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Osaka J. Math.
33 (1996), 47-55

BOUNDARY SLOPES FOR KNOTS

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(Received November 7, 1994)

Let T be a torus. By the *slope* of an essential simple closed curve on T we mean its isotopy class. The *distance* $\Delta(r_1, r_2)$ between two slopes r_1 and r_2 is defined to be $|\gamma_1 \cdot \gamma_2|$, where γ_1 and γ_2 are curves with slopes r_1 and r_2 and $\cdot$ denotes homological intersection number. (Note that this is independent of all orientations. Note also that Δ is not a metric on the set of slopes; the triangle inequality does not hold.)

Now let M be an irreducible, orientable 3-manifold and T a torus component of ∂M. Let $(F, \partial F) \subset (M, T)$ be an incompressible, boundary incompressible, orientable, genus g surface. Then the components of ∂F all have the same slope on T, and we call this the *boundary slope* of F. Let $S(M)_g$ denote the set of boundary slopes of such genus g surfaces. When M is an exterior $E(K)$ of a knot K, we write $S(E(K))_g$ as $S(K)_g$.

Gordon and Luecke gave estimations of ∂-slopes in $S(M)_0$ and $S(M)_1$, and showed that their estimations are the best possible (see [1], [3], [4]). So far, however, there is no estimation of ∂-slopes in $S(M)_g$ for $g \geq 2$.

In this paper, we give some estimation of ∂-slopes in $S(M)_g$ for arbitrary g when M has a certain geometric restriction, and we give an example which estimates the strength of the theorem.

Our main results are then the following.

Theorem 1. *If M has no essential annulus, then for any $g_1, g_2 \geq 1$, $r_1 \in S(M)_{g_1}$, $r_2 \in S(M)_{g_2}$, we have $\Delta(r_1, r_2) < 36(2g_1-1)(2g_2-1)$.*

Theorem 2. *Suppose a knot K has an m-string ∂-irreducible tangle decomposition.*

(i) *Let a/b $(\neq 0/1)$ be an element of $S(K)_g$, where a and b are coprime integers. Then $|b| \leq g/m$.*

(ii) *$g(K) \geq (m+1)/2$, where $g(K)$ is the genus of K.*

Theorem 3. *For any n non-trivial knots $K_1, \cdots, K_n$ and $a/b \in S(K_1 \# \cdots \#$*

$K_n)_g$, we have $|b| \leq g/(n-1)$.

The organization is as follows. In sections 1 and 2 we prove above theorems. In section 3 we give an example which concerns Theorem 2 and construct ∂-irreducible tangles systematically.

1. Proof of Theorem 1

Let $N(\cdot\cdot)$ denote a tublar neighbourhood. Let G be a finite graph in a closed surface S. We take edges and faces of G to be open edges and faces, i.e., components of $G-\{vertices\}$ and $S-G$, respectively. Then an edge e belongs to a face f if $e \subset cl(f)$, where $cl(f)$ denotes the closure of f in S. A face is 1-*sided* if it has only one edge (and one vertex).

To prove Theorem 1 we need the following lemma ([2, Lemma 6.2]).

Lemma 1.1. *Let Γ be a finite graph in a closed surface S, with V vertices and no* 1-*sided faces which are open discs. Suppose that, for some integer $n \geq 2$, every vertex of Γ has order greater than $(max\{[6(1-\chi(S)/V)], 1\})(n-1)$. Then Γ has n mutually parallel edges.*

