

## THE FIXED POINT SET OF $C$ ACTIONS ON A COMPACT COMPLEX SPACE

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(Received April 21, 1994)

### Introduction

Let  $X$  be a connected complete algebraic variety defined over the complex number field  $C$ . Suppose that a connected unipotent linear algebraic group  $G$  acts regularly on  $X$ . Let  $X^G$  be the set of fixed points of this action. Then in [3] Horrocks has shown that  $X^G$  is connected and that the inclusion  $X^G \hookrightarrow X$  induces an isomorphism of the algebraic fundamental groups, i.e., the profinite completions of the fundamental groups, of both spaces. In the complex analytic category, by Carrell and Sommese this result was partially generalized in [1], where they have shown that if  $X$  is a general connected complex manifold and  $G$  is a unipotent algebraic group as above which acts biholomorphically and meromorphically on  $X$  (cf. 1.2 for the definition), then the set of fixed points  $X^G$  is connected.

In this note we push this analogy a little further to one on the comparison of the fundamental groups. Namely, we show that when  $X$  is a compact complex space, the inclusion  $X^G \hookrightarrow X$  induces an isomorphism of the (topological) fundamental groups of these spaces (cf. Theorem 3.1).

### 1. Preliminaries

In this section we summarize some of the known results on the fundamental groups and the meromorphic actions of algebraic groups on a complex space.

1.1. First we consider the fundamental groups. Let  $X$  be a topological space. We denote by  $\pi_0(X)$  (resp.  $\pi_1(X)$ ) the set of connected components (resp. the fundamental group with respect to some reference point) of  $X$ . (We shall be sloppy about the base points of fundamental groups throughout the paper.) We shall use the following terminology:

DEFINITION 1.1. Let  $f: X \rightarrow Y$  be a continuous map of  $X$  into another topological space  $Y$ . Then we say that

- 1)  $f$  is *0-connected* if the induced map  $f_{0*}: \pi_0(X) \rightarrow \pi_0(Y)$  is bijective, and

2)  $f$  is  $\frac{1}{2}$ -conncted (resp. 1-connected) if it is 0-connected and the induced homomorphism of the fundamental groups of the corresponding connected components is surjective (resp. isomorphic).

In general we denote by  $\mathcal{UC}(X)$  the category of all the unramified coverings of  $X$  and continuous maps over  $X$ . Then a continuous map  $f$  as above defines a contravariant functor  $f^*: \mathcal{UC}(Y) \rightarrow \mathcal{UC}(X)$  by taking the fibered products. Let  $\tilde{Y} \rightarrow Y$  be the universal covering of  $Y$  and  $\tilde{X} := f^*(\tilde{Y}) = X \times_Y \tilde{Y}$  its pull-back to  $X$ . The follwing is then standard.

**Lemma 1.1.** 1) *The following conditions are equivalent:*

- a)  $f$  is  $\frac{1}{2}$ -connected,
- b) the natural map  $\tilde{X} \rightarrow \tilde{Y}$  is 0-connected, and
- c) the functor  $f^*: \mathcal{UC}(Y) \rightarrow \mathcal{UC}(X)$  is fully faithful.

2) *The following two conditions are also equivalent:*

- a)  $f$  is 1-connected, and
- b) the functor  $f^*: \mathcal{UC}(Y) \rightarrow \mathcal{UC}(X)$  is an equivalence of categories. □

Using this criterion, as in [3, Prop. 1.2] we can easily prove the following:

**Lemma 1.2.** *Let  $f: X \rightarrow Y$  be as above. Suppose that  $X$  and  $Y$  are unions of two subspaces  $X_1$  and  $X_2$  and  $Y_1$  and  $Y_2$  respectively such that  $f(X_i) \subseteq Y_i, i=1,2$ . Suppose further that the induced maps  $f_i: X_i \rightarrow Y_i$  are 1-connected and the induced map  $f_{12}: X_1 \cap X_2 \rightarrow Y_1 \cap Y_2$  is  $\frac{1}{2}$ -connected, then  $f$  is 1-connected. □*

Now we pass to the category of complex spaces. (In this note all the complex spaces are assumed to have countable topology.) The following is more or less well-known and can be proved by a dirct topological method. Here, we give an analytic proof using Lemma 1.1 above.

