proposition 6.1. Thus $d_3^{AN}(v_3^t/v_1^A)$ is divisible by v_1. Furthermore it implies that $v_3^{2t+1}h_{30}h_{31}\rho/v_1$ cannot be a target of d_3^{AN}. In fact, it is not divisible by v_1 by Proposition 4.15. Now the lemma follows from Lemma 4.15. q.e.d.

Theorem 6.3. *The Adams-Novikov spectral sequence for computing $\pi_*(L_2X)$ collapses from the E_2-term.*

Proof. By proposition 4.15, the Adams-Novikov differentials are all trivial except for $d_3^{AN}: E_2^0(X) \to E_2^3(X)$. So it is sufficient to show that $d_3^{AN}(v_3^t/v_1^j)=0$ for each $v_3^t/v_1^j \in E_2^0(X)$. By Lemma 6.2,

$$d_3^{AN}(v_3^t/v_1^{A-k}) = \sum_{u,i} \lambda_{u,i} v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2 \tag{6.4}$$

for some $k \geq 0$, where $\lambda_{u,i} \in \boldsymbol{Z}/2$. Since

$$d_3(v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2) = v_3^{2u} h_{20}^2 h_{3i} \rho / v_1 \neq 0$$

in the cobar complex $\Omega_\Gamma^4 BP_*(C)$, we see that

$$j_*(\sum_{u,i} \lambda_{u,i} v_3^{2u+1} h_{21} h_{3i} \rho / v_1^2) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho / v_1 \neq 0. \tag{6.5}$$

Now send (6.4) by j_* and we have a contradiction to Lemma 5.2, which says $j_*(v_3^t/v_1^{A-k})=0$ if $k>0$. If $k=0$ and $j_*(v_3^t/v_1^A) \neq 0$, then

$$j_*(v_3^t/v_1^A) = v_3^{2u}/v_1$$

for some $u \geq 0$ as is seen in the proof of Lemma 5.2. Therefore, (6.4) and (6.5) yield

$$d_3^{AN}(v_3^{2u}/v_1) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho / v_1 \neq 0$$

in $E_2^*(C)$ for some $\lambda_{u,i} \in \boldsymbol{Z}/2$. Now pull this back to $E_2^*(D)$ under the map $i_*: E_2^*(D) \to E_2^*(C)$ to obtain the non-trivial differential

$$d_3^{AN}(v_3^{2u}) = \sum_{u,i} \lambda_{u,i} v_3^{2u} h_{20} h_{3i} \rho \neq 0$$

in $E_2^*(D)$, which again contradicts to a result of [7] which says $d_3^{AN}(v_3^{4k})=0$ and $d_3^{AN}(v_3^{4k+2}) = v_3^{4k} h_{20}^3$ for $k>0$. q.e.d.

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Department of Mathematics,
Faculty of Science,
Okayama University,
Okayama, 700,
Japan

Faculty of Education,
Tottori University,
Tottori, 680,
Japan