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PSEUDODIFFERENTIAL CALCULUS FOR OSCILLATING SYMBOLS

NAMIK KASUMOV

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0. Introduction

The theory of pseudodifferential operators (ψDO) is a powerful tool for describing those properties of solutions to elliptic problems that are not affected by an addition of smoothing operators, notably the pseudolocality and L^p-regularity properties. Its natural generalization, the theory of Fourier integral operators (FIO), develops the property of pseudolocality into a refined analysis of wave propagation whereas the L^p-theory of FIOs (cf. [1],[2],[6],[8]) is centering around the famous formula $\mu \leq -(N-1)|1/p-1/2|$ that relates the parameter p of the L^p-space on which an FIO is bounded with the latter's order μ and with the dimension of the underlying space $\boldsymbol{R}^N$. (For more recent results, as well as a short overview, see [10] where sharp L^p estimates are applied to hyperbolic equations.)

An immediate consequence of the above fomula is that when one goes over from ψsc dos bounded on all L^p, $1<p<\infty$, to general FIOs the scope of possible p's is significantly limited, e.g. for zero order operators $p=2$. This is apparently an inevitable price of generalization, given the well-known fact that solution to the wave equation is *not* bounded on L^p for $p \neq 2$.

Here, we propose a new extension of ψDO theory that combines propagation of singularities with L^p results for all $p \in (1,\infty)$. Although by the very nature of these values of p, the wave equation will be necessarily excluded from possible applications, in turns out that hyperbolic equations with essentially nonzero minor terms can be covered, and we get sharp estimates, in terms of the rate of decay of these minor coefficients, that still tolerates the full range of $1<p<\infty$.

Another point of difference with standard FIOs is the property of expansion of a singular support from a bounded to an unbounded one. In terms of solutions to hyperbolic equations, normally considered as time-dependent operators, this would mean an infinite rate of propagation. However, it is possible to handle finite propagation of singularities by slightly changing the viewpoint and regarding time and space variables together. Since propagation mode becomes too specialized for higher dimensions, this point will be best illustrated in the case $N=2$.

Theorem. *Let $u \in \mathcal{S}'(\boldsymbol{R}^2)$ and let $P: \mathcal{S}'(\boldsymbol{R}^2) \to \mathcal{S}'(\boldsymbol{R}^2)$ be a hyperbolic operator $(Pu)(x,y) \equiv u_{xx} - u_{yy} + a_1 u_x + a_2 u_y + a_0 u$ with real-valued $a_j(x,y) \in C^\infty \cap \mathcal{S}'(\boldsymbol{R}^2)$ satisfying*

$$\inf_{x,y}\{|a_1(x,y) \pm a_2(x,y)|(1+|x|+|y|)^r\} > 0$$

for some $r < 1$. Then: (i) *$u \in L^p_{loc}(\boldsymbol{R}^2)$ whenever $Pu \in L^p_{comp}(\boldsymbol{R}^2)$;* (ii) *$u \in C^\infty(\boldsymbol{R}^2)$ whenever $Pu \in C^\infty(\boldsymbol{R}^2)$;* (iii) *$P$ is invertible modulo the space* $\mathrm{OP}S^{-\infty}(\boldsymbol{R}^2)$ *of smoothing operators.*

If the condition $r < 1$ is relaxed to $r < d$ with $d > 1$, then all implications fail.

Thus, we have a sharp estimate on the rate of decay of minor terms at infinity, that guarantees L^p regularity with a full range of p. The case $a_1 \equiv a_2 \equiv 0$ of the wave operator is clearly outside the hypothesis of the theorem, and we see that the last assertion puts upper bounds, final in terms of powers of distance, on deviations of minor coefficients from identical zero that preclude the full L^p-theory.

We start with a very explicit and straightforward construction: multiply a one-parameter family of symbols by pure oscillations and integrate. Its early unwanted consequence, however, is the fact that such an integration applied to smoothing symbols (undistinguished from zero in ψDO theory) may result in any operator whatsoever. In Section 1 operators arising from these symbols are studied along the lines of comparison with the standard ψDO theory. While the theorems on continuity in Sobolev- and Lebesgue-type spaces are still true (Theorem 1.2), the extension proposed here enjoys, in contrast to ψDOs, the properties of anisotropic smoothing (Theorem 1.3) and of the predictable expansion of a singular support (Theorem 1.5).

A more significant deviation from the classical theory is that factorization modulo smoothing operators (an indispensable tool for ψDOs) cannot be applied indiscriminately to the symbolic integral construction above (Proposition 1.4). In particular, the definition of operator composition in general requires a new approach, different from the usual approximation by operators with proper kernels. For this purpose, we fix an intermediate space, which happens to be the Schwartz space of tempered distributions, that will be left invariant under the action of any admissible operator. (Theorems 2.3 and 2.6). Thus, the main construction must be applied in a more resticted framework: symbols are of polynomial growth whereas the regions of integration are planar varieties (Definition 2.4).

With this, an appropriate symbolic calculus can be developed that includes an asymptotic expansion formula (Theorem 2.7), a well-defined symbolic composition (Proposition 2.8), as well as invariance under linear transformations (Proposition 2.9). These properties are applied in Section 3, where an explicit construction of parametrices is given for a certain class of hyperbolic operators exemplified by the above theorem. All proofs are collected in the final Section 4.

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1. Supersingular pseudodifferential operators

We fix a natural N throughout the paper. Recall that $q \in C^\infty(\boldsymbol{R}^N \times \boldsymbol{R}^N)$ belongs to the symbol space S^m, $m \in \boldsymbol{R}$, if for all $(\alpha,\beta,n) \in \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+$

$$\|q:\alpha,\beta,n;S^m\| \equiv \sup_{|x|\le n} \sup_{\xi \in \boldsymbol{R}^N} |\partial_\xi^\alpha \partial_x^\beta q(x,\xi)|(1+|\xi|)^{-m+|\alpha|} < \infty. \tag{1.1}$$

Pseudodifferential operators are generated from S^m-symbols by the formula

$$\mathrm{OP}q:\mathscr{E}'(\boldsymbol{R}^N) \ni f \mapsto \int e^{ix\xi} q(x,\xi)\hat{f}(\xi)d\xi \in \mathscr{D}'(\boldsymbol{R}^N). \tag{1.2}$$

They are continuous in Sobolev and Lebesgue spaces:

$$\begin{aligned} &\mathrm{OP}:S^m \to \mathscr{L}(H^s_{comp}(\boldsymbol{R}^N), H^{s-m}_{loc}(\boldsymbol{R}^N)), \qquad s \in \boldsymbol{R}, \\ &\mathrm{OP}:S^0 \to \mathscr{L}(L^p_{comp}(\boldsymbol{R}^N), H^p_{loc}(\boldsymbol{R}^N)), \qquad 1/p \in (0,1). \end{aligned} \tag{1.3}$$

Moreover, both of OP above are continuous [11, §2.7, §9.2], the former in all Frechet spaces S^m, $m \in \boldsymbol{R}$, with seminorms (1.1). The space $S^{-\infty}$ defined as $\cap_m S^m$ is also Frechet (we can always restrict $m \in \boldsymbol{Z}_-$ whenever the countability of index sets will be needed).

DEFINITION 1.1. Let σ be a complex (and maybe infinite) measure on $\boldsymbol{R}^N$. We call a mapping $\psi:\boldsymbol{R}^N \to S^m$, $m \in \boldsymbol{R}$, a supersymbol and write $\psi \in SS^m(\sigma)$, or $(\psi,\sigma) \in SS^m$ if $\forall(\alpha,\beta,n) \in \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+$

$$\int \|\psi(t):\alpha,\beta,n;S^m\| d\sigma(t) < \infty. \tag{1.4}$$

Each supersymbol, regarded together with the measure, generates a supersingular pseudodifferential operator (abbr. SuperψDO)

$$T(\psi,\sigma) \equiv \int \mathrm{OP}\psi(t) \circ Sh_t d\sigma(t) \tag{1.5}$$

(here $Sh_t: f(x) \mapsto f(x-t)$ is the translation, or shift operator, cf. §4.1.1).

As usual, notations $\mathrm{OP}SS^m(\sigma)$, $\mathrm{OP}SS^{-\infty}$ etc. stand for the space of operators generated by the corresponding space of supersymbols, i.e. of $SS^m(\sigma), SS^{-\infty} \equiv \cap_m SS^m$ etc.

In the trivial case when σ is the unit measure $\delta(t)$ supported at the origin, $T(\psi,\sigma)$ is the pseudodifferential operator $\mathrm{OP}\psi(0)$. We note also (cf. 2.4) that the integral (1.5) does not depend on the values of ψ away from $\operatorname{supp}\sigma$.

Theorem 1.2. *Each SuperψDO $T(\psi,\sigma)$ is continuous in the Sobolev and Lebesgue spaces below*:

$$
(1.6)\qquad
\begin{aligned}
&T(\psi,\sigma): H^s_{comp}(\boldsymbol{R}^N) \to H^{s-m}_{loc}(\boldsymbol{R}^N), \quad \psi \in SS^m(\sigma), \quad s \in \boldsymbol{R},\\
&T(\psi,\sigma): L^p_{comp}(\boldsymbol{R}^N) \to L^p_{loc}(\boldsymbol{R}^N), \quad \psi \in SS^0(\sigma), \quad 1/p \in (0,1)
\end{aligned}
$$

Observe that PDOs $\partial/\partial t_j$ are well-defined for each Frechet-space-valued mapping ψ as well as for each distribution σ on $\boldsymbol{R}^N$, and that $\partial\psi/\partial x_j: t \mapsto \partial\psi(t)(x,\xi)/\partial x_j$ belongs to $SS^m(\sigma)$ whenever ψ does so.