Suppose, for a contradiction, $\Delta = \Delta(r_1, r_2) \geq 36(2g_1-1)(2g_2-1)$. Let F_i be an incompressible, ∂-incompressible, orientable, connected, genus g_i surface with ∂-slope r_i $(i=1, 2)$. After an isotopy of F_i we may assume that F_1 and F_2 intersect transversely, and each component of ∂F_1 [resp. ∂F_2] intersects that of ∂F_2 [resp. ∂F_2] exactly $\Delta(r_1, r_2)$ times. Then $F_1 \cap F_2 = A \amalg S$, where A is a disjoint union of properly embedded arcs and S is a disjoint union of simple closed curves. By a standard disc swapping argument, using the incompressibility of F_1 [resp. F_2], we may assume that no component of S bounds a disc on F_2 [resp. F_1]. As in [2], we form graphs G_{F_1}, G_{F_2} as follows. Let $\widehat{F}_i$ be the closed surface obtained by capping off the boundary components of F_i by disc $(i=1, 2)$. We obtain a graph G_{F_1} in $\widehat{F}_1$ by taking as the "fat" vertices of G_{F_1} the discs attached as above, and as the edges of G_{F_1}, the arcs in A. Similarly we obtain the graph G_{F_2} in $\widehat{F}_2$. Since F_1 [resp. F_2] is ∂-incompressible, we may assume (again by a standard disc swapping argument) that G_{F_2} [resp. G_{F_1}] has no 1-sided faces. Let n_i denote the number of boundary components of G_{F_i} $(i=1, 2)$. Then G_{F_i} has n_i vertices, each of order Δn_j $(i \neq j)$. By a homological argument, we may assume $n_2 \geq 2$. (If $n_1 = 1$, then ∂F_i is null-homologus in $H_1(E(K))$. Hence ∂-slope of F_i is 0/1.) Then by the assumption, $\Delta n_2 \geq 36(2g_1-1)(2g_2-1)n_2 > 6(2g_1-1)\{6(2g_2-1)n_2-1\} \geq [6(1-(2-2g_1)/n_1)]\{6(2g_2-1)n_2-1\}$. Hence by Lemma 1.1, G_{F_1} has $6(2g_2-1)n_2$ mutually parallel edges. Let Γ be the subgraph of G_{F_2} arising from these edges. Then the order of each vertex of Γ is $6(2g_2-1)$. Since $6(2g_2-1) > [6(1-(2-2g_2)/n_2)]$, by Lemma 1.1 again, Γ has parallel edges. Let e_1 and e_2 be edges of Γ which are parallel and adjacent in F_2, and let B [resp. E] be the disc in F_1 [resp. F_2] cut off by e_1 and e_2. Put $A=B\cup E$, then A is either an annulus or a Möbius band properly embedded in M.

Case 1. $A=B\cup E$ is an annulus: Then ∂A is a union of two essential simple loops on T.

CLAIM. *A is not boundary parallel.*

Proof. If A is boundary parallel, then e_1 partially bounds a boundary compression disc of F_1, a contradiction.

By Claim and the assumption of the theorem, A is compressible, and hence T is compressible. Then it follows that M is a solid torus by the irreducibility of M. This is a contradiction, since a solid torus has only one ∂-slope.

Case 2. $A=B\cup E$ is a Möbius band: If ∂A is an inessential loop on T, then the disc on T bounded by ∂A and A make P^2 in M; therefore $M=M'\#P^3$, for some 3-manifold M', a contradiction. Hence, ∂A is an essential loop on T. Since M is orientable, $N(A)$ is a twisted I-bundle over A. Therefore $\tilde{A}=FrN(A)$ is an annulus properly embedded in M, where $F_rN(A)$ is the frontier of $N(A)$ in M. By the assumption, $\tilde{A}$ is compressible or boundary compressible. If $\tilde{A}$ is compressible, then by the argument in Case 1, we have a contradiction. Hence $\tilde{A}$ is boundary compressible. Thus we see $\tilde{A}$ is boundary parallel by using the irreducibility of M and the fact that $\tilde{A}=FrN(A)$ is separating. Therefore M is a union of $N(A)$ and $\tilde{A}\times I$ along $\tilde{A}\times 0$; so $M\cong N(A)\cong S^1\times D^2$, a contradiction. This completes the proof of Theorem 1.

2. Proof of Theorem 2 and Theorem 3

Let K be a knot in S^3. The exterior of K is $E(K)=S^3-intN(K)$. A *tangle* (B, t) is a pair that consists of a 3-ball B and a 1-dimensional manifold t properly embedded in B. A tangle (B, t) is an *m-string tangle* if t consists of m number of arcs. A tangle (B, t) is called *∂-irreducible* if $\partial(cl(B-N(t)))$ is incompressible in $cl(B-N(t))$. We say that K has an m-string ∂-irreducible tangle decomposition if it can be expressed as a sum of two m-string ∂-irreducible tangles, i.e., there is a sphere S meeting K transversely in $2m$ points, such that each of the balls bounded by S determines, with its intersection with K, an m-string ∂-irreducible tangle.