**Lemma 1.3.** *Let  $X$  and  $Y$  be complex spaces and  $f: X \rightarrow Y$  a proper morphism of complex spaces.*

- 1) *Suppose that each fiber of  $f$  is connected. Then  $f$  is  $\frac{1}{2}$ -connected.*
- 2) *If, further, each fiber of  $f$  is simply connected, then  $f$  is 1-connected.*

**Proof.** 1) We may assume that  $Y$  is connected. Let  $r_Y: \tilde{Y} \rightarrow Y$  be the universal covering of  $Y$ . Then  $\tilde{X} = f^*\tilde{Y}$  is a fiber space over  $\tilde{Y}$  with connected fibers. Hence,  $\tilde{X}$  also is connected. The assertion then follows from Lemma 1.1.

2) Let  $r_X: \tilde{X} \rightarrow X$  be any unramified covering of  $X$  and  $\tilde{f} = fr_X$  the induced map. Again by Lemma 1.1 it suffices to show that  $r_X$  descends to an unramified covering  $\tilde{Y} \rightarrow Y$  so that  $f^*(\tilde{Y}) = \tilde{X}$ . For this purpose we put an equivalence relation  $\sim$  on the space  $\tilde{X}$  as follows; for any two points  $x$  and  $x'$  of  $\tilde{X}$ ,  $x \sim x'$  if

and only if they both belong to one and the same connected component of some fiber of  $\tilde{f}$ . Let  $\tilde{Y}$  be the quotient space of  $\tilde{X}$  (as a topological space) with respect to this equivalence relation. We then claim that  $\tilde{Y}$  gives an unramified covering of  $Y$  to which  $\tilde{X}$  descends. Indeed, by construction we have a natural continuous map  $r_Y: \tilde{Y} \rightarrow Y$ . Since the map  $f$  is proper and the fibers are simply connected, for any point  $y$  of  $Y$  there exists a neighborhood  $V$  of  $y$  such that  $\tilde{X}$  is trivial over  $f^{-1}(V)$ , i.e., isomorphic to the product  $f^{-1}(V) \times \pi_1(X)$  over  $f^{-1}(V)$ . Hence, over each  $V$ ,  $r_Y$  is homomorphically to the projection  $V \times \pi_1(X) \rightarrow V$ ; this implies that  $r_Y$  is a desired unramified covering.  $\square$

**1.2.** We next recall some definitions and results on the actions of algebraic groups on complex spaces. Let  $G$  be a (complex) linear algebraic group acting biholomorphically on a complex space  $X$ . Let  $G^*$  be any algebraic compactification of  $G$ . We say that the action of  $G$  on  $X$  is *meromorphic* if the morphism of complex spaces  $\sigma: G \times X \rightarrow X$  defining the action extends to a meromorphic map  $\sigma^*: G^* \times X \rightarrow X$ . (The condition is independent of the compactification  $G^*$ .) We denote by  $C$  the additive group of complex numbers considered as a 1-dimensional complex algebraic group. In this case we can take  $G^* = P$ , the complex projective line.

We shall use the following lemma which is essentially proved in [2, §4] (cf. also [1]).

**Lemma 1.4.** *Let  $G$  be a connected algebraic group acting biholomorphically and meromorphically on a compact irreducible complex space  $X$ . Then there exists a diagram of morphisms of irreducible compact complex spaces*

$$\begin{array}{ccc} & \pi & \\ & Z \rightarrow X & \\ f \downarrow & & \\ & T & \end{array}$$

with the following properties:

1)  $G$  acts on each space biholomorphically and meromorphically, making the morphisms  $G$ -equivariant, where the action on  $X$  is the given one and that on  $T$  is the trivial one.