Theorem 1.3. *Let $(\psi,\sigma) \in SS^m$, $m \in \boldsymbol{R}$, $j = 1, \cdots, N$. Assume that the distribution $\partial\sigma/\partial t_j$ is a (complex and maybe infinite) measure, that $\partial\psi/\partial t_j$ exists everywhere, and that both ψ and $\partial\psi/\partial t_j$ are continuous. Then the following formulas for compositions of a SuperψDO with a partial differential operator ∂_j hold, provided all pairs involved are elements of SS^m*:

$$
(1.7)\qquad
\begin{aligned}
&T(\psi,\sigma) \circ \partial_j = T(\psi, \partial\sigma/\partial t_j) + T(\partial\psi/\partial t_j, \sigma)\\
&\partial_j \circ T(\psi,\sigma) = T(\psi, \partial\sigma/\partial t_j) + T(\partial\psi/\partial t_j, \sigma) + T(\partial\psi/\partial x_j, \sigma)
\end{aligned}
$$

An application of (1.3) to (1.7) shows that such SuperψDOs send H^s into a proper (anisotropic) Sobolev-type subspace in H^{s-m}. In particular, if σ is supported and uniformly distributed on the half-axis $\boldsymbol{R}_j^+ \equiv \{(0,.,0,t_j,0,.,0) \in \boldsymbol{R}^N | t_j \geq 0\}$, i.e.

$$
(1.8)\qquad d\sigma(t) = \delta(x_1) \otimes \cdots \otimes \delta(x_{j-1}) \otimes (1/2)(\operatorname{sgn} t_j) + 1) dt_j \otimes \delta(x_{j+1}) \otimes \cdots \otimes \delta(x_N),
$$

then $T(\psi, \partial\sigma/\partial t_j)$ in (1.7) reduces to $\mathrm{OP}\psi(0)$. In order to employ this observation in §3, we note here a refinement of Theorem 1.3 in the case of (1.8)

Remark 1.3A. If σ is a uniformly distributed on $\boldsymbol{R}_j^+$ measure, the continuity requirements in Theorem 1.3 can be relaxed as follows: $\psi(0, \cdots, t_j, \cdots, 0)$ as a function of one real variable is continuously differentiable on positive half-axis, with continuity, resp. differentiability, at the end-point (zero) understood as one-sided limits.

The theory of pseudodifferential operators extensively uses the factorizations of S^m w.r.t. its subspace $S^{-\infty}$, and of $\mathrm{OP}S^m$ w.r.t. its subspace $\mathrm{OP}S^{-\infty}$. However, if the above construction of a SuperψDO is applied to equivalent modulo $S^{-\infty}$ symbols, it does not necessarily yield equivalent modulo $\mathrm{OP}S^{-\infty}$ operators: dependence of seminorms (1.1) on m becomes crucial.

Proposition 1.4. (i) *For each real $m \in \boldsymbol{R}$ there exists a pair $(\psi,\sigma) \in SS^m$ such that $T(\psi,\sigma) \notin \mathrm{OP}S^{-\infty}$ whereas $\psi(t) \in S^{-\infty}$ for all t.* (ii) *If $(\psi,\sigma) \in SS^m$ for all $m \in \boldsymbol{R}$, then $T(\psi,\sigma) \in \mathrm{OP}S^{-\infty}$, provided* $\operatorname{supp}\sigma$ *is compact.*

REMARK 1.4A. The second statement is true for non-compactly supported measures as well if ψ is a key-controlled supersymbol (see Def. 2.4):

$$\mathrm{OP}KS^{-\infty}V \subseteq \mathrm{OP}S^{-\infty}. \tag{1.9}$$

Although the notations will be introduced later, we state the result here for the sake of further reference. The so-called smoothing operators $T_0 \in \mathrm{OP}S^{-\infty}$ are characterized by the continuity $T_0 : \mathscr{E}'(\boldsymbol{R}^N) \to C^\infty(\boldsymbol{R}^N)$. In particular, $\operatorname{sing\,supp} T_0 f = \emptyset$ for each compactly supported f. The so-called pseudolocality property states that the singular support of a function (distribution) does not increase after an application of a ψDO.

For SuperψDOs this is not true in general, while an actual expansion $\operatorname{sing\,supp} f \subset \operatorname{sing\,supp} Tf$, may well be observed for some f (see §4.1.0). However, certain regularity of the expansion can be described by the set $\operatorname{sing\,supp}(\psi,\sigma)$ which we define as an intersection of all complements to open and σ-measurable subsets $U \subseteq \boldsymbol{R}^N$ with $T(\psi,\sigma|_U) \in \mathrm{OP}S^{-\infty}$, where $\sigma|_U$ is a restriction of a measure (e.g. $(\psi,\sigma|_U) \in SS^{-\infty}$ with a bounded U, cf. 1.4(ii)).

Theorem 1.5. *If $(\psi,\sigma) \in SS^m$, $m \in \boldsymbol{R}$, then for each $f \in \mathscr{E}'(\boldsymbol{R}^N)$*

$$\operatorname{sing\,supp} T(\psi,\sigma) f \subseteq \operatorname{sing\,supp} f + \operatorname{sing\,supp}(\psi,\sigma).$$

2. Calculus of *KS*-symbols

Recall that factorization is particularly helpful in an attempt to provide a correct definition for the composition of two operators since an arbitrary ψDO, which in general offers no more than the $\mathscr{E}' \to \mathscr{D}'$-continuity, happens to be equivalent modulo $\mathrm{OP}S^{-\infty}$ to an $\mathscr{E}'$-preserving ψDO. The composition $\mathrm{OP}q' \circ \mathrm{OP}q'' = \mathrm{OP}(q' \circ q'')$ of $\mathscr{E}'(\boldsymbol{R}^N)$-preserving ψDOs is defined modulo $\mathrm{OP}S^{-\infty}$ via a symbolic composition by

$$q' \circ q'' \equiv \sum_{j=0}^{\infty} (-1)^j \sum_{|\alpha|=j} (\alpha!)^{-1} \partial_x^\alpha q' \partial_\xi^\alpha q'' \qquad (\text{mod } S^{-\infty}). \tag{2.1}$$

It turns out that this asymptotic expansion formula is inapplicable for a correct definition of a composition supersymbol without certain reservations. The point is that, in view of Prop. 1.4.(i), an indiscriminate substitution of OP$\psi(t)$ by an equivalent modulo OP$S^{-\infty}$ operator for each t may result, after integration w.r.t. t, in non-equivalent operators. Moreover, an $\mathscr{E}' \to \mathscr{E}'$-continuous operator may not always be available for a given SuperψDO. Indeed, an addition of a smoothing term T_0 cannot guarantee a compact sing supp$(T+T_0)f$ for a general non-compact sing supp Tf (see §4.1.0).

Thus, the most indispensable tool of ψDO theory, factorization w.r.t. $S^{-\infty}$, must be abandoned when dealing with SuperψDOs. To overcome this difficulty, we shall extend the domain $\mathscr{E}'(\boldsymbol{R}^N)$ and reduce the range $\mathscr{D}'(\boldsymbol{R}^N)$ in (1.2) to a common intermediate subspace that would be left invariant under the action of some SuperψDOs $T(\psi,\sigma)$ taken from a broad enough class. This leads to certain restraints on both ψ and σ.

DEFINITION 2.1. A non-decreasing sequence of natural numbers will be called a key. Let a symbol $q \in S^m$ and a key $k: \boldsymbol{Z}_+ \to \boldsymbol{Z}_+$ be related by

$$\|q: \alpha,\beta; kS^m\| \equiv \sup_n \sup_{\xi \in \boldsymbol{R}^N, |x| \le n} |\partial_\xi^\alpha \partial_x^\beta q(x,\xi)|(1+|\xi|)^{-m+|\alpha|}(1+|x|)^{-k(|\alpha+\beta|)} < \infty \tag{2.2}$$

for all $(\alpha,\beta) \in \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+^N$. We say that such a symbol q is key-controlled (by k) and we denote (i) by kS^m the space of all key-controlled q with a fixed key k, (ii) by KS^m the union of kS^m over all possible keys, (iii) by $KS^{-\infty}$ the intersection of KS^m over all $m \in \boldsymbol{R}$, or all $m \in \boldsymbol{Z}_-$, cf. §4.2.1. (Note that while the lower-case k denotes a specific key, the upper-case K above is used merely as a generic for 'key-controlled').

Clearly, kS^m, KS^m, and $KS^{-\infty}$ are linear subspaces of Frechet space S^m.

Proposition 2.2. (i) *The subspace kS^m of the space S^m is a Frechet space with seminorms* (2.2). (ii) *KS^m is an LF-space of an inductive limit of kS^m w.r.t. the set of keys partially ordered by pointwise domination.* (iii) *$KS^{-\infty}$ is an LF-space of an inductive limit of analogous Frechet subspaces* (*see* §4.2.2) *w.r.t. the same partial ordering.*

Key-controlled symbols enjoy the following important property.

Theorem 2.3. *A ψDO OPq generated by a key-controlled symbol $q \in KS^m$ is continuous regarded as an operator on the space of tempered distributions:*

$$\mathrm{OP}q : \mathscr{S}'(\boldsymbol{R}^N) \to \mathscr{S}'(\boldsymbol{R}^N). \tag{2.3}$$

If we wish this property to hold for SuperψDOS we must consider, in view of Prop. 2.2. (ii), variable key-controlled symbols $\psi(t) \in kS^m \subseteq KS^m$ with the controlling key k independent of t. At the same time, for the Schwartz space of rapidly decreasing functions, the operator seminorms of $Sh_t : \mathscr{S}(\boldsymbol{R}^N) \ni f(x) \mapsto f(x-t) \in \mathscr{S}(\boldsymbol{R}^N)$ are bounded at best by polynomials in $|t|$, and in order to take care of this fact, a rapid decrease of kS^m-seminorms of $\psi(t)$ shall be required in the next definition.

