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DIFFUSION PROCESSES ON MANDALA

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1. Introduction

The concept of "fractal" is fairly broad. A mathematical framework of fractals were given by Hutchinson [5] (he call them *strictly self-similar sets*). His self-similar set K is a compact subset of a complete metric space X, and *invariant* with respect to a collection $\{F_1,\cdots,F_N\}$ of contraction maps on X: that is, $K=\cup_{i=1}^{N}F_i(K)$. One way to obtain the physical properties of these media is to construct Brownian motion on them. The study of diffusion processes on fractals was initiated by Kusuoka [11], Goldstein [3], and Barlow-Perkins [1]. They constructed Brownian motion on the Sierpinski gasket, and investigated it in detail. The Sierpinski gasket is one of Hutchinson's fractals of finitely ramified type (i.e., $\max_{i\neq j}\#(F_i(K)\cap F_j(K))<\infty$), which have been studied by many probabilists. Lindstrøm [13] introduced a class of finitely ramified fractals, containing the Sierpinski gasket, called "Nested fractals", and constructed Brownian motion on them. Kusuoka [12] and Fukushima [2] studied these processes by using (regular local) Dirichlet forms. Also, there are many works on nested fractals (for example see Shima [15] and Kumagai [10]). Post critically finite (P.C.F. for short) self-similar sets, a generalization of nested fractals, were introduced by Kigami [6], and he considered Laplace operators and Dirichlet forms on them.

Hutchinson's fractal (i.e., strictly self-similar set) K is associated with the full-shift symbolic space, i.e., there is a natural surjective map π: $\{1,\cdots,N\}^N\to K$, cf. Kigami [6]. In this paper, our object is a finitely ramified fractal which is not associated with full-shift, but with a Markov sub-shift. Let C be a unite circle in $\boldsymbol{R}^2$ having the origin as its center, and a collection $\{F_1,\cdots,F_5\}$ of 3-similitudes with fixed points (1,0), (0,1), $(-1,0)$, $(0,-1)$, (0,0), respectively. There exists a unique compact set $K\subset\boldsymbol{R}^2$ such that $K=\cup_{i=1}^{5}F_i(K)\cup C$, cf. Hata [4], which we call *the Mandala* (see Figure 1): this name is taken from the Buddhist magic diagram. However, we will be exclusively concerned with *the plain Mandala* which is a simplification of the Mandala, in order to avoid notational complications. The Mandala and the plain Mandala are not included in Hutchinson's framework.

We shall give a mathematical definition of the plain Mandala in Section 2. Our method of constructing diffusion processes is a modification of Kigami's method for P.C.F. selfsimilar sets [6] (see also Kumagai [9]). The plain Mandala is

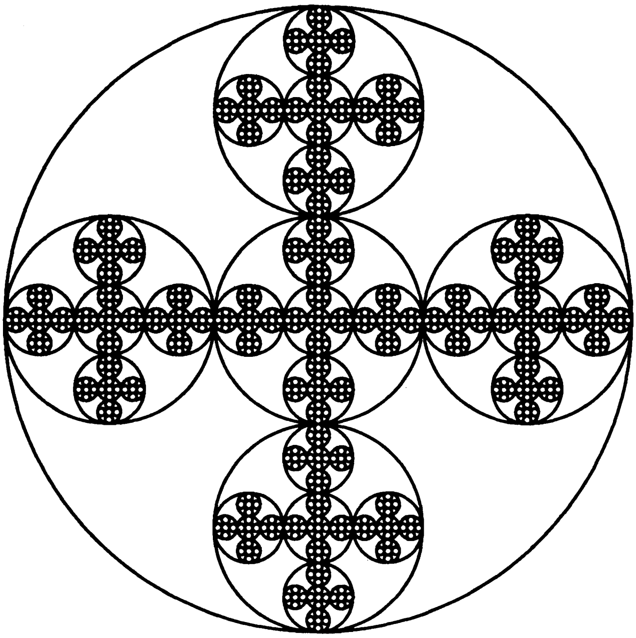

Fig. 1.

denoted by K. We define a sequence $\{V_m\}_{m\geq 0}$ of finite subsets of K such that $V_\infty = \cup_{m\geq 0} V_m$ is dense in K. In Section 3, we first define a difference operator H_m on the space $l(V_m)$ of functions on V_m and then, we introduce a bilinear form $\mathscr{E}^m$ on $l(V_m)$ using H_m. Under some assumption (cited as Basic Assumption), it is shown that, for each $u \in l(V_\infty)$, $\mathscr{E}^m(u,u)$ is increasing in m. This limit bilinear form on $l(V_\infty)$ is denoted by $\mathscr{E}$. Then, we show the domain of $\mathscr{E}$ is embedded in $L^2(K)$ relative to some measure, and finally we prove this embedded bilinear form is a regular local Dirichlet form. Our bilinear form depends on several parameters, and in order to obtain our assertion, we ought to restrain these parameters (say, Condition A or Condition B). In Section 4, under Condition A, we consider a measure μ of Bernoulli type which is naturally associated with the bilinear form $\mathscr{E}$ in a certain sense, and then we give an injective map from the domain of $\mathscr{E}$ to $L^2(K,\mu)$, proving that this embedded bilinear form is a regular local Dirichlet form on $L^2(K,\mu)$. In Section 5, under Condition B, we prove that any function of the domain of the bilinear form defined in Section 3 is extended to be a continuous function on the plain mandala K, and that this embedded bilinear form is a regular local Dirichlet form on $L^2(K,\nu)$, where ν can be any everywhere dense probability measure on K. Moreover, we see that the diffusion process associated with the Dirichlet form on $L^2(K,\nu)$ is point recurrent, using the argument established by Fukushima [2]. In Section 6, under Condition A, we determine the spectral dimension of the diffusion process on the plain Mandala, which has been constructed in Section 4. In Section 7, we consider to what extent the argument in Section 4 and Section 6 works if we take a more general measure of Bernoulli type instead of the measure defined in Section 4.

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2. Plain Mandala

In this section, we give a mathematical definition of the plain Mandala, and explain the notation and the terminology. As will be seen in the sequel, the plain Mandala is characterized as a kind of self-similar set associated with a Markov sub-shift.

Let L be an (upper) unit semicircle in $\boldsymbol{R}^2$ having the origin as its center. We denote $(-1,0)$, $(1,0)$ by ξ_1, ξ_2, respectively. We give a collection $\{F_1,F_2,F_3,F_4\}$ of "contraction maps". F_1 and F_2 are the mappings from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$ defined by

$$F_i(x)=\frac{1}{2}(x-\xi_i)+\xi_i \qquad \text{for all } x\in\boldsymbol{R}^2,\ i=1,2.$$

F_3 and F_4 are the mappings from L to L defined by

$$F_i(x)=\varphi\left(\frac{1}{2}(\psi(x)-\psi(\xi_{i-2}))+\psi(\xi_{i-2})\right) \qquad \text{for all } x\in L,\ i=3,4,$$

where $\varphi:[0,\pi]\to L$ is defined by $\varphi(\theta)=(\cos\theta,\sin\theta)$, and $\psi=\varphi^{-1}$.

Then there exists a unique compact set K in $\boldsymbol{R}^2$ such that $K=F_1(K)\cup F_2(K)\cup L$, (see Figure 2); cf. Hata [4].

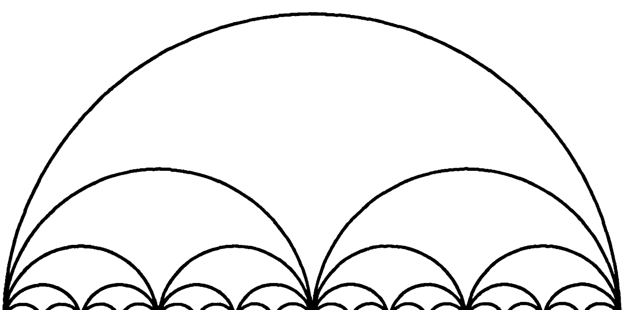

Fig. 2.

