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UNIVERSAL R-MATRICES FOR THE QUANTUM GROUP $Uq(sl(N+1, C))$: THE ROOT OF UNITY CASE

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Intoroduction

The aim of this paper is to construct a universal R-matirix for a certain quotient of the quantized universal enveloping algebra $U_q(sl(N+1,C))$ in the sense of Drinfel'd [2] and Jimbo [5][6] at roots of unity. The notion of universal R-matrix is due to Drinfel'd. A universal R-matrix for a Hopf algebra A over C is an invertible element $R \in A \otimes A$ with the following properties: (1) $R\Delta(a)R^{-1} = \tilde{\Delta}(a)$, for $a \in A$, (2) $(\Delta \otimes id)(R) = R_{13}R_{23}$, $(id \otimes \Delta)(R) = R_{13}R_{12}$. Here $\Delta: A \rightarrow A \otimes A$ is the comultiplication, and $\tilde{\Delta}$ is the opposite comultiplication $\tilde{\Delta} = P \circ \Delta$ for the permutation P in $A \otimes A$, $P(a \otimes b) = b \otimes a$. The map Δ is not in general symmetric in the sense that $\tilde{\Delta} \neq \Delta$, but from the property (1) of this universal R-matrix, there arises an A-module isomorphism $V \otimes W \rightarrow W \otimes V$ for A-modules V and W. It follows from two properties (1) and (2) that it satisfies the Yang-Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where R_{ij} is the embedding of R into the i-th and j-th factor of $A \otimes A \otimes A$.

In [14], Rosso gave an explicit formula of universal R-matrix for $U_q(sl(N+1,C))$ for generic q, and in [15], he obtained a universal R-matrix for a quotient of $U_q(sl(N+1,C))$ when q is a primitive r-th root of unity for an integer r satisfying that $r \geq N+1$ and that r and $N+1$ are coprime. The result was independently obtained in [17]. In [23],[24],[25], and [26], Yamane introduced quasi-triangular Hopf algebras associated to complex simple Lie superalgebras of types A-G, and gave explicit formulas of their universal R-matrices, both in generic and non-generic cases. In particular, he got an explicit formula of a universal R-matrix for a quotient of $U_q(sl(N+1,C))$.

In the present paper, we give an explicit formula of a universal R-matrix for a quotient of $U_q(sl(N+1,C))$ for a primitive r-th root of q of unity, $r \neq 1,2,4$. Let E_i, F_i, and K_i, $1 \leq i \leq N$, be the generators of the Hopf algebra $U_q(sl(N+1,C))$. Let U^+ be the Hopf subalgebra $U_q(sl(N+1,C))$ generated by E_i, K_i, $1 \leq i \leq N$ and U^- the Hopf subalgebra generated by F_i, K_i, $1 \leq i \leq N$. The construction of the universal R-matrix is based on the quantum double construction due to Drinfel'd [2]. An essential point of this construction is the existence of a non-degenerate pairing $U^+ \times U^- \to \boldsymbol{C}$ compatible with the Hopf algebra structures of U^+ and U^-. Since a pairing naturally defined degenerates when q is a root of unity, we consider, following Yamane [25], a certain quotient of $U_q(sl(N+1,\boldsymbol{C}))$.

For $N \in \boldsymbol{N}$ and $1 < r \in \boldsymbol{N}$, we put $d=(r,N+1)$, $a=\frac{r}{d}$, $\bar{r}=\frac{r}{(r,2)}$. Let ζ be a primitive r-th root of unity with $(\zeta+\bar{\zeta})(\zeta-\bar{\zeta}) \neq 0$. We remark that ζ^{N+1} is a primitive a-th root of unity, and ζ^2 is a primitive $\bar{r}$-th root of unity. Let $(a_{ij})_{1 \leq i,j \leq N}$ be the Cartan matrix for $sl(N+1,\boldsymbol{C})$. In the present paper, we consider the Hopf algebra U_ζ which is a quotient Hopf algebra of $U_\zeta(sl(N+1,\boldsymbol{C}))$.

As an algebra U_ζ is generated by E_i, F_i, K_i, K_i^{-1}, $\Lambda=\Pi_{i=1}^N K_i^i$ for $1 \leq i \leq N$ with the relations:

$$K_iK_j=K_jK_i,\ K_iK_i^{-1}=K_i^{-1}K_i=1,$$

$$K_iE_j=\zeta^{(\alpha_i,\alpha_j)}E_jK_i,\ K_iF_j=\zeta^{-(\alpha_i,\alpha_j)}F_jK_i,$$

$$[E_i,F_j]=\delta_{ij}\frac{K_i-K_i^{-1}}{\zeta-\zeta^{-1}},$$

$$E_i^2E_j-(\zeta+\zeta^{-1})E_iE_jE_i+E_iE_j^2=0 \quad (|i-j|=1),$$

$$E_iE_j=E_jE_i \quad (|i-j|\geq 2),$$

$$F_i^2F_j-(\zeta+\zeta^{-1})F_iF_jF_i+F_iF_j^2=0 \quad (|i-j|=1),$$

$$F_iF_j=F_jF_i \quad (|i-j|\geq 2),$$

$$E_{ij}^{\bar{r}}=F_{ij}^{\bar{r}}=0,$$

$$K_i^r=1,\ \Lambda^a=1,$$

where $(\alpha_i,\alpha_j)=\alpha_{ij}$, and for $1 \leq i \leq j \leq N+1$ and $X=E$ or F, the element X_{ij} is inductively defined by

$$X_{ij}=\begin{cases} X_i & \text{if } j=i+1, \\ X_{ij-1}X_{j-1}-\zeta X_{j-1}X_{ij-1} & \text{if } j>i+1. \end{cases}$$

Let U_ζ^+ be the Hopf subalgebra of U_ζ generated by E_i, $K_i^{\pm}$, $1 \leq i \leq N$, U_ζ^- the Hopf subalgebra of U_ζ generated by F_i, $K_i^{\pm}$, $1 \leq i \leq N$, and $(U_\zeta^+)^o$ the dual algebra of U_ζ^+ with the opposite comultiplication. We construct a Hopf algebra isomorphism $\varphi: U_\zeta^- \to (U_\zeta^+)^o$, and give an explicit formula of an orthonormal basis with respect to the pairing Φ.

Applying the quantum double construction to the Hopf algebra U_ζ^+, we see that the Hopf algebra isomorphism φ induces a Hopf algebra epimorphism ψ from the quantum double $D(U_\zeta^+)$ to the Hopf algebra U_ζ. The image of the universal R of $D(U_\zeta^+)$ under $\psi \otimes \psi$ is a universal R of U_ζ.

As well-known, a universal R can be used in producing tangle invariants obtained from the representations of the quantized universal enveloping algebras for classical simple Lie algebras (see for example [11][12][13][18][19]). As an application of our universal R, we can calculate some tangle invariants, which are essential in the construction of Witten's 3-manifold invariants [21].

For any positive integer K, let $P_+(K)$ be the set of the dominant integral weights λ with $0\leq(\lambda,\theta)\leq K$, where θ denotes the longest root. We consider the family of finite dimensional irreducible representations of U_ζ whose highest weight λ is contained in $P_+(K)$, in the case $\bar{r}=K+N+1$. For an oriented framed link L, we denote by $J(L)$ the tangle invariant obtained by using these irreducible representations. Using our explicit formula of universal R for U_ζ in the case $\bar{r}=K+N+1$, one can calculate $J(H_{\lambda\mu})$, where $H_{\lambda\mu}$ denotes Hopf link with two components assigned with V_λ and V_μ:

$$J(H_{\lambda\mu})=\frac{\Sigma_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\mu+\rho))}}{\Sigma_{w\in W}(\det w)\bar{q}^{(\rho,w(\rho))}}.$$

Here ρ is half the sum of positive roots. Let $S=(S_{\lambda\mu})$ be the modular transformation S matrix for characters of the integrable highest weight modules due to Kac and Peterson [7]. Using the equality $S_{\lambda\mu}=S_{00}J(H_{\lambda\mu})$, we show Verlinde's formula for the fusion algebra of type $A_N^{(1)}$. The fusion algebra is an associative commutative ring with basis labelled by $P_+(K)$ and the product $w_\lambda\cdot w_\mu$ of two basis elements can be written as a sum $\Sigma N_{\lambda\mu}^{\nu}w_\nu$ with structure constants $N_{\lambda\mu}^{\nu}\in N$ called the fusion rule. The modular transfomation S-matrix and the fusion rules $N_{\lambda\mu}^{\nu}$'s are related by Verlinde's formula [20]:

$$N_{\lambda\mu}^{\nu}=\sum_{\varepsilon\in P_+(K)}\frac{S_{\lambda\varepsilon}S_{\mu\varepsilon}S_{\nu\varepsilon}^*}{S_{0\varepsilon}}.$$

The paper is organized as follows: In §1, we recall the quantum double construction due to Drinfel'd and define the Hopf algebra U_ζ. In §2, a universal R for U_ζ is obtained, applying the quantum double construction to the Hopf subalgebra U_ζ^+ of U_ζ. In §3, we state tangle operators derived from irreducible representations of U_ζ, and calculate some tangle invariants. As an application of the tangle invariants, we prove Verlinde's formula for the fusion algebra of type $A_N^{(1)}$.

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1. Hopf algebra U_ζ and quantum double construction

In this section, we define the Hopf algebra and recall the quantum double construction due to Drinfel'd [2].

Let A be a Hopf algebra over $\mathbf{C}$. A universal R-matrix for A is an invertible element $R \in A \otimes A$ such that

$$(1)\ R\Delta(a)R^{-1} = \check{\Delta}(a) \quad \text{for } a \in A, \tag{1.1}$$

$$(2)\ (\Delta \otimes id)(R) = R_{13}R_{23}, (id \otimes \Delta)(R) = R_{13}R_{12}, \tag{1.2}$$

where Δ is the comultiplication and $\check{\Delta} = P \circ \Delta$ for the permutation $P, P(a \otimes b) = b \otimes a$. Here $R_{12} = \Sigma_i a_i \otimes b_i \otimes 1$, $R_{13} = \Sigma_i a_i \otimes 1 \otimes b_i$, and $R_{23} = \Sigma_i 1 \otimes a_i \otimes b_i$, where the components of the universal R are given by $R = \Sigma_i a_i \otimes b_i$. The pair (A,R) is called a quasitriangular Hopf algebra.

The so-called quantum double construction due to Drinfel'd allows us to produce quasitriangular Hopf algebras from Hopf algebras. It can be used to construct a univeasal R. The method can be sketched as follows. Let A be a finite dimensional Hopf algebra and A^o its dual with opposite comultiplication. Then, the quantum double $D(A)$ is isomorphic to $A \otimes A^o$ as a vector space, and it contains A and A^o as Hopf subalgebras via the natural embeddings, and the universal R of $D(A)$ is the image of the canonical element of $A \otimes A^o$ i.e. $\Sigma_i e_i \otimes 1 \otimes 1 \otimes e^i$, if $\{e_i\}$ is a basis of A and $\{e^i\}$ the dual basis in A^* which is the dual space of A.

For $N \in \mathbf{N}$ and $1 < r \in \mathbf{N}$, we put $d = (r, N+1)$, $a = \frac{r}{d}$, and $\bar{r} = \frac{r}{(r,2)}$. Let $((\alpha_i, \alpha_j))_{1 \le i,j \le N}$ be the Cartan matrix of type A_N:

$$((\alpha_i,\alpha_j)) = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}.$$

Let ζ be a primitive r-th root of unity with $(\zeta + \bar{\zeta})(\zeta - \bar{\zeta}) \neq 0$. We remark that ζ^{N+1} is a primitive a-th root of unity, and that ζ^2 is a primitive $\bar{r}$-th root of unity.

We define the Hopf algebra U_ζ which is a quotient Hopf algebra of $U_\zeta(sl(N+1, \mathbf{C}))$.

