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ON THE *K*-THEORY OF PE_6

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Introduction

Let E_6 be the exceptional compact simply-connected simple Lie group and let PE_6 be the projective group associated with E_6. In other words $PE_6 = E_6/Z(E_6)$ with $Z(E_6) \cong \boldsymbol{Z}/3$ where $Z(E_6)$ denotes the center of E_6. The complex K-group $K^*(PE_6)$ of PE_6 has been calculated by Held and Suter in [5] and by Hodgkin in [7] independently. In this paper we calculate the real K-group $KO^*(PE_6)$ of PE_6. To our aim, however, we begin with the computation of $K^*(PE_6)$ by a different method from [5, 7] and we compute $KO^*(PE_6)$ by applying the techniques parallel to $K^*(PE_6)$ and using some results obtained in course of calculation as well as the result on $K^*(PE_6)$.

We study these K-groups along the way of getting the K-groups of PE_7 in [10]. In the case of E_7 we used the $\boldsymbol{Z}/2$-equivariant K-theories because of $Z(E_7) \cong \boldsymbol{Z}/2$. In the present case we make use of the $\boldsymbol{Z}/3$-equivariant K-theories and we reduce the structures of the K-groups of PE_6 to those of K-groups of E_6 and $L^n(3)$, the usual lens spaces, for $1 \leq n \leq 6$. We refer to [6, 12] for information about the K-groups of E_6.

In Section 1 we review some basic materials and give the ring structures of K-groups of the relevant lens spaces. In Section 2 and in Sections 3, 4 we determine the structures of $K^*(PE_6)$ and $KO^*(PE_6)$ respectively. The main results are Theorems 2.1 and 3.1.

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1. Preliminaries

By Γ we denote the center of E_6 which is a cyclic group of order 3 and set

$$\Gamma = \{\gamma \mid \gamma^3 = 1\}.$$

Consider the symmetric pair $(E_6, Spin(10) \cdot S^1)$ with the subgroup of maximal rank. Then we see that Γ coincides with the central subgroup of $S^1 \subset Spin(10) \cdot S^1$ or order 3.

According to [13] we have the following irreducible representations

$$\rho: E_6 \to U(78),\ \rho_1: E_6 \to U(27) \text{ and } \rho_1^*: E_6 \to U(27)$$

where ρ_1^* denotes the complex conjugate of ρ_1 and ρ the adjoint representation of E_6. Moreover

$$\operatorname{Ker}\rho = \Gamma \quad \text{and} \quad \operatorname{Ker}\rho_1 = \operatorname{Ker}\rho_1^* = \{1\}.$$

And the fundamental representations of E_6 are ρ, ρ_1, $\lambda^2\rho_1$, $\lambda^3\rho_1(=\lambda^3\rho_1^*)$, $\lambda^2\rho_1^*$ and ρ_1^*, in which in particular ρ and $\lambda^3\rho_1$ are the complexification of real representations. The same symbols ρ and $\lambda^3\rho_1$ are used to denote also these real representations hereafter.

By Lemma of [9] (see also [1], Chap. 10) we have

(1.1) The restrictions of the fundamental representations to $Spin(10)\cdot S^1$ are

$$\rho = \lambda^2\rho_{10}\otimes 1 + \Delta^+\otimes t^3 + \Delta^-\otimes t^{-3} + 1,$$

$$\rho_1 = 1\otimes t^4 + \Delta^-\otimes t + \rho_{10}\otimes t^{-2},$$

$$\lambda^2\rho_1 = \Delta^-\otimes t^5 + \lambda^3\rho_{10}\otimes t^2 + \rho_{10}\otimes t^2 + \Delta^-\rho_{10}\otimes t^{-1} + \lambda^2\rho_{10}\otimes t^{-4},$$

and

$$\lambda^3\rho_1 = \lambda^3\rho_{10}\otimes t^6 + \lambda^3\rho_{10}\otimes t^{-6} + \Delta^+\lambda^2\rho_{10}\otimes t^3 + \Delta^-\lambda^2\rho_{10}\otimes t^{-3} + \rho_{10}\lambda^3\rho_{10}\otimes 1 + \lambda^2\rho_{10}\otimes 1$$

where ρ_{10} and t are the canonical non-trivial 10- and 1-dimensional representations of $Spin(10)$ and S^1 respectively, and $\Delta^\pm$ are the half-spin representations of $Spin(10)$. The restrictions of ρ_1^* and $\lambda^2\rho_1^*$ are immediate from (1.1) since $(\Delta^\pm)^* = \Delta^\mp$.

Let V be the representation space of the canonical non-trivial complex 1-dimensional representation of Γ. We write nV for the direct sum of n copies of V. Let $B(nV\oplus C^k)$ and $S(nV\oplus C^k)$ denote the unit ball and unit sphere in $nV\oplus C^k$ centered at the origin o, and let $\Sigma^{nV+2k} = B(nV\oplus C^k)/S(nV\oplus C^k)$ with the collapsed $S(nV\oplus C^k)$ as base point. And then the lens space $L^n(3)$ is defined to be the orbit space $S((n+1)V)/\Gamma$.

Let nV be embedded in $(n+k)V = nV\oplus kV$ by the assignment $v\mapsto(v, 0)$. Then there is an equivariant homeomorphism $S((n+k)V)/S(nV) \approx \Sigma^{nV}\wedge S(kV)_+$ via which these spaces are identified below. For our computation we use mainly the following exact sequences, which are obtained from applying the equivariant K-functor to the cofibrations

$$S(nV)\times X \xrightarrow{i} B(nV)\times X \xrightarrow{j} \Sigma^{nV}\wedge X_+$$

and

$$S(nV)\times X \xrightarrow{i} S((n+k)V)\times X \xrightarrow{j} \Sigma^{nV}\wedge (S(kV)\times X)_+$$

where i's and j's are the canonical inclusions and projections and Y_+ denotes the disjoint union of a Γ-space Y and a point.

(1.2) (i) $\cdots \to \tilde{h}_\Gamma^*(\Sigma^{nV}\wedge X_+) \xrightarrow{j^*} h_\Gamma^*(B(nV)\times X) \xrightarrow{i^*} h_\Gamma^*(S(nV)\times X) \xrightarrow{\delta}$

$\tilde{h}_\Gamma^*(\Sigma^{nV}\wedge X_+) \to \cdots$

and (ii) $\cdots \to \tilde{h}_\Gamma^*(\Sigma^{nV}\wedge (S(kV)\times X)_+) \xrightarrow{j^*} h_\Gamma^*(S((n+k)V)\times X) \xrightarrow{i^*} h_\Gamma^*(S(nV)\times X)$

$\xrightarrow{\delta} \tilde{h}_\Gamma^*(\Sigma^{nV}\wedge (S(kV)\times X)_+) \to \cdots$

for $h=K$, KO, in which there holds $\delta(xi^*(y))=\delta(x)y$.

If X is a compact *free* Γ-space then we have a canonical isomorphism $h^*(X/\Gamma)\cong h_\Gamma^*(X)$ which we identify in the following.

Especially we consider (1.2) (ii) when $k=1$ and $X=$a point, E_6. Then we have a homeomorphism

$$\varphi:(\Sigma^{nV}\wedge(S(V)\times X)_+)/\Gamma \approx \Sigma^{2n}\wedge(S^1\times X)_+$$

arising from the map from $B(nV)\times S(V)\times X$ to $B(C^n)\times S^1\times X$ given by the assignment $((z_1,\cdots,z_n),z,x)\mapsto((z^{-1}z_1,\cdots,z^{-1}z_n),z^3,z^{-1}x)$ where $z^{-1}x$ is x if $X=$a point and denotes the product of z^{-1} and x in E_6 if $X=E_6$, under the identification $S(V)=S^1$, the circle subgroup of E_6 which is a factor of $Spin(10)\cdot S^1$ stated above. Therefore we see that (1.2) (ii) yields the following exact sequence

(1.3) $\cdots \to h^*(S^1\times X) \xrightarrow{J} h_\Gamma^*(S((n+1)V)\times X) \xrightarrow{i^*} h_\Gamma^*(S(nV)\times X) \xrightarrow{\bar{\delta}} h^*(S^1\times X) \to \cdots$

for $X=$a point, E_6, in which $J=j^*\varphi^*$, $\bar{\delta}=\varphi^{*-1}\delta$ (up to the suspension isomorphism) and so there holds $\bar{\delta}(xi^*(y))=\bar{\delta}(x)y$.

For later use we write $A\cdot g$ for the module over a ring A generated by g. We recall from [11] the Thom isomorphism theorem in complex K-theory. Let $\mu\in\tilde{K}(S^2)$ be the Bott element. Then $\tilde{K}(S^{2n})=\boldsymbol{Z}\cdot\mu^n$ and we have by [11] the following.

(1.4) There exists an element τ_{nV} of $\tilde{K}_\Gamma(\Sigma^{nV})$ such that multiplication by τ_{nV}, $x\mapsto\tau_{nV}\wedge x$, induces an isomorphism $K_\Gamma^*(X)\cong\tilde{K}_\Gamma^*(\Sigma^{nV}\wedge X_+)$ for any Γ-space X, the restriction of τ_{nV} to $K_\Gamma(o)=R(\Gamma)$ is $(1-V)^n$ and forgetting action τ_{nV} becomes μ^n, where $R(\Gamma)$ is the complex representation ring of Γ.