From now on, we shall consider only $\sigma = \sigma_V$ that are dim V-dimensional Lebesgue measures supported on, and consequently determined uniquely by, varieties

$$V = V(v_1, \cdots, v_r) \equiv \{\sum_j t_j v_j | t_j \geq 0,\ v_j \in \boldsymbol{R}^N\}. \tag{2.4}$$

Accordingly, we shall use V instead of σ_V in notations, e.g. $T(\psi, V)$, and whenever we do so, we shall also assume that $\psi(t) = 0$ for $t \notin V$ (cf. Remark 1.3A).

DEFINITION 2.4. A supersymbol $\psi \in SS^m(V)$, $m \in \boldsymbol{R}$, will be called a KS-symbol (Key-controlled Supersymbol) if for each $l \in \boldsymbol{Z}_+$ a key $k = k_l$ may be found so that for each $(\alpha, \beta) \in \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+^N$ thereis a σ_V-integrable function $z_{\alpha\beta l}(t)$, $t \in V$, with

$$|\partial_\xi^\alpha \partial_x^\beta \psi(t)(x, \xi)| \leq z_{\alpha\beta l}(t)(1 + |t|)^{-l}(1 + |x|)^{k_l(|\alpha+\beta|)}(1 + |\xi|)^{m - |\alpha|}. \tag{2.5}$$

We denote by $KS^m V$ the space of all such supersymbols, and by $KS^{-\infty} V$ the intersection of $KS^m V$ over all $m \in \boldsymbol{R}$ (equivalently over all $m \in \boldsymbol{Z}_-$).

Although it may look as if a host of keys is required to verify that ψ is a KS-symbol, a special single key is actually sufficent here if used cumulatively as explained in §4.2.4.

Proposition 2.5. *All $KS^m V$, as well as $KS^{-\infty} V$, are LF-spaces.*

Theorem 2.6. *If $\psi \in KS^m V$, then the corresponding SuperψDO $T(\psi, V)$ sends the space $\mathscr{S}'(\boldsymbol{R}^N)$ of tempered distributions continuously into itself.*

We are now prepared to state the main results of this section. The next assertion reduces, for $V = \{0\}$, to a variant of asymptotic expansion lemma [11, ch.2, §3].

Theorem 2.7. *For each sequence $\psi_j \in KS^{m-j}V$ of KS-symbols there is a KS-symbol $\psi \in KS^m V$, defined uniquely up to $KS^{-\infty}V$, such that for all $J \in \boldsymbol{Z}_+$*

$$\psi - \sum_{j=0}^{J-1} \psi_j \in KS^{m-J}V. \tag{2.6}$$

We shall denote this KS-symbol, or any representative of its class of equivalence modulo $KS^{-\infty}V$ for that matter, by $\Sigma\psi_j$. Like in the classical symbolic calculus of standard ψDOs, we can term by term add, differentiate, and integrate w.r.t. additional parameters, the series of KS-symbols of decreasing order (§4.2.7 and Remark 3.6).

This provides the right tool to revisit the problem of composition operator. If we now put $q' = \psi'(t') \in KS^{m'}$, $q'' = \psi''(t'') \in KS^{m''}$ into (2.1), the composition $T(\psi', V') \circ T(\psi'', V'')$ can be represented by an infinite sum of decreasing order involving with integration over $V' \cap V''$. This way the most general formula is obtainable but, since it is too involved to produce here, we concentrate on the simpler cases needed in §3.

Proposition 2.8. *Let $\psi' \in KS^{m'}V'$ and $\psi'' \in KS^{m''}V''$. Then*

$$T(\psi', V') \circ T(\psi'', V'') \equiv T(\psi' * \psi'', V' + V'') \quad (\mathrm{mod}\, KS^{-\infty}V)$$

where $\psi' * \psi'' \in KS^{m'+m''}$ is explicitly defined below at three special cases.

(i) *If $V'' = \{0\}$, $\psi' = \psi$, $\psi'' = q$, then*

$$(\psi * q)(t)(x, \xi) \equiv \psi(t)(x, \xi) \circ q(x - t, \xi). \tag{2.7}$$

(ii) *If $V' = V'' = V, t \in V$, then*

$$(\psi' * \psi'')(t)(x, \xi) \equiv \int \psi'(s)(x, \xi) \circ \psi''(t - s)(x - s, \xi) d\sigma_V(s). \tag{2.7'}$$

(iii) *If $\dim(V' + V'') = \dim V' + \dim V''$, so that each $t \in V' + V''$ determines uniquely a pair of $t' \in V'$ and $t'' \in V''$ with $t' + t'' = t$, then*

$$(\psi' * \psi'')(t)(x, \xi) \equiv \psi'(t')(x, \xi) \circ \psi''(t'')(x - t', \xi). \tag{2.7'}$$

Formula (2.7′) establishes the closedness of $KS^m V$-algebra expressed as

$$\begin{aligned} KS^{m'}V * KS^{m''}V &\to KS^{m'+m''}V \quad (\mathrm{mod}\, KS^{-\infty}V), \\ \mathrm{OP}KS^{m'}V \circ \mathrm{OP}KS^{m''}V &\to \mathrm{OP}KS^{m'+m''}V \quad (\mathrm{mod}\, \mathrm{OP}KS^{-\infty}V), \end{aligned} \tag{2.8}$$

Finally, classes $KS^m V$ enjoy invariance under linear transformations.

Proposition 2.9. *Let $L: \boldsymbol{R}^N \to \boldsymbol{R}^N$ be a nondegenerate linear map, let $V_L \equiv L(V)$ and let $\psi \in KS^m V$. Then the transformation $T_L: f \mapsto T(\psi, V)(f \circ L) \circ L^{-1}$ of the corresponding SuperψDO under L is also a SuperψDO $T(\psi_L, V_L)$ where $\psi_L \in KS^m V_L$ with the same m.*

3. Parametrix construction

An immediate consequence of the calculus above is that the sum $I+R$ of the identity operator $I: \mathscr{S}'(\boldsymbol{R}^N) \to \mathscr{S}'(\boldsymbol{R}^N)$ and of a SuperψDO $R \in \mathrm{OP}KS^{-1}V$ has a parametrix in the $KS^m V$-algebra, i.e. an inverse modulo $\mathrm{OP}KS^{-\infty}V$, which may be represented by the series $\Sigma(-R)^j \in \mathrm{OP}KS^0 V$. In the next statement we replace the identity operator by a more general one.

Proposition 3.1. *Let $J: \mathscr{S}'(\boldsymbol{R}^N) \to \mathscr{S}'(\boldsymbol{R}^N)$ and $T \in \mathrm{OP}KS^0 V$ be related by*

$$T \circ J = J \circ T = I \qquad (\mathrm{mod}\, \mathrm{OP}KS^{-1}V). \tag{3.1}$$

Assume that R is either from $\mathrm{OP}KS^{-1}V$, or from $\mathrm{OP}KS^{-1}$. Then there exists a $KS^m V$-parametrix $T_R \in \mathrm{OP}KS^0 V$ for $J+R$, i.e.

$$T_R \circ (J+R) = (J+R) \circ T_R = I \qquad (\mathrm{mod}\, \mathrm{OP}KS^{-\infty}V). \tag{3.2}$$

Observe that any $KS^m V$-parametrix, in view of (1.9), is also a parametrix in the standard sense of factorization w.r.t. smoothing operators [11, ch.3, §1]. In order to clarify the alternative hypotheses in 3.1, we note that the more general case (which we don't need here) $R \in \mathrm{OP}KS^{-1}V_1$ with $0 \subseteq V_1 \subseteq V$ is taken care of by an easy analogue of (2.8) unifying (i) and (ii) of Prop. 2.8:

$$\mathrm{OP}KS^{m'}V \circ \mathrm{OP}KS^{m''}V_1 \to \mathrm{OP}KS^{m'+m''}V \qquad (\mathrm{mod}\, \mathrm{OP}KS^{-\infty}V).$$

Let us now apply formula (1.7) of anisotropic smoothing to $J = \partial_j + \mathrm{OP}q$, $q \in KS^0$, and $T(\psi, \boldsymbol{R}_j^+)$ as in (1.8). Elementary transformation reduce (3.1) to a Cauchy problem (cf. (2.7))

$$\begin{cases} \partial / \partial t_j \psi(t)(x,\xi) = -\psi(t)(x,\xi) \circ q(x-t,\xi) & (\mathrm{mod}\, KS^{-1}V) \\ \psi(0)(x,\xi) \equiv 1 & (\mathrm{mod}\, KS^{-1}V) \end{cases} \tag{3.3}$$

with a solution $\psi: t = (0, \cdots, t_j, \cdots, 0) \mapsto KS^0$ sought in $KS^0 \boldsymbol{R}_j^{\pm}$. The solvability is determined by $\mathrm{Re}\, q(x,\xi)$ and its rate of decay to, or rather of "keeping away from" zero.

Theorem 3.2. *Let $J = \partial_j + \mathrm{OP}q$, $(j = 1, \cdots, N)$. Assume that $q = q(x,\xi) = q_0 + q_1$, with $q_0 \in KS^0$, $q_1 \in KS^{-1}$, satisfies*

(3.4) $$\exists r<1,\ \exists B>0:\forall x\in \boldsymbol{R}^N \quad |\operatorname{Re} q_0(x,\xi)|\geq B(1+|x|)^{-r},$$

so that, in particular, $\operatorname{sgn}\operatorname{Re} q_0(x,\xi)$ *is constant. Then the solution to* (3.3), *resp.* (3.1) *is given, for* $\operatorname{sgn}\operatorname{Re} q_0=\pm 1$, *by a Super*$\psi$DO $T(\psi,\boldsymbol{R}_j^{\pm})$ *with*

$$\psi(t)(x,\xi)=\exp\left\{-\int_0^{t_j} q_0(x-t',\xi)dt'_j\right\},\ t'=(0,\cdots,t'_j,\cdots,0),\ t=(0,\cdots,t_j,\cdots,0).$$

The sharpness of the upper bound for r is treated in the next example which shows that Theorem fails if we relax the requirement $r<1$ to $r<d$ with any $d>1$.