The algebra U_ζ is generated by E_i, F_i, K_i, K_i^{-1}, $\Lambda = \Pi_{i=1}^N$ for $1 \le i \le N$ with the relations:

$$K_i K_j = K_j K_i, K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{1.3}$$

$$K_i E_j = \zeta^{(\alpha_i,\alpha_j)} E_j K_i, K_i F_j = \zeta^{-(\alpha_i,\alpha_j)} F_j K_i, \tag{1.4}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{\zeta - \zeta^{-1}}, \tag{1.5}$$

$$E_i^2E_j-(\zeta+\zeta^{-1})E_iE_jE_i+E_iE_j^2=0 \quad (|i-j|=1), \tag{1.6}$$

$$E_iE_j=E_jE_i \quad (|i-j|\geq 2), \tag{1.7}$$

$$F_i^2F_j-(\zeta+\zeta^{-1})F_iF_jF_i+F_iF_j^2=0 \quad (|i-j|=1), \tag{1.8}$$

$$F_iF_j=F_jF_i \quad (|i-j|\geq 2), \tag{1.9}$$

$$E_{ij}^{\bar{r}}=F_{ij}^{\bar{r}}=0, \tag{1.10}$$

$$K_i^r=1, \Lambda^a=1, \tag{1.11}$$

where, for integers i and j with $1\leq i<j\leq N+1$ and $X=E$ or F, the element X_{ij} is inductively defined by

$$X_{ij}=\begin{cases} X_i & \text{if } j=i+1, \\ X_{ij-1}X_{j-1}-\zeta X_{j-1}X_{ij-1} & \text{if } j>i+1. \end{cases}$$

The algebra U_ζ has a Hopf algebra structure with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(E_i)=E_i\otimes 1+K_i\otimes E_i, \ \Delta(F_i)=F_i\otimes K_i^{-1}+1\otimes F_i,$$
$$\Delta(K_i^{\pm})=K_i^{\pm}\otimes K_i^{\pm},$$
$$\varepsilon(E_i)=\varepsilon(F_i)=0, \ \varepsilon(K_i^{\pm})=1,$$
$$S(E_i)=-K_i^{-1}E_i, \ S(F_i)=-F_iK_i, \ S(K_i^{\pm})=K_i^{\mp}.$$

Let us show that the definitions of Δ and S are compatible with (1.10). We prove some Lemmas.

We put

$$[X,Y]_\zeta=XY-\zeta YX, \ [X,Y]_{\bar{\zeta}}=XY-\zeta^{-1}YX.$$

Lemma 1.1. *Let M be the $\mathbf{C}$-algebra generated by A and B with the relations:*

$$A^2B-(\zeta+\zeta^{-1})ABA+BA^2=0, \tag{1.12}$$

$$B^2A-(\zeta+\zeta^{-1})BAB+AB^2=0. \tag{1.13}$$

We put

$$C=[A,B]_\zeta, \ C'=[A,B]_{\bar{\zeta}}.$$

Then it holds:

$$C'^{\bar{r}}=C^{\bar{r}}+(1-\zeta^{-2})^{\bar{r}}\zeta^{-\frac{\bar{r}(\bar{r}-1)}{2}}A^{\bar{r}}B^{\bar{r}}.$$

Proof. When $C'=AB-\zeta^{-1}BA$, we have, for any positive integer n,

$$(C')^n=(\zeta^{-2}C+(1-\zeta^{-2})AB)^n$$
$$=\sum_{i=0}^{n}\binom{n}{i}(1-\zeta^{-2})^i\zeta^{-\frac{i(i-1)}{2}+(i-2)(n-i)}A^iC^{n-i}B^i, \tag{1.14}$$

where

$$\binom{n}{i}_\zeta=\frac{[n]\cdots[n-i+1]}{[i]\cdots[1]}, \quad [n]=\frac{1-\zeta^{-2n}}{1-\zeta^{-2}}$$

The equality is shown as follows. We have the following equalities for any non-negative integer n:

$$B^nAB=\zeta^{-n}AB=\zeta^{-n}AB^{n+1}-\zeta^{n-2}[n]CB^n, \tag{1.15}$$

$$(1-\zeta^{-2})[n]=1-\zeta^{-2n}, \tag{1.16}$$

$$\binom{n}{i-1}_\zeta+\binom{n}{i}_\zeta\zeta^{-2i}=\binom{n+1}{i}_\zeta. \tag{1.17}$$

We show the equality (1.14) by induction on n. We suppose that the equality (1.14) holds for n, and then it follows from (1.15),(1.16),(1.17) that

$$(C')^{n+1}$$
$$=(C')^n(\zeta^{-2}C+(1-\zeta^{-2})AB)$$
$$=\sum_{i=0}^{n}\binom{n}{i}_\zeta(1-\zeta^{-2})^i\zeta^{-\frac{i(i-1)}{2}+(i-2)(n-i)+i-2}A^iC^{n+1-i}B^i$$
$$+\sum_{i=0}^{n}\binom{n}{i}_\zeta(1-\zeta^{-2})^{i+1}\zeta^{-\frac{i(i-1)}{2}+(i-1)(n-i)}A^{i+1}C^{n-i}B^{i+1}$$
$$-\sum_{i=0}^{n}\binom{n}{i}_\zeta(1-\zeta^{-2})^{i+1}\zeta^{-\frac{i(i-1)}{2}+(i-2)(n-i+1)}[i]A^iC^{n+1-i}B^i$$
$$=\zeta^{-2(n+1)}C^{n+1}+(1-\zeta^{-2})^{n+1}\zeta^{-\frac{n(n+1)}{2}}A^{n+1}B^{n+1}$$
$$+\sum_{i=1}^{n}\left\{\binom{n}{i-1}_\zeta(1-\zeta^{-2})\zeta^{-\frac{i(i-1)}{2}+(i-2)(n+1-i)}\right.$$
$$\left.+\binom{n}{i}_\zeta(1-\zeta^{-2})^i\zeta^{-\frac{i(i-1)}{2}+(i-2)(n+1-i)}(1-(1-\zeta^{-2})[i])\right\}A^iC^{n+1-i}B^i$$
$$=\sum_{i=0}^{n+1}\binom{n+1}{i}_\zeta(1-\zeta^{-2})^i\zeta^{-\frac{i(i-1)}{2}+(i-2)(n+1-i)}A^iC^{n+1-i}B^i.$$

So the equality (1.14) holds. As ζ^2 is a primitive $\bar{r}$-th root of unity, we obtain the claim, putting $n=\bar{r}$ in the equality (1.14).

For $1\leq i<j\leq N+1$, the elements E_{ij} and E'_{ij} are inductively defined by

$$E_{ij}=\begin{cases} E_i & \text{if } j=i+1, \\ [E_{i\,j-1},E_{j-1}]_{\zeta} & \text{if } j>i+1. \end{cases}$$

$$E'_{ij}=\begin{cases} E_i & \text{if } j=i+1, \\ [E'_{i\,j-1},E_{j-1}]_{\bar{\zeta}} & \text{if } j>i+1. \end{cases}$$

Lemma 1.2. (i) *For $i<p<j$, we put $A=E_{ip}$ and $B=E'_{pj}$. Then these A and B satisfy the relations* (1.12) *and* (1.13).

(ii) *We have that* $[E_{ip},E'_{pj}]_{\zeta}=[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}$.

Proof. (i) We show by induction on p that $[E_i,E_{ip}]_{\bar{\zeta}}=0$, for $p\geq i+2$. It follows from relation (1.6) that $[E_i,E_{ii+2}]_{\bar{\zeta}}=0$. We suppose that $[E_i,E_{ip}]_{\bar{\zeta}}=0$. Then we obtain from the relation (1.7),

$$[E_i,E_{ip+1}]_{\bar{\zeta}}=[E_i,E_{ip}]_{\bar{\zeta}}E_p-\zeta E_p[E_i,E_{ip}]_{\bar{\zeta}}=0.$$

Similarly, using the relation (1.7) and the equality

$$[E'_{p-1j},E_{j-1}]_{\zeta}=E_{p-1}[E'_{pj},E_{j-1}]_{\zeta}-\bar{\zeta}[E'_{pj},E_{j-1}]_{\bar{\zeta}}E_{p-1},$$

we obtain by induction on p that $[E'_{pj},E_{j-1}]_{\zeta}=0$ for $p\leq j-2$.

We put $X=[E_{ip},[E_{ip},E'_{pj}]_{\zeta}]_{\bar{\zeta}}$ and $Y=[[E_{ip},E'_{pj}]_{\zeta},E'_{pj}]_{\bar{\zeta}}$. Computing $[E_{i-1},[E_{i-1},X]_{\zeta^2}]$, $[X,E_j]_{\bar{\zeta}}$, $[[Y,E_j]_{\bar{\zeta}^2},E_j]$ and $[E_{i-1},Y]$, we prove that E_{ip} and E'_{pj} satisfy the relations (1.12) and (1.13). Noting that $[E_i,E_{ip}]_{\bar{\zeta}}=0$ and

$$E_{i-1}^2[E_{ip},E'_{pj}]_{\zeta}-(\zeta+\bar{\zeta})E_{i-1}[E_{ip},E'_{pj}]_{\zeta}E_{i-1}-[E_{ip},E'_{pj}]_{\zeta}E_{i-1}^2=0,$$

it follows that

$$\begin{aligned}
&[E_{i-1},[E_{i-1},X]_{\zeta^2}]\\
&=E_{i-1}^2E_{ip}[E_{ip},E'_{pj}]_{\zeta}-\bar{\zeta}E_{i-1}^2[E_{ip},E'_{pj}]_{\zeta}E_{ip}\\
&\quad-\zeta^2E_{i-1}E_{ip}[E_{ip},E'_{pj}]_{\zeta}E_{i-1}+\zeta E_{i-1}[E_{ip},E'_{pj}]_{\zeta}E_{ip}E_{i-1}\\
&\quad-E_{i-1}E_{ip}[E_{ip},E'_{pj}]_{\zeta}+\bar{\zeta}E_{i-1}[E_{ip},E'_{pj}]_{\zeta}E_{ip}E_{i-1}\\
&\quad+\zeta^2E_{ip}[E_{ip},E'_{pj}]E_{i-1}^2-\zeta[E_{ip},E'_{pj}]_{\zeta}E_{ip}E_{-1}^2\\
&=(\zeta+\bar{\zeta})E_{i-1}E_{ip}E_{i-1}[E_{ip},E'_{pj}]_{\zeta}-E_{ip}E_{i-1}^2[E_{ip},E'_{pj}]_{\zeta}
\end{aligned}$$

$$
\begin{aligned}
&-\bar{\zeta}\{(\zeta+\bar{\zeta})E_{i-1}[E_{ip},E'_{pj}]_\zeta E_{i-1}-[E_{ip},E'_{pj}]_\zeta E_{i-1}^2\}E_{ip}\\
&-(\zeta^2+1)E_{i-1}E_{ip}[E_{ip},E'_{pj}]_\zeta E_{i-1}+(\zeta+\bar{\zeta})E_{i-1}[E_{ip},E'_{pj}]_\zeta E_{ip}E_{i-1}\\
&+\zeta^2E_{ip}\{(\zeta+\bar{\zeta})E_{i-1}[E_{ip},E'_{pj}]_\zeta E_{i-1}-E_{i-1}^2[E_{ip},E'_{pj}]_\zeta\}\\
&-\zeta[E_{ip},E'_{pj}]_\zeta\{(\zeta+\bar{\zeta})E_{i-1}E_{ip}E_{i-1}-E_{i-1}^2E_{ip}\}\\
=&(\zeta+\bar{\zeta})E_{i-1}E_{ip}[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta+(\zeta+\bar{\zeta})[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta E_{ip}E_{i-1}\\
&-\zeta(\zeta+\bar{\zeta})E_{ip}E_{i-1}[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta-\bar{\zeta}(\zeta+\bar{\zeta})[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta E_{i-1}E_{ip}\\
=&(\zeta+\bar{\zeta})[E_{i-1\,p},[E_{i-1\,p},E'_{pj}]_\zeta]_{\bar{\zeta}}.
\end{aligned}
$$

Here we have used $[E_{i-1\,p},E'_{pj}]_\zeta=[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta$. So, when $\zeta+\bar{\zeta}\neq 0$ and $X=0$, it turns out that $[E_{i-1\,p},[E_{i-1\,p},E'_{pj}]_\zeta]_{\bar{\zeta}}=0$. From the formula $[E_{ip},E'_{pj+1}]_\zeta=[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}$ and the relation (1.7), we have

$$
\begin{aligned}
[X,E_j]_{\bar{\zeta}}&=E_{ip}[E_{ip},E'_{pj}]_\zeta E_j-\bar{\zeta}[E_{ip},E'_{pj}]_\zeta E_{ip}E_j\\
&\quad-\zeta E_jE_{ip}[E_{ip},E'_{pj}]_\zeta+\bar{\zeta}^2[E_{ip},E'_{pj}]_\zeta E_{ip}\\
&=E_{ip}[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}-\bar{\zeta}[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}E_{ip}\\
&=[E_{ip},[E_{ip},E'_{pj+1}]_\zeta]_{\bar{\zeta}}.
\end{aligned}
$$

So, if $[E_{ip},[E_{ip},E'_{pj}]_\zeta]_{\bar{\zeta}}=0$, then $[E_{ip},[E_{ip},E'_{pj+1}]_\zeta]_{\bar{\zeta}}=0$. Thus the elements E_{ip} and E'_{pj} satisfy the relations (1.12).