As is well known, given a map $f:X\to U(n)$ (resp. $O(n)$), the homotopy class of the composite of this with an inclusion $U(n)\subset U$ (resp. $O(n)\subset O$) can be viewed as an element of $K^{-1}(X)$ (resp. $KO^{-1}(X)$) for which $\beta(f)$ we write in any case where U (resp. O) is the infinite unitary (resp. orthogonal) group. According to [6], then

(1.5) $K^*(E_6)=\Lambda(\beta(\rho),\beta(\rho_1),\beta(\lambda^2\rho_1),\beta(\lambda^3\rho_1),\beta(\lambda^2\rho_1^*),\beta(\rho_1^*))$ as a ring.

When we deal with the real K-theory, we consider the complex K-theory to be $Z/8$-graded. The coefficient ring of each theory is given by $KO^*(+)=Z[\eta_1,\eta_4]/(2\eta_1,\eta_1^3,\eta_1\eta_4,\eta_4^2-4)$ where $\eta_i\in KO^{-i}(+)$ and $K^*(+)=Z[\mu]/(\mu^4-1)$ ($+=$ point). Let us denote by r and c the realification and complexification homomorphisms as usual. In [12], Theorem 5.6 $KO^*(E_6)$ is determined by using (1.5) as follows.

(1.6) There exist elements $\lambda_1,\lambda_2\in KO^0(E_6)$ such that $c(\lambda_1)=\mu^3\beta(\rho_1)\beta(\rho_1^*)$, $c(\lambda_2)=\mu^3\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)$ and as a $KO^*(+)$-module

$$KO^*(E_6)=F\oplus r(T).$$

Here F is the subalgebra of $KO^*(E_6)$ generated by

$$\beta(\rho),\ \beta(\lambda^3\rho_1),\ \lambda_1,\ \lambda_2$$

and is a free $KO^*(+)$-module, and T is the submodule of $K^*(E_6)$ generated by the monomials

$$\boldsymbol{n}\beta(\rho_1),\ \boldsymbol{n}\beta(\lambda^2\rho_1),\ \boldsymbol{n}\beta(\rho_1)\beta(\lambda^2\rho_1),$$
$$\boldsymbol{n}\beta(\rho_1)\beta(\lambda^2\rho_1^*),\ \boldsymbol{n}\beta(\rho_1)\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),\ \boldsymbol{n}\beta(\rho_1)\beta(\rho_1^*)\beta(\lambda^2\rho_1)$$

where $\boldsymbol{n}$ is a monomial in $\beta(\rho)$, $\beta(\lambda^3\rho_1)$ with coefficients in $K^*(+)$. Further, $\lambda_1^2=\lambda_2^2=0$, and $\beta(\rho)^2$ and $\beta(\lambda^3\rho_1)$ are divisible by η_1.

REMARKS 1. In fact it follows from the square formula of [4], §6 that $\beta(\rho)^2=\eta_1(\beta(\lambda^2\rho)+78\beta(\rho))$ and $\beta(\lambda^3\rho_1)^2=\eta_1(\beta(\lambda^2(\lambda^3\rho_1))+27\beta(\lambda^3\rho_1))$. And we have $\lambda^2\rho=\lambda^3\rho_1+\rho$ by (1.1), so that $\beta(\rho)^2=\eta_1(\beta(\lambda^3\rho_1)+\beta(\rho))$. Using $\eta_1 r(x)=0$ stated in the subsequent remark we see that $\eta_1\beta(\lambda^2(\lambda^3\rho_1))$ is only a linear combination of $\eta_1\beta(\lambda^3\rho_1)$ and $\eta_1\beta(\rho)$, and further observation of the restriction of $\lambda^2(\lambda^3\rho_1)$ to $Spin(10)\cdot S^1$ leads to $\eta_1\beta(\lambda^2(\lambda^3\rho_1))=0$ which therefore implies $\beta(\lambda^3\rho_1)^2=\eta_1\beta(\lambda^3\rho_1)$. As is noted in [12] all the other relations can be easily obtained from making use of the equality

$$r(x)r(y)=r(xcr(y))=r(xy)+r(xy^*)\quad \text{for}\quad x,\ y\in T$$

where y^* denotes the complex conjugate of y.

2. The elements λ_1, λ_2 described above are unique. For example, if there exists another element λ_1' such that $c(\lambda_1)=c(\lambda_1')$ then, considering the Bott exact sequence

$$\cdots\to KO^*(E_6)\xrightarrow{\chi} KO^*(E_6)\xrightarrow{c} K^*(E_6)\xrightarrow{\delta}\cdots$$

where χ is multiplication by η_1 and δ is given by $\delta(\mu x)=r(x)$ [2], we see that $\lambda_1'-\lambda_1$ can be written as $\lambda_1'-\lambda_1=\eta_1 a$ for some $a\in KO^{-7}(E_6)$. But we may assume that $a \in F$ because of $\chi\delta=0$ and the odd dimensional generators of F are only $\beta(\rho)$, $\beta(\lambda^3\rho_1)$. Hence we see that a is divisible by η_1^2, so that $\eta_1 a$ must be zero. This is quite similar to λ_2.

We next recall the Bott element of the equivariant KO-theory associated with Γ. Let $W=r(V)$, the realification of V, and we write nW to denote the direct sum of n copies of W as before. We show that $W \oplus W$ is provided with a *Spin* Γ-module structure. It suffices to prove that the composite homomorphism $i: \Gamma \to U(1) \to SO(2) \xrightarrow{d} SO(2) \times SO(2) \to SO(4)$, where the unlabelled arrows are canonical inclusions and d is the diagonal map, may be lifted to a homomorphism $\tilde{i}$ from Γ to $Spin(4)$, satisfying $\pi\tilde{i}=i$ where π denotes the canonical projection from $Spin(4)$ to $SO(4)$. Now we see that the map $\gamma \mapsto (\cos\frac{\pi}{3}+e_1e_2\sin\frac{\pi}{3})(\cos\frac{\pi}{3}+e_3e_4\sin\frac{\pi}{3})$, where $e_1,\cdots,e_4$ is an orthonormal basis of $\boldsymbol{R}^4$ such that $e_i^2=-1$, $e_ie_j=-e_je_i$ if $i \neq j$, defines a required lifting $\tilde{i}$. So we see further that $2nW$ in general can be provided with a *Spin* Γ-module structure. To state the Thom isomorphism theorem in the equivariant KO-theory moreover we need the following fact [11].

(1.7) Let X be a compact *trivial* Γ-space. Then for a real Γ-vector bundle E over X the assignment $E \mapsto \mathrm{Hom}_\Gamma(X \times \boldsymbol{R}, E) \oplus W \otimes_{\boldsymbol{C}} \mathrm{Hom}(X \times W, E)$ induces an isomorphism

$$KO_\Gamma^*(X) \cong KO^*(X) \oplus \boldsymbol{Z} \cdot W \otimes K^*(X)$$

where $\boldsymbol{C}$ is identified with $\mathrm{Hom}_\Gamma(W, W)$ normally. In fact the 2nd direct summand of this equality is equal to $r(\boldsymbol{Z} \cdot V \otimes K^*(X))$.

From [3] we then have

(1.8) There is an element $\tau_{(4n+2\varepsilon)W+4\varepsilon} \in \widetilde{KO}_\Gamma(\Sigma^{(4n+2\varepsilon)V+4\varepsilon})$ for $\varepsilon=0,1$ such that the assignment $x \mapsto \tau_{(4n+2\varepsilon)W+4\varepsilon} \wedge x$ induces an isomorphism $KO_\Gamma^*(X) \cong KO_\Gamma^*(\Sigma^{(4n+2\varepsilon)V+4\varepsilon} \wedge X_+)$ for any Γ-space X and the restriction of $\tau_{(4n+2\varepsilon)W+4\varepsilon}$ to $\widetilde{KO}_\Gamma(\Sigma^{4\varepsilon}) = \boldsymbol{Z} \cdot \eta_4^\varepsilon \oplus \boldsymbol{Z} \cdot W\mu^{2\varepsilon}$ is $3^n(r(V-1))^n(r(\mu^2-V\mu^2))^\varepsilon$.

Finally we mention the structure of the K-groups of lens spaces $L^n(3)$ for $1 \leq n \leq 6$. This can be obtained by easy calculations using (1.3) when $X=$a point. As for the 0-terms it can be found in [8] for any lens space $L^n(p)$ with p, prime. But the technique used here is essential for our computation in the following sections. In order to describe the results we introduce the ring generators. By ξ_n we denote the complex line bundle $S((n+1)V) \times_\Gamma V \to L^n(3)$. And we set

$$\sigma_n=\xi_n-1 \in \tilde{K}(L^n(3)) \quad \text{and} \quad \bar{\sigma}_{n,i}=r(\mu^i\sigma_n) \in \widetilde{KO}^{-2i}(L^n(3)).$$

Let p be the composite $L^n(3) \to L^n(3)/(L^n(3)-N) \approx S^{2n+1}$ of canonical projection and homeomorphism where N is a coordinates neighborhood of some element of $L^n(3)$.

Then we set

$$v_n = p^*(\iota_n) \in \tilde{K}^{2n+1}(L^n(3)) \quad \text{and} \quad \bar{v}_n = p^*(\iota_n) \in \widetilde{KO}^{2n+1}(L^n(3))$$

where $p^*: \tilde{h}^{2n+1}(S^{2n+1}) \to \tilde{h}^{2n+1}(L^n(3))$ and ι_n denotes a generator of $\tilde{h}^{2n+1}(S^{2n+1}) \cong \boldsymbol{Z}$.