DEFINITION 2.1. The compact set K is said to be *the plain Mandala.*

We give an increasing sequence $\{V_m\}_{m\geq 0}$ of finite sets which approximate K. We start with $V_0=V_{L,0}=\{\xi_1,\xi_2\}$. For $m\geq 1$, let

$$V_m=\cup_{i=1,2}F_i(V_{m-1})\cup V_{L,m},$$
$$V_{L,m}=\cup_{i=3,4}F_i(V_{L,m-1}).$$

The set $\cup_{m\geq 0}V_m$ is denoted by V_∞. We see that V_∞ is a countable dense subset of K: i.e., $K=Cl(V_\infty)$. We also define the subset V_m^s of V_m by

$$V_0^s=V_0,\quad V_m^s=\cup_{i=1,2}F_i(V_{m-1}^s),$$

which will be used often.

DEFINITION 2.2. Let $I_s=\{1,2\}$, $I_b=\{3,4\}$ and $I=I_s\cup I_b$. (s and b stand for self-similar and bridge, respectively.)

(i) $A=(a_{ij})_{i,j\in I}$ is the *structure matrix* given by

$$a_{ij}=\begin{cases}0, & \text{if } i\in I_b \text{ and } j\in I_s,\\ 1, & \text{otherwise.}\end{cases}$$

(ii) The collection $W_m(I,A)$ (W_m for short) of words of length m is defined by

$$W_m(I,A)=\{\omega_1\cdots\omega_m\in I^m; a_{\omega_k\omega_{k+1}}=1 \text{ for all } k=1,\cdots,m-1\}.$$

(iii) The Makov sub-shift $\Sigma(I,A)$ (Σ for short) is defined by

$$\Sigma(I,A)=\{\omega=\omega_1\omega_2\cdots\in I^{\mathbf{N}}; a_{\omega_n\omega_{n+1}}=1 \text{ for all } n\geq 1\}.$$

(iv) For $\omega\in W_m$, *m-complex* K_ω is defined by

$$K_\omega=\begin{cases}F_\omega(K), & \text{if } \omega\in I_s^m,\\ F_\omega(L), & \text{if } \omega\in W_m\backslash I_s^m,\end{cases}$$

where $F_\omega=F_{\omega_1}\circ F_{\omega_2}\circ\cdots\circ F_{\omega_m}$.

REMARK. The collection of words W_m is associated with V_m in a sense. Indeed, $V_m\cup_{\omega\in W_m}F_\omega(V_0)$ and $V_m^s=\cup_{\omega\in I_s^m}F_\omega(V_0)$ holds.

Proposition 2.3. *There exists a surjective mapping π from Σ to K.*

Proof. For each $\omega\in\Sigma$, the sequence $\{K_{\omega_1\cdots\omega_n}\}_{n\geq 1}$ is decreasing in n, and the diameter of $K_{\omega_1\cdots\omega_n}\to 0$. This implies $\cap_{n\geq 1}K_{\omega_1\cdots\omega_n}$ consists of a single point. Let π be a mapping from Σ to K given by $\{\pi(\omega)\}=\cap_{n\geq 1}K_{\omega_1\cdots\omega_n}$. Then we can verify that π is surjective. □

REMARK. In case of Hutchinson's fractal, the "symbolic space" Σ coinsides with I^N. However, in case of the plain Mandala, Σ is a proper subset of I^N.

For $k \geq 1$ (resp. $k=0$), the subset $\cup_{\eta \in I_s^k} F_\eta(L)$ of K (resp. L) is said to be *the k-th bridge* (resp. *the 0-th bridge*). For any $m \geq k$, an element ω of W_m is said to be in the k-th bridge if the set $F_\omega(L)$ is included in the k-th bridge. It is obvious that $\omega \in W_m$ is in the k-th bridge if and only if $\omega \in I_s^k I_b^{m-k}$. For any $x \in V_m$, *the m-neighborhood* $N_m(x)$ of x is defined by

$$N_m(x) = \{y \in V_m \mid \exists\, \omega \in W_m \text{ such that } F_\omega(V_0) = \{x,y\}\}.$$

Then, $\sharp N_m(x) = 2$ for $x \in V_m \setminus V_m^s$, while $\sharp N_m(x) \to \infty$ as $m \to \infty$ for $x \in V_m^s$.

3. Bilinear form $\mathscr{E}$ on $l(V_\infty)$

Let $l(V)$ be the set of real-valued functions on at most countable set V. In this section, we shall define a bilinear form $\mathscr{E}$ on $l(V_\infty)$. First of all, we give a bilinear form $\mathscr{E}^m$ on $l(V_m)$ for each approximating set V_m. Then, under some condition, we see that the bilinear form $\mathscr{E}^m$ converges as $m \to \infty$; we define $\mathscr{E}$ as this limit form.

The difference operator D on $l(V_0)$ is defined by

$$D = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let t, r_1, r_2, r_3, r_4 be positive numbers ($t \neq 1$), and let $\bar{r}_i$ equal tr_i for $i=1,2$, and equal r_i for $i=3,4$.

DEFINITION 3.1. For each $m \geq 1$, a difference operator H_m on $l(V_m)$ is defined by

$$H_m = \sum_{\omega \in W_m \setminus I_s^m} \bar{r}_\omega^{-1}\, {}^t R_\omega D R_\omega + \frac{t}{t-1} \sum_{\omega \in I_s^m} \bar{r}_\omega^{-1}\, {}^t R_\omega D R_\omega, \tag{3.1}$$

where R_ω is a mapping from $l(V_m)$ to $l(V_0)$ defined by $R_\omega(u) = u \circ F_\omega$ for all $u \in l(V_m)$, and $\bar{r}_\omega = \bar{r}_{\omega_1} \bar{r}_{\omega_2} \cdots \bar{r}_{\omega_m}$.

For any $\omega \in W_m$, we see that

$$({}^t R_\omega D R_\omega)_{xy} = \begin{cases} 1, & \text{if } F_\omega(V_0) = \{x,y\}, \\ -1, & \text{if } x = y \in F_\omega(V_0), \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

For any $x,y \in V_m$, if x and y are m-neighbors to each other, then there exists a unique element ω of W_m such that $F_\omega(V_0) = \{x,y\}$. Let us denote the (x,y)-component of H_m by $h_{xy}^{(m)}$. By (3.2) we have

$$h_{xy}^{(m)} = \begin{cases} \bar{r}_\omega^{-1}, & \text{if } \exists \omega \in W_m \backslash I_s^m \text{ such that } F_\omega(V_0)=\{x,y\}, \\ \frac{t}{t-1}\bar{r}_\omega^{-1}, & \text{if } \exists \omega \in I_s^m \text{ such that } F_\omega(V_0)=\{x,y\}, \\ -\sum_{\substack{\omega\in W_m\backslash I_s^m \\ x\in F_\omega(V_0)}} \bar{r}_\omega^{-1} - \frac{t}{t-1}\sum_{\substack{\omega\in I_s^m \\ x\in F_\omega(V_0)}} \bar{r}_\omega^{-1}, & \text{if } x=y, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Let u be a function defined on a set containing V_m (resp. $V_m \cap L$). We will write $H_m u$ (resp. $H_m|_{V_m \cap L} u$) for $H_m(u|_{V_m})$ (resp. $H_m|_{V_m\cap L}(u|_{V_m\cap L})$).