From the equalities $[E'_{pj},E_{i-1}]=0$ and

$$[E_{ip},E'_{pj}]_\zeta E_j^2-(\zeta+\bar{\zeta})E_j[E_{ip},E'_{pj}]_\zeta E_j+E_j^2[E_{ip},E'_{pj}]_\zeta=0.$$

it follows that

$$
\begin{aligned}
&[[Y,E_j]_{\bar{\zeta}^2},E_j]\\
=&[E_{ip},E'_{pj}]_\zeta E'_{pj}E_j^2-\zeta E'_{pj}[E_{ip},E'_{pj}]_\zeta E_j^2\\
&-\bar{\zeta}^2E_j[E_{ip},E'_{pj}]_\zeta E'_{pj}E_j+\zeta^3E_jE'_{pj}[E_{ip},E'_{pj}]_\zeta E_j\\
&-E_j[E_{ip},E'_{pj}]_\zeta E'_{pj}E_j+\bar{\zeta}E_jE'_{pj}[E_{ip},E'_{pj}]_\zeta E_j\\
&+\bar{\zeta}^2E_j^2[E_{ip},E'_{pj}]_\zeta E'_{pj}-\zeta^3E_j^2E'_{pj}[E_{ip},E'_{pj}]_\zeta\\
=&[E_{ip},E'_{pj}]_\zeta\{(\zeta+\bar{\zeta})E_jE'_{pj}E_j-E_j^2E'_{pj}\}\\
&-\zeta E'_{pj}\{(\zeta+\bar{\zeta})E_j[E_{ip},E'_{pj}]_\zeta E_j-E_j^2[E_{ip},E'_{pj}]_\zeta\}\\
&-(\bar{\zeta}^2+1)E_j[E_{ip},E'_{pj}]_\zeta E'_{pj}E_j+(\bar{\zeta}^3+\bar{\zeta})E_jE'_{pj}[E_{ip},E'_{pj}]_\zeta E_j\\
&+\bar{\zeta}^2\{(\zeta+\bar{\zeta})E_j[E_{ip},E'_{pj}]_\zeta E_j-[E_{ip},E'_{pj}]_\zeta E_j^2\}E'_{pj}\\
&-\bar{\zeta}^3\{(\zeta+\bar{\zeta})E_jE'_{pj}E_j[E_{ip},E'_{pj}]_\zeta-E'_{pj}E_j^2[E_{ip},E'_{pj}]_\zeta\}
\end{aligned}
$$

$$=(\zeta+\bar{\zeta})[[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}E'_{pj}E_j+\bar{\zeta}^2(\zeta+\bar{\zeta})E_jE'_{pj}[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}$$
$$-\bar{\zeta}(\zeta+\bar{\zeta})[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}E_jE'_{pj}-\bar{\zeta}(\zeta+\bar{\zeta})E'_{pj}E_j[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}$$
$$=(\zeta+\bar{\zeta})[[E_{ip},E'_{pj+1}]_\zeta,E'_{pj+1}]_{\bar{\zeta}}.$$

Here we have used the equality $[E_{ip},E'_{pj+1}]_\zeta=[[E_{ip},E'_{pj}]_\zeta,E_j]_{\bar{\zeta}}$.

So, if $\zeta+\bar{\zeta}\neq 0$ and $[[E_{ip},E'_{pj}]_\zeta,E'_{pj}]_{\bar{\zeta}}=0$, then $[[E_{ip},E'_{pj+1}]_\zeta,E_{pj+1}]_{\bar{\zeta}}=0$. From the equality $[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta=[E_{i-1p},E'_{pj}]_\zeta$, we have

$$[E_{i-1},Y]_\zeta=E_{i-1}[E_{ip},E'_{pj}]_\zeta E'_{pj}-\bar{\zeta}E_{i-1}E'_{pj}[E_{ip},E'_{pj}]_\zeta$$
$$-\zeta[E_{ip},E'_{pj}]_\zeta E'_{pj}E_{i-1}+E'_{pj}[E_{ip},E'_{pj}]_\zeta E_{i-1}$$
$$=[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta E'_{pj}-\bar{\zeta}E'_{pj}[E_{i-1},[E_{ip},E'_{pj}]_\zeta]_\zeta$$
$$=[[E_{i-1,p},E'_{pj}]_\zeta,E'_{pj}]_{\bar{\zeta}}.$$

If $[[E_{ip},E'_{pj}]_\zeta,E'_{jp}]_{\bar{\zeta}}=0$, then $[[E_{i-1p},E'_{pj}]_\zeta,E'_{pj}]_{\bar{\zeta}}=0$. Thus the elements E_{ip} and E'_{pj} satisfy the relations (1.13).

(ii) Let us show that $[E_{ip},E'_{pj}]_\zeta=[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}$.

We have

$$[E_{ii+1},E'_{i+1i+3}]_\zeta=E_iE_{i+1}E_{i+2}-\zeta E_{i+1}E_iE_{i+2}-\bar{\zeta}E_{i+2}E_iE_{i+1}+E_{i+2}E_{i+1}E_i$$
$$=[E_{ii+2},E_{i+2}]_{\bar{\zeta}}.$$

We suppose that $[E_{ip},E'_{pj}]_\zeta=[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}$. Then we obtain

$$[E_{ip},E'_{pj+1}]_\zeta=[E_{ip},E'_{pj}]_\zeta E_j-\bar{\zeta}E_j[E_{ip},E'_{pj}]_\zeta$$
$$=[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}E_j-\bar{\zeta}E_j[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}$$
$$=E_{ip+1}(E'_{p+1j}E_j-\bar{\zeta}E_jE'_{p+1j})-\bar{\zeta}(E'_{p+1j}E_j-\bar{\zeta}E_jE'_{p+1j})E_{ip+1}$$
$$=[E_{ip+1},E'_{p+1j+1}]_{\bar{\zeta}}$$

and

$$[E_{i-1p},E'_{pj}]_\zeta$$
$$=E_{i-1}[E_{ip},E'_{pj}]_\zeta-\zeta[E_{ip},E'_{pj}]_\zeta E_{i-1}$$
$$=E_{i-1}[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}-\zeta[E_{ip+1},E'_{p+1j}]_{\bar{\zeta}}E_{i-1}$$
$$=(E_{i-1}E_{ip+1}-\zeta E_{ip+1}E_{i-1})E'_{p+1j}-\bar{\zeta}E'_{p+1j}(E_{i-1}E_{ip+1}-\zeta E_{ip+1}E_{i-1})$$
$$=[E_{i-1p+1},E'_{p+1j}]_{\bar{\zeta}}.$$

So the claim holds.

By Lemma 1.1 and Lemma 1.2, we have the equality

$$\begin{aligned} E'^{\bar r}_{ij} &= ([E_{ii+1}, E'_{i+1j}]_{\bar\zeta})^{\bar r} \\ &= ([E_{ii+1}, E'_{i+1j}]_{\zeta})^{\bar r} + (1-\zeta^{-2})^{\bar r}\zeta^{-\frac{\bar r(\bar r-1)}{2}} E_{ii+1}{}^{\bar r} E'_{i+1j}{}^{\bar r} \\ &= ([E_{ii+2}, E'_{i+2j}]_{\bar\zeta})^{\bar r} + (1-\zeta^{-2})^{\bar r}\zeta^{-\frac{\bar r(\bar r-1)}{2}} E_{ii+1}{}^{\bar r} E'_{i+1j}{}^{\bar r}. \end{aligned}$$

Lemma 1.3. *We have the formula*

$$E'^{\bar r}_{ij} = E_{ij}{}^{\bar r} + \sum_{i<p_1<\cdots<p_s<j} ((1-\zeta^{-2})^{\bar r}\zeta^{-\frac{\bar r(\bar r-1)}{2}} E_{ip_1}{}^{\bar r} \cdots E_{p_s j}{}^{\bar r}.$$

Proof. From the equality stated just before the lemma repeatedly, we have that

$$E'^{\bar r}_{ij} = E^{\bar r}_{ij} + \sum_{k=i+1}^{j-1} (1-\zeta^{-2})^{\bar r}\zeta^{-\frac{\bar r(\bar r-1)}{2}} E^{\bar r}_{ik} E'^{\bar r}_{kj}.$$

By induction on $j-i$, we get the claim.

By Lemma 1.3, we obtain $E'^{\bar r}_{ij}=0$ and similarly, $F'^{\bar r}_{ij}=0$.

Now we prove that the definition of the coproduct Δ is compatible with the relation (1.10). We can prove the following formula

$$\Delta(E_{ij}) = E_{ij}\otimes 1 + (1-\zeta^2)\sum_{i<k<j} K_{ik}E_{kj}\otimes E_{ik} + K_{ij}\otimes E_{ij},$$

where $K_{ij}=K_i\cdots K_{j-1}$. We put

$$\begin{aligned} u_1 &= E_{ij}\otimes 1, \\ u_2 &= K_{ii+1}E_{i+1j}\otimes E_{ii+1}, \\ &\vdots \\ u_{j-i} &= K_{ij-1}E_{j-1j}\otimes E_{ij-1}, \\ u_{j-i+1} &= K_{ij}\otimes E_{ij}. \end{aligned}$$

It follows that if $k>l$, then $u_k u_l=\zeta^2 u_l u_k$. As we can write that $\Delta(E_{ij}) = u_1 + (1-\zeta^2)(u_2+\cdots+u_{j-i}) + u_{j-i+1}$, we have

$$\Delta(E_{ij})^m = \sum_{m_1+\cdots+m_{j-i+1}=m} \frac{\phi_m(\zeta^2)}{\phi_{m_1}(\zeta^2)\cdots\phi_{m_{j-i+1}}(\zeta^2)} (1-\zeta^2)^{m_2+\cdots+m_{j-i}} u_1^{m_1}\cdots u_{j-i+1}^{m_{j-i+1}},$$

where $\phi_m(\zeta^2)=(1-\zeta^2)(1-\zeta^4)\cdots(1-\zeta^{2m})$ (see [14]). Putting $m=\bar r$, we can obtain the equality $\Delta(E_{ij})^{\bar r}=0$.

By induction, it follows that $S(E_{ij}) = -K_{ij}^{-1}E'_{ij}$ and $S(F_{ij}) = -\zeta^{2(j-i-1)}F'_{ij}$. We recall that $E'^{\bar{r}}_{ij}=F'^{\bar{r}}_{ij}=0$ and so one can obtain that $S(E_{ij})^{\bar{r}}=S(F_{ij})^{\bar{r}}=0$.

2. A construction of a universal R-matrix for U_ζ

In this section, we construct a universal R-matrix for U_ζ, using the quantum double construction due to Drinfel'd [2]. Our method is similar to that of the construction of the universal R-matrix in [23] and [26].

Let U_ζ^+ be the Hopf subalgebra of U_ζ generated by E_i, $K_i^{\pm}$, $1\le i\le N$ and U_ζ^- the Hopf subalgebra of U_ζ generated by F_i, K_i, $1\le i\le N$ and $(U_\zeta^+)^o$ be the dual algebra of U_ζ^+ with the opposite comultiplication.