EXAMPLE 3.3. Suppose $q(x)=(1+|x|)^{-d}$, $d>1$. Then the kernel of the operator $J=\partial_j+\mathrm{OP}q:\mathscr{S}'(\boldsymbol{R}^N)\to\mathscr{S}'(\boldsymbol{R}^N)$ is infinite-dimensional. According to Fredholm criterion, such an operator cannot be invertible modulo the space $\mathrm{OP}S^{-\infty}$, let alone its subspace $\mathrm{OP}KS^{-\infty}V$ (cf. 3.1 and (1.9)).

As suggested by the property of expansion of singular supports, SuperψDOs may describe the propagation of singularities, and we illustrate this point in a very special case of $q(x,\xi)$ independent of ξ and $N=2$ by considering hyperbolic PDOS

(3.5) $$(Pu)(x,y)\equiv u_{xx}-u_{yy}+a_1(x,y)u_x+a_2(x,y)u_y+a_0(x,y)u$$

with minor coefficients "keeping off zero" in the sense of (3.4).

Theorem 3.4. *Let P be given by* (3.5) *where* $a_j\in\mathscr{S}'\cap C^\infty(\boldsymbol{R}^2)$, $j=0,1,2$, *are real-valued and satisfy*

$$\exists r<1,\ \exists B>0:\forall x\in\boldsymbol{R}^N \qquad |a_1(x,y)\pm a_2(x,y)|\geq B(1+\sqrt{|x|^2+|y|^2})^{-r}.$$

Then there is a SuperψDO $T(\psi,V)\in\mathrm{OP}KS^0V$ *such that*

(3.6) $$P\circ T=T\circ P=I \qquad (\operatorname{mod}\mathrm{OP}KS^{-\infty}V),$$

where V is an oblique quadrant

$$\{(x,y)\}\in\boldsymbol{R}^2|\ \begin{matrix} x=t'\operatorname{sgn}(a_1-a_2)+t''\operatorname{sgn}(a_1+a_2), \\ y=t'\operatorname{sgn}(a_1-a_2)-t''\operatorname{sgn}(a_1+a_2), \end{matrix}\ t'\geq 0,\ t''\geq 0\}.$$

Corollary 3.5. *Let P and V be as above and assume that* $u\in\mathscr{S}'(\boldsymbol{R}^N)$. (i) *If* $Pu\in L^p_{comp}(\boldsymbol{R}^N)$ *then* $u\in L^p_{loc}(\boldsymbol{R}^N)$. (ii) *If* $u\in\mathscr{E}'(\boldsymbol{R}^N)$, *then*

(3.7) $$\operatorname{sing\,supp} u\subseteq \operatorname{sing\,supp} Pu+V.$$

Both statements 3.4 and 3.5 fail if we relax $r<1$ to $r<d$, $d>1$ (see §4.3.3). Inclusion (3.7) describes the propagation of singularities for solutions of hyperbolic equations from an $(n+1)$-dimensional viewpoint (i.e. in terms of functional spaces over $\boldsymbol{R}^{n+1}=\{(x_1,\cdots,x_n,t)\}$) rather than from the standard approach of time cuts (i.e. of t-dependent operators $\mathscr{E}'(\boldsymbol{R}^n(0))\to\mathscr{D}'(\boldsymbol{R}^n(t))$).

For $N>2$, the almost diagonalization (cf. §4.3.4.) yields an analogue of Theorem 3.4 for a class of Nth order operators in $\boldsymbol{R}^N$ described by technically more loaded restrictions of "keeping-off-zero" type on minor terms. The analogue of Corollary 3.5 then describes a very special propagation cone V, namely the one representable as a sum of rays (cf. (2.4)).

REMARK 3.6. The theory developed above essentially carries over to $S^m_{\rho\delta}$-valued super-symbols ($0\le\delta<\rho\le 1$, $\delta+\rho\ge 1$) and to asymptotic expansions over $\psi_j\in KS^{m_j}V$ with monotone $m_j\to-\infty$. Accordingly, in statements 3.1, 3.2, 3.3, resp. in (3.1)–(3.4), we may replace each appearance of $KS^{-1}V$ by $KS^{-\varepsilon}V, \varepsilon>0$.

REMARK 3.7. The analysis of the case of continuity in $L^p(\boldsymbol{R}^N)$ goes along the same lines. In particular, using the results of [7](see also [3]), one can choose an appropriate subspace in KS^mV, closed w.r.t. the $*$-convolution of 2.8, and prove the 3.5 for the $L^p(\boldsymbol{R}^N)\to L^p(\boldsymbol{R}^N)$ case under an additional assumption of boundedness on $\boldsymbol{R}^N$ of all derivatives $\partial^\beta a_j$, with $|\beta|\le 3, j=0,1,2$.

4. Proofs

Subsection 4.S.n. ($n\ge 1$) refers to statement S.n. in Section S, whereas the case 4.S.0 treats other, i.e. non-enumerated, assertions within Section S ($S=0,\cdots,4$).

4.0.0. Part (i) is precisely Corollary 3.5 (i). Part (ii) follows from Part (iii). Part (iii) is contained in (3.6) and (1.9).

4.1.0. If $\psi(t)$ assumes the constant value of a symbol from $S^m\backslash S^{-\infty}$, and σ is as in (1.8) then $\operatorname{sing\,supp} T(\psi,\sigma)f$ strictly includes $\operatorname{sing\,supp} f$ even for most simple $f\in\mathscr{E}'(\boldsymbol{R}^N)$. E.g. for $f=\delta$ it coincides with $\operatorname{supp}\sigma$.

4.1.1. Formula (1.5) contains a composition of two operator-valued functions in $t\in\boldsymbol{R}^N$. The first one is the composition of a continuous OP from (1.3) with a σ-measurable ψ, hence it is σ-measurable. The second one, $Sh: t\mapsto\mathscr{L}(X,X)$, is continuous and bounded in a number of spaces X of interest (e.g. $X=L^p$, $X=H^s$). Therefore, the result is σ-measurable and the Bochner integral $T(\psi,\sigma)$ is meaningful. Its finiteness in some standard cases follows from Lemma 4.1. below (cf. also [5]).

4.1.2. We concentrate on the L^p case. The continuity of an $A: L^p_{comp} \to L^p_{loc}$ means that for every two compacts W and W' in $\boldsymbol{R}^N$ there exists a constant C such that $\|Af\|_{Lp(W')} \leq C\|f\|_{Lp(W)}$, for all $f \in L^p(W)$. Defining $\|A: W, W'\|$ as the greatest lower bound of all such constants C, the continuity in (1.3) means that for each (W'', W') there is a multi-index $(\alpha, \beta, n) \in \boldsymbol{Z}_+^{2N+1}$ such that

$$\|\mathrm{OP}q: W'', W'\| \leq C(W'', W') \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} \sum_{n' \leq n} \|q: \alpha', \beta', n', S^m\|.$$

Consider first a compactly supported σ. Fix a pair of compacts W and W' and let $f \in L^p(W)$ be arbitrary. We have $Sh_t f \in L^p(W'')$ with $W'' = W + \operatorname{supp} \sigma$. Therefore,

$$\|\mathrm{OP}\psi(t) f(\cdot - t)\|_{Lp(W')} \leq \|f\|_{Lp(W)} C(W'', W') \sum_{\alpha' \leq \alpha} \sum_{\beta' \leq \beta} \sum_{n' \leq n} \|\psi(t): \alpha', \beta', n', S^m\|.$$

In order to obtain the required continuity of $T(\psi, \sigma)$ all we need is to integrate the above and apply the following extension of Bochner criterion.

Lemma 4.1. *Let σ be a (complex) measure on $\boldsymbol{R}^N$ and let F be a Frechet space. A σ-measurable function $x: \boldsymbol{R}^N \to F$ is integrable w.r.t. σ iff for each seminorm $\|\cdot\|_j$ in F the real-valued functions $\|x(t)\|_j$ are $|\sigma|$-integrable. Moreover, Minkowsky inequality $\|\int x(t) d\sigma(t)\|_j \leq \int \|x(t)\|_j d|\sigma|(t)$ holds in each seminorm. (Proof in §4.4.1 below.)*

For non-compactly supported measures, decompose $\sigma = \sigma' + \sigma''$ into two others in such a way that $\operatorname{supp} \sigma'$ is compact (hence the continuity of $T(\psi, \sigma')$ is already proven), whereas $\operatorname{supp} \sigma''$ lies outside the ball of radius $\geq \operatorname{diam} W' + \operatorname{diam} W + \operatorname{dist}(W', W) + 1$ which guarantees that $|x - y| \geq 1$ for all $x \in W + \operatorname{supp} \sigma''$, $y \in W'$. Then the kernel of $\mathrm{OP}\psi(t) Sh_t$ is smoothing by Lemma 4.2 (§4.1.5, §4.4.2) and moreover, putting $V = W + \{t\}, U = W', \alpha = \beta = 0, k = N + 1 + |m| > N$ into (4.11) of §4.4.2, we see that

$$\|\chi' \mathrm{OP}\psi(t) Sh_t f\|_p \leq C'' \left\| \int \chi(x-y)(Sh_t f)(y)|x-y|^{-k} dy \right\|_p \sum_{|\alpha'| \leq k} \|\psi(t): \alpha', 0, n; S^m\|$$

where χ' and χ are characteristic functions of W' and of the outside of the unit ball respectively (recall that $|x-y| \geq 1$), and $\|\cdot\|_p$ is the $L^p(\boldsymbol{R}^N)$-norm. Now use Hausdorff-Young inequality $\|\int \chi(x-y)(Sh_t f)(y)|x-y|^{-k} dy\|_p \leq \|Sh_t f\|_p \|\chi(x) x^{-k}\|_1 \leq C'_k \|f\|_p$ and observe that $C'' = C_{mNW'}$ is independent of $V = W + \{t\}$ hence of t. Integrating the resulting inequality along σ'' we finish, as before, with Lemma 4.1.