2) There exists a Zariski open subset  $V$  of  $T$  such that for any  $v \in V$  the fiber  $Z_v := f^{-1}(v)$  of  $f$  is mapped isomorphically by  $\pi$  onto a closed subspace of  $X$  which is the closure of a  $G$ -orbit in  $X$ .

3)  $\pi$  is bimeromorphic.

**Proof.** The proof is essentially the same as in [2, Th. 4.1]. So we shall recall only briefly the construction of the diagram. Let  $\sigma^*: G^* \times X \rightarrow X$  define the

meromorphic action of  $G$  on  $X$  with respect to the natural open embedding  $G \hookrightarrow G^*$ . Take the graph  $\Gamma \subseteq G^* \times X \times X$  of  $\sigma^*$ , and  $\Gamma'$  its projection in  $X \times X$ . Regard  $\Gamma' \subseteq X \times X$  as defining a family of complex subspaces of  $X$  parametrized by  $X$  with respect to the first projection. Then we get the universal meromorphic map  $\tau: X \rightarrow D_X$  of  $X$  into the Douady space  $D_X$  of compact subspaces of  $X$ . Then  $T$  is by definition the minimal analytic subspace of  $D_X$  containing the (meromorphic) image of  $\tau$ , and the diagram is obtained by restricting the universal family over  $D_X$  to  $T$ .

Roughly speaking, for a general point  $x$  of  $X$  the corresponding subspace  $\Gamma'_x$  of  $X$  is the closure  $\overline{Gx}$  of the orbit  $Gx$  of  $x$ , and  $\tau$  is defined at such a point  $x$  with the point  $\tau(x) \in D_X$  corresponding to the subspace  $\overline{Gx}$  itself. It follows that with respect to the natural action of  $G$  on  $D_X$ ,  $\tau(x)$  belongs to the fixed point set  $D_X^G$ . Therefore, we conclude that  $T$  also is contained in  $D_X^G$ . Finally, since  $\pi$  is bijective on the union of  $Gx \subseteq Z_{\tau(x)}$  for all the general points  $x$  as above, we can deduce that  $\pi$  is bimeromorphic. □

## 2. Simple connectivity of fibers

**2.1.** First we give a general lemma relating the fundamental group of a special fiber to that of a general fiber in a certain fiber space.

Let  $B$  be a polycylinder of sufficiently small radius in  $\mathbb{C}^n = \mathbb{C}^n(z_1, \dots, z_n)$  for some  $n > 0$ . Define a hypersurface  $A$  in  $B$  by the equation  $z_1 \cdots z_l = 0$  for some  $1 \leq l \leq n$ . We set  $U = B - A$ . Let  $Z$  be an irreducible complex space and  $f: Z \rightarrow B$  a proper morphism of complex spaces with connected fibers. We assume that  $f$  is smooth over  $U$ . For any point  $b \in B$  we write  $Z_b = f^{-1}(b)$ . We write  $Y = Z_{0,red}$ , where  $0$  is the origin of  $B$ , and  $_{red}$  means taking the underlying reduced subspace. Fixing any point  $b_0 \in U$  we set  $F = Z_{b_0}$ .

In order to state the lemma exactly we introduce the notion of multiplicity of the map  $f$  along each irreducible component  $A_i := \{z_i = 0\}$ ,  $1 \leq i \leq l$ , of  $A$ . Take any resolution  $r: \hat{Z} \rightarrow Z$  of the singularities of  $Z$  according to Hironaka so that the exceptional set of  $r$  is a divisor with only normal crossings in  $\hat{Z}$ . We take a point  $a^i = (a_1^i, \dots, a_n^i)$  of  $A_i - \cup_{j \neq i} A_j$  and consider the restriction  $f_i: \hat{Z}_i \rightarrow D_i$  of  $fr$  over the 1-dimensional disc  $D_i = \{z_j = a_j^i; j \neq i\}$ . If we take  $a^i$  sufficiently general, then  $\hat{Z}_i$  is nonsingular and the fiber  $Y_i := f_i^{-1}(a^i)$  over  $a^i$  of  $f_i$  is a divisor with only normal crossings in  $\hat{Z}_i$ . Let  $Y_{i\mu}$ ,  $1 \leq \mu \leq d_i$ , be the irreducible components of  $Y_i$  and  $m_{i\mu}$  the multiplicity of  $Y_{i\mu}$  in  $f_i^{-1}(a^i)$ . Then we call the greatest common divisor  $m_i$  of  $m_{i\mu}$  the *multiplicity* of  $f$  along  $A_i$ . (It is standard to check that  $m_i$  is independent of the choice of a resolution  $r$  and a point  $a^i$  as above (and even of the coordinates  $z_i$ .) Then our lemma is stated as follows.