Let $u \in l(V_\infty)$ and $n > m$. Then for each $x \in V_n \backslash V_m$, there exists a unique $\omega \in W_m$ such that $x \in K_\omega$. We can see that

$$H_n u(x) = \bar{r}_\omega^{-1} H_{n-m}(u \circ F_\omega)(\tilde{x}), \quad \text{if } \omega \in I_s^m, \tag{3.4}$$

$$H_n u(x) = \bar{r}_\omega^{-1} H_{n-m}|_{V_{n-m}\cap L}(u \circ F_\omega)(\tilde{x}), \quad \text{if } \omega \in W_m \backslash I_s^m, \tag{3.5}$$

where $\tilde{x} = F_\omega^{-1}(x) \in V_{n-m} \backslash V_0$.

Let $u \in C(K;\boldsymbol{R})$. Then u is said to be a *harmonic function* if $(H_n u)|_{V_n\backslash V_0} = 0$ for all $n \geq 1$. Let $v \in C(L;\boldsymbol{R})$. Then v is said to be a *bridge harmonic function* if $(H_m|_{V_m\cap L} v)|_{(V_m\cap L)\backslash V_0} = 0$ for all $m \geq 1$. In the same way as Kigami [5], we can solve the following *Dirichlet problem*.

Proposition 3.2. *For each $g \in l(V_0)$, there exists a unique harmonic function u such that $u|_{V_0} = g$. The same result holds for the bridge harmonic function.*

Let $m \geq 1$. An element u of $C(K;\boldsymbol{R})$ is said to be an m-harmonic function if the following two conditions are satisfied:

(i) $u \circ F_\omega$ is a harmonic function for any $\omega \in I_s^m$,

(ii) $u \circ F_\omega$ is a bridge harmonic function for any $\omega \in W_m \backslash I_s^m$.

A harmonic function is called *0-harmonic function.*

Corollary 3.3. *Let m be a non-negative integer and $g \in l(V_m)$. Then there exists a unique m-harmonic function u such that $u|_{V_m} = g$.*

We denote by $\mathscr{H}_m$ the set of all m-harmonic functions. Then $\mathscr{H}_m$ is increasing in m. Indeed, from (3.4) and (3.5), it is easily shown that u is m-harmonic if and only if

$$H_n u(x) = 0 \qquad \text{for all } n > m \text{ and } x \in V_n \backslash V_m. \tag{3.6}$$

Thus, if $m<m'$, any m-harmonic function u is m'-harmonic.

DEFINITION 3.4. For $m\geq 1$, $\mathscr{E}^m : l(V_m)\times l(V_m)\to \boldsymbol{R}$ is defined by

$$\mathscr{E}^m(u,v)=-{}^t u H_m v \quad \text{for } u,v\in l(V_m).$$

From (3.3), we have

$$\begin{aligned}\mathscr{E}^m(u,v)&=\frac{1}{2}\sum_{x,y\in V_m} h_{xy}^{(m)}(u(x)-u(y))(v(x)-v(y))\\ &=\sum_{\omega\in W_m\backslash I_s^m} \bar{r}_\omega^{-1}\{u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))\}\{v(\pi(\omega\dot{3}))-v(\pi(\omega\dot{4}))\}\\ &\quad+\frac{t}{t-1}\sum_{\omega\in I_s^m} \bar{r}_\omega^{-1}\{u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))\}\{v(\pi(\omega\dot{3}))-v(\pi(\omega\dot{4}))\}\end{aligned}$$

where $\dot{3}=333\cdots$ and $\dot{4}=444\cdots$.

For every $u,v\in l(V_\infty)$ and $m\geq 1$, we denote $\mathscr{E}^m(u|_{V_m},v|_{V_m})$ by $\mathscr{E}^m(u,v)$.

Lemma 3.5. *Let* $u\in l(V_\infty)$.
(i) *For any* $\omega\in I_s^m$,

$$\{u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))\}^2\leq \sum_{i\in I_s} r_i \sum_{i\in I_s} r_i^{-1}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}^2. \tag{3.7}$$

In particular, equality holds if $u\in\mathscr{H}_m$.
(ii) For any $\omega\in W_m$,

$$\{u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))\}^2\leq \sum_{i\in I_b} r_i \sum_{i\in I_b} r_i^{-1}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}^2. \tag{3.8}$$

In particular, equality holds if $u\in\mathscr{H}_m$.

Proof. For $\omega\in I_s^m$, obviously $\pi(\omega 1\dot{4})=\pi(\omega 2\dot{3})$, $\pi(\omega 1\dot{3})=\pi(\omega\dot{3})$, $\pi(\omega 2\dot{4})=\pi(\omega\dot{4})$, so

$$u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))=\sum_{i\in I_s}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}.$$

By Schwarz' inequality, we have

$$\left[\sum_{i\in I_s}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}\right]^2\leq \sum_{i\in I_s} r_i \sum_{i\in I_s} r_i^{-1}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}^2.$$

Thus we obtain the inequality (3.7). Equality holds in (3.7) if and only if

$$r_i^{-\frac{1}{2}}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}/r_i^{\frac{1}{2}}=r_i^{-1}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}$$

is independent of $i\in I_s$, i.e.,

(3.9) $$r_1^{-1}\{u(\pi(\omega\dot{3}))-u(\pi(\omega 1\dot{4}))\}=r_2^{-1}\{u(\pi(\omega 2\dot{3}))-u(\pi(\omega\dot{4}))\}.$$

Let $\xi_3=\pi(1\dot{4})=\pi(2\dot{3})$. For any $u\in\mathscr{H}_m$, $u\circ F_\omega$ is a harmonic function. Thus we see that

$$0=H_1(u\circ F_\omega)(\xi_3)=t^{-1}r_1^{-1}(u\circ F_\omega(\xi_1)-u\circ F_\omega(\xi_3))+t^{-1}r_2^{-1}(u\circ F_\omega(\xi_2)-u\circ F_\omega(\xi_3)).$$

Noticing that $F_\omega(\xi_1)=\pi(\omega\dot{3})$ and $F_\omega(\xi_2)=\pi(\omega\dot{4})$, $F_\omega(\xi_3)=\pi(\omega 1\dot{4})=\pi(\omega 2\dot{3})$, we have (3.9). To see the second assertion, let $\omega\in W_m$. It is also obvious that $\pi(\omega 3\dot{4})=\pi(\omega 4\dot{3})$.
Using the same argument above, we obtain (3.8). Equality holds in (3.8) if and only if

(3.10) $$r_3^{-1}\{u(\pi(\omega\dot{3}))-u(\pi(\omega 3\dot{4}))\}=r_4^{-1}\{u(\pi(\omega 4\dot{3}))-u(\pi(\omega\dot{4}))\}.$$

For any $u\in\mathscr{H}_m$, $u\circ F_\omega$ is a bridge harmonic function. Hence for $\xi_4=\pi(\omega 3\dot{4})=\pi(\omega 4\dot{3})$,

$$0=H_1|_{V_1\cap L}(u\circ F_\omega)(\xi_4)=r_3^{-1}(u\circ F_\omega(\xi_1)-u\circ F_\omega(\xi_4))+r_4^{-1}(u\circ F_\omega(\xi_2)-u\circ F_\omega(\xi_4)),$$

which proves (3.10). □

Corollary 3.6. *For any $n\geq 0$ and any $x,y\in L\cap V_n$, the following halds*

$$(u(x)-u(y))^2\leq\left(\sum_{i\in I_b}r_i\right)^n\sum_{\eta\in I_b^n}r_\eta^{-1}\{u(\pi(\eta\dot{3}))-v(\pi(\eta\dot{4}))\}^2.$$

Proof. Use Lemma 3.5 and the fact that $(r_3+r_4)/r_i>1$ for $i\in I_b$ repeatedly. We omit the details of the proof. □

So far, $\{r_i\}_{i\in I}$ and t are arbitrary positive numbers. For the sake of further discussion, we need the folloing assumption.