Argumentation for the Sobolev case in (1.6) is omitted here since it follows easily from the above (e.g. apply $p = 2$ for partial derivatives, then interpolate).

4.1.3. For Frechet-space-valued functions $G: \boldsymbol{R}^N \to F$ and complex measures σ one can verify that integration by parts formula

$$\int \partial_j G(t) d\sigma(t) = -\int G(t) d(\partial_j \sigma)(t) \tag{4.1}$$

is true under the following assumptions (cf. [5] for the Banach- space-valued case): (a) the three integrals $\int G d\sigma$, $\int \partial_j G d\sigma$, $\int G d(\partial_j \sigma)$ are well-defined and finite; and (b) both G and $\partial_j G$ are continuous (Frechet-space-valued) functions on $\boldsymbol{R}^N$. Setting $G(t) = \mathrm{OP}\psi(t) \circ Sh_t$ and F equal to either one of the target spaces in (1.3), we see that both assumptions hold and we get (1.7).

4.1.3A. All the results that the proof of 1.3 refers to, are trivially modified as required in the special case of a singular measure (1.8).

4.1.4. (i) For σ as in (1.8), set $\psi(t) \equiv e^{-tA}$ for $0 \neq t \in \boldsymbol{R}_j^+$ and $\psi(t) \equiv 0$ otherwise, with an elliptic $A \equiv 1 + \sqrt{1 + |\xi|^2} \in S^1$ and $e^{-tA} \in S^{-\infty}$. Then $T(\psi,\sigma) = \int_0^\infty \mathrm{OP} e^{-t(A + i\xi_j)} dt = \mathrm{OP}(A + i\xi_j)^{-1}$ is also elliptic and therefore cannot be smoothing. Here $\psi \in SS^m(\sigma)$ iff $m > -1$. (A compactly supported measure can be obtained by taking $\int_0^1$ rather than $\int_0^\infty$ above since $\int_1^\infty \mathrm{OP} e^{-t(A + i\xi_j)} dt = \mathrm{OP}(e^{-(A + i\xi_j)}(A + i\xi_j)^{-1})$ is smoothing.) To verify the first statement in full, fix an $m \in \boldsymbol{R}$, take the pointwise composition of the above $\psi(t)$ with an elliptic symbol of appropriate order and get a supersymbol lying in $SS^m(\sigma)$.

(ii) Since $\|e^{-it\xi}\psi(t): \alpha,\beta,n; S^m\| \leq (1 + |t|)^{|\alpha|} \max_{\alpha' \leq \alpha} \|\psi(t): \alpha - \alpha',\beta,n; S^{m - |\alpha'|}\|$, then for $\psi \in \cap_m SS^m(\sigma)$ we get $e^{-it\xi}\psi(t) \in \cap_m SS^m(\sigma)$ in e.g. the following two cases: (a) $\operatorname{supp}\sigma$ is compact, so that $(1 + |t|)^{|\alpha|}$ is bounded; or (b) all S^m-seminorms of $\psi(t)$ are rapidy decreasing w.r.t t, so that it compensates the growth of $(1 + |t|)^{|\alpha|}$ at infinity (which is the subject of Def. 2.4). Now we can apply Lemma 4.1 from §4.1.2 to $x(t) = \mathrm{OP}\psi(t) \circ Sh_t \in \mathrm{OP}S^{-\infty}$ to see that the Bochner-Frechet integral $\int x(t) d\sigma(t) = T(\psi,\sigma)$ defines an element of $\mathrm{OP}S^{-\infty}$.

4.1.4A. For a given $\psi \in KS^{-\infty}V = \cap_m \cup_k kS^m V$ (see Def. 2.4) and fixed m,α,α' there is a key k such that $\psi \in kS^{m - |\alpha'|}V$ and therefore, setting $l = |\alpha|$ in (2.5) we see that $(1 + |t|)^{|\alpha|} \|\psi(t): \alpha - \alpha',\beta,n; S^{m - |\alpha'|}\|$ is σ-integrable. Now we can proceed as in §4.1.4(ii)(b) above.

4.1.5. Assuming that a weaker inclusion

$$\operatorname{sing\,supp} T(\psi,\sigma) f \subseteq \operatorname{sing\,supp} f + \operatorname{supp}\sigma \tag{4.2}$$

holds, decompose $T(\psi,\sigma)$ into a sum $T(\psi,\sigma|_U) + T(\psi,\sigma|_{\boldsymbol{R}^N \setminus U})$. If the first term is smoothing, the singular support of the sum is included into any set of the form $\operatorname{sing\,supp} f + \operatorname{supp}(\sigma|_{\boldsymbol{R}^N \setminus U})$ hence into intersection thereof over all possible U's which yields precisely sing supp(ψ,σ). Now, the inclusion (4.2) is obtained by standard arguments (cf. [11, ch.2, §2] for the case supp $\sigma=\{0\}$) from the following refinement of the singular support lemma.

Lemma 4.2. *Let* $K_T\in\mathscr{D}'(\boldsymbol{R}^N\times\boldsymbol{R}^N)$ *be the distribution kernel of a SuperψDO* $T(\psi,\sigma)\in \mathrm{OP}SS^m$, *i.e.* $\langle v,T(\psi,\sigma)u\rangle=\langle v(x)u(y),K_T(x,y)\rangle$, $u\in C_0^\infty$, $v\in C_0^\infty$. *Then* sing supp $K_T\subseteq\{(x+t,x)\in\boldsymbol{R}^N\times\boldsymbol{R}^N|t\in\operatorname{supp}\sigma\}$. *(Proof in §4.4.2 below.)*

4.2.1. Observe that when (2.2) is satisfied by positive reals $k_j^*(\alpha,\beta)$ (rather than by natural $k(|\alpha|+|\beta|)$), possibly depending on additional integer parameter(s) j, we can always construct a monotone $k(r)\equiv 1+[\max_{|\alpha|+|\beta|+|j|\le r}k_j^*(\alpha,\beta)]:\boldsymbol{Z}_+\to\boldsymbol{Z}_+$.

4.2.2. Each $q\in KS^{-\infty}=\cap\{KS^m:m\in\boldsymbol{Z}_-\}$, determines a one-parameter set of keys $k_m=k_m^{(q)}(\cdot)$. For a given key k, we can define the space

$$kS\equiv\{q\in S^{-\infty}|\forall m\in\boldsymbol{Z}_-,\ \sup_n\|q:\alpha,\beta,n;S^m\|(1+n)^{-k(|\alpha+\beta|+|m|)}<\infty\}. \tag{4.3}$$

As before, a sequence $k_m(r)\equiv k(r+|m|)$ is determined uniquely by the given key k. Now we see that $kS\subseteq k_mS^m\subseteq KS^m$, hence $kS\subseteq\cap_m KS^m$, and therefore $\cap_m KS^m=\cup_k kS$. The rest of the assertion is verified by routine calculations.

4.2.3. Fix a monotone system $\|v:r;\mathscr{S}\|\equiv\max_{|\alpha|+s\le r}\sup_x(1+|x|)^s|\partial^\alpha v(x)|$ of seminorms in $\mathscr{S}(\boldsymbol{R}^N)$, and consider $I_v(\xi)\equiv\int q(x,\xi)e^{ix\xi}v(x)dx$. First we shall verify, for $q\in kS^m$, $r\in\boldsymbol{Z}_+$, $v\in\mathscr{S}(\boldsymbol{R}^N)$, the estimate

$$|I_v(\xi)|\le C(r,m,N)(1+|\xi|)^{-r}\times$$

$$\sum_{|\beta|\le 2m+2r}\|q:0,\beta;kS^m\|\cdot\|v:2m+2r+k(2m+2r)+N+1;\mathscr{S}\|. \tag{4.4}$$

Indeed, for arbitrary $\alpha\in\boldsymbol{Z}_+^N$, $\xi\in\boldsymbol{R}^N$, integration by parts yields

$$|\xi^\alpha I_v(\xi)|\le C(\alpha,N)\sum_{\beta\le\alpha}\Big|\int\partial_x^\beta q(x,\xi)e^{ix\xi}\partial^{\alpha-\beta}v(x)dx\Big|,$$

and each term in the sum is estimated, for $|\alpha|\le s$, by

$$\|q:0,\beta;kS^m\|\cdot\int(1+|x|)^{k(|\beta|)}(1+|\xi|)^m|\partial^{\alpha-\beta}v(x)|dx$$

$$\le C_N\cdot\|q:0,\beta;kS^m\|(1+|\xi|)^m\cdot\|v:s+k(s)+N+1;\mathscr{S}\|$$

Since $(1+|\xi|)^s\le C(s,N)\Sigma_{|\alpha|\le 2s}|\xi^\alpha|$, we get, by appropriately summing and putting $s=r+m$, the desired inequality (4.4).

We estimate partial derivatives of $I_v(\xi)$ in a similar fashion and get

$$(4.4)' \qquad (1+|\xi|)^r|\partial_\xi^\alpha I_v(\xi)| \le C \sum_{\alpha' \le \alpha} \sum_{|\beta| \le 2m+2r} \|q: \alpha',\beta; kS^m\| \cdot \|v: r^*; \mathscr{S}\|$$

with $r^* = |\alpha| + 2m + 2r + k(2m+2r) + N + 1$. Now the functional $v \mapsto \langle I_v, \hat{u} \rangle = \langle v, (\mathrm{OP}q)u \rangle$ is well-defined for $v \in \mathscr{S}(\boldsymbol{R}^N)$, $u \in \mathscr{S}'(\boldsymbol{R}^N)$, $q \in kS^m$ and therefore $\mathrm{OP}q$ sends $\mathscr{S}'(\boldsymbol{R}^N)$ continuously into itself. (For more details on continuity cf. §4.2.6.)