Observing the exact sequence (1.3) where X= a point we see that

(1.9) $\delta(v_{n-1}) = 3 \in K^0(S^1) = \boldsymbol{Z} \cdot 1$, $J(\iota_0) = v_n$ (up to sign) and $J(1) = (-\sigma_n)^n$.

Forgetting the action of Γ, the v_{n-1} and τ_{nV} become $3\iota_{n-1}$ and μ^n respectively. So we have $\delta(v_{n-1}) = 3\tau_{nV} \wedge 1$ (up to sign), so that the 1st formula follows. The 2nd formula is immediate from the definition of v_{n-1} and the 3rd also follows from (1.4) immediately. We ignore the sign below because it may be exchanged if necessary. Then from this it follows that

(1.10) $\delta(\bar{v}_{n-1}) = 3 \in KO^0(S^1) = \boldsymbol{Z} \cdot 1$, $J(\iota_0) = \bar{v}_n$ and $J(r(\mu^{i+n})) = r(\mu^i(-\sigma_n)^n)$.

Making use of (1.3) when X= a point together with these two facts (1.9), (1.10) we can get the following results inductively by taking n in turn to be $0, 1, \cdots, 6$.

(1.11) (i) $\tilde{K}^0(L^n(3)) = \boldsymbol{Z}/3^{s+r} \cdot \sigma_n \oplus \boldsymbol{Z}/3^s \cdot \sigma_n^2$ and $K^{-1}(L^n(3)) = \boldsymbol{Z} \cdot v_n$

for $0 \leq n \leq 6$ where $s = [\frac{n}{2}]$, $r = ((-1)^{n-1}+1)/2$ and the ring structure is given by

$$\sigma_n^3 + 3\sigma_n^2 + 3\sigma_n = 0 \quad \text{and} \quad v_n^2 = 0.$$

(ii)
$$\widetilde{KO}^0(L^n(3)) = \begin{cases} \boldsymbol{Z}/3^s \cdot \bar{\sigma}_{n,0} \oplus \boldsymbol{Z}/2 \cdot \eta_1 \bar{v}_n & (n=0, 4) \\ \boldsymbol{Z}/3^s \cdot \bar{\sigma}_{n,0} & (\text{otherwise}), \end{cases}$$

$$\widetilde{KO}^{-1}(L^n(3)) = \begin{cases} \boldsymbol{Z} \cdot \eta_4 \bar{v}_n & (n=1, 5) \\ 0 & (n=2, 6) \\ \boldsymbol{Z} \cdot \bar{v}_n & (n=3) \\ \boldsymbol{Z}/2 \cdot \eta_1^2 \bar{v}_n & (n=0, 4), \end{cases}$$

$$KO^{-2}(L^n(3)) = \begin{cases} \boldsymbol{Z}/3^t \cdot \bar{\sigma}_{n,1} \oplus \boldsymbol{Z}/2 \cdot \eta_1 \bar{v}_n & (n=3) \\ \boldsymbol{Z}/3^t \cdot \bar{\sigma}_{n,1} & (\text{otherwise}), \end{cases}$$

$$\widetilde{KO}^{-3}(L^n(3)) = \begin{cases} 0 & (n=1, 5) \\ \boldsymbol{Z} \cdot \bar{v}_n & (n=2, 6) \\ \boldsymbol{Z}/2 \cdot \eta_1^2 \bar{v}_n & (n=3) \\ \boldsymbol{Z} \cdot \eta_4 \bar{v}_n & (n=0, 4), \end{cases}$$

$$\widetilde{KO}^{-4}(L^n(3)) = \begin{cases} \boldsymbol{Z}/3^s \cdot \bar{\sigma}_{n,2} \oplus \boldsymbol{Z}/2 \cdot \eta_1 \bar{v}_n & (n=2, 6) \\ \boldsymbol{Z}/3^s \cdot \bar{\sigma}_{n,2} & (\text{otherwise}), \end{cases}$$

$$\widetilde{KO}^{-5}(L^n(3))=\begin{cases} Z\cdot\bar{v}_n & (n=1,5)\\ Z/2\cdot\eta_1^2\bar{v}_n & (n=2,6)\\ Z\cdot\eta_4\bar{v}_n & (n=3)\\ 0 & (n=0,4)\end{cases}$$

$$\widetilde{KO}^{-6}(L^n(3))=\begin{cases} Z/3^t\cdot\bar{\sigma}_{n,3}\oplus Z/2\cdot\eta_1\bar{v}_n & (n=1,5)\\ Z/3^s\cdot\bar{\sigma}_{n,3} & (n=0,2,4,6)\\ Z/3^t\cdot\bar{\sigma}_{n,3} & (n=3)\end{cases}$$

and
$$\widetilde{KO}^{-7}(L^n(3))=\begin{cases} Z/2\cdot\eta_1^2\bar{v}_n & (n=1,5)\\ Z\cdot\eta_4\bar{v}_n & (n=2,6)\\ 0 & (n=3)\\ Z\cdot\bar{v}_n & (n=0,4)\end{cases}$$

for $0\leq n\leq 6$ where $s=[\frac{n}{2}]$, $t=[\frac{n+1}{2}]$ and the ring structure is given by

$$\bar{\sigma}_{n,i}\bar{\sigma}_{n,j}=((-1)^{i+j}+(-1)^{i+1}+(-1)^{j+1}-2)\bar{\sigma}_{n,i+j}+((-1)^{i+j}+(-1)^{i+1}+(-1)^{j}-1)r(\mu^{i+j})$$

$$\eta_4\bar{\sigma}_{n,i}=2\bar{\sigma}_{n,i+2}\quad\text{and}\quad \bar{v}_n^2=0.$$

2. The complex K-group of PE_6

In this section we give the structure of $K^*(PE_6)$.

We denote a canonical complex line bundle $E_6\times_\Gamma V\to PE_6$ by ξ and set

$$\sigma=\xi-1\in K(PE_6).$$

Since ρ and $\lambda^3\rho_1$ are trivial on $Z(E_6)=\Gamma$, these can be regarded as representations of PE_6 and so the elements

$$\beta(\rho),\ \beta(\lambda^3\rho_1)\in K^{-1}(PE_6)$$

can be defined in the manner as mentioned in the preceding section. From (1.1) we see that $\rho_1(\gamma)$ is a 27×27 scalar matrix with all diagonal entries $\omega=\exp(\frac{2\pi i}{3})$ where $\gamma\in\Gamma$. Hence it follows that the assignments $g\mapsto\rho_1^*(g)\rho_1(g)$, $g\mapsto\lambda^2\rho_1(g)13\rho_1(g)$ and $g\mapsto\lambda^2\rho_1^*(g)13\rho_1^*(g)$ induce three maps from PE_6 to U where $g\in E_6$. We denote also the homotopy classes of maps by

$$\beta(\rho_1+\rho_1^*),\ \beta(13\rho_1+\lambda^2\rho_1),\ \beta(13\rho_1^*+\lambda^2\rho_1^*)\in K^{-1}(PE_6)$$

respectively. In order to describe the result we need one more element. Let N be the representation space of the (regular) representation $\Gamma\to SO(3)$ of N given by the assignment

$$\gamma \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and put $F = E_6 \times C^{27} \otimes N$ which is viewed as a product bundle over E_6. We define a Γ-equivariant bundle isomorphism $f: F \to F$ by the assignment $(g,(v_1,v_2,v_3)) \mapsto (g,(\rho_1(\gamma^2 g)v_1, \rho_1(\gamma g)v_2, \rho_1(g)v_3))$. Then f defines an element of $K_\Gamma^{-1}(E_6)$ in the usual way, which we denote by

$$\beta(\rho_1, \Gamma) \in K_\Gamma^{-1}(E_6) = K^{-1}(PE_6).$$

In fact, this coincides with $t(\beta(\rho_1))$ where $t: K^{-1}(E_6) \to K_\Gamma^{-1}(E_6)$ is the transfer map.

Then we have

Theorem 2.1 ([5, 7]). *With the notation as above*

$$K^*(PE_6) = \Lambda(\beta(\rho), \beta(\lambda^3\rho_1), \beta(\rho_1 + \rho_1^*), \beta(13\rho_1 + \lambda^2\rho_1), \beta(13\rho_1^* + \lambda^2\rho_1^*), \beta(\rho_1, \Gamma)) \otimes P / (\beta(\rho_1, \Gamma)\sigma)$$

as a ring. Here P is the subring of $K^*(PE_6)$ generated by σ such that

$$P \cong Z \cdot 1 \oplus Z/27 \cdot \sigma \oplus Z/27 \cdot \sigma^2,$$

where the ring structure is given by

$$\sigma^3 + 3\sigma^2 + 3\sigma = 0.$$

We prepare a lemma for a proof of the theorem. According to (1.4) the restriction of $\tau_{7V} \in \tilde{K}_\Gamma(\Sigma^{7V})$ to $R(\Gamma)$ is $27(V-1)$. From this fact we see that τ_{7V} yields an equivariant bundle isomorphim α from $S(7V) \times E_6 \times (27V \oplus S)$ to $S(7V) \times E_6 \times (C^{27} \oplus S)$ for some Γ-module S. On the other hand, ρ_1 induces an equivariant bundle isomorphism f from $S(7V) \times E_6 \times (C^{27} \oplus S)$ to $S(7V) \times E_6 \times (27V \oplus S)$ given by $f(x,g,(u,v)) = (x,g,(\rho_1(g)u,v))$. Then, in the usual way, the composite αf defines an element of $K_\Gamma^{-1}(S(7V) \times E_6) = K^{-1}(S(7V) \times_\Gamma E_6)$ which we denote by $\tilde{\beta}(\rho_1)$. Similarly, by taking $\lambda^2\rho_1$ and $\lambda^2\alpha$, ρ_1^* and α^*, and $\lambda^2\rho_1^*$ and $\lambda^2\alpha^*$ instead of ρ_1 and α respectively we get the elements $\tilde{\beta}(\lambda^2\rho_1), \tilde{\beta}(\rho_1^*), \tilde{\beta}(\lambda\rho_1^*) \in K_\Gamma^{-1}(S(7V) \times E_6)$. Also we denote by the same symbols the restrictions of these elements to $K_\Gamma^{-1}(S(nV) \times E_6)$ for $1 \le n \le 6$.