First we fix some notations. Let $\{\alpha_i|1\le i\le N\}$ be the system of simple roots and Π_+ the set of positive roots $\alpha_i+\cdots+\alpha_{j-1}$ with $1\le i<j\le N+1$ of $sl(N+1, C)$. We denote by $Q=\oplus Z\alpha_i$ the root lattice and let $(,):Q\times Q\to Z$ be the pairing defined by $(\alpha_i,\alpha_j)=a_{ij}$, where $(a_{ij})_{1\le i,j\le N}$ is the Cartan matrix of type A_N.

We shall put on the set $\{E_{ij}|1\le i<j\le N+1\}$ a total order $\prec$ defining $E_{kl}\prec E_{ij}$ if $k<i$, or $k=i$ and $l<j$. We also denote E_{ij} by E_α for $\alpha\in\Pi_+$ if $\alpha=\alpha_i+\cdots+\alpha_{j-1}$. The following notation will be used in describing a C-basis of U_ζ^+:

$$I=\{(m_\alpha)_{\alpha\in\Pi_+}|0\le m_\alpha<\bar{r}\},$$

$$J=\{(v_i)_{1\le i\le N}|0\le v_p<r,p=1,\cdots,N-1,0\le v_N<a\},$$

$$P=\{v|v=\sum_{i=1}^{N}v_i\alpha_i,(v_i)\in J\}.$$

Moreover, we denote by $\Pi_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}$ for $(m_\alpha)\in I$ ordered monomials of the E_α's according to the total order defined above, $E_{12}^{m_{12}}E_{13}^{m_{13}}\cdots E_{NN+1}^{m_{NN+1}}$, and for $v=\Sigma_{i=1}^N v_i\alpha_i$ with $(v_i)_{1\le i\le N}\in J$, set $K_v=\Pi_{i=1}^N K_i^{v_i}$. In a way similar to Lemma 4.2 in [22], we can derive a system of generators of U_ζ^+.

Proposition 2.1. *The algebra U_ζ^+ is generated by $\{\Pi_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v|(m_\alpha)\in I,(v_i)\in J\}$ as a C-vector space.*

Proof. Using ths relations (1.3),(1.4) and (1.11), any element x of U_ζ^+ can be written as a C-lenear combination of the elements $E_{i_1}\cdots E_{i_m}K_v$ with $1\le i_k\le N$ and $0\le v_i<r$. Let L be the subalgebra generated by K_i, $0\le i\le N$. We remark that L is generated by $\{K_\lambda|\lambda\in P\}$ as a C-vector space. In fact, it follows, from the relations $(\Pi_{i=1}^N K_i^i)^a=1$, $K_N^{ad}=1$, that $K_N^a=(\Pi_{i=1}^{N-1}K_i^i)^a$. So we can write K_N^b for $a\le b\le r-1$ as a product of elements in $\{K_v|(v_i)\in J\}$. Let $P_N=\{((i_1,j_1),\cdots,(i_k,j_k))|(i_p,j_p)\in N\times N,\ 1\le i_p<j_p\le N+1\}\cup\{\phi\}$. For $\Sigma=((i_1,j_1),\cdots,(i_k,j_k))\in P_N$, we put $E_\Sigma=E_{i_1j_1}\cdots E_{i_kj_k}$. We define a map $\eta:P_N\to Z$ given by

$$\eta(\Sigma)=i_1(j_1-i_1)+\cdots+i_k(j_k-i_k) \text{ for } \Sigma\in P_N,\ \eta(\phi)=0.$$

We consider the subspace W_m generated by $\{E_\Sigma|\eta(\Sigma)\leq m\}$. A sequence $\Sigma=((i_1,j_1),\cdots,(i_k,j_k))\in P_N$ is called increasing if $(i_1,j_1)\leq(i_2,j_2)\leq\cdots(i_k,j_k)$. In particular, ϕ is increasing. From [22], for a pair $(s,t)<(x,y)$, we can show

$$E_{xy}E_{st}=\zeta^{\delta_{xs}-\delta_{xt}-\delta_{ys}+\delta_{yt}}E_{st}E_{xy}+\sum_{\substack{\eta(\Sigma)<\eta(((s,t),(x,y)))\\ \Sigma=((i_1,j_1),\cdots,(i_n,j_n))}}c_\Sigma E_{i_1j_1}\cdots E_{i_nj_n} \tag{$*$}$$

for some $c_\Sigma\in \boldsymbol{C}$. By induction on m, we can show that for any m, any element in W_m is written as a $\boldsymbol{C}$-linear combination of the elements in the set $\{E_\Sigma|\eta(\Sigma)\leq m,\ \Sigma \text{ is increasing}\}$ (see [22]).

We give a triangular decomposition of U_ζ, using a way similar to one in [22].

Let us prepare some notations.

· $\tilde{U}_\zeta$ is the algebra over $\boldsymbol{C}$ generated by E_i, F_i, $K_i^{\pm}$, $1\leq i\leq N$ with relations (1.3), (1.4), (1.5).

· $\mathscr{N}_+$ (resp. $\tilde{\mathscr{N}}_+$) is the subalgebra of U_ζ (resp. $\tilde{U}_\zeta$) generated by E_i, $1\leq i\leq N$ along with 1.

· $\mathscr{N}_-$ (resp. $\tilde{\mathscr{N}}_-$) is the subalgebra of U_ζ (resp. $\tilde{U}_\zeta$) generated by F_i, $1\leq i\leq N$ along with 1.

· T (resp. $\tilde{T}$) is the subalgebra of U_ζ (resp. $\tilde{U}_\zeta$) generated by $K_i^{\pm}$, $1\leq i\leq N$ along with 1.

· ϕ_{ij}^+, ϕ_{ij}^-, $1\leq i\neq j\leq N$ are the elements of $\tilde{U}_\zeta$, defined

$$\phi_{ij}^+=\begin{cases}E_iE_j-E_jE_i & \text{if } |i-j|\geq 2,\\ E_i^2E_j-(\zeta+\zeta^{-1})E_iE_jE_i+E_iE_j^2 & \text{if } |i-j|=1,\end{cases}$$

$$\phi_{ij}^-=\begin{cases}F_iF_j-F_jF_i & \text{if } |i-j|\geq 2,\\ F_i^2F_j-(\zeta+\zeta^{-1})F_iF_jF_i+F_iF_j^2 & \text{if } |i-j|=1.\end{cases}$$

· $\mathscr{I}_+$ (resp. $\mathscr{I}_-$) is the two sided ideal of $\tilde{\mathscr{N}}_+$ (resp. $\tilde{\mathscr{N}}_-$) generated by ϕ_{ij}^+, $1\leq i\neq j\leq N$, $E_{ij}^{\bar{r}}$, $1\leq i<j\leq N+1$ (resp. ϕ_{ij}^-, $1\leq i\neq j\leq N$, $F_{ij}^{\bar{r}}$, $1\leq i<j\leq N+1$).

· $\mathscr{I}_0$ is the two sided ideal of $\tilde{T}$ generated by K_i^r-1, $1\leq i\leq N$, Λ^a-1.

· $\mathscr{I}$ is the two sided ideal of $\tilde{U}_\zeta$ generated by ϕ_{ij}^+, $1\leq i\neq j\leq N$, $E_{ij}^{\bar{r}}$, $1\leq i<j\leq N+1$, ϕ_{ij}^-, $1\leq i\neq j\leq N$, $F_{ij}^{\bar{r}}$, $1\leq i<j\leq N+1$, K_i^r-1, $1\leq i\leq N$, Λ^a-1.

We investigate the structute of $\tilde{U}_\zeta$ as a vector space, in a way similar to the proof in Lemma 2.1 and 2.2 in [22].

Let $\mathscr{X}_+$ (resp. $\mathscr{X}_-$) be the free associative $\boldsymbol{C}$-algebra with 1 generators e_i, $1\leq i\leq N$ (resp. f_i, $1\leq i\leq N$). Let $\boldsymbol{C}[k_1^{\pm},\cdots,k_N^{\pm}]$ be the $\boldsymbol{C}$-algebra of Laurent polynomials in indeterminates $k_1,\cdots,k_N$. Let $\mathscr{M}=\mathscr{X}_-\otimes_{\boldsymbol{C}}\boldsymbol{C}[k_1^{\pm},\cdots,k_N^{\pm}]\otimes_{\boldsymbol{C}}\mathscr{X}_+$. The elements $f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}$, $v_1,\cdots,v_N\in\boldsymbol{Z}$, $1\leq i_1,\cdots,i_s,j_1,\cdots,j_t\leq N$, form an $\boldsymbol{C}$-basis of $\mathscr{M}$.

$\mathscr{M}$ has a left U_ζ-module structure defined by

$$K_p\cdot f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}$$
$$=\zeta^{-(\alpha_p,\alpha_{i_1}+\cdots+\alpha_{i_s})}f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_p^{v_p+1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t},$$

$$F_p\cdot f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}$$
$$=f_pf_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t},$$

$$E_p\cdot f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}$$
$$=\zeta^{-(\alpha_p,v_1\alpha_1+\cdots+v_N\alpha_N)}f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_pe_{j_1}\cdots e_{j_t}$$
$$+\frac{1}{\zeta-\bar{\zeta}}\sum_{i_u=p}\{\zeta^{-(\alpha_p,\alpha_{i_{u+1}}+\cdots+\alpha_{i_s})}f_{i_1}\cdots\hat{f}_{i_u}\cdots f_{i_s}k_1^{v_1}\cdots k_p^{v_p+1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}$$
$$-\zeta^{-(\alpha_p,\alpha_{i_{u+1}}+\cdots+\alpha_{i_s})}f_{i_1}\cdots\hat{f}_{i_u}\cdots f_{i_s}k_1^{v_1}\cdots k_p^{v_p-1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}\},$$

where $\hat{f}_{i_u}$ means that f_{i_u} is omitted.

By this fact, it follows that the elements $F_{i_1}\cdots F_{i_s}K_1^{v_1}\cdots K_N^{v_N}E_{j_1}\cdots E_{j_t}$, $v_1,\cdots,v_N\in\mathbf{Z}$, $1\le i_1,\cdots,i_s,j_1,\cdots,j_t\le N$, form a basis of $\tilde{U}_\zeta$. In fact, we have the left $\tilde{U}_\zeta$-module isomorphism $\tau:\tilde{U}_\zeta\to\mathscr{M}$ defined by

$$\tau(F_{i_1}\cdots F_{i_s}K_1^{v_1}\cdots K_N^{v_N}E_{j_1}\cdots E_{j_t})=F_{i_1}\cdots F_{i_s}K_1^{v_1}\cdots K_N^{v_N}E_{j_1}\cdots E_{j_t}\cdot(1\otimes 1\otimes 1)$$
$$=f_{i_1}\cdots f_{i_s}k_1^{v_1}\cdots k_N^{v_N}e_{j_1}\cdots e_{j_t}.$$

So we have $\tilde{U}_\zeta\cong\tilde{\mathscr{N}}_-\otimes\tilde{T}\otimes\tilde{\mathscr{N}}_+$ as a vector space, $\tilde{\mathscr{N}}_+$ (resp. $\tilde{\mathscr{N}}_-$) is a free algebra in the variables E_i (resp. F_i), and $\tilde{T}$ is the Laurent polynomial ring in the variables $K_i^\pm$.

We have $U_\zeta\cong\tilde{U}_\zeta/\mathscr{I}$ as an algebra over $\mathbf{C}$.