4.2.4. As in §4.2.1., we construct a single key $k(r) \equiv \max\{k_l(s+l): s+l \le r\}$ and verify the following: for a KS-symbol ψ definition 2.4 entails σ_V-integrability, for all $(\alpha,\beta,l) \in \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+^N \times \boldsymbol{Z}_+$, of the function

$$(4.5) \qquad \|\psi: \alpha,\beta,l; kS^mV\|(t) \equiv$$

$$\sup_n \sup_{|x| \le n} \sup_{\xi \in \boldsymbol{R}^N} |\partial_\xi^\alpha \partial_x^\beta \psi(t)(x,\xi)|(1+|t|)^l (1+|x|)^{-k(|\alpha+\beta|+l)}(1+|\xi|)^{-m+|\alpha|}$$

and vice versa, this latter condition generates (2.5) with $k_l(r) \equiv k(l+r)$. We shall denote by kS^mV the space of all such KS-symbols. Clearly, it is a Frechet space with integrals of functions in (4.5) as seminorms.

4.2.5. For finite m, KS^mV is an LF-space of an inductive limit of the Frechet spaces $\cup kS^mV$ introduced just above. Introduce, for a fixed key k, the space $kSV = \{\psi \in KS^{-\infty}V | \int \|\psi: \alpha,\beta,l; k_mS^mV\|(t)d\sigma_V(t) < \infty \forall (\alpha,\beta,l,-m) \in \boldsymbol{Z}_+^{2N+2}\}$ where $k_m(\cdot) \equiv k(\cdot + |m|)$ (cf. (4.5)). Each $\psi \in \cap KS^mV$ determines a common key k as in §4.2.1 and lies in $\cap k_mS^mV$. Now $\cap_m \cup_k kS^mV = \cup_k kSV$ are two coinciding subspaces and $KS^{-\infty}V$ is an inductive limit of the spaces kSV w.r.t. pointwise dominated keys(cf. §4.2.2).

4.2.6. We will show that $J_v(\xi) \equiv \int e^{-it\xi} \int \psi(t)(x,\xi) e^{ix\xi} v(x) dx d\sigma_V(t)$ is rapidly decreasing when $v \in \mathscr{S}$. In view of $\langle T(\psi,V)u,v \rangle = \int \hat{u}(\xi) J_v(\xi) d\xi$, this would mean that operator $T(\psi,V)$ preserves $\mathscr{S}'(\boldsymbol{R}^N)$. The continuity in $\mathscr{S}'$ means exactly the following: if v varies in a bounded subset B of $\mathscr{S}(\boldsymbol{R}^N)$, i.e. if $\forall r \in \boldsymbol{Z}_+, \sup_{v \in B} \|v: r; \mathscr{S}\| < \infty$ (notations in 4.2.3), then so does J_v. Since

$$(4.6) \qquad \partial^\gamma J_v(\xi) = \sum_{\alpha \le \gamma} C_\alpha \int t^{\gamma-\alpha} e^{-it\xi} \{\partial_\xi^\alpha \int \psi(t)(x,\xi) e^{ix\xi} v(x) dx\} d\sigma(t)$$

we can put $q = \psi(t)$ into (4.4)′ and integrate w.r.t. $\sigma_V(t)$. For $\psi \in kS^mV$ we have $\|\psi(t): \alpha',\beta; k_lS^m\| \le (1+|t|)^{-l} \|\psi: \alpha',\beta,l; kS^mV\|(t)$ (notation (4.5) is employed) with $k_l(r) = k(r+l)$ and, setting $l = |\gamma| + N + 1$, we get the boundedness of $\{J_v: v \in B\}$.

4.2.7. We choose a sequence of cutoff functions $\chi_j(x,\xi) = \chi(\xi/\rho_j(x))$ with

$\chi \in C^{\infty}(\boldsymbol{R}^N)$, $\chi(\xi) \equiv 0$ for $|\xi| \leq 1/2$, and $\chi(\xi) \equiv 1$ for $|\xi| \geq 1$, whereas $\rho_j(x)$ will be taken in the form $\rho_j(x) = C(1+|x|^2)^m$ with $C = C_{j\chi}$, $m = m_{jk}$ to be determined at the final stage. We can verify by double induction, the inner on $|\beta| \geq 0$, $|\alpha| = 0$ and the outer on $|\alpha| \geq 0$, the formula

$$\partial_\xi^\alpha \partial_x^\beta \chi(\xi \rho_j^{-1}(x)) = \sum_{|\eta| \leq |\alpha+\beta|} \sum_{|\gamma| \leq |\beta|} \sum_{\alpha' \leq \min(\gamma,\alpha)} \partial^\eta \chi(\xi \rho_j^{-1}) \xi^{\gamma-\alpha'} \rho_j^{-|\gamma-\alpha'+\alpha|} F_{j\eta\gamma\alpha'}(x)$$

where $F(x)$ is a finite linear combination of products of derivatives of $x_i/(1+|x|^2)$ (observe that $\partial_i \rho_j^{-r}(x) = C_{jr} \rho_j^{-r}(x) \cdot (x_j/(1+|x|^2)))$. Since we have $1/2 \leq |\xi| \rho_j^{-1} \leq 1$ on the support of $\partial_\xi^\alpha \partial_x^\beta \chi_j$, $\alpha + \beta > 0$, and since each derivative of $x_i/(1+|x|^2)$ hence each $F(x)$ is uniformly bounded, we see that $|\partial_\xi^\alpha \partial_x^\beta \chi_j(x,\xi)| \leq C^*_{j\alpha\beta\chi} \rho_j^{-|\alpha|}(x)$. We may assume without loss of generality that all appearing constants are positive and non-decreasing w.r.t. integer parameters (cf. §4.2.1), in particular, w.r.t. j.

For the collection of $\psi_j \in KS^{m-j}V$ choose, as in §4.2.1, a common key k independent of j, so that $k(\cdot + j) = k_j(\cdot)$ while $\psi_j \in k_j S^{m-j} V$. Direct differentiation of the "remainder" of $\Sigma \chi_j \psi_j$ yields

$$(4.7) \qquad |\partial_\xi^\alpha \partial_x^\beta \sum_{j=J}^{\infty} \chi_j(x,\xi) \psi_j(t)(x,\xi)| \leq C''(1+|t|)^{-l}(1+|\xi|)^{m-J-|\alpha|} \times$$

$$\left(\sum_J^{J'} + \sum_{J'+1}^{\infty} \right) \rho_j^{J-j}(x) C_j'(1+|x|)^{k(|\alpha+\beta|+l+j)} \max_{\alpha' \leq \alpha} \max_{\beta' \leq \beta} \|\psi_j : \alpha', \beta', l; k_j S^{m-j} V\|(t)$$

with some $C'' = C''_{\alpha\beta\chi}$ and $C_j' = C'_{j\alpha\beta\chi}$ (notation (4.5) is employed). Set now $J' = |\alpha+\beta| + l + J$, so that the power of $(1+|x|)$ in each term of the infinite sum becomes $\leq k(2j)$ and observe that $2(1+|x|^2) \geq (1+|x|)^2$ and that $C'''_{j\alpha\beta\chi} \equiv 2^{k(2j)/2} C'' C_j'$ is non-decreasing w.r.t. j. Finally, choose $\rho_j(x)$ as a pointwise *strictly* monotone sequence of smooth functions $C_{j\chi}(1+|x|^2)^{k(2j)/2}$ where $C_{j\chi}$ are defined for $j = -1$ as $\equiv 0$ and for $j \geq 0$ consecutively by

$$C_{j\chi} \equiv C_{j-1\chi} + 2^j \max_{|\alpha+\beta|+l \leq j} C'''_{j\alpha\beta\chi} \max_{\alpha' \leq \alpha} \max_{\beta' \leq \beta} \{1 + \int \|\psi_j : \alpha', \beta', l; k_j S^{m-j} V\|(t) d\sigma_V(t)\}.$$

Now we can integrate the inequality w.r.t. σ and read off the convergence of the series of σ-integrals of terms in (4.7) and the desired inclusion $\Sigma_{j=J}^{\infty} \chi_j \psi_j \in k^* S^{m-J} V$ with a key $k^*(r) \equiv k(2r+J)$.

The rest of the proof, i.e. $\Sigma_0^{J-1}(1-\chi_j)\psi_j \in KS^{-\infty}V$, the independence of $\Sigma \chi_j \psi_j$ modulo $KS^{-\infty}V$ from the choice of the cutoffs, etc. follow the standard argumentation and are virtually identical to the proof of the corresponding classical result in e.g. [12, ch.1, §4]. The same applies to the remark on termwise addition, differentiation, and integration w.r.t. supplementary parameters.

4.2.8. We concentrate on the most difficult part of (ii). First we note that $\forall q \in S^m$

$$Sh_t \circ \mathrm{OP}q(x,\xi) = \mathrm{OP}q(x-t,\xi) \circ Sh_t$$

and that if $\psi \in kS^m V$ then $\partial_\xi^\alpha \partial_x^\beta \psi \in k^* S^{m-|\alpha|} V$ with $k^*(r) = k(|\alpha+\beta|+r)$.