Let π_1 (resp. π_2) denote the projection from $S(nV) \times E_6$ to the 1st (resp. 2nd) factor. Put $\tilde{\beta}(\rho) = \pi_2^*(\beta(\rho))$, $\tilde{\beta}(\lambda^3\rho_1) = \pi_2^*(\beta(\lambda^3\rho_1))$, $\tilde{\sigma} = \pi_1^*(\sigma_{n-1}) = \pi_2^*(\sigma)$ and $\tilde{v}_{n-1} = \pi_2^*(v_{n-1})$.

Then we have

Lemma 2.2. *With the notation as above*

$$K_\Gamma^*(S((n+1)V)\times E_6)=P_n\otimes\Lambda_n/(\tilde{\sigma}\otimes\tilde{v}_n)$$

as a ring for $0\leq n\leq 6$. *Here* P_n *is the subring generated by* $\tilde{\sigma}$ *such that*

$$P_n=\boldsymbol{Z}\cdot 1\oplus\boldsymbol{Z}/3^{s+t}\cdot\tilde{\sigma}\oplus\boldsymbol{Z}/3^s\cdot\tilde{\sigma}^2$$

where $s=[\frac{n}{2}]$, $r=((-1)^{n-1}+1)/2$ *and the ring structure is given by*

$$\tilde{\sigma}^3+3\tilde{\sigma}^2+3\tilde{\sigma}=0,$$

and $$\Lambda_n=\Lambda(\tilde{\beta}(\rho),\tilde{\beta}(\rho_1),\tilde{\beta}(\lambda^2\rho_1),\tilde{\beta}(\lambda^3\rho_1),\tilde{\beta}(\lambda^2\rho_1^*),\tilde{\beta}(\rho_1^*),\tilde{v}_n).$$

In other words,

$$K^*(S((n+1)V)\times E_6)\cong\Lambda(\tilde{\beta}(\rho),\tilde{\beta}(\rho_1),\tilde{\beta}(\lambda^2\rho_1),\tilde{\beta}(\lambda^3\rho_1),\tilde{\beta}(\lambda^2\rho_1^*),\tilde{\beta}(\rho_1^*))\otimes K^*(L^n(3))$$

as a ring canonically.

Proof. For a proof we make use of (1.3) when $X=E_6$ and we show this inductively on n. In this case the exact sequence (1.3) is as follows.

$$\cdots\to K^*(S^1\times E_6)\xrightarrow{J}K_\Gamma^*(S((n+1)V)\times E_6)\xrightarrow{i^*}K_\Gamma^*(S(nV)\times E_6)\xrightarrow{\delta}\cdots$$

in which the maps satisfy $\delta(xi^*(y))=\delta(x)y$. Furthermore we see by (1.9) that there hold the equalities $\delta(\tilde{v}_{n-1})=3$, $J(\iota_0\times 1)=\tilde{v}_n$ and $J(1)=(-\tilde{\sigma})^n$. We now check the 1st stage of our induction. Because $S(V)$ may be viewed as a Γ-invariant subspace of E_6 as noted in the preceding of (1.3), it follows that $S(V)\times_\Gamma E_6\approx S^1\times E_6$ which is induced by the assignment $(z,g)\mapsto(z^3,z^{-1}g)$ where $z\in S(V)$ and $g\in E_6$, and so

$$\begin{aligned}K_\Gamma^*(S(V)\times E_6)&\cong K^*(S^1\times E_6)\\&\cong\Lambda(\iota_0)\otimes\Lambda(\beta(\rho),\beta(\rho_1),\beta(\lambda^2\rho_1),\beta(\lambda^3\rho_1),\beta(\lambda^2\rho_2^*),\beta(\rho_1^*))\end{aligned}$$

by (1.5).

We consider the elements of $K_\Gamma^*(S(V)\times E_6)$ corresponding to the generators of $K^*(S^1\times E_6)$ via this isomorphism. By definition we see that $\tilde{\beta}(\rho_1)$ of $K_\Gamma^*(S(V)\times E_6)$ can be decomposed into the form $\beta(\rho_1)+n\mu$ for some $n\in\boldsymbol{Z}$ via this isomorphism where $n\mu$ is constructed with $\rho_1|S^1$ and α described in the preceding of Lemma 2.2. Now as mentioned above α arises from τ_{7V} and $\rho_1|S^1=t^4+16t+10t^{-2}$ which follows from the 2nd formula of (1.1). So we get the case when $n=0$ by an inspection of the construction of $\tilde{\beta}(\rho_1)$. For the same reasons the $\tilde{\beta}(a)$'s correspond to $\beta(a)$'s respectively. In particular, it is immediate as for $a=\rho$, $\lambda^3\rho_1$. And also it is straightforward that $\tilde{v}_0$ corresponds to ι_0 up to sign. Hence we conclude that

$$K_\Gamma^*(S(V)\times E_6)=\Lambda_0\ (=P_0\otimes\Lambda_0/(\tilde{\sigma}\otimes\tilde{v}_0)).$$

For the next stage of induction we observe the above exact sequence when $n=1$. Then clearly $i^*(\tilde{\beta}(a))=\tilde{\beta}(a)$, $i^*(\tilde{\sigma})=0$ and from the discussion above it follows that

$$\tilde{\delta}(\tilde{v}_0)=3,\ J(\iota_0\times \boldsymbol{n})=\tilde{v}_1\tilde{\boldsymbol{n}} \text{ and } J(\boldsymbol{n})=-\tilde{\sigma}\tilde{\boldsymbol{n}}$$

where $\boldsymbol{n}$ is a monomial in $\beta(\rho)$, $\beta(\rho_1)$, $\beta(\lambda^2\rho_1)$, $\beta(\lambda^3\rho_1)$, $\beta(\lambda^2\rho_1^*)$, $\beta(\rho_1^*)$ and $\tilde{\boldsymbol{n}}$ the monomial obtained by replacing by $\beta(a)$'s by $\tilde{\beta}(a)$'s in $\boldsymbol{n}$. Furthermore we have

$$\tilde{\delta}(\tilde{v}_0\boldsymbol{n})=3\tilde{\boldsymbol{n}}$$

using the equality $\tilde{\delta}(xi^*(y))=\tilde{\delta}(x)y$. By applying these formulas and the result for $S(V)\times E_6$ to the exact sequence above we can get $K_\Gamma^*(S(2V)\times E_6)=P_1\otimes\Lambda_1/(\tilde{\sigma}\otimes\tilde{v}_1)$. Similarly we see that the remaining stages of induction can be done in turn as in the computation of $K^*(L^n(3))$.

From this result and (1.11) (i) we infer that the last isomorphism is given by using the canonical action of $K_\Gamma^*(S((n+1)V))$ on $K_\Gamma^*(S((n+1)V)\times E_6)$ induced by the external tensor product, and the proof is completed.

Proof of Theorem 2.1. According to (1.2) (i) where $X=E_6$ and $n=7$ we have an exact sequence

$$\cdots\to \tilde{K}_\Gamma^*(\Sigma^{7V}\wedge E_{6+})\xrightarrow{j^*} K^*(PE_6)\xrightarrow{i^*} K_\Gamma^*(S(7V)\times E_6)\xrightarrow{\delta}\cdots.$$

Here we have $j^*(\tau_{7V}\wedge 1)=27\sigma$ by (1.3). But ρ_1 induces a bundle isomorphism $E_6\times_\Gamma 27V\cong PE_6\times \boldsymbol{C}^{27}$ in a canonical way because $\rho_1(\gamma)$ is the 27×27 scalar matrix with entries $\omega=\exp(\frac{2\pi i}{3})$ where γ is the generator of Γ. So $27\sigma=0$ which implies $j^*=0$. Therefore the above exact sequence becomes the short exact sequence

$$0\to K^*(PE_6)\xrightarrow{i^*} K_\Gamma^*(S(7V)\times E_6)\xrightarrow{\delta} K^*(PE_6)\to 0. \tag{2.3}$$

where δ also denotes the composition of the δ as above with the inverse of the Thom isomorphism.