We obtain a triangular decomposition of U_ζ. It follows that $U_\zeta\cong\mathscr{N}_-\otimes T\otimes\mathscr{N}_+$ as a vector space, $\mathscr{N}_\pm\cong\tilde{\mathscr{N}}_\pm/\mathscr{I}_\pm$ and $T\cong\tilde{T}/\mathscr{I}_0$ as an algebra over $\mathbf{C}$. It is proved in the following way, which is analogous to the proof of Proposition 2.3 in [22]. It suffices to prove:

$$\mathscr{I}=\tilde{\mathscr{N}}_-\tilde{T}\mathscr{I}_++\tilde{\mathscr{N}}_-\mathscr{I}_0\tilde{\mathscr{N}}_++\mathscr{I}_-\tilde{T}\tilde{\mathscr{N}}_+.$$

To prove it, we show that $\tilde{\mathscr{N}}_-\tilde{T}\mathscr{I}_+$, $\tilde{\mathscr{N}}_-\mathscr{I}_0\tilde{\mathscr{N}}_+$, and $\mathscr{I}_-\tilde{T}\tilde{\mathscr{N}}_+$ are ideals of U_ζ. Firstly, we consider $\mathscr{I}_-\tilde{T}\tilde{\mathscr{N}}_+$. The argument for $\tilde{\mathscr{N}}_-\tilde{T}\mathscr{I}_+$ is analogous. Let $Y=\tilde{\mathscr{N}}_-\tilde{T}\mathscr{I}_+$. It is clear that $K_i^\pm Y\subset Y$, $YK_i^\pm\subset Y$, $F_iY\subset Y$, $YF_i\subset Y$, $YE_i\subset Y$. Let us show that $E_iY\subset Y$. We define the two $\mathbf{C}$-linear maps $E_i^\pm:\tilde{\mathscr{N}}_-\to\tilde{\mathscr{N}}_-$ by

$$E_i^{\pm}(F_{i_1}\cdots F_{i_s})=\sum_{i_u=i}\zeta^{\pm a_u}F_{i_1}\cdots \check{F}_{i_u}\cdots F_{i_s},$$

where $a_u=(\alpha_i,\alpha_{i_{u+1}}+\cdots+\alpha_{i_s})$, so that

$$\begin{aligned}&E_i\cdot F_{i_1}\cdots F_{i_s}K_1^{l_1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\\ &=\zeta^{-(\alpha_i,l_1\alpha_1+\cdots+l_N\alpha_N)}F_{i_1}\cdots F_{i_s}K_1^{l_1}\cdots K_N^{l_N}E_iE_{j_1}\cdots E_{j_t}\\ &+\frac{1}{\zeta-\zeta^{-1}}\sum_{i_u=i}\{E_i^-(F_{i_1}\cdots F_{i_s})K_1^{l_1}\cdots K_i^{l_i+1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\\ &-E_i^+F_{i_1}\cdots F_{i_s})K_1^{l_1}\cdots K_i^{l_i-1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\}.\end{aligned}$$

We can show

$$E_i^{\pm}(F_{i_1}\cdots F_{i_p}\phi_{lm}^-F_{i_s}\cdots F_{i_{s+l}})\in\mathcal{I}_-$$

(see Proposition 2.3 in [22]). Moreover we have

$$\begin{aligned}&E_pF_{i_1}\cdots F_{i_s}F_{ij}^{\bar r}F_{i_k}\cdots F_{i_{k+1}}K_1^{l_1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\\ &=F_{i_1}\cdots F_{i_s}E_pF_{ij}^{\bar r}F_{i_k}\cdots F_{i_{k+1}}K_1^{l_1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\\ &+\frac{1}{\zeta-\zeta^{-1}}\sum_{i_u=p}\{E_p^-(F_{i_1}\cdots F_{i_s})K_pF_{ij}^{\bar r}F_{i_k}\cdots F_{i_{k+1}}K_1^{l_1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\\ &-E_p^+(F_{i_1}\cdots F_{i_s})K_p^{-1}F_{ij}^{\bar r}F_{i_k}\cdots F_{i_{k+1}}K_1^{l_1}\cdots K_N^{l_N}E_{j_1}\cdots E_{j_t}\},\end{aligned}$$

for $1\le p\le N$ and $1\le i<j\le N+1$.

Let us show that $[E_p,F_{ij}^{\bar r}]=0$, for $1\le p\le N$ and $1\le i<j\le N+1$.

If $i<p<j-1$, then we can obtain

$$\begin{aligned}E_pF_{ij}&=E_p(F_{ip}-F_{pj}-\zeta F_{pj}F_{ip})\\ &=F_{ip}E_pF_{pj}-\zeta E_pF_{pj}F_{ip}\\ &=F_{ij}E_p+\zeta F_{ip}K_p^{-1}F_{p+1j}-\zeta^2K_p^{-1}F_{p+1j}F_{ip}\\ &=F_{ij}E_p,\end{aligned}$$

using the equality $E_pF_{pj}=F_{pj}E_p+\zeta K_p^{-1}F_{p+1j}$ and so it follows that $[E_p,F_{ij}^{\bar r}]=0$.

We consider the case $p=i$. We have

$$\begin{aligned}E_iF_{ij}^{\bar r}&=(E_iF_iF_{i+1j}-\zeta F_{i+1j}E_iF_i)F_{ij}^{\bar r-1}\\ &=(F_{ij}E_i+\zeta K_i^{-1}F_{i+1j})F_{ij}^{\bar r-1}\end{aligned}$$

$$=F_{ij}(F_{ij}E_iF_{ij}^{\bar{r}-2}+\zeta K_i^{-1}F_{i+1j}F_{ij}^{\bar{r}-2})+\zeta K_i^{-1}F_{i+1}F_{ij}^{\bar{r}-1}.$$

Here we used the equality $F_{ij}+F_{i+1j}=\zeta^{-1}F_{i+1j}F_{ij}$. By induction, we can obtain that

$$E_iF_{ij}^{\bar{r}}=F_{ij}^{\bar{r}}E_i+\zeta(1+\zeta^{-2}+\cdots+\zeta^{-2(\bar{r}-1)})=F_{ij}^{\bar{r}}E_i.$$

Similarly, we can prove that $[E_{j-1},F_{ij}^{\bar{r}}]=0$.

Thus, we obtain that $E_iY\subset Y$.

Nextly, we consider $\tilde{\mathcal{N}}_-\mathcal{I}_0\tilde{\mathcal{N}}_+$. It suffices to prove that for $X=E$ or F and $1\le i,j\le N$,

$$[X_i,K_j^r-1]=0 \text{ and } [X_i,\Lambda^a-1]=0.$$

Let us show the formulas for E_i. Indeed, we have

$$E_iK_j^r=\zeta^{r}K_j^rE_i=K_j^rE_i$$

and

$$\begin{aligned}E_i\left(\prod_{j=1}^{N}K_j^j\right)^a&=\zeta^{a(\alpha_k,\Sigma j\alpha_j)}\left(\prod_{j=1}^{N}K_j^j\right)^aE_i\\&=\zeta^{\delta_{iN}a(N+1)}\Lambda^aE_i\\&=\zeta^{\delta_{iN}r\frac{N+1}{d}}\Lambda^aE_i\\&=\Lambda^aE_i.\end{aligned}$$

Similarly, we can prove the formulas $[F_i,K_j^r-1]=0$ and $[F_i,\Lambda^a-1]=0$.

The following map $\varphi: U_\zeta^-\to(U_\zeta^+)^o$ plays an important role.

Proposition 2.2. *There is a Hopf algebra homomorphism $\varphi: U_\zeta^-\to(U_\zeta^+)^o$ such that for $X=E_{i_1}\cdots E_{i_m}K_v$,*

$$\varphi(F_i)(X)=\begin{cases}b & \text{if } X=E_iK_v,\\ 0 & \text{otherwise,}\end{cases}$$

$$\varphi(K_i^{\pm})(X)=\begin{cases}\zeta^{\mp(\alpha_i,v)} & \text{if } X=K_v,\\ 0 & \text{otherwise,}\end{cases}$$

where $b=-\frac{1}{\zeta-\zeta^{-1}}$.

Proof. We put $\varphi(F_i)=\xi_i,\varphi(K_i^{\pm})=\eta_i^{\pm}$ and $\eta_{ij}=\eta_i\cdots\eta_{j-1}$. We define ξ_{ij} inductively by

$$\xi_{ij}=\begin{cases}\xi_i & \text{if } j=i+1,\\ \xi_{ij-1}\xi_{j-1}-\zeta\xi_{j-1}\xi_{ij-1} & \text{if } j>i+1.\end{cases}$$

We remark that if $\{i_1,\cdots,i_m\}\neq\{j_1,\cdots,j_n\}$, then

$$\xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}(E_{j_1}E_{j_2}\cdots E_{j_n})=0. \tag{**}$$

Let us prove the fact by induction on. We assume that it holds for $m-1$. Then we have

$$\begin{aligned}&\xi_{i_1}\xi_{i_2}\cdots\xi_{i_m}(E_{j_1}E_{j_2}\cdots E_{j_n})\\ &=\xi_{i_1}\cdots\xi_{i_{m-1}}\otimes\xi_{i_m}(\Delta(E_{j_1})\cdots\Delta(E_{j_n}))\\ &=\xi_{i_1}\xi_{i_{m-1}}\otimes\xi_{i_m}(\sum_{1\le p\le n}E_{j_1}\cdots E_{j_{p-1}}K_{j_p}E_{j_{p+1}}\cdots E_{j_n}E_{j_p})\\ &=\sum_{1\le p\le n}\delta_{i_m j_p}\zeta^{(\alpha_{j_{p+1}}+\cdots+\alpha_{j_n},\alpha_{j_p})}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{m-1}}(E_{j_1}\cdots\hat{E}_{j_p}\cdots E_{j_n})\xi_{i_m}(E_{i_p}).\end{aligned}$$

By the hypothesis of induction, if $\{i_1,\cdots,i_{m-1}\}\neq\{j_1,\cdots,j_{p-1}j_{p+1},\cdots,j_n\}$, then

$$\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{m-1}}(E_{j_1}\cdots\hat{E}_{j_p}\cdots E_{j_n})=0.$$

We consider the pair $(i'j')$ satisfying that $(i'j')<(ij)$ and that there is no pair $(i''j'')$ with $(i'j')<(i''j'')<(ij)$. It follows that

$$\xi_{12}^{m_{12}}\xi_{13}^{m_{13}}\cdots\xi_{i'j'}^{m_{i'j'}}(E_{ij})=0, \tag{1}$$

$$\xi_{ij}(E_{12}^{m_{12}}E_{13}^{m_{13}}\cdots E_{i'j'}^{m_{i'j'}})=0. \tag{2}$$

In fact, $\xi_{12}^{m_{12}}\xi_{13}^{m_{13}}\cdots\xi_{i'j'}^{m_{i'j'}}(E_{ij})$ and $\xi_{ij}(E_{12}^{m_{12}}E_{13}^{m_{13}}\cdots E_{i'j'}^{m_{i'j'}})$are $\boldsymbol{C}$-linear combinations of the elements in (**).

We note that

$$\Delta(E_{ij})=E_{ij}\otimes 1+(1-\zeta^2)\sum_{i<k<j}K_{ik}E_{kj}\otimes E_{ik}+K_{ij}\otimes E_{ij},$$

$$\Delta(\xi_{ij})=\xi_{ij}\otimes\eta_{ij}^{-1}+(1-\zeta^2)\sum_{i<k<j}\xi_{ik}\otimes\eta_{ik}^{-1}\xi_{kj}+1\otimes\xi_{ij}.$$

From these facts, it follows that if $m_\alpha>n_\alpha$ and for any β with $E_\alpha\prec E_\beta$, $m_\beta=0$ or $n_\beta=0$, then

$$\left(\prod_{\alpha\in\Pi_+}\xi_\alpha{}^{n_\alpha}\eta_w\right)\left(\prod_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v\right)=Y\xi_\alpha^{n_\alpha}\otimes\eta_w((XE_\alpha^{m_\alpha}\otimes 1)(K_v\otimes K_v))$$

$$= Y\xi_\alpha{}^{n_\alpha}\otimes(XE_\alpha{}^{m_\alpha}K_v)\eta_w(K_v)$$
$$= Y\xi_\alpha{}^{n_\alpha}(XE_\alpha{}^{m_\alpha})\eta_w(K_v),$$

where $X=\Pi_{E_\beta<E_\alpha}E_\beta^{m_\beta}$ and $Y=\Pi_{E_\beta<E_\alpha}\xi_\beta^{n_\beta}$. By the equality $K_{ij}E_{ij}=\zeta^2E_{ij}K_{ij}$, we obtain

$$\begin{aligned} &Y\xi_\alpha^{n_\alpha}(XE_\alpha^{m_\alpha})\\ &=Y\xi_\alpha^{n_\alpha-1}\otimes\xi_\alpha((X\otimes 1)\Delta(E_\alpha)^{m_\alpha})\\ &=Y\xi_\alpha^{n_\alpha-1}(XE_\alpha^{m_\alpha-1})\xi_\alpha(E_\alpha)[m_\alpha]\\ &=\prod_{\alpha\in\Pi_+}\delta_{m_\alpha n_\alpha}\xi_\alpha(E_\alpha)^{m_\alpha}[m_\alpha]!,\end{aligned}$$

where $[m]=\frac{\zeta^{2m}-1}{\zeta^2-1}$. Here we have used the formula (*). Similarly, for $m_\alpha<n_\alpha$, the similar equality holds. Thus, we compute

$$\left(\prod_{\alpha\in\Pi_+}\xi_\alpha{}^{n_\alpha}\eta_w\right)\left(\prod_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v\right)=\prod_{\alpha\in\Pi_+}\delta_{m_\alpha n_\alpha}\xi_\alpha(E_\alpha)^{m_\alpha}[m_\alpha]!\bar{\zeta}^{(v,w)}.$$

It follows that any element $\Pi_{\alpha\in\Pi_+}\xi_\alpha{}^{n_\alpha}\eta_w$ is zero on ϕ_{ij}, $1\le i\ne j\le N$, $E_{ij}^{\bar{r}}$, $1\le i<j\le N+1$, K_i^r-1, $1\le i\le N$, and Λ^a-1, and from the triangular decomposition of U_ζ, φ is well-defined.