Assuming that $\psi' \in k'S^{m'}V$, $\psi'' \in k''S^{m''}V$, we wish to apply, with the help of (2.1) and the above formula, Theorem 2.7 and the remarks thereafter to $\Sigma \int \psi_j(s,t) d\sigma_V(s)$ with

$$\psi_j(s,t) \equiv \sum_{|\gamma|=j} c_\gamma \partial_\xi^\gamma \psi'(s)(x,\xi) \partial_x^\gamma \psi''(t-s)(x-s,\xi).$$

Recalling that k' and k'' are chosen in §4.2.4 independent of l', resp. of l'', and omitting subindexes in $z(t) \equiv z_{\alpha\beta l}(t)$ from (2.5)-type definitions of KS-symbols ψ' and ψ'', we can directly work out, by using repeatedly the monotonicity of keys and the pair of inequalities

$$(1+|x-s|) \le (1+|x|)(1+|s|), \quad (1+|t-s|)^{-1} \le (1+|s|)(1+|t|)^{-1}, \tag{4.8}$$

the estimate

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \psi_j(s,t)(x,\xi)| \le C_{\alpha\beta j} \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \sum_{|\gamma|=j} z'(s)(1+|s|)^{-l'} \times \\ (1+|x|)^{k'(l'+|\alpha'+\beta'+\gamma|)}(1+|\xi|)^{m'-|\alpha'+\gamma|} z''(t-s)(1+|s|)^{l''}(1+|t|)^{-l''} \times \\ ((1+|x|)(1+|s|))^{k''(l''+|\alpha-\alpha'+\beta-\beta'+\gamma|)}(1+|\xi|)^{m''-|\alpha-\alpha'|} \\ \le C_{\alpha\beta jN}(1+|t|)^{-l''}(1+|x|)^{(k'+k'')(l'+l''+j+|\alpha+\beta|)}(1+|\xi|)^{m'+m''-j-|\alpha|} \times \\ \left(\sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \sum_{|\gamma|=j} z'(s) z''(t-s)(1+|s|)^{k''(l''+j+|\alpha+\beta|+l''-l'} \right) \end{aligned}$$

for arbitrary $l' \in \boldsymbol{Z}_+$ and $l'' \in \boldsymbol{Z}_+$. Choose now $l' = k''(l''+j+|\alpha+\beta|)+l''$ so that integral w.r.t. $\sigma(s)$ of the triple sum, being a finite sum of convolutions of $L^1(V)$-functions z, is itself integrable on V. This proves that $\int(\psi_j(s,\cdot)ds \in kS^{m-j}V$ with $m \equiv m'+m''$ and $k(r) \equiv (k'+k'')(2r+k''(r))$.

The rest of assertion (ii) is now evident. The proofs of (i) and (iii) follow the same lines, and are omitted here as technically subordinate to the one presented.

4.2.9. The statement follows from the invariance under non-degenerate linear transformations (a) of the space $\mathrm{OP}S^m$, (b) of the class (2.4) of varieties V, and (c) of the property of a polynomial growth of derivatives.

4.3.1. For $R_1 \equiv I - T(J+R) \in \mathrm{OP}KS^{-1}V$, operators $R_1^j \equiv R_1 \circ \cdots \circ R_1$ lie in

OP$KS^{-j}V$, so the series $\Sigma_0^\infty R_1^j$ represents (cf.(2.8) and (2.6)) an element of OPKS^0V uniquely modulo OP$KS^{-\infty}V$. Therefore $T_R \equiv (\Sigma R_1^j) \circ T \in$ OPKS^0V is the left parametrix. The right parametrix is found in an analogous way and the equivalence modulo OP$KS^{-\infty}V$ of the two follows from standard arguments.

4.3.2. We shall consider the case $\operatorname{Re} q_0 > 0$ throughout. Observe that the S^m-derivative $\partial\psi/\partial t_j$ coincides with the pointwise (partial) derivative $\partial\psi/\partial t_j(t)(x,\xi)$.

Step 1. The real-valued symbol $q(x,\xi) = q_0(x)$ does not depend on ξ. The equalities in (3.3) are verified directly, and all we need to show is the inclusion $\psi \in KS^0\boldsymbol{R}_j^+$. The latter means that for each $(\beta, l) \in \boldsymbol{Z}_+^{N+1}$ we must find $k_{l\beta} \in \boldsymbol{Z}_+$ such that

$$\sup_{n\in\boldsymbol{Z}_+}\sup_{|x|\le n}(1+|x|)^{-k_{l\beta}}\int(1+|t|)^l|\partial_x^\beta\psi(t)(x,\xi)|dt_j < \infty. \tag{4.9}$$

Assume without loss of generality that the parameter r in the hypotheses (3.4) is positive, so that (cf. (4.8))

$$\int_0^{t_j} q(x-t',\xi)dt_j' \ge B|t|(1+|x|)^{-r}(1+|t|)^{-r} \tag{4.10}$$

holds with a positive B. We can verify by induction the following finite representation

$$\partial_x^\beta \exp\left\{-\int_0^{t_j} q\right\} = \exp\left\{-\int_0^{t_j} q\right\}\sum_{\beta_i}\prod_i C(\beta_i)\partial_x^{\beta_i}\int_0^{t_i} q,$$

where the summation is over all collections of $\beta_i > 0$ with $\Sigma\beta_i = \beta$ (a note important for Step 2 later on). Each factor above may be estimated by $C|t|\cdot\sup_{0\le t_j'\le t_j}|\partial_x^{\beta_i} q(x-t',\xi)|$ and, in view of the polynomial increase of the derivatives of q and the convexity of $(1+(\cdot)^2)^k$, the appearing sup's are to be estimated at the end-points (0 or else t_j), and we get

$$|\partial_x^\beta \exp\left\{-\int_0^{t_j} q(x-t')dt_j'\right\}| \le C_\beta|t|^k(1+|t|)^k(1+|x|)^k\exp\left\{-\int_0^{t_j} q(x-t')dt_j'\right\}$$

for some $k = k^{(\beta)}$ depending on β only. Putting this together with (4.10) we obtain an interim estimate

$$(1+|t|)^l\left|\partial_x^\beta \exp\left\{-\int_0^{t_j} q(x-t')dt_j'\right\}\right|$$
$$\le C_\beta|t|^k(1+|t|)^{k+l}(1+|x|)^k\exp\{-B|t|(1+|x|)^{-r}(1+|t|)^{-r}\}.$$

While integrating both sides of it, we will use $e^{-y}\le 1$, $y>0$, for the finite interval and $e^{-y}\le (m/ey)^m$, $m>0$, $y>0$, for the infinite one, with the choice of m to be done later. We get

$$\int_0^\infty (1+|t|)^l \left|\partial_x^\beta \exp\left\{-\int_0^{t_j} q(x-t')dt'_j\right\}\right| dt_j$$

$$\le C(1+|x|)^k \int_0^1 |t|^k(1+|t|)^{k+l}dt_j + C(1+|x|)^{k+rm}\int_1^\infty (1+|t|)^{k+l+rm}|t|^{k-m}dt_j$$

$$\le C'(1+|x|)^k + C'(1+|x|)^{k+rm}\int_1^\infty |t|^{2k+l-(1-r)m}dt_j$$

and setting $m=(2k+l+2)/(1-r)$, we arrive at the desired estimate (4.9) with $k_{l\beta}=k+(2k+l+2)r/(1-r)\to +\infty$ when $r\to 1$.

Step 2. The real-valued symbol $q(x,\xi)=q_0(x,\xi)$ itself satisfies (3.4). The arguments of Step 1 are still valid. Indeed, an application of ∂_ξ^α will simply create an additional factor $(1+|\xi|)^{-|\alpha|}$ in the estimates plus a possible dependence of k and, ultimately, of m, on the multi-index α. Therefore $\psi \in KS^0V$, and each term except for the first one in the asymptotic expansion of $\psi(t)(x,\xi)\circ q_0(x-t,\xi)$ lies in $KS^{-j}V$, $j>0$. Thus, $\partial\psi(t)(x,\xi)/\partial t+\psi\cdot q_0(x-t,\xi)$ is an element of $KS^{-1}V$. (By way of an aside, observe that if q_0 is homogeneous of order 0, then $B(\xi)\equiv \inf_x q_0(x,\xi)\cdot(1+|x|)^r$ is positive and continuous on the unit sphere, hence B may be taken as $\inf_\xi B(\xi)>0$.)

Step 3. The real-valued $q=q_0+q_1$ where q_0 is discussed in Step 2 and $q_1\in KS^{-1}$. The very last part of the proof of Step 2 applies as well here: $\partial\psi/\partial t(t)(x,\xi)\equiv\psi(t)(x,\xi)\cdot q_0(x-t,\xi)\equiv\psi(t)(x,\xi)\cdot(q_0+q_1)(x-t,\xi) \bmod KS^{-1}V$. Let us also note that with the Remark 3.6 we can substitute, in Steps 2 and 3, $KS^{-1}V$ by $KS^{-\varepsilon}V$, $\varepsilon>0$, throughout.

Step 4. The symbol q is complex-valued. Again equalities (3.3) are straightforward, while the above argumentation holds: indeed, $|\psi(t)|=\exp\{-\int \mathrm{Re}\, q\}$ with real-valued $\mathrm{Re}\, q$ treated above.

4.3.3. Equality $\partial_j u+qu=0$ holds for all distributions u of the form

$$u(x_1,\cdots,x_N)=v(x_1,\cdots x_{j-1},x_{j+1},\cdots,x_N)\exp\left\{-\int_{-\infty}^{x_j} q(x_1,\cdots,t_j,\cdots,x_N)dt_j\right\}.$$

Clearly, $u\in\mathcal{S}'(\boldsymbol{R}^N)$ as soon as $v\in\mathcal{S}'(\boldsymbol{R}^{N-1})$, and the image of the latter set in $\mathcal{S}'(\boldsymbol{R}^N)$ is infinite-dimensional. Furthermore, choosing $v_0\notin L^p(\boldsymbol{R}^{N-1})$ with a non-empty singular support, we see that $Pu_0=(\partial_j+\mathrm{OP}q)u_0=0\in L^p(\boldsymbol{R}^N)$ but $u_0\notin L^p(\boldsymbol{R}^N)$. Also, $\operatorname{sing\,supp} u_0\neq\emptyset$ whereas $\operatorname{sing\,supp} Pu_0+V=\emptyset+V=\emptyset$. Therefore, if we permit the upper bound for r to be greater than 1 then, not just implications in Theorem

3.2, but also those of Corollary 3.5 fail.