**Lemma 2.1.** *Let the notations and assumptions be as above. Suppose that  $Z$  is locally irreducible. Then, possibly after restricting  $B$  around the origin, there*

exist a quotient group  $H$  of  $\pi_1(F)$  and positive integers  $k_i$  which are divisors of  $m_i$ ,  $1 \leq i \leq l$ , such that  $\pi_1(Y)$  is isomorphic to a group which is an extension by  $H$  of the finite abelian group  $\bigoplus_i \mathbb{Z}/k_i \mathbb{Z}$  so that we have an exact sequence (of non-commutative groups) of the form

$$1 \rightarrow H \rightarrow \pi_1(Y) \xrightarrow{q} \bigoplus_i \mathbb{Z}/k_i \mathbb{Z} \rightarrow 1. \tag{1}$$

**Corollary 2.2.** *If  $\pi_1(F)$  is finite, then so is  $\pi_1(Y)$ . Moreover, if  $F$  is simply connected,  $\pi_1(Y)$  is a finite abelian group with at most  $l$  generators.*

**REMARK 2.1.** 1) If  $Z$  is nonsingular, we can actually take  $k_i = m_i$ . In general,  $k_i$  can be smaller than  $m_i$ .

2) The abelian unramified covering corresponding to the quotient map  $q$  is described as follows. Define a covering map  $h: B' \rightarrow B$  by  $h(z_1, \dots, z_n) = (z_1^{k_1}, \dots, z_1^{k_1}, z_{i+1}, \dots, z_n)$  for an appropriate polycylinder  $B'$ . Then, if  $f': Z' \rightarrow B$  is the normalized pullback of  $Z \rightarrow B$  by  $h$ , the induced morphism  $Z' \rightarrow Z$  turns out to be unramified and the induced unramified covering  $Y' \rightarrow Y$  is the desired abelian covering, where  $Y' = Z'_{0,red}$ ;  $\pi_1(Y')$  is thus isomorphic to the group  $H$  of the above lemma.

**Proof of Lemma 2.1.** Restricting  $B$  if necessary, we may assume that the natural map  $\pi_1(Y) \rightarrow \pi_1(Z)$  induced by the inclusion is isomorphic. Let  $V = f^{-1}(U)$ . Then the natural map  $t: \pi_1(V) \rightarrow \pi_1(Z)$  is surjective. Indeed, let  $w: \tilde{Z} \rightarrow Z$  be the universal covering map. Then, since  $Z$  is locally irreducible,  $\tilde{Z}$  is irreducible and hence  $w^{-1}(V)$  is connected. Then by Lemma 1.1, 1) applied to the inclusion  $V \hookrightarrow Z$  it follows that  $t$  is surjective. On the other hand, since the restriction  $f_U: V \rightarrow U$  of  $f$  to  $V$  is topologically a fiber bundle, we get an exact sequence of the form

$$1 \rightarrow L \rightarrow \pi_1(V) \xrightarrow{u} \pi_1(U) \rightarrow 1, \tag{2}$$