Moreover, the elements ξ_i and $\eta_i{}^{\pm}$ satisfy the following relations:

(1) $\eta_i\eta_j=\eta_j\eta_i, \eta_i^{-1}\eta_i=\eta_i\eta_i^{-1}=\varepsilon,$ (2.1)

(2) $\eta_i\xi_j=\zeta^{-(\alpha_i,\alpha_j)}\xi_j\eta_i,$ (2.2)

(3) $\xi_i{}^2\xi_j-(\zeta+\zeta^{-1})\xi_i\xi_j\xi_i+\xi_i\xi_j^2=0 \quad (|i-j|=1),$ (2.3)

(4) $\xi_i\xi_j=\xi_j\xi_i \quad (|i-j|\ge 2),$ (2.4)

(5) $\xi_{ij}{}^{\bar{r}}=0,$ (2.5)

(6) $\eta_i{}^r=\varepsilon, \left(\prod_{j=1}^N\eta_i{}^i\right)^a=\varepsilon,$ (2.6)

(7) $\Delta(\xi_i)=\xi_i\otimes\eta_i{}^{-1}+1\otimes\xi_i, \Delta(\eta_i^{\pm})=\eta_i{}^{\pm}\otimes\eta_i{}^{\pm},$ (2.7)

(8) $\varepsilon(\xi_i)=0, \varepsilon(\eta_i{}^{\pm})=1,$ (2.8)

(9) $S(\xi_i)=-\xi_i\eta_i, S(\eta_i{}^{\pm})=\eta_i{}^{\mp}.$ (2.9)

One can prove these formulas by easy computations. In the following, we show only the formulas (2.2), (2.5), (2.6) and (2.7). For (2.2), $\eta_i\xi_j$ is non-zero only on E_jK_v

where its value is $\bar{\zeta}^{(\alpha_i,\alpha_j)}b\bar{\zeta}^{(\alpha_i,v)}$ and $\xi_j\eta_i$ is non-zero only on E_jK_v where its value is $b\bar{\zeta}^{(\alpha_i,v)}$. For (2.5), it follows from the above equality that $\xi_{ij}{}^{\bar{r}}(\Pi_{\alpha\in\Pi_+}E_\alpha^{m_\alpha}K_v)=0$. For (2.6), $(\Pi_{j=1}^N\eta_i{}^i)^a$ is non-zero only on K_v. We have that for $1\leq p\leq N$, $(\Pi_{j=1}^N\eta_i{}^i)^a(K_p)=1$. In fact, by the definition of η_i, we have

$$\left(\prod_{j=1}^{N}\eta_i{}^i\right)^a(K_p)=\bar{\zeta}^{a(\Sigma j\alpha_j,\alpha_p)}=\bar{\zeta}^{\delta_{Np}a(N+1)}=1.$$

For (2.7), $\Delta(\xi_i)$ is non-zero only on $E_iK_v\otimes K_w$ and $K_v\otimes E_iK_w$, where their values are respectively $b\zeta^{(\alpha_i,w)}$ and b. On the other hand, $\xi_i\otimes\eta_i$ is non-zero only on $E_iK_v\otimes K_w$, where its value is $b\zeta^{(\alpha_i,w)}$, and $\eta_i^{-1}\otimes\xi_i$ is non-zero only on $K_v\otimes E_iK_w$, where its value is b. The map φ is a Hopf algebra homomorphism.

Proposition 2.3. *We define* $\Phi: U_\zeta^+\times U_\zeta^-\to \boldsymbol{C}$ *by* $\Phi(x,y)=\varphi(y)(x)$ *for* $(x,y)\in U_\zeta^+\times U_\zeta^-$. *Then* Φ *is non-degenerate. Moreover,* $\{\Pi_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v|(m_\alpha)\in I,\ (v_i)\in J\}$ *in proposition* 2.1 *is a* $\boldsymbol{C}$*-basis of* U_ζ^+ *and the Hopf algebra homomorphism* φ *is an isomorphism.*

Proof. By the discussion in the proof of Proposition 2.2, it follows that

$$\Phi\left(\prod_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v,\ \prod_{\alpha\in\Pi_+}F_\alpha{}^{n_\alpha}K_w\right)=\prod_{\alpha\in\Pi_+}\delta_{m_\alpha n_\alpha}\xi_\alpha(E_\alpha)^{m_\alpha}[m_\alpha]!\bar{\zeta}^{(v,w)},$$

where $[m]=\frac{\zeta^{2m}-1}{\zeta^2-1}$, and $[m]!=[m][m-1]\cdots[1]$.

For $v,w\in P$, we put

$$h_{v-w}=\sum_{\mu\in P}\zeta^{(\mu,v-w)}$$

and

$$v-w=x_1\alpha_1+\cdots+x_{N-1}\alpha_{N-1}x_N\alpha_N\ \ ((x_i)=(v_i)-(w_i),(v_i),(w_i)\in J).$$

We have that

$$h_{v-w}=\sum_{\substack{0\leq u_1,\cdots,u_{N-1}\leq r-1\\ 0\leq u_N\leq a-1}}\zeta^{\Sigma_{i=1}^N u_i(-x_{i-1}+2x_i-x_{i+1})}$$

$$=\left(\prod_{i=1}^{N-1}\sum_{u_i=0}^{r-1}\zeta^{u_i(-x_{i-1}+2x_i-x_{i+1})}\right)\sum_{u_N=0}^{a-1}\zeta^{u_N(-x_{N-1}+2x_N)}.$$

We assume $h_{v-w}\neq 0$. Then $\Pi_{i=1}^{N-1}\Sigma_{u_i=0}^{r-1}\zeta^{u_i(-x_{i-1}+2x_i-x_{i+1})}\neq 0$. Hence we have that $-x_{i-1}+2x_i-x_{i+1}\equiv 0\ (\mathrm{mod}\ r)$, $2\leq i\leq N-1$ and $x_2\equiv 2x_1\ (\mathrm{mod}\ r)$. So, it follows that

$$x_{i+1}\equiv 2x_i-x_{i-1}\equiv 2ix_1-(i-1)x_1\equiv(i+1)x_1 \pmod r.$$

Thus we obtain $x_i\equiv ix_1 \pmod r$, $1\leq i\leq N$. From the equality

$$\sum_{u_N=0}^{a-1}\zeta^{u_N(-x_{N-1}+2x_N)}=\sum_{u_N=0}^{a-1}\zeta^{u_N(N+1)x_1}=\sum_{u_N=0}^{a-1}\zeta^{nu_Nx_1}\neq 0,$$

we obtain that $x_1\equiv 0$, (mod a), noting that ζ^n is a primitive a-th root unity. While $x_N\equiv Nx_1$ (mod r) and $ad=r$, we have that $x_N\equiv 0$ (mod a). As $|x_N|<a$, it follows that $x_N=0$. From the formulas $x_i\equiv ix_1$ (mod r) and $x_1\equiv 0$ (mod a), we have that $-x_{N-1}+2x_N\equiv(N+1)x_1\equiv 0$, (mod r) and so $x_{N-1}\equiv 2x_N$ (mod r). From the equality $x_{i-1}\equiv 2x_i-x_{i+1}$ (mod r), by induction, we have that $x_i\equiv(N-i+1)x_N=0$, (mod r). As $|x_i|<r$ for $1\leq i\leq N-1$, we obtain that $x_i=0$ for $1\leq i\leq N-1$. Thus we obtain that $h_{v-w}\neq 0$ if and only if $v=w$. Let $L=|J|$, and then

$$\Phi\left(\frac{1}{L}\sum_{(u_i)\in J}\zeta^{(v,u)}K_u, K_w\right)=\delta_{vw}.$$

For $m=(m_\alpha)_{\alpha\in\Pi_+}$, we put

$$c_m=\prod_{\alpha\in\Pi_+}(\xi_\alpha(E_\alpha))^{m_\alpha}[m_\alpha]!=\prod_{\alpha\in\Pi_+}\left(-\frac{1}{\zeta-\zeta^{-1}}(-\zeta)^{\mathrm{ht}(\alpha)-1}\right)^{m_\alpha}[m_\alpha]!,$$

where c_m is non-zero. From the above discussion,

$$\left\{\frac{1}{Lc_m}\sum_{(u_i)\in J}\zeta^{(v,u)}\prod_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v\right\}_{(m_\alpha)\in I,(v_i)\in J},\ \left\{\prod_{\alpha\in\Pi_+}F_\alpha^{n_\alpha}K_w\right\}_{(n_\alpha)\in I,(w_i)\in J}$$

is a basis for U_ζ^+ and U_ζ^-, and they are orthonormal for the pairing Φ. Thus Φ is non-degenerate and $\{\Pi_{\alpha\in\Pi_+}E_\alpha{}^{m_\alpha}K_v\}_{(m_\alpha)\in I,(v_i)\in J}$ is a C-basis of U_ζ^+, by Proposition 2.1. From the definition of Φ, the homomorphism φ is an isomorphism.

Now we apply the quantum double construction to the Hopf algebra U_ζ^+. By the definition of the multiplication of the quantum double, one can derive the following Lemma.

Lemma 2.4. *Let $e_i=E_i\otimes 1$, $k_i^\pm=K_i^\pm\otimes 1$, $f_i=1\otimes\varphi(F_i)$, and $h_i^\pm=1\otimes\varphi(K_i^\pm)$ in the quantum double $D(U_\zeta^+)$. These elements satisfy the following commutation relations*:

$$(1)\quad k_ih_j=h_jk_i, k_ih_i^{-1}=k_i^{-1}h_i=1, \tag{2.11}$$

$$(2)\quad h_ie_j=\zeta^{(\alpha_i,\alpha_j)}e_jh_i, k_if_j=\zeta^{-(\alpha_i,\alpha_j)}f_jk_i, \tag{2.12}$$

$$(3)\quad [e_i, f_j] = \delta_{ij}\frac{k_i - h_i^{-1}}{\zeta - \zeta^{-1}}. \tag{2.13}$$

Proof. For (2.13), we have

$$f_j e_i = S(\xi_j)(E_i)\cdot h_j^{-1}\cdot \eta_j^{-1}(1) + S(1)(K_i)\cdot e_i f_j \cdot \eta_j^{-1}(1) + S(1)(K_i)\cdot k_j \cdot \xi_j(E_i)$$
$$= \delta_{ij}\frac{h_i^{-1}}{\zeta - \zeta^{-1}} + e_i f_j - \delta_{ij}\frac{k_i}{\zeta - \zeta^{-1}},$$

where $\xi_j = \varphi(F_j)$ and $\eta_i = \varphi(K_i)$. The other relations are also immediately obtained.

The Hopf algebra strucrture on $D(U_\zeta^+)$ induces the one on U_ζ.