4.3.4. Applying Proposition 3.1 we reduce the proof to solving (3.1) with $J=P$ from (3.5). This, in turn, will be reduced to two other problems of exactly the same type, i.e. $J=P_+$ and $J=P_-$, as soon as P will be represented modulo $\mathrm{OP}KS^{-1}$ via a composition $P=P_+\circ P_-$. Indeed, put $P_\pm u=u_x\pm u_y+(a_1\mp a_2)u/2+b(x,y)\cdot A_\pm u$ with $b(x,y)=(\partial/\partial x+\partial/\partial y)(a_1+a_2)/2+(a_1^2-a_2^2)/4-a$, $A_\pm=\mathrm{OP}((i/2)(\xi_1\mp\xi_2)|\xi|^{-2})\in\mathrm{OP}KS^{-1}$. Applying Proposition 3.1, this time to each of $P_\pm$, we further reduce the problem to solving (3.1) with $J=P'_\pm\equiv\partial_x\pm\partial_y+(a_1\mp a_2)/2$. By a suitable choice of linear mappings $L_\pm$ of the coordinate space, these $P'_\pm$ may be transformed to some $P''_\pm$ satisfying all the hypotheses of Theorem 3.2. Its application yields two $KS^mR_j^\mp$-parametrices for $P''_\pm$ whereas an application of Proposition 2.9 sends these to $KS^mV_\pm$-parametrices for $P''_\pm$. Observe now that if $T_\pm=T(\psi_\pm,V_\pm)$ are $KS^mV_\pm$-parametrices for $P'_\pm$, then $T_-\circ T_+$ is a KS^mV-parametrix for $P=P_+\circ P_-$ with V equal to the sum of oblique rays $V_\pm$ and we may apply Proposition 2.8(iii) to get a $T(\psi,V)$ from $\mathrm{OP}KS^0V$. The precise form of V is verified by simple computation.

4.3.5. Observe that elements of $\mathrm{OP}KS^{-\infty}V$ are smoothing and therefore they eliminate singular supports and also send $L^p_{comp}(\boldsymbol{R}^N)$ to $L^p_{loc}(\boldsymbol{R}^N)$. Thus, assertions (i) and (ii) are consequences of invertibility of P modulo $\mathrm{OP}KS^{-\infty}V$ and of Theorems 1.2 and 1.5 respectively.

4.4.1. Proof of Lemma 4.1 from §4.1.2. The necessity is evident, so we prove sufficiency. Assume, without loss of generality, that σ is finite, that seminorms are indexed by natural numbers $j\in\boldsymbol{Z}_+$, and $\|\cdot\|_j\le\|\cdot\|_{j+1}$ for all j. Fix a seminorm (i.e. $j\in\boldsymbol{Z}_+$) and suppose that $\|x(t)\|_j$ is $|\sigma|$-integrable. Choose a sequence $\{x_n\}$ of plain functions converging $|\sigma|$-a.e. to $x(t)$. By plain functions we understand those x_n that are representable as a finite linear combination in F of the characteristic functions of $|\sigma|$-measurable sets, i.e. $x_n(t)\equiv x_{nk}\in F$ when $t\in T_k$, where $\{T_k\}$ is a disjoint covering of $\boldsymbol{R}^N$ by $|\sigma|$-measurable subsets.

Since each Frechet space is metrizable, we may apply Egorov's theorem (or its appropriate extension) and find a subset S_j with $|\sigma|(\boldsymbol{R}^N\backslash S_j)$ less than $\varepsilon=1/j$ upon which $x_n\to x$ uniformly. Hence $\|x_n(s)-x_j(s)\|_j\to 0$ also uniformly in S_j and therefore $\exists n(j)$: $\forall s\in S_j$ $\forall n\ge n(j)$ $\|x_n(s)-x(s)\|_j<1/j$. We can always choose the subsets and the numbering in such a way that they are monotonous w.r.t. j, i.e. $S_j\subseteq S_{j+1}$, and $n(j)\le n(j+1)$.

Define $y_k(s)\equiv x_{n(k)}(s)$ on S_k and $\equiv 0$ outside S_k. Clearly, y_k is a plain function. While $|\sigma|(\boldsymbol{R}^N\backslash\cup_m S_m)=0$ we have: $\forall s\in\cup_m S_m$, $\exists l=l(s):s\in S_l$

$$Z_{kj}(s)\equiv\|y_k(s)-x(s)\|_j\le\|y_k(s)-x(s)\|_k\le\|x_{n(k)}(s)-x(s)\|_k\le 1/k$$

as soon as $k \geq \max\{j,l\}$ (so that $s \in S_l \subseteq S_k$). Hence Z_{kj} pointwise tends to zero as k tends to infinity. On the other hand, as soon as $k \geq j$, the above-mentioned monotonicity yields $Z_{kj}(s) \leq \text{const} = 1/j$ for $s \in S_k$, and $Z_{kj}(s) \leq \|x(s)\|_j$ for $s \in \cup_m S_m \backslash S_k$. Thus, $Z_{kj}(s)$ is $|\sigma|$-a.e. dominated (uniformly in k) by a sum of two $|\sigma|$-integrable functions and we can apply Lebesgue-Fatou lemma to get

$$\lim_{k \to \infty} \int \|y_k(s) - x(s)\|_j d|\sigma|(s) = \int \lim_{k \to \infty} Z_{kj}(s) d|\sigma|(s) = 0.$$

Since plain functions $y_k(s)$ tend $|\sigma|$-a.e. to $x(s)$ and since $y_k(s)$ does not depend on j, we see that $x(s)$ is Bochner-Frechet-integrable w.r.t. σ.

4.4.2. Proof of Lemma 4.2 from §4.1.5. Let $K_t(x,y) = \int \psi(t)(x,\xi) e^{i(x-y)\xi} d\xi$ be the distribution kernel for $\mathrm{OP}\psi(t)$. It is known to be smooth away from the diagonal and, moreover, if k is large enough then $|\partial_y^\alpha \partial_x^\beta K_t(x,y)| \leq C(K_t)|x-y|^{-k}$ for all $x \in U$, $y \in V$ in any two disjoint compacts U and V in $\boldsymbol{R}^N$ (see [11, ch.2, §2]). We are going to expose the dependence of the constant on S^m-seminorms and prove that, whenever $k \geq |\alpha+\beta| + m + N + 1$ and $U \subseteq \{x : |x| \leq n\}$

$$|\partial_y^\alpha \partial_x^\beta K_t(x,y)| \leq \tag{4.11}$$

$$C(\alpha,\beta,k,m,n,N)|x-y|^{-k} \sum_{|\alpha'| \leq k} \sum_{|\beta'| \leq |\beta|} \|\psi(t) : \alpha', \beta', n; S^m\|.$$

with a constant C independent on V (a note important for §4.1.2). With integrals understood in distributional sense, $K_T(x,y) = \int K_t(x, y+t) d\sigma(t)$ is evident, and the required inclusion $K_T \in C^\infty(\boldsymbol{R}^N \backslash \{(x + \operatorname{supp} \sigma, x)\})$ follows from (4.11) by

$$|\partial_y^\alpha \partial_x^\beta K_T(x,y)| \leq C' \operatorname{dist}(x-y, \operatorname{supp} \sigma)^{-k} \int \sum_{\alpha',\beta'} \|\psi(t) : \alpha', \beta', n; S^m\| d|\sigma|(t).$$

Assume, as we may, that k is even, so that $|x-y|^k = \Sigma_{|\eta|=k} C_{\eta N}(x-y)^\eta$ and fix a $\eta \in \boldsymbol{Z}_+^N$ with $|\eta| = k$. Using $(x-y)^\eta e^{i(x-y)\xi} = \partial_\xi^\eta e^{i(x-y)\xi}$ and integrating by parts at a later stage, we get

$$(x-y)^\eta \partial_x^\beta \partial_y^\alpha K_t(x,y) = C_N (x-y)^\eta \partial_x^\beta \int \psi(t)(x,\xi) \partial_y^\alpha e^{i(x-y)\xi} d\xi$$

$$= (x-y)^\eta \sum_{\beta' \leq \beta} C_{\alpha\beta N'} \int (\partial_x^{\beta'} \psi(t)(x,\xi))(\partial_x^{\beta-\beta'} e^{i(x-y)\xi}) \xi^\alpha d\xi$$

$$= \sum_{\beta' \leq \beta} \sum_{\eta' \leq \eta} C_{\alpha\beta'\eta' N} \int (\partial_\xi^{\eta'} \partial_x^{\beta'} \psi(t)(x,\xi))(\partial_\xi^{\eta-\eta'} \xi^{\beta-\beta'+\alpha}) e^{i(x-y)\xi} d\xi.$$

The validity of these distributional transformations rests, as usual, on the final estimate. Summing over $|\eta|=k$ with appropriate coefficients we see that

$$|x-y|^k|\partial_y^\alpha\partial_x^\beta K_T(x,y)|\leq C'(\alpha,\beta,k,N)\sum_{\beta'\leq\beta}\sum_{|\eta'|\leq k}\|\psi(t):\eta',\beta',n;S^m\|$$

since the result of integration w.r.t. ξ is bounded when $k\geq|\alpha|+|\beta|+m+N+1$.

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Institute of Mathematics and Mechanics
Academy of Sciences
Baku, AZERBAIJAN
e-mail: namik-kasumov@metu.edu.tr