where  $L$  is the natural image of  $\pi_1(F)$  in  $\pi_1(V)$ . We claim that the image by  $u$  of the kernel  $K$  of  $t$  in  $\pi_1(U) (\cong \bigoplus_i \mathbb{Z})$  contains the subgroup  $\bigoplus_i m_i \mathbb{Z}$ . (Note that the class  $\gamma_i$  defined by a closed path which turns once around the  $a^i$  counterclockwise in the punctured disc  $D_i - a^i$  with  $D_i$  and  $a^i$  as above give canonical generators of  $\pi_1(U)$ .) Using the notations introduced before the lemma, take a 1-dimensional disc  $D_{i\mu}$  in  $\hat{Z}_i$  which intersects transversally with  $Y_{i\mu}$  at a general point  $y_{i\mu}$  and then consider the class in  $\pi_1(V)$  defined by a closed path in  $D_{i\mu}$  turning once around  $y_{i\mu}$ . Since the induced map  $D_{i\mu} \rightarrow D_i$  is an  $m_{i\mu}$ -ple covering ramified at  $y_{i\mu}$ , the image by  $u$  of this class in  $\pi_1(U)$  generate the subgroup  $m_{i\mu} \mathbb{Z}$  of the  $i$ -th component of  $\pi_1(U)$ . This consideration for all  $i$  and  $\mu$  gives the desired

claim. Then by taking the quotient of the sequence (2) by  $K$  we get an exact sequence of the form (1) with  $H=L/L \cap K$ .

When the general fiber  $F$  is a nonsingular rational curve, we can obtain a little more precise result.

**Corollary 2.3.** *In Lemma 2.1 suppose that  $F$  is a nonsingular rational curve. Then  $Y$  is simply connected.*

*Proof.* It is well-known that  $Y$  is a union of irreducible rational curves, say  $Y_1, \dots, Y_m$ . Suppose that  $Y$  is not simply connected. Then either the dual graph associated to  $Y$  contains a cycle, or some irreducible component  $Y_i$  has a node as its singularity. In any case, the fundamental group  $\pi_1(Y)$  of  $Y$  is necessarily infinite. This contradicts Corollary 2.2. (Note that we may also show directly that  $m_i=1$  for all  $i$ .)  $\square$

**2.2.** We now apply the above corollary to a quotient map of a  $C$ -action. Namely, combining Corollary 2.3 and Lemma 1.4 we shall prove the following:

**Lemma 2.4.** *Let  $X$  be a compact irreducible complex space on which a 1-dimensional connected algebraic group  $G$  acts biholomorphically and meromorphically. Then there exists a diagram of morphisms of irreducible compact complex spaces*

$$\begin{array}{ccc} & \pi & \\ & Z \rightarrow X & \\ f \downarrow & & \\ & T & \end{array}$$

with  $Z$  normal and with the following properties:

- 1)  $G$  acts on each space biholomorphically and meromorphically making the morphisms  $G$ -equivariant, where the action on  $X$  is the given one and that on  $T$  is the trivial one,
- 2) a general fiber of  $f$  is a nonsingular rational curve, and every fiber of  $f$  is a connected and simply connected 1-dimensional subspace of  $Z$ , and
- 3)  $\pi$  is bimeromorphic.

*Proof.* Consider the diagram of complex spaces

$$\begin{array}{ccc} & \pi_0 & \\ & Z_0 \rightarrow X & \\ f_0 \downarrow & & \\ & T_0 & \end{array}$$

associated to the given action as in Lemma 1.4. Let  $n: Z_1 \rightarrow Z_0$  be the normalization of  $Z_0$  and  $f_1: Z_1 \rightarrow T_0$  the induced morphism. By the property 2) in Lemma 1.4 of the above diagram we see that the general fibers of  $f_1$  are (connected) nonsingular rational curves. Let  $N$  be the maximal Zariski open subset of  $T$  such that  $f_1$  is smooth over  $N$ . Let  $S = T_0 - N$ . Take by Hironaka a proper bimeromorphic morphism  $g: T \rightarrow T_0$  such that  $T$  is nonsingular and  $\hat{S} := g^{-1}(S)_{red}$  is a divisor with only normal crossings in  $T$ . Let  $Z$  be the normalization of the pull-back  $Z_1 \times_{T_0} T$ , and  $f: Z \rightarrow T$  and  $\pi: Z \rightarrow X$  the induced morphisms. We have the naturally induced  $G$ -action on  $Z$  which makes the morphisms  $f$  and  $\pi$   $G$ -equivariant, where the action on  $T$  is the trivial one. Clearly every fiber of  $f$  is connected and 1-dimensional and  $\pi$  is bimeromorphic. So what remains to be shown is that every fiber of  $f$  is simply connected. The problem is local with respect to  $T$ . So take any point  $t$  of  $T$ . Then, if we restrict the morphism  $f$  over a polycylindrical neighborhood of  $t$ , the resulting morphism satisfies all the conditions of Corollary 2.3. Thus the desired simple connectivity follows from that corollary.  $\square$