Proposition 2.5. *Let us define a map* $\psi: D(U_\zeta^+) \to U_\zeta$ *by* $\psi(x\otimes y) = x\varphi^{-1}(y)$ *for* $x\otimes y \in U_\zeta^+ \otimes (U_\zeta^+)^o \cong D(U_\zeta^+)$. *Then the map* ψ *is a Hopf algebra epimorphism.*

Proof. Comparing Lemma 2.4 with the commutation relations between E_i, F_i and K_i, $1\le i\le N$, one can easily show that ψ is an algebra homomorphism. From the fact that φ^{-1} is a Hopf algebra isomorphism, due to the Hopf algebra structure of $D(U_\zeta^+)$, it follows that ψ is a Hopf algebra homomorphism. The surjectivity of ψ follows from the fact that any element $X_1\cdots X_p$, $X_i \in \{E_i, F_i, K_i^{\pm} \mid 1\le i\le N\}$ is written as a $\mathbf{C}$-linear combination of the elements X_+Y_-, $X_+\in U_\zeta^+$, $Y_-\in U_\zeta^-$, using the relations (1.4) and (1.5).

Now, we obtain an explicit formula for a universal R of U_ζ, as the image of the universal R of $D(U_\zeta^+)$ under $\psi\otimes\psi$.

Theorem 2.6. *A universal R-matirx for* U_ζ *is given by*

$$R = \frac{1}{L}\sum_{\substack{(m_\alpha)\in I\\ (v_i),(w_i)\in J}} \frac{1}{c_m}\zeta^{(v,w)} \prod_{\alpha\in\Pi_+} E_\alpha^{m_\alpha} K_v \otimes \prod_{\alpha\in\Pi_+} F_\alpha^{m_\alpha} K_w, \tag{2.14}$$

where

$$I = \{(m_\alpha)_{\alpha\in\Pi_+} \mid 0\le m_\alpha < \bar{r}\},$$
$$J = \{(v_i)_{1\le i\le N} \mid 0\le v_p < r, p = 1,\cdots,N-1, 0\le v_N < a\},$$
$$L = r^{N-1}a,$$
$$c_m = \prod_{\alpha\in\Pi_+}\left(-\frac{1}{\zeta-\zeta^{-1}}(-\zeta)^{\mathrm{ht}(\alpha)-1}\right)^{m_\alpha}[m_\alpha]! \quad \textit{for } m = (m_\alpha)_{\alpha\in\Pi_+}.$$

Proof. Since the universal R of $D(U_\zeta^+)$ satisfies (1.1) and (1.2), and ψ is a Hopf algebra epimorphism, R also satisfies (1.1) and (1.2).

3. Results from the universal R-matrix for U_ζ

We recall how one can obtain tangle operators from representations of the quasitriangulra Hopf algebra (U_ζ,R), where R is the universal R-matrix for U_ζ in the previous section [13].

For non negative integers k and l, a (k,l)-tangle T is a smooth 1-manifold in $\boldsymbol{R}^2\times[0,1]$ such that its boundary $\partial T=\{(i,0,0)|1\leq i\leq k\}\cup\{(j,0,1)|1\leq j\leq l\}$. We put $\partial T_+=\{(i,0,0)|0\leq i\leq k\}$ and $\partial T_-=\{(j,0,1)|1\leq j\leq l\}$. All tangles are assumed to be oriented.

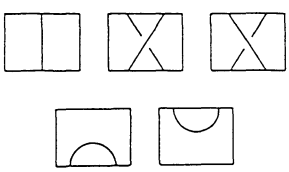
It is well-known that every tangle diagram can be reconstructed from the elementary diagrams in Fig.3.1, using the composition $\circ$ (when defined) ant the tensor product $\otimes$ in the Fig.3.2.

A coloring of a tangle T is defined to be an assignment of a U_ζ-module to each component of T. According to a coloring, we assign U_ζ-modules $T_\pm$ to $\partial T_\pm$ as follows: if an arc S of T has a color V, then to each boundary point in $\boldsymbol{R}^2\times\{0,1\}$ associate V if the orientation is downwards and associate V^* if it is upwards. Then the U_ζ-module T_+ (resp. T_-) is the tensor product from left to right of the U_ζ-modules associated to ∂T_+ (resp. ∂T_-). By convention, $T_\pm=\boldsymbol{C}$ if T is a link.

In this paper, we consider the following family of irreducible representations of U_ζ with $\bar{r}=K+N+1$ for a positive integer K. Let $\alpha_1,\cdots,\alpha_N$ be the simple roots of $sl(N+1,C)$ and we put

$$P_+(K)=\{\lambda\in\mathfrak{h}^*|(\lambda,\alpha_i)\in\boldsymbol{Z},0\leq(\lambda,\alpha_i),i=1,\cdots,N,\ 0\leq(\lambda,\theta)\leq K\},$$

where θ is the longest root, $\mathfrak{h}$ is the Cartan subalgebra of $sl(N+1,C)$. Let $\lambda_1,\cdots,\lambda_N$ be the fundamental dominant integral weight: each λ_i satisfies $(\lambda_i,\alpha_j)=\delta_{ij}$ for any α_j. We see that $\lambda=\Sigma_{i=1}^N m_i\lambda_i$ for integers $m_1,\cdots,m_N$. For each $\lambda\in P_+(K)$, there exists an irreducible highest weight module V_λ of U_ζ with highest weigth λ and



elementary diagrams

Fig. 3.1

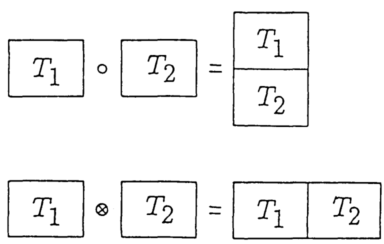


Fig. 3.2

highest weight vector e_λ such that

$$\mathcal{N}_+ e_\lambda = 0,\ V_\lambda = \mathcal{N}_- e_\lambda,\ K_v e_\lambda = \zeta^{(\lambda, v)} e_\lambda.$$

Here $\mathcal{N}_+$ is the subalgebra of U_ζ generated by E_i, $1 \leq i \leq N$ and $\mathcal{N}_-$ is the subalgebra of U_ζ generated by F_i, $1 \leq i \leq N$.

Let T be a colored tangle such that each color of a component of T is contained in the set $\{V_\lambda | \lambda \in P_+(K)\}$. When $S_1, \cdots, S_n$ are the components of T, a coloring of T can be viewed as the map $\{1, \cdots, n\} \rightarrow P_+(K)$. As is shown in [13], there exists a U_ζ-linear map $F_T: T_- \rightarrow T_+$ such that it satisfies $F_{T \circ T'} = F_T \circ F_{T'}$ and $F_{T \otimes T'} = F_T \otimes F_{T'}$, and for elementary diagrams,

$$F_{\downarrow} = \mathrm{id}_{V_\lambda},\ F_{\downarrow} = \mathrm{id}_{V_{\lambda^*}},$$

$$F_{\times}(x \otimes y) = \sum_k \beta_k y \otimes \alpha_k x, \text{ where } R = \sum_k \alpha_k \otimes \beta_k,$$

$$F_{\times}(x \otimes y) = \sum_k \beta'_k y \otimes \alpha'_k x, \text{ where } R^{-1} = \sum_k \alpha'_k \otimes \beta'_k,$$

$$F_{\cap}(f \otimes x) = f(x),\ F_{\cap}(x \otimes f) = f(K_\rho^{-1} x),$$

$$F_{\cup}(1) = \sum_i e_i \otimes e^i,\ F_{\cup}(1) = \sum_i e^i \otimes K_\rho e_i, \text{ (for any basis } \{e_i\}),$$

where $K_\rho = \Pi_{\alpha \in \Pi_+} K_\alpha$. If L is a colored oriented link with coloring v, F_L is a scalar map. We denote this scalar by $J(L, v)$.

In the following proposition, using the explicit formula (2.14) of the universal R for U_ζ, we shall compute two values, which are essential in the construction of 3-manifold invariants. We put $q = \zeta^2$.

Proposition 3.1. (1) *Let $H_{\lambda\mu}$ be a colored Hopf link such that the colors of the two components are V_λ and V_μ drawn in Fig.3.3. Then we have*

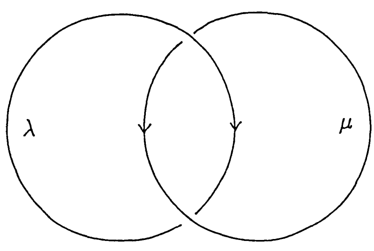


Fig. 3.3

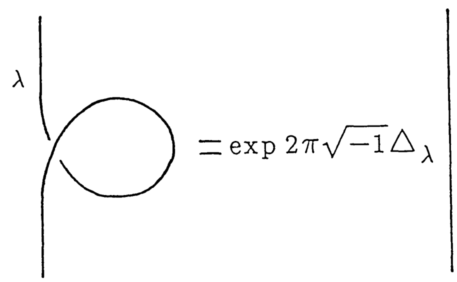


Fig. 3.4

$$J(H_{\lambda\mu})=\frac{\Sigma_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\mu+\rho))}}{\Sigma_{w\in W}(\det w)\bar{q}^{(\rho,w(\rho))}}, \tag{3.1}$$

where ρ is half the sum of positive roots and W is the Weyl group.

(2) *Let T be a colored* (1,1)*-tangle such that the one component has a color V_λ in* Fig.3.4. *Then F_T is the multiplication by* $\exp 2\pi\sqrt{-1}\Delta_\lambda$, *where* $\Delta_\lambda=\frac{(\lambda,\lambda+2\rho)}{2r}$.

Proof. (1) We conside the colored (1,1)-tangle in Fig.3.5. Since V_λ is irreducible, F_Γ is a scalar map. We denote this scalar by $b_{\lambda\mu}$. To compute $b_{\lambda\mu}$, it is enough to evaluate $F_\Gamma(e_\lambda)$ for the highest weight vector e_λ. If $R=\Sigma_k\alpha_k\otimes\beta_k$, then we see $R^{-1}=(\mathrm{id}\otimes S)(R)$. From the definitions of tangle operators, one can obtain

$$F_\Gamma(e_\lambda)=b_{\lambda\mu}e_\lambda=\sum_{k,l}S(\beta_k)\alpha_l\mathrm{Tr}_\mu(K_\rho^{-1}\alpha_k S(\beta_l))e_\lambda.$$

By the formula (2.14), one has

$$b_{\lambda\mu}e_\lambda=\frac{1}{L^2}\sum_{\substack{(m_\alpha)(n_\alpha)\in I\\(v_i)(u_i)(w_i)(u_i')\in J}} S\left(\prod_{\alpha\in\Pi_+}F_\alpha^{n_\alpha}K_w\right)\left(\frac{1}{C_m}\zeta^{(u,v)}\prod_{\alpha\in\Pi_+}E_\alpha^{m_\alpha}K_u\right)$$

$$\mathrm{Tr}_\mu\left(K_\rho^{-1}\frac{1}{C_n}\zeta^{(u',w)}\prod_{\alpha\in\Pi_+}E_\alpha^{n_\alpha}K_{u'}S\left(\prod_{\alpha\in\Pi_+}F_\alpha^{m_\alpha}K_v\right)\right)e_\lambda.$$

Since e_λ is the highest weight vector, the only terms with $m_\alpha=n_\alpha=0$ for any $\alpha\in\Pi_+$ are non zero. Thus one can get

$$b_{\lambda\mu}e_\lambda=\frac{1}{L^2}\sum_{(v_i)(u_i)(w_i)(u_i')\in J}K_w^{-1}\zeta^{(v,u)}K_u\mathrm{Tr}_\mu(K_\rho^{-1}\zeta^{(w,u')}K_{u'}K_v^{-1})e_\lambda$$

$$=\frac{1}{L^2}\sum\zeta^{(w,\lambda)}\zeta^{(v,u)}\bar{\zeta}^{(u,\lambda)}\mathrm{Tr}_\mu(K_\rho^{-1}\zeta^{(w,u')}K_{u'}K_v^{-1})e_\lambda.$$

Noting that $\Sigma_{(u_i)\in J}\zeta^{(u,\lambda-v)}\neq 0$ if and only if $\lambda=v$, we can compute

$$b_{\lambda\mu}e_\lambda=\frac{1}{L}\sum\zeta^{(w,\lambda)}\mathrm{Tr}_\mu(K_\rho^{-1}\zeta^{(w,u')}K_{u'}K_v^{-1})e_\lambda$$

$$=\sum_{\mu_s}\zeta^{(\mu_s,\lambda)}\zeta^{(2\rho,\mu_s)}\zeta^{(\mu_s,\lambda)}e_\lambda$$

$$=\sum_{\mu_s}\bar{q}^{(\lambda+\rho,\mu_s)}e_\lambda,$$

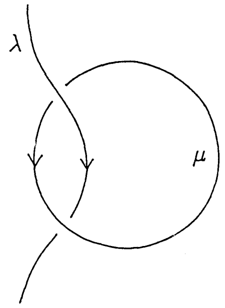


Fig. 3.5

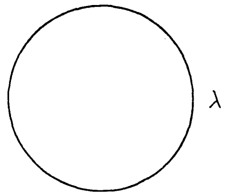


Fig. 3.6

where $\{\mu_s\}$ is the set of weights of V_μ with multiplicity and $\bar{\zeta}^2=q$. It follows from the character formula of Weyl (see for example [10]) that

$$b_{\lambda\mu}=\frac{\Sigma_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\mu+\rho))}}{\Sigma_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\rho))}}.$$

Let L_λ be a colored unknot with a color λ in Fig.3.6. Then we see

$$J(L_\lambda)=\frac{\Sigma_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\rho))}}{\Sigma_{w\in W}(\det w)\bar{q}^{(\rho,w(\rho))}},$$

which is called the quantum dimension of V_λ and we write it by $\dim_q V_\lambda$. Since $J(H_{\lambda\mu})=b_{\lambda\mu}\dim_q V_\lambda$ according to [13, Lemma 2.6], the formula (3.1) holds.