Consider the images of the elements given in the beginning of this section by i^*. Then by an inspection of definition we have

(2.4) $\quad i^*(\sigma)=\tilde{\sigma},\ i^*(\beta(\rho))=\tilde{\beta}(\rho),\ i^*(\beta(\lambda^3\rho_1))=\tilde{\beta}(\lambda^3\rho_1),$

$i^*(\beta(\rho_1+\rho_1^*))=\tilde{\beta}(\rho_1)+(\tilde{\sigma}+1)\tilde{\beta}(\rho_1^*),\ i^*(\beta(13\rho_1+\lambda^2\rho_1))$

$=13\tilde{\beta}(\rho_1)+(\tilde{\sigma}+1)\tilde{\beta}(\lambda^2\rho_1),\ i^*(\beta(13\rho_1^*+\lambda^2\rho_1^*))=13\tilde{\beta}(\rho_1^*)+(\tilde{\sigma}+1)^2\tilde{\beta}(\lambda^2\rho_1^*)$

and $\quad i^*(\beta(\rho_1,\Gamma))=(\tilde{\sigma}^2+3\tilde{\sigma}+3)\tilde{\beta}(\rho_1)-\tilde{v}_6.$

By these formulas and Lemma 2.2 when $n=6$ we see easily that the right-hand side R of the equality of Theorem 2.1 becomes a subalgebra of $K^*(PE_6)$, since i^* is injective. Moreover by definition it follows that

$$\delta(\tilde{\beta}(\rho_1))=1 \quad \text{and} \quad \delta(\tilde{v}_6)=\tilde{\sigma}^2+3\tilde{\sigma}+3. \tag{2.5}$$

Using (2.4), (2.5) together with the equality $\delta(xi^*(y))=\delta(x)y$ we can verify easily that R fills $K^*(PE_6)$, because of the surjectivity of δ. This completes the proof of Theorem 2.1.

3. The real K-group of PE_6

In this section and the following we study the real K-group of PE_6. To begin with we recall the convention done in Section 1. The representations ρ and $\lambda^3\rho_1$ of E_6 are indeed real and are trivial on the center of E_6. So we view these as real representations of PE_6 and for these the same notation is used. Furthermore the complex K-theory is regarded as a $\boldsymbol{Z}/8$-graded cohomology theory with the coefficient ring $K^*(+)=\boldsymbol{Z}[\mu]/(\mu^4-1)$. Now we set

$$\bar{\sigma}_i=r(\mu^i\sigma) \quad \text{for} \quad 0\leq i\leq 3.$$

Then we have

Theorem 3.1. *There exist elements* λ, $\bar{\lambda}_1\in\widetilde{KO}^0(PE_6)$ *such that* $c(\lambda)=\mu^3\beta(13\rho_1+\lambda^2\rho_1)\beta(13\rho_1^*+\lambda^2\rho_1^*)$, $c(\bar{\lambda}_1)=\mu^3\beta(\rho_1,\Gamma)\beta(\rho_1+\rho_1^*)$, *and as a* $KO^*(+)$*-module*

$$KO^*(PE_6)=P\otimes F\oplus r(T).$$

Here

$$P=\boldsymbol{Z}/27[\bar{\sigma}_0,\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3]/I$$

where I *denotes the ideal of* $\boldsymbol{Z}/27[\bar{\sigma}_0,\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3]$ *generated by*

$$\bar{\sigma}_i\bar{\sigma}_j-((-1)^{i+j}+(-1)^{i+1}+(-1)^{j+1}-2)\bar{\sigma}_{i+j}-((-1)^{i+j}+(-1)^{i+1}+(-1)^j-1)r(\mu^{i+j})),$$

F *denotes the subalgebra of* $KO^*(PE_6)$ *generated by* $\beta(\rho)$, $\beta(\lambda^3\rho_1)$, λ, $\bar{\lambda}_1$, *which is a free* $KO^*(+)$*-module, and* T *the submodule in* $K^*(PE_6)$ *generated by the monomials*

$$\boldsymbol{n}\beta(\rho_1,\Gamma),\ \boldsymbol{n}\beta(\rho_1+\rho_1^*),\ \boldsymbol{n}\beta(13\rho_1+\lambda^2\rho_1^*),\ \boldsymbol{n}\beta(\rho_1,\Gamma)\beta(13\rho_1+\lambda^2\rho_1),$$
$$\boldsymbol{n}\beta(\rho_1,\Gamma)\beta(\rho_1+\rho_1^*)\beta(13\rho_1+\lambda^2\rho_1),\ \beta(\rho_1,\Gamma)\beta(13\rho_1+\lambda^2\rho_1)\beta(13\rho_1^*+\lambda^2\rho_1^*)$$

where $\boldsymbol{n}$ *is a monomial in* σ, $\beta(\rho)$, $\beta(\lambda^3\rho_1)$ *with coefficients in* $K^*(+)$. *Further,* $\lambda^2=\bar{\lambda}_1^2=\bar{\sigma}_i\bar{\lambda}_1=0$, $\beta(\rho)^2=\eta_1(\beta(\lambda^3\rho_1)+\beta(\rho))$ *and* $\beta(\lambda^3\rho_1)^2=\eta_1\beta(\lambda^3\rho_1)$.

REMARK. All the other relations can be obtained from the relations in $K^*(PE_6)$, $K^*(L^6(3))$ and $KO^*(L^6(3))$ by using the equalities $r(x)r(y)=r(xy)+r(xy^*)$, $r(x^*)=r(x)$

and (2.4). The following is a sample calculation. For $x \in T$

$$r(x)r(\beta(\rho_1,\Gamma))=(\bar{\sigma}_0+3)r(x\beta(\rho_1+\rho_1^*)),\ r(x^*\beta(\rho_1+\rho_1^*))=r((\sigma+1)^2x\beta(\rho_1+\rho_1^*)),$$

$$r(x\sigma\beta(\rho_1,\Gamma))=0,\ \bar{\sigma}_0 r(\mu^i\beta(\rho,\Gamma))=0,\ (\bar{\sigma}_0+3)r(\mu^i\beta(\rho_1+\rho_1^*))=0,\ i=1,\ 3,$$

$$(\bar{\sigma}_0+3)r(\beta(\rho_1+\rho_1^*))=2r(\beta(\rho_1,\Gamma)),\ (\bar{\sigma}_0+3)r(\mu^2\beta(\rho_1+\rho_1^*))=\eta_4 r(\beta(\rho_1,\Gamma)).$$

We are now going to prove the theorem. The proof is done parallel to that of the complex case. However we have a difference between the complex and real cases in the real version of (2.3) for reasons of the real Thom isomorphism theorem.

Apply (1.2) (i) to $X=E_6$, $n=7$, then we have an exact sequence

$$\cdots \to \widetilde{KO}_\Gamma^*(\Sigma^{7V}\wedge E_{6+}) \xrightarrow{j^*} KO^*(PE_6) \xrightarrow{i^*} KO_\Gamma^*(S(7V)\times E_6) \xrightarrow{\delta} \cdots.$$

Combining this with the Thom isomorphism (1.8) such that $\widetilde{KO}_\Gamma^{k+4}(\Sigma^V\wedge E_{6+}) \cong \widetilde{KO}_\Gamma^k(\Sigma^{7V}\wedge E_{6+})$ gives the following.

Lemma 3.2. *We have a short exact sequence*

$$0 \to KO^*(PE_6) \xrightarrow{i^*} KO_\Gamma^*(S(7V)\times E_6) \xrightarrow{\bar{\delta}} \widetilde{KO}_\Gamma^*(\Sigma^V\wedge E_{6+}) \to 0$$

where $\bar{\delta}$ is the composite of δ with the inverse of the Thom isomorphism, so that $\bar{\delta}$ is of degree 5 *and satisfies $\bar{\delta}(xi^*(y))=\bar{\delta}(x)y$.*

Proof. The Thom isomorphism is given by multiplication by τ_{6W+4}. So any element of $\widetilde{KO}_\Gamma^*(\Sigma^{7V}\wedge E_{6+})$ may be written as $x=\tau_{6W+4}\wedge x'$ for some $x' \in \widetilde{KO}_\Gamma^*(\Sigma^{V+4}\wedge E_{6+})$. Now by (1.8) the restriction of τ_{6W+4} to $\widetilde{KO}_\Gamma(\Sigma^4)$ is $9r(\mu^2V-\mu^2)$ and by Theorem 2.1 $27\sigma=0$. Therefore we see that $3j^*(x)=0$.

Consider $c(x)\in\widetilde{K}_\Gamma^*(\Sigma^{7V}\wedge E_{6+})$. Then $c(x)$ may be written in the form $c(x)=\tau_{7V}\wedge y$ for some $y\in K_\Gamma^*(E_6)=K^*(PE_6)$. So the restriction of $c(x)$ to $K^*(PE_6)$ is $27\sigma y$ which is, of course, zero. This shows that $c(j^*(x))=0$, so that applying r to this equality yields $2j^*(x)=0$. By comparing these two results we see that $j^*=0$ whence the assertion follows.

We are in need of $KO_\Gamma^*(S(7V)\times E_6)$, which is given inductively as in the complex case by changing 7 for $0,1,\cdots,6$ in turn.