For the total space  $Z$  of the fiber space  $f: Z \rightarrow T$  as in the above lemma we can now prove the 1-connectivity of the inclusion  $Z^G \hookrightarrow Z$ , to which the proof in the general case will be reduced by using the above lemma.

**Lemma 2.5.** *Let  $f: Z \rightarrow T$  be a morphism of compact irreducible complex spaces such that its general fibers are nonsingular and every fiber is connected, simply connected and is of dimension one. Suppose that there exist biholomorphic and meromorphic actions of the additive group  $G = C$  on  $Z$  and  $T$  making  $f$   $G$ -equivariant, where the action on  $T$  is the trivial one. Then the natural inclusion  $\iota: Z^G \hookrightarrow Z$  is 1-connected.*

**Proof.** We may assume that the action is not trivial. First by Lemma 1.3 and by our assumption we get that  $f: Z \rightarrow T$  is 1-connected. So it suffices to show that the induced morphism  $f_G: Z^G \rightarrow T$  also is 1-connected. Let  $U$  be a Zariski open subset of  $T$  over which  $f$  is smooth. Then  $f_G^{-1}(U)$  is mapped isomorphically onto  $U$  by  $f_G$ . Hence, there exists a unique irreducible component  $Z_0^G$  of  $Z^G$  which is mapped surjectively onto  $T$ . On the other hand, for any point  $t \in T$  each fiber  $(Z^G)_t = (Z_t)^G$  of  $f_G$  is a connected subspace of  $Z_t$ ,  $Z_t$  being of dimension one. It follows that  $Z^G$  itself is connected. Moreover, as a connected subspace of the 1-dimensional simply connected space  $Z_t$ ,  $(Z^G)_t$  also is simply connected. Thus, again by Lemma

1.3  $f_G$  is 1-connected, which completes the proof of the lemma. □

**3. Theorem**

We consider a connected linear algebraic group  $G$  which admits an increasing sequence of connected normal algebraic subgroups

$$G_0 = \{e\} \subseteq \dots \subseteq G_i \subseteq \dots \subseteq G_m = G \tag{3}$$

such that all the quotient groups  $G_i/G_{i+1}$  are isomorphic to the additive group  $C$ . Such a group is necessarily unipotent and vice versa. So we refer to such a group simply as a connected unipotent algebraic group in what follows. On the fixed point set of the action of such a group we prove the following result which is a generalization of a result of Horrocks in [3] (cf. Remark 3.1 below).

**Theorem 3.1.** *Let  $G$  be a connected unipotent linear algebraic group and  $X$  a compact complex space. Suppose that  $G$  acts biholomorphically and meromorphically on  $X$ . Let  $X^G$  be the set of fixed points of this action. Then the natural inclusion  $\iota: X^G \rightarrow X$  is 1-connected (cf. Def. 1.1).*

**REMARK 3.1.** When  $X$  is an analytic space underlying some complete algebraic variety (defined over  $C$ ), Horrocks [3] has proved that  $\iota$  is 0-connected and induces an isomorphism of the algebraic fundamental groups (i.e., the profinite completions of  $\pi_1$ ) of the corresponding connected components of  $X^G$  and  $X$ . As a partial generalization of this, Carrell and Sommese [1] have shown that  $\iota$  is 0-connected even if  $X$  is a general complex space which is not necessarily compact.