Proof of Theorem 2. Suppose K is expressed as the sum of two m-string ∂-irreducible tangles (B_1, t_1) and (B_2, t_2). Let P denote $\partial B_1\cap E(K)$ $(=\partial B_2\cap E(K))$, then P is incompressible and ∂-incompressible by the definition of a ∂-irreducible tangle and an argument in [6, Lemma 1.10]. Let F be an incompressible, ∂-incompressible, orientable, connected, genus g surface with ∂-slope a/b. As in the proof of Theorem 1, we may assume that each component of ∂F [resp. ∂P] intersects that of ∂P [resp. ∂F] exactly $|b|$ times, and we define $\hat{F}$, $\hat{P}$, G_F, and G_P. Again we may assume that G_F and G_P have no 1-sided face.

Lemma 2.1. *There is no disc face in G_F.*

Proof. Suppose there is a disc face D in G_F, and D is contained in B_1. Then $cl(D)\cap\partial(cl(B_1-N(t_1)))$ is a simple loop in $\partial(cl(B_1-N(t_1)))$. By the definition of a ∂-irreducible tangle, $cl(D)\cap\partial(cl(B_1-N(t_1)))$ bounds a disc D' in $\partial(cl(B_1-N(t_1)))$. Let α be a component of $\partial D'\cap\partial P$ which is outermost disc in D', and let d be the (outermost) disc in D' cut off by α. Then d is contained in P and it produces a 1-sided face in G_P, a contradiction.

Let V and E, respectively, be the numbers of the vertices and the edges of G_F. Note that $E=m|b|V$.

Lemma 2.2. $g\geq V(m|b|-1)/2+1$

Proof. Note that $2-2g=\chi(\hat{F})=V-E+\sum_i\chi(F_i)$, where F_i runs over all faces of G_F. Since $\chi(F_i)\leq 0$ by Lemma 2.1, we have $2-2g\leq V-E=(1-m|b|)V$.

If F is a Seifert surface, then $a/b=0/1$ and $V=1$. Therefore, by Lemma 2.2, $g\geq(m+1)/2$.

If F is not a Seifert surface, then $V\geq 2$. Therefore, again by Lemma 2.2, $g\geq m|b|$.

This completes the proof of Theorem 2.

Proof of Theorem 3. Let $A_1,\cdots,A_{n-1}$ denote the annuli in $E(K)$ defining the connected sum as illustrated in Figure 2.1, and put $P=\bigcup_{i=1}^{n-1}A_i$.

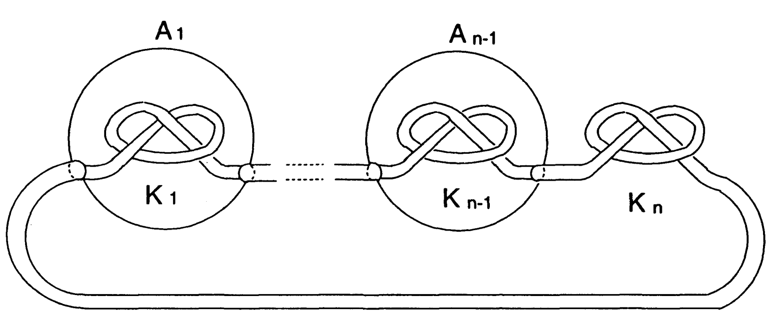

Figure 2.1

We define $\hat{F}$, $\hat{P}$, G_F, G_P, E, and V as in the proof of Theorem 2. Then G_F does not have a disc face. To show this note that P cuts $E(K)$ into the disjoint union $\coprod_{i=1}^{n} E(K_i)$. Suppose G_F has a disc face D. Then D is a properly embedded disc in some $E(K_i)$, and we can see that ∂D is essential in $\partial E(K_i)$. This implies that K_i is a trivial knot, a contradiction.