Basic Assumption. *Positive numbers $\{r_i\}_{i\in I}$ and t satisfy*

(B.A.1) $$r_1+r_2=1,\ r_3+r_4=1,$$

(B.A.2) $$t>1.$$

Lemma 3.7. *Suppose that* (B.A.1) *is satisfied. Then for any $u\in l(V_\infty)$,*

(3.11) $$\mathscr{E}^{m+1}(u,u)\geq\mathscr{E}^m(u,u).$$

In particular, equality holds if u is an m-harmonic function.

Proof. Using Lemma 3.5, we have

$$\mathscr{E}^{m+1}(u,u)=\sum_{\omega\in W_m}\bar{r}_\omega^{-1}\sum_{i\in I_b}r_i^{-1}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}^2$$

$$+\frac{t}{t-1}\sum_{\omega\in I_s^m}\bar{r}_\omega^{-1}\sum_{i\in I_s}\bar{r}_i^{-1}\{u(\pi(\omega i\dot{3}))-u(\pi(\omega i\dot{4}))\}^2\geq\mathscr{E}^m(u,u).\quad\square$$

By Lemma 3.7, $\mathscr{E}^m(u,u)$ is increasing as $m\to\infty$, so here we can define the bilinear form on $l(V_\infty)$ as follows.

DEFINITION 3.8. A bilinear form $\mathscr{E}$ on $l(V_\infty)$ is given by

$$\mathscr{E}(u,u)=\lim_{m\to\infty}\mathscr{E}^m(u,u)\quad\text{for all } u\in l(V_\infty).$$

$$Dom(\mathscr{E})=\{u\in l(V_\infty)\mid\mathscr{E}(u,u)<\infty\}.$$

$$\mathscr{E}(u,v)=\lim_{m\to\infty}\mathscr{E}^m(u,v)\quad\text{for all } u,v\in Dom(\mathscr{E}).$$

By Lemma 3.7, we have the following proposition.

Proposition 3.9. *Under Basic Assumption, the following holds*

$$\cup_{m\geq 0}\mathscr{H}_m\subset Dom(\mathscr{E}).\tag{3.12}$$

The proposition above plays an important role in proving *the regularity* condition of the Dirichlet form in Secion 4 and Section 5. In the rest of this section, we assume Basic Assumption.

In order to obtain the scaling property of the bilinear forms, we ought to define the bilinear form restricted to a subset of K.

DEFINITION 3.10. For any subset $\tilde{K}$ of K,

$$\mathscr{E}^m_{\tilde{K}}(u,v)=\frac{1}{2}\sum_{x,y\in V_m\cap\tilde{K}}h^{(m)}_{xy}(u(x)-u(y))(v(x)-v(y))$$

$$=\sum_{\substack{\omega\in W_m\setminus I_s^m\\ K\omega\subset\tilde{K}}}\bar{r}_\omega^{-1}\{u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))\}\{v(\pi(\omega\dot{3}))-v(\pi(\omega\dot{4}))\}$$

$$+\frac{t}{t-1}\sum_{\substack{\omega\in I_s^m\\ K_\omega\subset\tilde{K}}}\bar{r}_\omega^{-1}\{u(\pi(\omega\dot{3}))-u(\pi(\omega\dot{4}))\}\{v(\pi(\omega\dot{3}))-v(\pi(\omega\dot{4}))\},$$

$$Dom(\mathscr{E}_{\tilde{K}})=\{u\in l(V_\infty)|\lim_{m\to\infty}\mathscr{E}^m_{\tilde{K}}(u,u)<\infty\}.$$

$$\mathscr{E}_{\tilde{K}}(u,v)=\lim_{m\to\infty}\mathscr{E}^m_{\tilde{K}}(u,v) \quad \text{for all } u,v\in Dom(\mathscr{E}_{\tilde{K}}).$$

REMARK. We also see that $\mathscr{E}^m_{\tilde{K}}(u,v)$ is increasing as $m\to\infty$. Since $\mathscr{E}_{\tilde{K}}(u,u)\le\mathscr{E}(u,u)$ for $u\in l(V_\infty)$, we see that $Dom(\mathscr{E})\subset Dom(\mathscr{E}_{\tilde{K}})$.

We see the following Lemma in the same way as Kigami-Lapidus [8].

Lemma 3.11. *(Scaling properties) Let $u,v\in Dom(\mathscr{E})$. Then $u\circ F_\omega\in Dom(\mathscr{E})$ for $\omega\in I^m_s$, and $u\circ F_\omega\in Dom(\mathscr{E}_L)$ for $\omega\in W_m\backslash I^m_s$, and the following hold*

$$\mathscr{E}(u\circ F_\omega, v\circ F_\omega)=\bar{r}_\omega\mathscr{E}_{K_\omega}(u,v) \quad \textit{for all } \omega\in I^m_s. \tag{3.13}$$

$$\mathscr{E}_L(u\circ F_\omega, v\circ F_\omega)=\bar{r}_\omega\mathscr{E}_{K_\omega}(u,v) \quad \textit{for all } \omega\in W_m\backslash I^m_s. \tag{3.14}$$

$$\mathscr{E}(u,v)=\sum_{i\in I_s}\bar{r}_i^{-1}\mathscr{E}(u\circ F_i, v\circ F_i)+\sum_{j\in I_b}\bar{r}_j^{-1}\mathscr{E}_L(u\circ F_j, v\circ F_j). \tag{3.15}$$

$$\mathscr{E}_L(u,v)=\sum_{j\in I_b}\bar{r}_j^{-1}\mathscr{E}_L(u\circ F_j, v\circ F_j). \tag{3.16}$$

For each $x\in V_m$, we denote by ψ^m_x the m-harmonic function with the boundary condition $\psi^m_x|_{V_m}=1_{\{x\}}$, where $1_{\{x\}}$ denote the indicator function of $\{x\}$.

DEFINITION 3.12. For $m\ge0$, let us define $P_m: l(V_\infty)\to\mathscr{H}_m$ by

$$P_m u=\sum_{x\in V_m}u(x)\psi^m_x \quad \text{for all } u\in l(V_\infty). \tag{3.17}$$

Noticing that $u|_{V_m}=(P_m u)|_{V_m}$.

By Lemma 3.7, we have the following Lemma.

Lemma 3.13. *Let $m\ge0$ and $u\in l(V_\infty)$. Then*

$$\mathscr{E}(P_m u, P_m u)\le\mathscr{E}(u,u). \tag{3.18}$$

4. Dirichlet form-(I)

In this section and the subsequent sections, we always assume that Basic Assumption is satisfied.

In this section, we see that there exists an invariant measure μ_m on V_m with respect to the remdom walk associated with *the conductance* $\{h_{xy}^{(m)}\}_{x\neq y}$. Under Condition A, we can verify that μ_m weakly converges as $m\to\infty$. The measure μ of Bernoulli type, to which we referred in the introduction, is defined by this limit measure. Then, we shall prove the bilinear form $(\mathscr{E}, Dom(\mathscr{E}))$ is regular local Dirichlet form on $L^2(K,\mu)$. The same result holds for other measures of Bernoulli type instead of μ: see section 7.

Let $P^{(m)}=\{p_{xy}^{(m)}\}_{x,y\in V_m}$ be the transition probability of the random walk associated with the conductance $\{h_{xy}^{(m)}\}_{x\neq y}$: i.e.,

$$p_{xy}^{(m)}=\begin{cases}\frac{h_{xy}^{(m)}}{h_x^{(m)}}, & \text{if } x\neq y,\\ 0, & \text{if } x=y,\end{cases}$$

where $h_x^{(m)}=\sum_{z\neq x} h_{xz}^{(m)}=-h_{xx}^{(m)}>0$.