**Proof of Theorem 3.1.** Using the sequence (3) and the fact that  $X^{G_i} = (X^{G_{i-1}})^{G_i/G_{i-1}}$  the proof is reduced easily to the case where  $G$  is the additive group  $C$  (cf. [3, proof of Theorem 6.2] and [1]). Then, we proceed by induction on  $n := \dim X$ . First of all, the assertion is trivial when  $n=0$ . So suppose that  $n>0$  and that the theorem is true for any meromorphic  $G$ -action on a complex space of smaller dimension.

The fact that  $\iota$  is  $\frac{1}{2}$ -connected follows just as in the proof of Theorem 5.1 of [3]: Let  $r: \tilde{X} \rightarrow X$  be the universal covering of  $X$ . Then the action of  $G$  on  $X$  lifts to a meromorphic action on  $\tilde{X}$ ,  $G$  being simply connected. Since  $r$  is a discrete map and  $G$  is connected, we have  $(\tilde{X})^G = r^{-1}(X^G)$ . Then by the above mentioned result of [1], we get that  $r^{-1}(X^G)$  is connected. Thus,  $\iota$  is  $\frac{1}{2}$ -connected by Lemma 1.1.

On the other hand, suppose that  $X$  is a union of two  $G$ -invariant closed analytic subsets  $X_1$  and  $X_2$  such that  $\dim X_1 \cap X_2 < n$ . Then by induction hypothesis the inclusion  $(X_1 \cap X_2)^G = X_1^G \cap X_2^G \hookrightarrow X_1 \cap X_2$  is 1-connected. Thus, if the theorem is true for the induced  $G$ -actions on  $X_i$ , by Lemma 1.2 the same is also true for our  $G$ -action on  $X$ . Using this inductively we may easily reduce the proof to the



case where  $X$  is irreducible. So suppose that  $X$  is irreducible. We then consider a diagram of  $G$ -equivariant morphisms of irreducible compact complex spaces

$$\begin{array}{ccc} & \pi & \\ & Z \rightarrow X & \\ f \downarrow & & \\ & T & \end{array}$$

obtained in Lemma 2.4 associated to the given  $G$ -action on  $X$ . In view of the properties 1) and 2) in that lemma, we conclude from Lemma 2.5 that the natural inclusion  $Z^G \hookrightarrow Z$  is 1-connected.

We shall now derive the 1-connectedness of the inclusion  $\iota : X^G \hookrightarrow X$  from this. Let  $r_G : \tilde{X}^G \rightarrow X^G$  be any unramified covering of  $X^G$ . Since we know already that  $\iota$  is  $\frac{1}{2}$ -connected, we have only to show that there exists an unramified covering  $r : \tilde{X} \rightarrow X$  which induces  $r_G$  over  $X^G$ .

First, consider the maximal Zariski open subset  $U$  of  $X$  such that  $\pi : \pi^{-1}(U) \rightarrow U$  is isomorphic. Set  $S = X - U$  and  $M = \pi^{-1}(S)$ . Since  $\pi$  is  $G$ -equivariant,  $S$  and  $M$  are  $G$ -invariant. Moreover, since the dimensions of  $S$  and  $M$  are less than  $n$ , by the induction hypothesis the inclusions  $S^G \hookrightarrow S$  and  $M^G \hookrightarrow M$  are both 1-connected. Next, let  $S' = S \cup X^G$  and  $M' = M \cup Z^G$  so that we have  $M' = \pi^{-1}(S')$ . Then the inclusions  $X^G = S'^G \hookrightarrow S'$  and  $Z^G = M'^G \hookrightarrow M'$  are also 1-connected by Lemma 1.2.