Next we remark $E=(n-1)|b|V$. Then as in the proof of Lemma 2.2, we see $2-2g\leq V-E=V-(n-1)|b|V$. Hence, $g\geq V\{(n-1)|b|-1\}/2+1$.

If $V\geq 2$, then we obtain $g\geq(n-1)|b|$. If $V=1$, then F is a Seifert surface, and hence $g(K_1\#\cdots\#K_n)\geq n>n-1$.

This completes the proof of Theorem 3.

3. Constructing ∂-irreducible Tangles

In this section, we give a systematic construction of ∂-irreducible tangles. And combining the results of [5] with this construction, we present examples of knots which estimate the strength of Theorem 2.

A *Montesinos tangle* $T(r_1, \cdots, r_n)$ ($r_i \in \boldsymbol{Q}\cup\{1/0\}$) is a tangle illustrated in Figure 3.1.

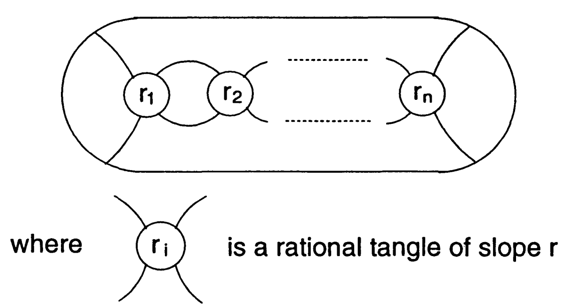

Figure 3.1

First we study which Montesions tangle is ∂-irreducible.

Theorem 4. *Suppose $n\geq 2$, $r_i\notin \boldsymbol{Z}\cup\{1/0\}$ $(1\leq i\leq n)$, and r_1, $r_n\notin\{q/2|q\in \boldsymbol{Z}\}$. Then $T(r_1, \cdots, r_n)$ is a ∂-irreducible tangle.*

REMARK.

(1) *$(B, t)=T(1/2, p/q)$ is not a ∂-irreducible tangle, indeed, $cl(B-N(t))$ is a genus 2 handlebody.*

(2) *After having done this work, the author learned that Wu [7] had proved that, except for trivial cases, a Montesinos tangle which is not ∂-irreducible*

is $T(1/2, p/q)$.

Proof of Theorem 4. Put $T(r_1, \cdots, r_n)=(B, t)$, $E(t)=cl(B-N(t))$, and let $A_1, \cdots, A_{n-1}$ be the surfaces in $E(t)$ as illustrated in Figure 3.2.

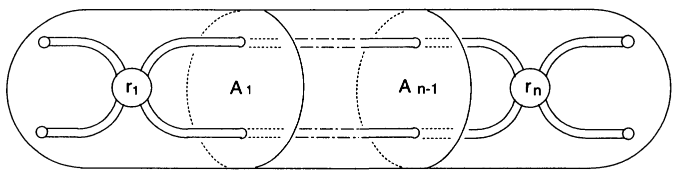

Figure 3.2

Then $\bigcup_{i=1}^{n-1} A_i$ decomposes $E(t)$ into $\prod_{i=1}^{n} E(t_i)$, where $E(t_i)$ is the exterior $cl(B_i - N(t_i))$ of a rational tangle (B_i, t_i) of slope r_i $(1 \leq i < n)$.

Suppose $E(t)$ has a compressing disc D. Then we may assume D intersects $\cup A_i$ transversely. By using the assumption that $r_i \neq 1/0$ $(1 \leq i \leq n)$, we can isotope D so that $D \cap (\cup A_i)$ consists of only arcs. In the following, we assume that $|D \cap (\cup A_i)|$ is minimized; we see this number is not zero by using the same assumption. Let α be a component of $D \cap (\cup A_i)$ which is outermost in D, and let E be the disc in D cut off by α, such that $(\text{int } E) \cap (\cup A_i) = \emptyset$. Then α lies in some A_i, and (A_i, α) is of one of the six types illustrated in Figure 3.3.