Proposition 4.1. *For each $m\geq 0$, the unique normalized invariant measure μ_m with respect to $P^{(m)}$ exists: i.e.,*

$$(4.1)\qquad (P^{(m)}u,v)_{L^2(V_m,\mu_m)}=(u,P^{(m)}v)_{L^2(V_m,\mu_m)} \qquad \textit{for all } u,v\in l(V_m).$$

More precisely, for all $x\in V_m$

$$(4.2)\qquad \mu_m(x)=\frac{h_x^{(m)}}{\sum_y h_y^{(m)}}=\frac{\sum_{\substack{\eta\in W_m\setminus I_S^m\\ x\in F_\eta(V_0)}} \bar{r}_\eta^{-1}+\frac{t}{t-1}\sum_{\substack{\eta\in I_S^m\\ x\in F_\eta(V_0)}}\bar{r}_\eta^{-1}}{2\sum_{\omega\in W_m\setminus I_S^m}\bar{r}_\omega^{-1}+2\frac{t}{t-1}\sum_{\omega\in I_S^m}\bar{r}_\omega^{-1}}.$$

Proof. We first note that the second equality in (4.2) follows from (3.3). Suppose that μ_m be a probability measure satisfying (4.1). Obviously, (4.1) is equivalent to

$$p_{xy}^{(m)}\mu_m(y)=p_{yx}^{(m)}\mu_m(x) \qquad \text{for all } x,y\in V_m.$$

Since H_m is symmetric and irreducible,

$$\frac{\mu_m(x)}{h_x^{(m)}}=\frac{\mu_m(y)}{h_y^{(m)}} \qquad \text{for all } x,y\in V_m.$$

Hence by the normality of μ_m, we have (4.2). This proves the uniqueness. It is obvious the measure μ_m in (4.2) satisfies (4.1). □

Let Θ^m be the self-adjoint operator on $l(V_m)$ associated with $\mathscr{E}^m(\cdot,\cdot)$ on

$L^2(V_m,\mu_m)$: for any $u,v\in Dom(\mathscr{E}^m)$,

$$\mathscr{E}^m(u,v)=-(\Theta^m u,v)_{L(V_m,\mu_m)}.$$

By the definition of $\mathscr{E}^m$,

$$\begin{aligned}\mathscr{E}^m(u,v)&=-\sum_{x\in V_m} h_{xy}^{(m)}u(y)v(x)\\ &=-\sum_{x\in V_m}\frac{h_{xy}^{(m)}}{\mu_m(x)}u(y)v(x)\mu_m(x).\end{aligned}$$

Thus, $\theta_{xy}^{(m)}=h_{xy}^{(m)}/\mu_m(x)$ where $\theta_{xy}^{(m)}$ the (x,y)-conponent of Θ^m. Let $\tau^{(m)}$ be the exponential holding time of the Markov chain associated with Θ^m. Then from Proposition 4.1, the average of $\tau^{(m)}$ dose not depend on the starting point $x\in V_m$ of the Markov chain. Indeed, we have

$$E_x[\tau^{(m)}]=\{-\theta_{xx}^{(m)}\}^{-1}=\{2\sum_{\omega\in W_m\backslash I_s^m}\bar{r}_\omega^{-1}+2\frac{t}{t-1}\sum_{\omega\in I_s^m}\bar{r}_\omega^{-1}\}^{-1}.$$

In the following, we think of μ_m as a measure on K. There exists a unique probability measure μ of Bernoulli type on K such that for any $m\geq 0$, the follwing two conditions are satisfied:

(i) Let k be a non-negative integer satisfying $k\leq m-1$. For any ω in the k-th brige,

$$\mu(K_\omega)=\left(1-\frac{1}{t}\right)(r_1r_2)^k(r_3r_4)^{m-k}\bar{r}_\omega^{-1}.$$

(ii) For any $\omega\in I_s^m$,

$$\mu(K_\omega)=(r_1r_2)^m\bar{r}_\omega^{-1}.$$

Condition A. *$r_1r_2=r_3r_4$: i.e., $r_1=r_3$ or r_4.*

Throghout this section, we assume Condition A. Let $c(r)=(r_1r_2)^{-1}=(r_3r_4)^{-1}$. Then we have

$$\mu(K_\omega)=\begin{cases}(1-t^{-1})c(r)^{-m}\bar{r}_\omega^{-1}, & \text{if } \omega\in W_m\backslash I_s^m,\\ c(r)^{-m}\bar{r}_\omega^{-1}, & \text{if } \omega\in I_s^m.\end{cases}\tag{4.3}$$

It is easy to show the following

Lemma 4.2. *For any measureable function f on K and any $m \geq 1$, the following hold*

$$\int_{K_\omega} f \circ F_\omega^{-1} d\mu = c(\boldsymbol{r})^{-m} \bar{r}_\omega^{-1} \int_K f d\mu \qquad \text{for } \omega \in I_s^m, \tag{4.4}$$

$$\int_{K_\omega} f \circ F_\omega^{-1} d\mu = c(\boldsymbol{r})^{-m} \bar{r}_\omega^{-1} \int_L f d\mu \qquad \text{for } \omega \in W_m \backslash I_s^m. \tag{4.5}$$

Proposition 4.3. *Under Condition A, μ_m converges weakly to μ.*

In order to prove Proposition 4.3, we need two lemmas.

Let $r_1^* = r_2$, $r_2^* = r_1$, $r_3^* = r_4$, $r_4^* = r_3$, and let $\bar{r}_i^*$ be tr_i^* if $i \in I_s$, be r_i^* otherwise. Let H_m^* be a difference operator on $l(V_m)$ given by

$$H_m^* = \sum_{\omega \in W_m} \bar{r}_\omega^{-1} {}^t R_\omega D R_\omega. \tag{4.6}$$

Then obviously Proposition 3.2 also holds if we substitute H_m^* for H_m. For $x \in V_m$, let $\tilde{\psi}_x^m$ be an m-harmonic function (with respect to $\{H_n^*\}_{n \geq 1}$) with the boundary condition $\tilde{\psi}_x^m|_{V^m} = 1_{\{x\}}$.

Lemma 4.4. *Let f be the bridge harmonic function with respect to $\{H_n^*\}_{n \geq 1}$ with the boundary condition $f(\xi_1) = 0$ and $f(\xi_2) = 1$. Then*

$$\int_L f d\mu = \frac{1}{2} \mu(L). \tag{4.7}$$

The same result holds if the boundary condition is replaced by $f(\xi_1) = 1$ and $f(\xi_2) = 0$.

Proof. Let $S_0 = \mu(L)$, $s_0 = 0$, and let

$$S_n = \sum_{\omega \in I_b^m} f(\pi(\omega\dot{4}))\mu(K_\omega), \quad s_n = \sum_{\omega \in I_b^m} f(\pi(\omega\dot{3}))\mu(K_\omega).$$

From the maximum principle, for any $n \geq 0$,

$$s_n \leq \int_L f d\mu \leq S_n.$$

For any $\omega \in I_b^n$, we have

$$\mu(K_{\omega 3}) = r_4 \mu(K_\omega), \quad \mu(K_{\omega 4}) = r_3 \mu(K_\omega), \tag{4.8}$$

$$f(\pi(\omega 3\dot{4}))=f(\pi(\omega 4\dot{3}))=r_3 f(\pi(\omega\dot{3}))+r_4 f(\pi(\omega\dot{4})). \tag{4.9}$$

(4.8) and (4.9) imply the following recursion formulae,

$$S_{n+1}=(r_3+r_4^2)S_n+r_3r_4s_n,$$

$$s_{n+1}=r_3r_4S_n+(r_3^2+r_4)s_n.$$

By an elementary calculation on the stochastic matrix $A=\begin{pmatrix} r_3+r_4^2 & r_3r_4 \\ r_3r_4 & r_3^2+r_4 \end{pmatrix}$, we have $\lim_{n\to\infty}A^n=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Hence, S_n and s_n converge to $\frac{1}{2}S_0=\frac{1}{2}\mu(L)$. □