Take now a small neighborhood  $W$  of  $S'$  in  $X$  (resp.  $Q$  of  $M'$  in  $Z$ ) such that  $\pi(Q) \subseteq W$  and that the inclusions  $S' \hookrightarrow W$  and  $M' \hookrightarrow Q$  are 1-connected. By what we have shown, it then follows that  $r_G$  extends to a (unique) unramified covering  $r_W : \tilde{W} \rightarrow W$  of  $W$  and that the pull-back  $r_Q : \tilde{Q} \rightarrow Q$  of  $r_W$  to  $Q$  by  $\pi$  extends to a unique unramified covering  $r_Z : \tilde{Z} \rightarrow Z$  of  $Z$ . Write  $U' = X - S'$  and  $V' = Z - M'$ . Since  $V' \cong U'$ , the restriction of  $r_Z$  to  $V'$  descends to an unramified covering  $r_{U'} : \tilde{U}' \rightarrow U'$ , which is isomorphic over  $\pi(Q) - S' (\subseteq W - S')$  to  $r_W$  by the definition of  $r_Z$ . Hence, we can patch together  $r_{U'}$  and  $r_W|_{\pi(Q)}$  over  $\pi(Q) - S'$  by some isomorphism and get a desired unramified covering which extends  $r_G$  to  $X$ . The proof of the theorem is now complete.  $\square$

Suppose that  $X$  is a manifold and  $G = C$ . Then a shorter proof using only Lemma 2.1 and without using induction is possible. In fact, in this case we can show the next lemma, which is enough for the proof of the theorem as one sees easily from the above proof in the general case.

**Lemma 3.2.** *Let  $Z$  be a normal compact complex space on which  $G$  acts biholomorphically and meromorphically. Let  $\pi : Z \rightarrow X$  be a  $G$ -equivariant bimeromorphic morphism. Suppose that the natural inclusion  $\iota_Z : Z^G \rightarrow Z$  is 1-connected. Then  $\iota_X : X^G \rightarrow X$  also is 1-connected.*

Proof. We may assume that  $X$  is connected. First we show that the homomorphism  $\pi_*: \pi_1(Z) \rightarrow \pi_1(X)$  is isomorphic. In fact, take any resolution  $r: \tilde{Z} \rightarrow Z$  and let  $\tilde{\pi}: \tilde{Z} \rightarrow X$  be the induced bimeromorphic morphism. Since both  $\tilde{Z}$  and  $X$  are nonsingular,  $\tilde{\pi}$  induces an isomorphism  $\pi_1(\tilde{Z}) \cong \pi_1(X)$  as is well-known. On the other hand, since  $Z$  is normal, the fibers of  $r$  is connected, and hence the induced homomorphism  $\pi_1(\tilde{Z}) \rightarrow \pi_1(Z)$  is surjective (cf. Lemma 1.3). This implies that  $\pi_*$  is isomorphic as claimed.

On the other hand, since  $Z \rightarrow X$  is  $G$ -equivariant, the induced map  $\pi_G: Z^G \rightarrow X^G$  is surjective with connected fibers. In fact, each fiber  $\pi^{-1}(x)$  of a point  $x \in X^G$  is connected and  $G$ -invariant, and hence by the  $\frac{1}{2}$ -connected theorem (which is proved independently of the induction)  $Z^G \cap \pi^{-1}(x)$  is nonempty and connected. Hence, again by Lemma 1.3.1), we get that  $\pi_{G*}: \pi_1(Z^G) \rightarrow \pi_1(X^G)$  is surjective. The lemma then follows from the following commutative diagram of homomorphisms of fundamental groups

$$\begin{array}{ccc} & \pi_{G*} & \\ \pi_1(Z^G) & \rightarrow & \pi_1(X^G) \\ \iota_{Z*} \downarrow & & \downarrow \iota_{X*} \\ \pi_1(Z) & \rightarrow & \pi_1(X). \end{array}$$

□

Once Theorem 3.1 is obtained, by the same argument as in [3, §7], where the case of algebraic fundamental group was treated, we get the following:

**Theorem 3.3.** *Let  $P^n$  be the complex projective space of dimension  $n$ . For nonnegative integers  $d$  and  $q$  with  $0 \leq q \leq n$  denote by  $C(n, d)$  the Chow variety of  $P^n$  of  $q$ -cycles of degree  $d$ . Then  $C(n, d)$  is connected and simply connected. □*

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