In order to describe the result we give some elements of $KO_\Gamma^*(S(nV)\times E_6)$ for $1\leq n\leq 7$. Similarly to the complex case we write $\tilde{a}$ for $\pi_1^*(a)$ (resp. $\pi_2^*(a)$) where $a\in KO_\Gamma^*(S(nV))=KO^*(L^{n-1}(3))$ (resp. $a\in KO_\Gamma^*(E_6)=KO^*(PE_6)$). Moreover, since $KO_\Gamma^*(S(7V)\times E_6)=KO^*(S(7V)\times_\Gamma E_6)$, by [12], Proposition 4.7 we have elements $\tilde{\lambda}_1,\tilde{\lambda}_2\in KO_\Gamma(S(7V)\times E_6)$ such that $c(\tilde{\lambda}_1)=\mu^3\tilde{\beta}(\rho_1)\tilde{\beta}(\rho_1^*)$, $c(\tilde{\lambda}_2)=\mu^3\tilde{\beta}(\lambda^2\rho_1)\tilde{\beta}(\lambda^2\rho_1^*)$, which satisfy $\tilde{\lambda}_1^2=\tilde{\lambda}_2^2=0$. For the restriction of these elements to $KO_\Gamma^*(S(nV)\times E_6)$ for

$1 \leq n \leq 6$ we use the same notation. We denote by $\tilde{F}$ the subalgebra of $KO^*_\Gamma(S(nV)\times E_6)$ generated by $\tilde{\beta}(\rho)$, $\tilde{\beta}(\lambda^3\rho_1)$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and by $\tilde{T}$ the submodule of $K^*_\Gamma(S(nV)\times E_6)$ generated by the monomials $\boldsymbol{n}\tilde{\beta}(\rho_1)$, $\boldsymbol{n}\tilde{\beta}(\lambda^2\rho_1)$, $\boldsymbol{n}\tilde{\beta}(\rho_1)\tilde{\beta}(\lambda^2\rho_1)$, $\boldsymbol{n}\tilde{\beta}(\rho_1)\tilde{\beta}(\lambda^2\rho_1^*)$, $\boldsymbol{n}\tilde{\beta}(\rho_1)\tilde{\beta}(\lambda^2\rho_1)\tilde{\beta}(\lambda^2\rho_1^*)$, $\boldsymbol{n}\beta(\rho_1)\tilde{\beta}(\rho_1^*)\tilde{\beta}(\lambda^2\rho_1)$ where $\boldsymbol{n}$ is a monomial in $\tilde{\beta}(\rho)$, $\tilde{\beta}(\lambda^3\rho_1)$.

Using the canonical action of $KO^*(L^n(3))=KO^*_\Gamma(S((n+1)V)$ on $KO^*_\Gamma(S((n+1)V)\times E_6)$ induced by the external product we obtain the following isomorphism.

Lemma 3.3. *With the notation as above*

$$KO^*_\Gamma(S((n+1)V)\times E_6)\cong KO^*(L^n(3))\otimes_{KO^*(+)}\tilde{F}\oplus r(K^*(L^n(3))\otimes\tilde{T})$$

for $0\leq n\leq 6$ *as a* $KO^*(+)$*-module and* $\tilde{F}$ *is a free* $KO^*(+)$*-module.*

Proof. The proof is quite similar to that of Lemma 2.2 and so proceeds inductively on n. Consider the exact sequence (1.3) when $X=E_6$

$$\cdots\to KO^*(S^1\times E_6)\xrightarrow{J} KO^*_\Gamma(S((n+1)V)\times E_6)\xrightarrow{i^*} KO^*_\Gamma(S(nV)\times E_6)\xrightarrow{\tilde{\delta}}\cdots$$

provided with the equality $\tilde{\delta}(xi^*(y))=\tilde{\delta}(x)y$. Viewing $S(V)$ as a Γ-invariant subspace of E_6 as in the proof of Lemma 2.2 yields $S(V)\times_\Gamma E_6\approx S^1\times E_6$ so that $KO^*_\Gamma(S(V)\times E_6)\cong KO^*(S^1)\otimes_{KO^*(+)}KO^*(E_6)$. So we may write $KO^*_\Gamma(S(V)\times E_6)=KO^*(E_6)\oplus KO^*(E_6)\cdot\iota_0$ where ι_0 is the generator of $\widetilde{KO}^1(S^1)$ as in Section 1. Hence by (1.6) and the argument as in the proof of Lemma 2.2 we get Lemma 3.3 when $n=0$. This is, of course, the 1st stage of our induction.

Next consider the maps of the above sequence. Then clearly $i^*(x)=x$ for $x\in\tilde{F}$, $x\in\tilde{T}$ and $i^*(\tilde{\bar{\sigma}}_{n,i})=\tilde{\bar{\sigma}}_{n-1,i}$. By (1.10) we have $\tilde{\delta}(\tilde{\bar{v}}_{n-1})=3$, $J(\iota_0)=\tilde{\bar{v}}_n$ and $J(r(\mu^{i+n}))=r(\mu^i(-\tilde{\sigma}^n))$. Moreover we note that the degree of v_n is considered to be -1, so that $c(\bar{v})=\mu^{3-n}v_n$. Using these formulas together with the equality $\tilde{\delta}(xi^*(y))=\tilde{\delta}(x)y$, (1.6) and (1.11) (ii) we can go on with our induction. Thus we get the lemma.

We are now ready to prove the theorem.

4. Proof of Theorem 3.1

We continue to prove the theorem. We identify the isomorphism of Lemma 3.3 below and consider the images of the elements of $KO^*(PE_6)$ described in Theorem 3.1 by i^* of Lemma 3.2. It is immediate by definition that $i^*(\bar{\sigma}_i)=\bar{\sigma}_{6,i}$, $i^*(\beta(\rho))=\tilde{\beta}(\rho)$, $i^*(\beta(\lambda^3\rho_1))=\tilde{\beta}(\lambda^3\rho_1)$. And by (2.4) $i^*(r(\mu^i\beta(\rho_1+\rho_1^*)))=r(\mu^i\tilde{\beta}(\rho_1)+(\sigma+1)\mu^i\tilde{\beta}(\rho_1^*))$, $i^*(r(\mu^i\beta(13\rho_1+\lambda^2\rho_1)))=r(13\mu^i\tilde{\beta}(\rho_1)+(\sigma+1)\mu^i\tilde{\beta}(\lambda^2\rho_1))$, $i^*(r(\mu^i\beta(\rho_1,\Gamma)))=r((\sigma^2+3\sigma+3)\mu^i\tilde{\beta}(\rho_1)-\mu^i v_6)$. Furthermore we may assume that

$$(4.1)\qquad i^*(\lambda)=13^2\tilde{\lambda}_1+\tilde{\lambda}_2+13r((\sigma_6+1)^2\mu^3\tilde{\beta}(\rho_1)\tilde{\beta}(\lambda^2\rho_1^*)).$$

Because, by using the Bott exact sequence we see that the difference between the elements on the both sides can be written as the form $\eta_1 a$ where $a\in KO_\Gamma^{-7}(S(7V)\times E_6)$ which satisfies $a^2=0$ by [4], Example (6.6) and hence if necessary it suffices to replace either $\tilde{\lambda}_1$ or $\tilde{\lambda}_2$ by $\tilde{\lambda}_1+\eta_1 a$ or $\tilde{\lambda}_2+\eta_1 a$. (In fact these a's above must be zero by the same reason as mentioned in Remark 2 for (1.6).) Similarly by definition we can write as $i^*(\bar{\lambda}_1)=(\bar{\sigma}_{6,0}+3)\tilde{\lambda}_1-\bar{v}_6 r(\mu^2\tilde{\beta}(\rho_1))+\eta_1 a$ for some $a\in KO_\Gamma^{-7}(S(7V)\times E_6)$. But the odd dimensional generators of the first direct summand of $KO_\Gamma^*(S(7V)\times E_6)$ in Lemma 3.3 is only $\tilde{\beta}(\rho)$, $\tilde{\beta}(\lambda^3\rho_1)$, $\bar{v}_6$ and so we see that the component of a which belongs to this direct summand is divisible by η_1^2. Therefore $\eta_1 a$ must be zero since $\eta_1 r(x)=0$, so that we have

$$(4.2)\qquad i^*(\bar{\lambda}_1)=(\bar{\sigma}_{6,0}+3)\tilde{\lambda}_1-\bar{v}_6 r(\mu^2\tilde{\beta}(\rho_1)).$$

Since i^* is injective by Lemma 3.2, it follows from this and the relation of (1.11) (ii) that $\bar{\sigma}_i\bar{\lambda}_1=0$.

Because of the injectivity of i^* of (2.3), we get by (2.4)

$$\begin{aligned}&\beta(\rho_1,\Gamma)+\beta(\rho_1,\Gamma)^*=(\sigma^2+3\sigma+3)\beta(\rho_1+\rho_1^*),\ \beta(\rho_1+\rho_1^*)^*\\&=(\sigma+1)^2\beta(\rho_1+\rho_1^*),\ \beta(\lambda^2\rho_1+\lambda^2\rho_1^*)=(\sigma+1)^2(\beta(13\rho_1+\lambda^2\rho_1)\\&-13\beta(\rho_1+\rho_1^*))+\beta(13\rho_1^*+\lambda^2\rho_1^*).\end{aligned}$$

(The last element can be defined analogously to $\beta(\rho_1+\rho_1^*)$.)

Denote by R the algebra over $KO^*(+)$ on the right-hand side of the equality of Theorem 3.1. In virtue of the formulas above and (1.11), Lemmas 3.2, 3.3 and Theorem 2.1 we can then verify that R is a subalgebra of $KO^*(PE_6)$. From now on we prove that $KO^*(PE_6)$ is filled with R. This is sufficient to show Theorem 3.1.