Lemma 4.5. *Let x be an element of $V_m\backslash V_m^s$. Then*

$$\mu_m(x)=\int_K \tilde{\psi}_x^m d\mu. \tag{4.10}$$

Proof. Note that

$$\int_K \tilde{\psi}_x^m d\mu=\sum_{\substack{\eta\in W_m \\ x\in F_\eta(V_0)}}\int_{K_\eta}\tilde{\psi}_x^m d\mu,$$

because $\tilde{\psi}_x^m=0$ an K_η if $x\notin F_\eta(V_0)$. Since $x\in V_m\backslash V_m^s$, if $x\in F_\eta(V_0)$ for $\eta\in W_m$, then $\eta\in W_m\backslash I_s^m$. By Lemma 4.2 and Lemma 4.4, we have

$$\int_{K_\eta}\tilde{\psi}_x^m d\mu=\frac{1}{2}\mu(K_\eta).$$

Thus we have

$$\bar{r}_\omega^{-1}=2c(\boldsymbol{r})^m\frac{t}{t-1}\int_{K_\omega}\tilde{\psi}_x^m d\mu.$$

On the other hand,

$$\sum_{\omega\in W_m\backslash I_s^m}\bar{r}_\omega^{-1}+\frac{t}{t-1}\sum_{\omega\in I_s^m}\bar{r}_\omega^{-1}=\sum_{k=0}^{m-1}t^{-k}\sum_{\omega\in I_s^k I_b^{m-k}}r_\omega^{-1}+\frac{t}{t-1}t^{-m}\sum_{\omega\in I_s^m}\bar{r}_\omega^{-1} \tag{4.11}$$

$$=c(\boldsymbol{r})^m\frac{1-t^{-m}}{1-t^{-1}}+\frac{t}{t-1}t^{-m}=c(\boldsymbol{r})^m\frac{t}{t-1}$$

Hence Propositon 4.1 gives

$$\mu_m(x)=\frac{1}{2}c(r)^{-m}\frac{t-1}{t}\sum_{\substack{\eta\in W_m\\ x\in F_\eta(V_0)}}\bar{r}_\eta^{-1}=\sum_{\substack{\eta\in W_m\\ x\in F_\eta(V_0)}}\int_{K_\eta}\tilde{\psi}_x^m d\mu.$$

Thus we have our assertion. □

Proof of Proposition 4.3. Let u be a continuous function on K. Then

$$|\sum_{\omega\in V_m^s}u(x)\mu_m(x)|\le\|u\|_\infty\mu_m(V_m^s).$$

Proposition 4.1 gives

$$\begin{aligned}(2\sum_{\omega\in W_m\setminus I_s^m}\bar{r}_\omega^{-1}+2\frac{t}{t-1}\sum_{\omega\in I_s^m}\bar{r}_\omega^{-1})\mu_m(V_m^s)&=\sum_{x\in V_m^s}\{\sum_{\substack{\eta\in W_m\setminus I_s^m\\ x\in F_\eta(V_0)}}\bar{r}_\eta^{-1}+\frac{t}{t-1}\sum_{\substack{\eta\in I_s^m\\ x\in F_\eta(V_0)}}\bar{r}_\eta^{-1}\}\\
&=\sum_{k=0}^{m-1}\sum_{\sigma\in I_s^k}(\bar{r}^{-1}_{\sigma\underbrace{3\cdots3}_{(m-k)\text{ times}}}+\bar{r}^{-1}_{\sigma\underbrace{4\cdots4}_{(m-k)\text{ times}}})+\frac{2t}{t-1}\sum_{\sigma\in I_s^m}\bar{r}_\sigma^{-1}\\
&=c(r)^m\{\sum_{k=0}^{m-1}t^{-k}(r_4^{m-k}+r_3^{m-k})+\frac{2t}{t-1}t^{-m}\}\le c(r)^m(t^{-1}\vee r_3\vee r_4)^m2(m+\frac{t}{t-1}).\end{aligned}$$

Noticing that $(t^{-1}\vee r_3\vee r_4)<1$. Form (4.11), we see that $\Sigma_{x\in V_m^s}u(x)\mu_m(x)$ converges to zero as $m\to\infty$. By Lemma 4.5, we have

$$\sum_{x\in V_m\setminus V_m^s}u(x)\mu_m(x)=\int_K\sum_{x\in V_m\setminus V_m^s}u(x)\tilde{\psi}_x^m d\mu.$$

Noticing that $\mu(cl(\cup_{m\ge0}V_m^s))=0$, we see that

$$\sum_{x\in V_m\setminus V_m^s}u(x)\tilde{\psi}_x^m\to u\quad \mu-a.s.$$

Since $|\Sigma_{x\in V_m\setminus V_m^s}u(x)\tilde{\psi}_x^m|\le\|u\|_\infty$, using Lebesgue's convergence theorem, we have

$$\sum_{x\in V_m\setminus V_m^s}u(x)\mu_m(x)\to\int_K ud\mu.$$

Observe that $\int_K ud\mu_m=\Sigma_{x\in V_m^s}u(x)\mu_m(x)+\Sigma_{x\in V_m\setminus V_m^s}u(x)\mu_m(x)$, we have thus proved the proposition. □

For $x\in K$ and $m\in N$, let us define the left edge $b_x^m\in V_m$ of an m-complex which includes x by

(i) If $x\in V_m$, then $b_x^m=x$.

(ii) If $x\in K\setminus V_m$, then there exists a unique $\omega\in W_m$ such

that $x \in K_\omega$, and we let $b_x^m = \pi(\omega\dot{3})$.
We denote by b_x^0 the left edge of K, i.e., $b_x^0 = \xi_1$.

Lemma 4.6. *Let u be a continuous function on K and $m \geq 0$. Then*

$$\int_K |u(b_x^m) - u(x)|^2 \mu(dx) \leq c(\boldsymbol{r})^{-m} \mathscr{E}(u,u) \tag{4.12}$$

Corollary 4.7. *Let $u \in Dom(\mathscr{E})$. Then $P_m u$ converges in $L^2(K,\mu)$.*

Let us denote by $\iota(u)$ the limit of $P_m u$.

Proof. Let $\omega \in W_m$, $n \in N$ and $U_n(\omega) = F_\omega(L) \cap (V_{m+n} \setminus V_m)$. Since

$$\sum_{x \in U_n(\omega)} \sum_{\substack{\eta \in W_{m+n} \\ x \in F_\eta(V_0)}} \bar{r}_\eta^{-1} \leq 2 \sum_{\sigma \in I_b^n} \bar{r}_{\omega\sigma}^{-1} = 2\bar{r}_\omega^{-1} c(\boldsymbol{r})^n,$$

(4.11) gives

$$\mu_{m+n}(U_n(\omega)) \leq c(\boldsymbol{r})^{-m} \bar{r}_\omega^{-1}. \tag{4.13}$$

Let $x \in U_n(\omega)$. By Corollary 3.6, we have

$$(u(b_x^m) - u(x))^2 \leq \sum_{\eta \in I_b^n} r_\eta^{-1} \{u(\pi(\omega\eta\dot{3})) - u(\pi(\omega\eta\dot{4}))\}^2. \tag{4.14}$$

Thus

$$\sum_{x \in U_n(\omega)} |u(b_x^m) - u(x)|^2 \mu_{m+n}(x) \leq c(\boldsymbol{r})^{-m} \mathscr{E}_{F_\omega(L)}^{m+n}(u,u). \tag{4.15}$$

Letting $n \to \infty$, by Proposition 4.3

$$\int_{F_\omega(L)} |u(b_x^m) - u(x)|^2 \mu(dx) \leq c(\boldsymbol{r})^{-m} \mathscr{E}_{F_\omega(L)}(u,u). \tag{4.16}$$

For any $\omega \in I_s^m$ and $h \geq 1$, let $L_\omega^h = K_\omega \cap (\cup_{\eta \in I_s^{m+h}} F_\eta(L))$. Then we also have

$$\int_{L_\omega^h} |u(b_x^m) - u(x)|^2 \mu(dx) \leq c(\boldsymbol{r})^{-m} \mathscr{E}_{L_\omega^h}(u,u). \tag{4.17}$$

Now sum up both sides of inequalities (4.16) and (4.17) over $\omega \in W_m$ and $h \geq 1$, respectively, we complete the proof of Lemma 4.6.