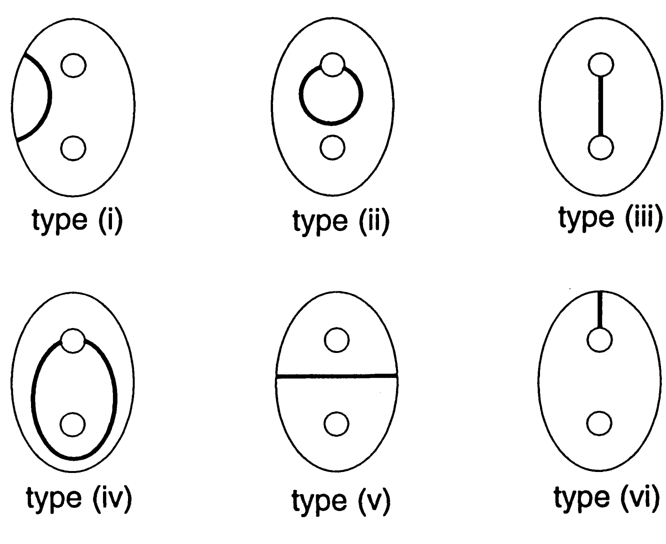

Figure 3.3

Then one of the following three cases occurs.

(1) E is contained in $E(t_1)$ (or $E(t_n)$) and $E(t_1)$ (or $E(t_n)$) is of X-type.

(2) E is contained in $E(t_1)$ (or $E(t_n)$) and $E(t_1)$ (or $E(t_n)$) is of Y-type.

(3) E is contained in $E(t_j)$, where $2 \leq j \leq n-1$.

Here, we say that $E(t_1)$ [resp. $E(t_n)$] is of X-type if each component of $FrN(t_1)$ [resp. $FrN(t_n)$] has one boundary component in A_1 [resp. A_{n-1}], Y-type otherwise. We show that we can find a contradiction in any case. We consider only Case (2), because the arguments for Cases (1) and (3) are similar to that for Case (2). It should be noted that (A_1, α) is not of type (vi) since $E(t_1)$ is of Y-type.

Without loss of generality we assume E is contained in $E(t_1)$. Put $\beta = cl(\partial E - \alpha)$, and T_1 and T_2 the components of $FrN(t_1)$; we assume that $T_1 \cap \alpha \neq \emptyset$ in case α is of type (ii), (iii), (iv), or (vi). By elementary but careful arguments, we may assume $(E(t_1), A_1, E)$ is as illustrated in Figure 3.4 (i)-(v) according as the type of (A_1, α). Here, in case (A_1, α) is of type (v), Figure 3.4 (v) illustrates $(E(t_1), A_1, E)$ only modulo integral twists of $E(t_1)$.

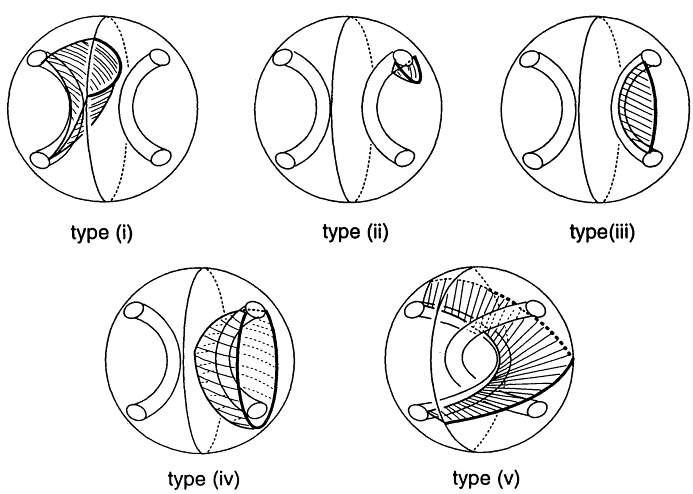

Figure 3.4

These figures imply that (1) if (A_1, α) is of type (i), (ii), (iii), or (iv), then $r_1 = 1/0$ and (2) if (A_1, α) is of type (v), then $r_1 = q/2$ with q an odd integer. This is a contradiction.