Observe the following exact sequence of (1.2) (i)

$$\cdots\to\widetilde{KO}_\Gamma^*(\Sigma^V\wedge E_{6+})\xrightarrow{j_1^*}KO^*(PE_6)\xrightarrow{i_1^*}KO_\Gamma^*(S(V)\times E_6)\xrightarrow{\delta_1}\cdots.$$

When we regard $S(V)$ as the circle group which is a factor of $Spin(10)\cdot S^1\subset E_6$ as before we have $S(V)\times_\Gamma E_6\approx S^1\times E_6$, so that $KO_\Gamma^*(S(V)\times E_6)\cong KO^*(S^1\times E_6)$, and so this sequence can be written as

$$(4.3)\qquad \cdots\to\widetilde{KO}_\Gamma^*(\Sigma^V\wedge E_{6+})\xrightarrow{j_1^*}KO^*(PE_6)\xrightarrow{i_1^*}KO^*(S^1\times E_6)\xrightarrow{\delta_1}\cdots.$$

Moreover we can write as

$$KO^*(S^1\times E_6)=KO^*(E_6)\oplus KO^*(E_6)\cdot\iota_0$$

where ι_0 denotes the generator of $KO^{-7}(S^1)\cong \boldsymbol{Z}$.

To investigate $\operatorname{Im} i_1^*$ under the identification above we consider $i_2^*: h_\Gamma^*(S(7V)\times E_6)$

$\to h^*_\Gamma(S(V)\times E_6)$ for $h=KO$, K where i_2 denotes an inclusion of $S(V)\times E_6$ into $S(7V)\times E_6$. From the arguments as in the proofs of Lemmas 2.2 and 3.3 it follows that $i_2^*(\tilde{\beta}(a))=\beta(a)$ for the fundamental representations a's of E_6 so that $i_2^*(\tilde{\lambda}_k)=\lambda_k$ $(k=1, 2)$, and $i_2^*(\sigma_6)=i_2^*(v_6)=0$ so that $i_2^*(\bar{\sigma}_{6,i})=i_2^*(\bar{v}_6)=0$. Therefore we have $i_2^*(\tilde{\beta}(\rho_1))=\beta(\rho_1)$. For the same reasons we get $i_2^*(\tilde{\beta}(\lambda^2\rho_1))=\beta(\lambda^2\rho_1)$. As to the other generators of $KO^*_\Gamma(S(7V)\times E_6)$ it follows immediately by definition that $i_2^*(\sigma_6)=i_2^*(v_6)=0$, $i_2^*(\bar{\sigma}_{6,i})=i_2^*(\bar{v}_6)=0$. These formulas, Lemma 3.3 and (1.6) show that

$$i_2^*(KO^*_\Gamma(S(7V)\times E_6))=KO^*(E_6)$$

and so because of $i_1^*=i_2^*i^*$ where i^* is as in Lemma 3.2 we have

$$i_1^*(KO^*(PE_6))\subset KO^*(E_6)$$

in (4.3). More precisely we have

Lemma 4.4. $$i_1^*(KO^*(PE_6))=i_1^*(R).$$

Proof. We use the same notation as in (4.3) below for the maps j_1^*, i_1^*, δ_1 of the same kind in the complex version of (4.3). Then by (2.4) we get

(4.5) $i_1^*(\beta(\rho_1,\Gamma))=3\beta(\rho_1)$, $i_1^*(\beta(\rho_1+\rho_1^*))=\beta(\rho_1)+\beta(\rho_1^*)$ and $i_1^*(\beta(13\rho_1+\lambda^2\rho_1))$
$=13\beta(\rho_1)+\beta(\lambda^2\rho_1)$.

For any $x\in\widetilde{KO}^*(PE_6)$ we see by Theorem 2.1 that $c(x)$ can be written as a polynomial in

$$\sigma,\ \beta(\rho),\ \beta(\lambda^3\rho_1),\ \beta(\rho_1+\rho_1^*),\ \beta(13\rho+\lambda^2\rho_1),\ \beta(13\rho_1^*+\lambda^2\rho_1^*),\ \beta(\rho_1,\Gamma)$$

with coefficients in $Z[\mu]/(\mu^4-1)$. Therefore using (4.5) it follows that $i_1^*(c(x))$ is written as a polynomial in

$$\beta(\rho),\ \beta(\lambda^3\rho_1),\ \beta(\rho_1)+\beta(\rho_1^*),\ \beta(\rho_1)+\beta(\lambda^2\rho_1),\ \beta(\rho_1^*)+\beta(\lambda^2\rho_1^*),\ 3\beta(\rho_1)$$

with coefficients in $Z[\mu]/(\mu^4-1)$.

On the other hand it follows from (1.5), (1.6) that $c(i_1^*(x))$ can be written as a sum of a polynomial in

$$\beta(\rho),\ \beta(\lambda^3\rho_1),\ \mu^3\beta(\rho_1)\beta(\rho_1^*),\ \mu^3\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),$$
$$2\mu^2\beta(\rho),\ 2\mu^2\beta(\lambda^3\rho_1),\ 2\mu\beta(\rho_1)\beta(\rho_1^*),\ 2\mu\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)$$

and the elements in the form

$n\mu^i(\beta(\rho_1)+(-1)^i\beta(\rho_1^*))$, $n\mu^i(\beta(\lambda^2\rho_1)+(-1)^i\beta(\lambda^2\rho_1^*))$,
$n\mu^i(\beta(\rho_1)\beta(\lambda^2\rho_1^*)+(-1)^i\beta(\rho_1^*)\beta(\lambda^2\rho_1))$,

$$\boldsymbol{n}\mu^i\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)(\beta(\rho_1)+(-1)^{i+1}\beta(\rho_1^*)),\ \boldsymbol{n}\mu^i\beta(\rho_1)\beta(\rho_1^*)(\beta(\lambda^2\rho_1)+(-1)^{i+1}\beta(\lambda^2\rho_1^*))$$

where $\boldsymbol{n}$ is a monomial in $\beta(\rho)$, $\beta(\lambda^3\rho_1)$ with coefficients in $\boldsymbol{Z}$. By combining these two facts we see that $i_1^*(c(x))$ must be written as a sum of a polynomial in

$$\beta(\rho),\ \beta(\lambda^3\rho),\ 2\mu^2\beta(\rho),\ 2\mu^2\beta(\lambda^3\rho_1),\ 3\mu^3\beta(\rho_1)\beta(\rho_1^*),\ 3\mu^3\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),$$
$$3\mu^2\beta(\rho_1)\beta(\rho_1^*)\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),\ 6\mu\beta(\rho_1)\beta(\rho_1^*),\ 6\mu\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),$$
$$6\mu\beta(\rho_1)\beta(\rho_1^*)\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),\ \mu^3(\beta(\rho_1)\beta(\rho_1^*)+\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)+\beta(\rho_1)\beta(\lambda^2\rho_1^*)$$
$$-\beta(\rho_1^*)\beta(\lambda^2\rho_1)),$$
$$2\mu(\beta(\rho_1)\beta(\rho_1^*)+\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)+\beta(\rho_1)\beta(\lambda^2\rho_1^*)-\beta(\rho_1^*)\beta(\lambda^2\rho_1))$$

and the elements in the form

$$\boldsymbol{n}\mu^{2i}(\beta(\rho_1)+\beta(\rho_1^*)),\ \boldsymbol{n}\mu^{2i}(\beta(\lambda^2\rho_1)+\beta(\lambda^2\rho_1^*)),\ 3\boldsymbol{n}\mu^{2i+1}(\beta(\rho_1)-\beta(\rho_1^*)),$$
$$3\boldsymbol{n}\mu^{2i+1}(\beta(\lambda^2\rho_1)-\beta(\lambda^2\rho_1^*)),\ 9\boldsymbol{n}\mu^{2i}(\beta(\rho_1)\beta(\lambda^2\rho_1^*)+\beta(\rho_1^*)\beta(\lambda^2\rho_1)),$$
$$3\boldsymbol{n}\mu^3(\beta(\rho_1)\beta(\lambda^2\rho_1^*)-\beta(\rho_1^*)\beta(\lambda^2\rho_1)),\ 6\boldsymbol{n}\mu(\beta(\rho_1)\beta(\lambda^2\rho_1^*)-\beta(\rho_1^*)\beta(\lambda^2\rho_1)),$$
$$3\boldsymbol{n}\mu^{2i+1}\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)(\beta(\rho_1)+\beta(\rho_1^*)),\ 9\boldsymbol{n}\mu^{2i}\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)(\beta(\rho_1)-\beta(\rho_1^*)),$$
$$3\boldsymbol{n}\mu^{2i+1}\beta(\rho_1)\beta(\rho_1^*)(\beta(\lambda^2\rho_1)+\beta(\lambda^2\rho_1^*)),\ 9\boldsymbol{n}\mu^{2i}\beta(\rho_1)\beta(\rho_1^*)(\beta(\lambda^2\rho_1)-\beta(\lambda^2\rho_1^*))$$

where $\boldsymbol{n}$ is as above.