To prove the corollary, observe that $P_m u(b_x^m) = u(b_x^m) = P_n u(b_x^m)$, for $n \geq m$ and $x \in K$. Lemma 3.13 and Lemma 4.6 give

$$\|P_m u - P_n u\|_{L^2(K,\mu)}$$
$$\leq \{\int_K |P_m u(x) - P_m u(b_x^m)|^2 \mu(dx)\}^{\frac{1}{2}} + \{\int_K |P_n u(b_x^m) - P_n u(x)|^2 \mu(dx)\}^{\frac{1}{2}}$$
$$\leq c(r)^{-\frac{m}{2}}\{\mathscr{E}(P_m u, P_m u)^{\frac{1}{2}} + \mathscr{E}(P_n u, P_n u)^{\frac{1}{2}}\} \leq 2c(r)^{-\frac{m}{2}}\mathscr{E}(u,u)^{\frac{1}{2}}.$$

Letting $m \to \infty$, we get the assertion. □

By virtue of Lemma 3.13 and Lemma 4.6, we see the following propositions and theorem in the same way as Kumagai [9].

Proposition 4.8. *The mapping ι from $Dom(\mathscr{E})$ to $L^2(K,\mu)$ is injective.*

We denote the image ι $(Dom(\mathscr{E}))$ by $\mathscr{F}$. For $u \in \mathscr{F}$, when no confusion can arise, we use $P_m u$ instead of $P_m \iota^{-1}(u)$, and $\mathscr{E}(u,u)$ instead of $\mathscr{E}(\iota^{-1}(u), \iota^{-1}(u))$.

Theorem 4.9. *$(\mathscr{E},\mathscr{F})$ is a regular local Dirichlet form on $L^2(K,\mu)$.*

Proposition 4.10. *For $\alpha > 0$, $(\mathscr{E},\mathscr{F})$ has the compact α-order resolvent.*

5. Dirichlet form-(II)

Throughout this section, we suppose Condition B instead of Condition A. In this case, every element of $Dom(\mathscr{E}) \subset l(V_\infty)$ has continuous extension on K. Then, we shall show the bilinear form $(\mathscr{E}, Dom(\mathscr{E}))$ is a regular local Dirichlet form on $L^2(K,\nu)$ for any eveywhere dense probability measure ν.

Let $T = r_1^{-1} \wedge r_2^{-1}$. Note that $1 < T \leq 2$.

Condition B. $T > t (> 1)$.

Proposition 5.1. *For any $u \in Dom(\mathscr{E})$,*

$$\sup_{x,y \in V_\infty} |u(x) - u(y)| \leq c(t)\mathscr{E}(u,u)^{\frac{1}{2}}, \tag{5.1}$$

where $c(t) = 4/(1 - \sqrt{(2t/T) - 1})$.

Lemma 5.2. *Let u be an m-harmonic function. Then for any $x \in K$,*

$$|u(\xi_1) - u(x)| \leq \frac{2}{1 - \sqrt{\frac{2t}{T} - 1}} \mathscr{E}(u,u)^{\frac{1}{2}}. \tag{5.2}$$

Proof. We prove this lemma by induction on m. Let us denote $(2/(1-\sqrt{(2t/T)-1}))^2$ by $D(t)$. Note that $D(t)\geq 4$. In case of $m=0$, by Lemma 3.5, we have

$$|u(\xi_1)-u(\xi_2)|^2=\mathscr{E}_L^1(u,u)\leq\mathscr{E}(u,u). \tag{5.3}$$

By the maximal principle, we get (5.2). Suppose that $m\geq 1$. We devide the plain Mandala into 3 parts: L, K_1 and K_2.

(Case I) Consider the case of $x\in L$. By the maximal principle, there exists $y\in L\cap V_m$ such that

$$|u(\xi_1)-u(x)|\leq|u(\xi_1)-u(y)|.$$

Appling Corollary 3.6 to the right hand side of the inequality above, we obtain

$$\begin{aligned}|u(\xi_1)-u(x)|^2&\leq\sum_{\eta\in I_b^m}r_\eta^{-1}\{u(\pi(\eta\dot{3}))-u(\pi(\eta\dot{4}))\}^2\\&=\mathscr{E}_L^m(u,u)\leq\mathscr{E}(u,u)\leq D(t)\mathscr{E}(u,u).\end{aligned} \tag{5.4}$$

(Case II) Consider the case of $x\in K_1$. Since $u\circ F_1\in\mathscr{H}_{m-1}$, by combining induction hypothesis and Lemma 3.11, we have

$$\begin{aligned}|u(\xi_1)-u(x)|^2&=|u\circ F_1(\xi_1)-u\circ F_1(F_1^{-1}(x))|^2\\&\leq D(t)\mathscr{E}(u\circ F_1,u\circ F_1)=\bar{r}_1D(t)\mathscr{E}_{K_1}(u,u)\\&\leq\frac{t}{T}D(t)\mathscr{E}(u,u).\end{aligned} \tag{5.5}$$

Since $t<T$, we have our assertion.

(Case III) Consider the case of $x\in K_2$. We denote $\pi(12)$ by ξ_3. Then, by using the same argument with (Case I) and (Case II), we can see that

$$|u(\xi_1)-u(\xi_3)|^2\leq\bar{r}_1\mathscr{E}_{F_1(L)}(u,u), \tag{5.6}$$

$$|u(\xi_3)-u(x)|^2\leq\bar{r}_2D(t)\mathscr{E}_{K_2}(u,u). \tag{5.7}$$

Let $\delta=|u(\xi_1)-u(\xi_3)|/|u(\xi_1)-u(x)|$.

(i) If $\delta>D(t)^{-\frac{1}{2}}$, then (5.6) gives

$$\begin{aligned}|u(\xi_1)-u(x)|^2&=\delta^{-2}|u(\xi_1)-u(\xi_3)|^2\\&\leq\bar{r}_1D(t)\mathscr{E}_{F_1(L)}(u,u).\end{aligned}$$

(ii) Suppose that $\delta\leq D(t)^{-\frac{1}{2}}$. Note that

$$|u(\xi_1)-u(\xi_3)|^2+|u(\xi_3)-u(x)|^2=(2\delta^2-2\delta+1)|u(\xi_1)-u(x)|^2.$$

Let $\lambda=(2\delta^2-2\delta+1)^{-1}$. Then by combining (5.6) and (5.7),

(5.8)
$$\begin{aligned}|u(\xi_1)-u(x)|^2&=\lambda\{|u(\xi_1)-u(\xi_3)|^2+|u(\xi_3)-u(x)|^2\}\\&\le\lambda\bar{r}_1\mathcal{E}_{F_1(L)}(u,u)+\lambda\bar{r}_2D(t)\mathcal{E}_{K_2}(u,u)\\&\le\lambda\frac{t}{T}\{\mathcal{E}_{K_1}(u,u)+D(t)\mathcal{E}_{K_1}(u,u)\}.\end{aligned}$$

Observe that $\delta<D(t)^{-\frac{1}{2}}$ is a part of the solution of the inequality $\lambda t/T<1$. This completes the proof. □

Proof of Proposition 5.1. For any $x,y\in V_\infty$, let $m=\max\{i(x),i(y)\}$. Recall that $i(x)=\min\{n\ge 0|x\in V_n\}$. Since $P_m u=u$ on V_m and $P_m u\in\mathcal{H}_m$, Lemma 5.2 gives

$$\begin{aligned}|u(x)-u(y)|&\le|P_mu(x)-P_mu(\xi_1)|+|P_mu(\xi_1)-P_mu(y)|\\&\le\frac{4}{1-\sqrt{\frac{2t}{T}-1}}\mathcal{E}(P_mu,P_mu)\le c(t)\mathcal{E}(u,u).\end{aligned}$$

□

By virtue of Proposition 5.1, we see the following theorem in the same way as Kigami [6] and Kusuoka [12].