This completes the proof of Theorem 4.□

From now, a thorough understanding of [5] is assumed, and we investigate the genus of the surface realizing a ∂-slope in the following proposition.

Proposition 1 ([5, Proposition 2.2]). *For each $p/q\in\boldsymbol{Q}$, there exists an incompressible, ∂-incompressible, orientable surface in the complement of some Montesinos knot, with ∂-slope p/q.*

Concerning Theorem 2, we look for surfaces whose ∂-slopes have denominators q. Then, following the first half of the proof of the above proposition, we obtain the following theorem.

Theorem 5. *For a natural number q, let K_q be the Montesinos knot $M(2/7, 1/(8q+13), -1/3, 5/18, 1/(8q+13), -1/3)$ or $M(2/7, 1/(8q+13), -1/3, 2/7, 1/(8q+13), -1/3, 5/18, 1/(8q+13), -1/3)$ according as q is odd or even. Then $E(K_q)$ contains an incompressible, ∂-incompressible, orientable surface S_q, such that the denominator of the ∂-slope of S_q is q, and the genus of S_q is at most cq where c is a constant independent of q.*

REMARK. *By* Theorem 4, *K_q has a 2-string ∂-irreducible tangle decomposition.*

We give the proof only for the case where q is odd, because the proof for the case where q is even is similar.

Proof of Theorem 5. We shall use notations of [5]. First we recall the construction of the K_q (see [5, pp. 455-456]). Viewing S^3 as the join of two circles A and B, let the circle B be subdivided as a six-sided polygon. Then the join of A with the ith edge of B is a ball B_i. Put $H_i=B_i\cap B_{i+1}=\partial B_i\cap\partial B_{i+1}$, then $\partial B_i=H_{i-1}\cup H_i$. The 6 balls B_i $(1\leq i\leq 6)$ cover S^3. Recall that (S^3, K_q) is constructed as the union $(B_1, t_1)\cup\cdots\cup(B_6, t_6)$, where $(B_1, t_1), \cdots, (B_6, t_6)$ are rational tangles of slopes $2/7, \cdots, -1/3$ respectively. In the proof of Proposition 1, an incompressible, ∂-incompressible, orientable, *candidate* surface S_q is constructed as follows. For each rational tangle (B_i, t_i), choose an *edgepath* γ_i as follows.

- γ_1 goes linearly from $(1, 6, 2)$ to the point $A=(4q-3)(1, 2, 1)+3(1, 6, 2)$.
- γ_2 and γ_5 go linearly from $(1, 8q+12, 1)$ to the point $B=(4q-1)(1, 0, 0)+(1, 8q+12, 1)$.
- γ_3 and γ_6 are constant, at the point $C=(4q, 8q+12, -4-4q)$.
- γ_4 first goes linearly from $(1, 17, 5)$ to $(1, 6, 2)$ and second goes linearly $(1, 6, 2)$ to the point A.

To each γ_i, a surface S_i in (B_i, t_i) is associated so that $S_i\cap H_i=S_{i+1}\cap H_i$. Then $S_q=\bigcup_{i=1}^{n}S_i$. The ∂-slope of S_q is given by $\tau(S_q)-\tau(S_0)$, where $\tau(S_q)=2/q-2$, and $\tau(S_0)$ is a certain integer associated with a Seifert surface S_0. Hence the denominator of the ∂-slope of S_q is q.

Finally we roughly estimate the genus of S_q.

(1) Through all S_i the number of saddles is less than c_1q, where c_1 is a

constant independent of q $(1 \leq i \leq 6)$.

(2) The number of arcs in $H_i \cap S_i$ is less than $c_2 q$, where c_2 is a constant independent of q $(1 \leq i \leq 6)$.

Hence we can see that the genus of S_q is at most cq, where c is a constant independent of q.

This completes the proof of Theorem 5.

REMARK. *In* Theorem 5 *we can take* $c=100$.

ACKNOWLEDEMENT. The author would like to thank Professor K. Kawakubo for his kind encouragement. He is also grateful for many pieces of useful advice offerd by Professor M. Sakuma and Professor T. Kobayashi.

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