From (4.1), (4.2) and (4.5) we get

$$ci_1^*(\beta(\rho))=\beta(\rho),\ ci_1^*(\eta_4\beta(\rho))=2\mu^2\beta(\rho),\ ci_1^*(\beta(\lambda^3\rho_1))=\beta(\lambda^3\rho_1),$$
$$ci_1^*(\eta_4\beta(\lambda^3\rho_1))=2\mu^2\beta(\lambda^3\rho_1),\ ci_1^*(\bar{\lambda}_1)=3\mu^3\beta(\rho_1)\beta(\rho_1^*),$$
$$ci_1^*(\eta_4\bar{\lambda}_1)=6\mu\beta(\rho_1)\beta(\rho_1^*),\ ci_1^*(r(\mu^i\beta(\rho_1,\Gamma)))=3\mu^i(\beta(\rho_1)+(-1)^i\beta(\rho_1^*)),$$
$$ci_1^*(r(\mu^{2i}(\beta(\rho_1,\Gamma)-\beta(\rho_1+\rho_1^*)))=\mu^{2i}(\beta(\rho_1)+\beta(\rho_1^*)),$$
$$ci_1^*(r(\mu^i(3\beta(13\rho_1+\lambda^2\rho_1)-13\beta(\rho_1,\Gamma))))=3\mu^i(\beta(\lambda^2\rho_1)+(-1)^i\beta(\lambda^2\rho_1^*)),$$
$$ci_1^*(r(\mu^{2i}(\beta(13\rho_1+\lambda^2\rho_1)-13\beta(\rho_1,\Gamma)+13\beta(\rho_1+\rho_1^*))))=\mu^{2i}(\beta(\lambda^2\rho_1)+\beta(\lambda^2\rho_1^*))$$

and furthermore setting

$$a=r(\mu^3\beta(\rho_1,\Gamma)\beta(13\rho_1+\rho_1^*))-13\bar{\lambda}_1,\ b=3\lambda-299\bar{\lambda}_1-13a,\ c=\lambda-121\bar{\lambda}_1-4a$$
$$d=r(\mu^{2i}(3\beta(\rho_1+\rho_1^*)-13\beta(\rho_1,\Gamma))(3\beta(13\rho_1+\lambda^2\rho_1)-13\beta(\rho_1,\Gamma)))$$

we get

$$ci_1^*(\mathrm{a})=3\mu^3(\beta(\rho_1)\beta(\lambda^2\rho_1^*)-\beta(\rho_1^*)\beta(\lambda^2\rho_1)),\ ci_1^*(b)=3\mu^3\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),$$

$$ci_1^*(c)=\mu^3(\beta(\rho_1)\beta(\rho_1^*)+\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)+\beta(\rho_1)\beta(\lambda^2\rho_1^*)-\beta(\rho_1^*)\beta(\lambda^2\rho_1)),$$

$$ci_1^*(ar(\mu^{2i}(\beta(\rho_1,\Gamma)-\beta(\rho_1+\rho_1^*)))=3\mu^{2i+3}\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*)(\beta(\rho_1)+\beta(\rho_1^*)),$$

$$ci_1^*(ac)=3\mu^2\beta(\rho_1)\beta(\rho_1^*)\beta(\lambda^2\rho_1)\beta(\lambda^2\rho_1^*),\ ci_1^*(d)=9\mu^{2i}(\beta(\rho_1)\beta(\lambda^2\rho_1^*)+\beta(\rho_1^*)\beta(\lambda^2\rho_1)).$$

By comparing these formulas with the above we obtain

(4.6) For any $x\in KO^*(PE_6)$ there exists an element $y\in R$ such that $ci_1^*(x)=ci_1^*(y)$.

By (4.6) and (1.6) we have $i_1^*(x-y)\in F\cdot\eta_1$ using the symbols of (4.6) where F is as in (1.6). But $\eta_1\lambda_1=i_1^*(\eta_1\bar{\lambda}_1)$, $\eta_1\lambda_2=i_1^*(\eta_1\lambda+\eta_1\bar{\lambda}_1)$ by (4.1), (4.2) and clearly $i_1^*(\beta(\rho))=\beta(\rho)$, $i_1^*(\beta(\lambda^3\rho_1))=\beta(\lambda^3\rho_1)$. So we see that for any $x\in KO^*(PE_6)$ there exist elements y, $z\in R$ such that $i_1^*(x)=i_2^*(y+\eta_1 z)$. This completes the proof of Lemma 4.4.

Finally we consider the image of of j_1^* of (4.3). Then we have

Lemma 4.7. $$j_1^*(\widetilde{KO}_\Gamma^*(\Sigma^V\wedge E_{6+}))\subset R.$$

Proof. Consider the composition of j_1^* with $\bar{\delta}$ of Lemma 3.2. Then $\mathrm{Im} j_1^*\bar{\delta}=\mathrm{Im} j_1^*$ because of the surjectivity of $\bar{\delta}$. So it suffices to check that

$$j_1^*\bar{\delta}(KO_\Gamma^*(S(7V)\times E_6))\subset R.$$

According to Lemma 3.3, $KO_\Gamma^*(S(7V)\times E_6)=KO^*(L^6(3))\otimes_{KO^*(+)}\tilde{F}\oplus r(K^*(L^6(3))\otimes\tilde{T})$. First we consider the image of the latter direct summand. Observe $\bar{\delta}(K^*(L^6(3))\otimes\tilde{T})$ where $\bar{\delta}$ is the coboundary homomorphism of the same kind in the complex case. From (2.4) and the equalities $c(\tau_{6W+4})=\tau_{6V}\mu^2$, $\tau_{7V}=\tau_{6V}\wedge\tau_V$ it follows that $\bar{\delta}(\tilde{\beta}(\rho_1))=-\tau_V\mu^2$, $\bar{\delta}(v_6)=(\sigma^2+3\sigma+3)\tau_V\mu^2$. Together with this, using the formulas in the preceding of (2.4) and the equality $\bar{\delta}(xi^*(y))=\bar{\delta}(x)y$ where i^* is as in (2.3) we can get $\bar{\delta}(K^*(L^6(3))\otimes\tilde{T})$ and so it can be easily verified that $j_1^*\bar{\delta}(r(K^*(L^6(3))\otimes\tilde{T}))\subset R$ by using $c(\tau_{6W+4})=\tau_{6V}\mu^2$.

We now observe the image of another direct summand. Clearly $j_1^*\bar{\delta}(x)=0$ for $x=\bar{\sigma}_{6,i}$, $\tilde{\beta}(\rho)$ and $\tilde{\beta}(\lambda^3\rho_1)$. As to the image of $\bar{v}_6\in KO^{-3}(L^6(3))=KO_\Gamma^{-3}(S(7V))$ by $j_1^*\bar{\delta}$ we see by definition that $j_1^*\bar{\delta}(\bar{v}_6)\in KO_\Gamma^{-6}(+)=\boldsymbol{Z}\cdot W\mu^3$ and $cj_1^*\bar{\delta}(\bar{v}_6)=0$ using $c(\bar{v}_6)=\mu v_6$. But $c(W\mu^3)\neq 0$, which shows that

$$j_1^*\bar{\delta}(\bar{v}_6)=0.$$

By definiton we can write as $c(\tilde{\lambda}_1)=-i^*(\mu(\sigma+1)^2\beta(\rho_1+\rho_1^*))\tilde{\beta}(\rho_1)$ where i^* is as in (2.3). Therefore $cj_1^*\bar{\delta}(\tilde{\lambda}_1)=-(\sigma^2+2\sigma)\mu\beta(\rho_1+\rho_1^*)$, so that $cj_1^*\bar{\delta}(\tilde{\lambda}_1)=cr(\mu\beta(\rho_1+\rho_1^*))$. Now $i_1^*r(\mu\beta(\rho_1+\rho_1^*))=0$. So we can construct an element $a_1\in\widetilde{KO}_\Gamma^{-3}(\Sigma^V\wedge E_{6+})$ such that $j_1^*(a_1)=r(\mu\beta(\rho_1+\rho_1^*))$ and $c(a_1)=-\tau_V(\sigma+1)^2\mu\beta(\rho_1+\rho_1^*)$. Then, from the surjectivity of $\bar{\delta}$ and the uniqueness of $\tilde{\lambda}_1$ it follows that $\bar{\delta}(\tilde{\lambda}_1)=a_1$, so that

$$j_1^*\bar{\delta}(\tilde{\lambda}_1)=r(\mu\beta(\rho_1+\rho_1^*)).$$

Similarly we obtain

$$j_1^*\delta(\tilde{\lambda}_2)=r(\mu(\sigma+1)^2(\beta(13\rho_1+\lambda^2\rho_1)-13\beta(\rho_1+\rho_1^*))+\mu\beta(13\rho_1^*+\lambda^2\rho_1^*)).$$

Using these three formulas we can easily prove that $j_1^*\delta(KO^*(L^6(3))\otimes_{KO^*(+)}\tilde{F})\subset R$. For example, since $\tilde{\lambda}_1 r((\sigma_6+1)\mu^3\tilde{\beta}(\rho_1)\tilde{\beta}(\lambda^2\rho_1^*))=r(c(\tilde{\lambda}_1)(\sigma_6+1)\mu^3\tilde{\beta}(\rho_1)\tilde{\beta}(\lambda^2\rho_1^*))=0$, we have $\tilde{\lambda}_1 i^*(\lambda)=\tilde{\lambda}_1\tilde{\lambda}_2$ by (4.1). Hence $j_1^*\delta(\tilde{\lambda}_1\tilde{\lambda}_2)=\lambda r(\mu\beta(\rho_1+\rho_1^*))$. Thus the proof is completed.

From Lemmas 4.4, 4.7 and the exactness of (4.3) it follows that $KO^*(PE_6)=R$ immediately. This completes the proof of Theorem 3.1.

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