(2) As the representation V_λ is irreducible, the tangle operator F_T is a scalar map. We denote this scalar by v_λ. To compute v_λ, it is enough to evaluate $F_T(e_\lambda)$ for the highest weight vector e_λ of V_λ. When $R=\Sigma\alpha_k\otimes\beta_k$, one can see

$$F_T(e_\lambda)=\alpha_k K_\rho e^\lambda(\beta_k e_\lambda)e_\lambda.$$

From computations similar to the one made in the proof of (1), it follows that

$$\begin{aligned}
v_\lambda e_\lambda &= \frac{1}{L}\sum_{(v_i)(w_i)\in J}\zeta^{(v,w)}K_v K_\rho e^\lambda(K_w e_\lambda)e_\lambda \\
&= \frac{1}{L}\sum_{(v_i)(w_i)\in J}\zeta^{(v,w)}\bar{\zeta}^{(v,\lambda)}\bar{\zeta}^{(\lambda,2\rho)}e^\lambda(\bar{\zeta}^{(\lambda,w)}e_\lambda)e_\lambda \\
&= \bar{\zeta}^{(\lambda,\lambda)}\bar{\zeta}^{(2\rho,\lambda)}e_\lambda \\
&= q^{\frac{1}{2}(\lambda,\lambda+2\rho)}e_\lambda
\end{aligned}$$

Thus the claim holds.

Let $S=(S_{\lambda\mu})$ be the so-called S-matrix due to Kac [7], which is given by

$$S_{\lambda\mu}=\frac{\sqrt{-1}^{N(N+1)/2}}{\sqrt{(N+1)\bar{r}^N}}\sum_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\mu+\rho))}. \tag{3.2}$$

Comparing (3.1) with (3.2), one easily sees that $S_{\lambda\mu}=S_{00}b_{\lambda\mu}$.

By the discussion in [9], for any closed oriented connected 3-manifold M,

$$Z_r(M)=C^{\sigma}\sum_{v\in col(L)}S_{0v(1)}\cdots S_{0v(n)}J(L,v)$$

is a topological invariant of M, where $C=(\exp 2\pi\sqrt{-1}\frac{c}{24})^{-3}$, $c=\frac{K\dim sl(N+1,C)}{r}$, L is a framed link with n components such that M is obtained by Dehn surgery of S^3 along L, σ is the signature of the linking matrix of L, and $col(L)$ means the set of colorings of L.

We denote by $Rep(sl(N+1,C))$ the representation ring of $sl(N+1,C)$). It is well-known that the representations of $sl(N+1,C)$ with fundamental weight λ_i, $1\leq i\leq N$, generate $Rep(sl(N+1,C))$. We put $\partial P_+(K)=P_+(K+1)\backslash P_+(K)$. Let I_K be the ideal of $Rep(sl(N+1,C))$ generated by the representations W_λ, $\lambda\in\partial P_+(K)$. We put $R_K=Rep(sl(N+1,C))/I_K$.

In [4], Goodman-Wenzl showed that the algebra R_K is a free Z- module with basis w_λ corresponding to $\lambda\in P_+(K)$ and that

$$w_\lambda\cdot w_\mu=\sum N_{\lambda\mu}^{v}w_v,$$

for non-negative integers $N_{\lambda\mu}^{v}$, which are called the fusion rule.

In $Rep(U_\zeta)$, the irreducible representation V_λ, $\lambda\in P_+(K)$, can be written as a formal sum of monomials in the fundamental representations V_{λ_i}, $1\leq i\leq N$ such that the monomials are in the span of $\{V_\omega|\omega\in P_+(K)\}$. This follows from the induction on the lexicographic order of Young diagrams, applying Littlewood-Richardson rule to the decomposition of the tensor products of V_λ and V_{λ_i}. Using the formal expressions, we can obtain the decomposition $V_\lambda\otimes V_\mu=\Sigma_{v\in P_+(K)}n_{\lambda v}^{v}V_v+Z_{\lambda\mu}$, for λ, μ, where $n_{\lambda\mu}^{v}$ are integers and $Z_{\lambda\mu}$ is contained in the ideal generated by the irreducible representations V_ω for $\omega\in\partial P_+(K)$. Since in decomposing tensor products of the fundamental representations and V_λ, $\lambda\in P_+(K)$, we can apply Littlewood-Richardson rule, in a way similar to the proof in Lemma 3.1 in [4], we get $n_{\lambda\mu}^{v}=N_{\lambda\mu}^{v}$. It follows that for $\lambda,v\in P_+(K)$,

$$V_\lambda\otimes V_\mu=\sum_{v\in P_+(K)}N_{\lambda\mu}^{v}V_v+Z_{\lambda\mu}. \tag{3.3}$$

We recall that the quantum dimension means the trace of the representation matrix of K_ρ and denote the quantum dimension of U_ζ module by $\dim_q V$. One can extend the definition of the quantum dimension to a C-linear map from $Rep(U_\zeta)$ to C. As the quantum dimension of V_ω, for $\omega \in P_+(K)$, is equal to 0 from the equality $[\bar{r}]=0$ (also see [3]), that of the tensor product of V_ω and any representation of U_ζ is also equal to 0. From these two facts, the extended quantum dimension of $Z_{\lambda\mu}$ is 0.

REMARK. It is shown in [1] that for λ, μ, we have a decomposition

$$V_\lambda \otimes V_\mu = \oplus (M^\nu_{\lambda\nu} \otimes V_\nu) \oplus Z_{\lambda\mu},$$

where the dimension of C-module $M^\nu_{\lambda\mu}$ is equal to $N^\nu_{\lambda\mu}$ and the quantum dimension of $Z_{\lambda\mu}$ is 0. Although, we don't need the fact.

As is shown in [13] for $sl(2,C)$ by Reshetikhin and Turaev, we extend $Z_r(M)$ to $Z_r(M,T)$ for M which contains a colored framed link L. Let T be a colored framed link in S^3 and we suppose that M is obtained by Dehn surgery on L. Then we think of $T \cup L$ as a framed link in S^3, and we put

$$Z_r(M,T) = C^\sigma \sum_{\nu \in col(L)} S_{0\nu(1)} \cdots S_{0\nu(n)} J(L \cup T, \nu).$$

From the above observation, one can get Verlinde's formula for the fusion algebra R_K with the fusion rule due to Goodman-Wenzl.

Proposition 3.2. *The S-matrix* $(S_{\lambda\mu})_{\lambda,\mu \in P_+(K)}$ *and the fusion rule* $N^\nu_{\lambda\mu}$ *satisfy Verlinde's formula:*

$$N^\nu_{\lambda\mu} = \sum_{\varepsilon \in P_+(K)} \frac{S_{\lambda\varepsilon} S_{\mu\varepsilon} S^*_{\nu\varepsilon}}{S_{0\varepsilon}},$$

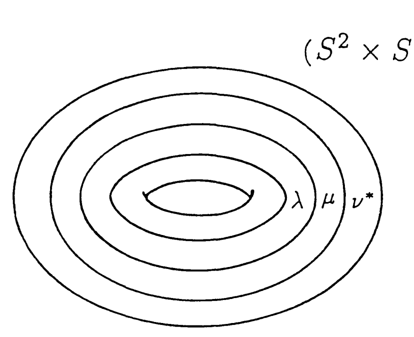


Fig. 3.7

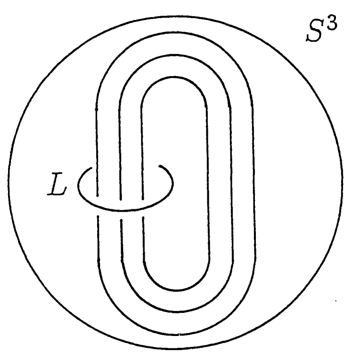


Fig. 3.8

where for $\lambda,\mu \in P_+(K)$,

$$S_{\lambda\mu}=\frac{\sqrt{-1}^{N(N+1)/2}}{\sqrt{(N+1)\bar{r}^N}}\sum_{w\in W}(\det w)\bar{q}^{(\lambda+\rho,w(\mu+\rho))}.$$

Proof. Let us consider $S^2\times S^1$ containing the 3-component link $L_{\lambda\mu\nu*}$ with colors λ, μ, ν^* drawn in Fig.3.7, where for the longest element w_0 in the Weyl group, $\lambda^*=-w_0(\lambda)$. Let L be an unknotted circle with the zero framing which links $L_{\lambda\mu\nu*}$drawn in Fig.3.8. By the Dehn surgery on S^3 along the circle L, one can obtain $(S^2\times S^1,L_{\lambda\mu\nu*})$. In a way similar to the proof in [16, §3], we prove the assertion, evaluating $Z_r(S^2\times S^1,L_{\lambda\mu\nu*})$ in two ways.

We note that for $\lambda\in\partial P_+(K)$, V_λ is irreducible and the quantum dimension $\dim_q V_\lambda=0$, and that a colored link with a component assigned with the tensor product of V_ω, $\omega\in\partial P_+(K)$ and the fundamental representations can be regarded as a colored link with a component assigned V_ω, $\omega\in\partial P_+(K)$. Then, by the formula (3.3) and the unitarity of the S-matrix $(S_{\lambda\mu})$ [7], we can compute

$$\begin{aligned}Z_r(S^2\times S^1,L_{\lambda\mu\nu*})&=\sum_{\varepsilon\in P_+(K)}S_{\varepsilon 0}\left(\sum_{\varepsilon'\in P_+(K)}\frac{S_{\varepsilon'\varepsilon}}{S_{\varepsilon 0}}\frac{S_{\varepsilon'\nu*}}{S_{\varepsilon 0}}\frac{S_{\varepsilon 0}}{S_{00}}N_{\lambda\mu}^{\varepsilon'}\right)\\&=\sum_{\varepsilon'\in P_+(K)}N_{\lambda\mu}^{\varepsilon'}\left(\frac{1}{S_{00}}\delta_{\varepsilon'\nu}\right)\\&=\frac{1}{S_{00}N_{\lambda\mu}^{\nu}}.\end{aligned}$$

On the other hand, a link $L_{\lambda\mu\nu*}\cup L$ can be regarded as the result of connecting 3 Hopf links in a way analogous to the proof in [16], and so we can directly compute from Proposition 3.1 (1)

$$Z_r(S^2\times S^1,L_{\lambda\mu\nu*})=\frac{1}{S_{00}}\sum_{\varepsilon}\frac{S_{\lambda\varepsilon}S_{\mu\varepsilon}S^*_{\nu\varepsilon}}{S_{0\varepsilon}}.$$

Thus the claim follows from the comparison of these two evaluations.

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