Theorem 5.3.

(i) *Every $u\in Dom(\mathcal{E})$ can be extended to a continuous function on K.*

(ii) *The bilinear form $(\mathcal{E}, Dom(\mathcal{E}))$ is a regular local Dirichlet form on $L^2(K,\nu)$, where ν is any everywhere dense probability measure on K.*

By using Theorem 5.3, we have the following proposition just in the same way as Fukushima [2, Theorem 2.3].

Proposition 5.4.

(i) *For $\alpha>0$, $(\mathcal{E}^{(\alpha)},Dom(\mathcal{E}))$ admits a positive continuous symmetric reproducing kernel $g_\alpha(x,y)$: for each $y\in E$ there exists $g_\alpha(\cdot,y)\in Dom(\mathcal{E})$ such that*

$$\mathcal{E}^{(\alpha)}(g_\alpha(\cdot,y),v)=v(y)\qquad \textit{for all } v\in Dom(\mathcal{E}).$$

(ii) *The associated diffuson on K is point recurrent:*

$$P_x(\sigma_{\{y\}}<\infty)=1\qquad \textit{for all } x,y\in K,$$

where $\sigma_{\{y\}}$ is the first hitting time for $\{y\}$.

Finally, we shall show that if $t>T$, then there exist a function in $Dom(\mathscr{E})$, which can not be extended to be a continuous function on K. Frist of all, we see that (5.1) fails, provided that $t>T$.

Lemma 5.5. *Suppose that $t>T$. Then for any positive constant c, there exists $u\in Dom(\mathscr{E})$ such that*

$$\sup_{x,y\in V_\infty}|u(x)-u(y)|>c\mathscr{E}(u,u)^{\frac{1}{2}}. \tag{5.9}$$

Proof. Without loss of generality, we may assume that $r_1\geq r_2$: i.e., $T=r_1^{-1}$. Let $m\geq 1$ and $z=\pi(\underbrace{1\cdots 1}_{m\text{ times}}\dot{2})\in V_m$. Obviously

$$\sup_{x,y\in V_\infty}|\psi_z^m(x)-\psi_z^m(y)|=1.$$

Since ψ_z^m is an m-hamonic function, Lemma 3.7 gives

$$\mathscr{E}(\psi_z^m,\psi_z^m)=\mathscr{E}^m(\psi_z^m,\psi_z^m)=\frac{c(r)}{t-1}\left(\frac{T}{t}\right)^{m-1}$$

This implies our assertion. □

Let f be a function on K defined by
(i) For any $n\geq 1$ and $\omega\in I_s^n$, $f\circ F_\omega$ is a bridge harmonic function.
(ii) $f(x)=1$ for any $x\in\pi(I_s^N)\backslash\{\xi_1\}$, and $f(\xi_1)=0$.

From Proposition 3.2, we see that f exists. Now, if we assume $T=r_1^{-1}<t$, then we have

$$\mathscr{E}(f,f)=\lim_{m\to\infty}\sum_{k=0}^{m-1}(tr_1)^{-k}=\frac{t}{t-T}<\infty.$$

Therefore $f\in Dom(\mathscr{E})$. On the other hand, it is obvious that f has no continuous version on $L^2(K,\nu)$.

6. Dimensions

We assume Condition A again throughout this section. Kigami-Lapidius [8] have studied the spectral dimensions of the P.C.F. self-similar sets with Bernoulli measures. This result suggests a way of determining the spectral dimensions of the plain Mandala (and the Mandala) with the self-similar measure μ defined at the end of Section 3.

Kigami [7] also obtained the Hausdorff dimensions of a class of self-similar sets associated with a non-transitive Markov subshift. First of all, we shall apply his result to our objects.

Theorem 6.1. *The Hausdorff dimension of the Mandala is log* $5/\log 3$, *and that of the plain Mandala is* 1.

By the proof of Lemma 4.6, we obtain

$$\int_L |u(b_x^m)-u(x)|^2\mu(dx)\leq c(r)^{-m}\mathscr{E}_L(u,u) \qquad \text{for } u\in C(L;\boldsymbol{R}).$$

For any $f\in l(V_\infty)$ and $y\in L$, let $P_m^{(L)}f(y)=\Sigma_{x\in V_m\cap L}f(x)\psi_x^m(y)$. Using the same argument with Cotollary 4.7 and Proposition 4.8, we see that $\{P_m^{(L)}g\}_{m\geq 0}$ is a $\boldsymbol{L}^2(L,\mu)$-Cauchy sequence for $g\in Dom(\mathscr{E}_L)$. We denote its limit by $\iota_L(g)$. Then, $\iota_L: Dom(\mathscr{E}_L)\to \boldsymbol{L}^2(L,\mu)$ is injective. Now, we define the domain $\mathscr{F}_L$ of $\mathscr{E}_L$ on $\boldsymbol{L}^2(L,\mu)$ by $\mathscr{F}_L=\iota_L(Dom(\mathscr{E}_L))$. For $u\in\mathscr{F}_L$, when no confusion can arise, we use $\mathscr{E}_L(u,u)$ instead of $\mathscr{E}_L(\iota^{-1}(u),\iota^{-1}(u))$. Let us introduce new Dirichlet forms following Fukushima [2].

DEFINITION 6.2.

(6.1)
$$\tilde{\mathscr{F}}=\{u\in \boldsymbol{L}^2(K,\mu)|\text{for } i\in I_s, u_i=u\circ F_i\,\mu-a.e., u_i\in\tilde{\mathscr{F}}$$
$$\text{for } j\in I_b, u_j=u\circ F_j\,\mu-a.e.(L), u_j\in\mathscr{F}_L\},$$

$$\tilde{\mathscr{E}}(u,v)=\sum_{i\in I_s}\bar{r}_i^{-1}\tilde{\mathscr{E}}(u_i,v_i)+\sum_{j\in I_b}r_j^{-1}\mathscr{E}_L(u_j,v_j) \quad \text{for } u,v\in\tilde{\mathscr{F}}.$$

(6.2)
$$\mathscr{F}^0=\{u\in\mathscr{F}\,|\,\iota^{-1}(u)(x)=0 \quad \text{for } x\in V_0\},$$
$$\mathscr{E}^0(u,v)=\mathscr{E}(u,v) \quad \text{for } u,v\in\mathscr{F}^0.$$

(6.3)
$$\tilde{\mathscr{F}}^0=\{u\in\mathscr{F}\,|\,\iota^{-1}(u)(x)=0 \quad \text{for } x\in V_1\},$$
$$\tilde{\mathscr{E}}^0(u,v)=\mathscr{E}(u,v) \quad \text{for } u,v\in\tilde{\mathscr{F}}^0.$$

(6.4)
$$\tilde{\mathscr{F}}_L=\{u\in \boldsymbol{L}^2(L,\mu)\mid \text{for } j\in I_b, u_j=u\circ F_j\,\mu-a.e.(L), u_j\in\mathscr{F}_L\},$$
$$\tilde{\mathscr{E}}_L(u,v)=\sum_{j\in I_b}r_j^{-1}\mathscr{E}_L(u_j,v_j) \quad \text{for } u,v\in\tilde{\mathscr{F}}_L.$$

(6.5)
$$\mathscr{F}_L^0=\{u\in\mathscr{F}_L\,|\,\iota_L^{-1}(u)(x)=0 \quad \text{for } x\in V_0\},$$

$$\mathscr{E}^0_L(u,v)=\mathscr{E}_L(u,v) \quad \text{for } u,v\in\mathscr{F}^0_L.$$

(6.6) $$\tilde{\mathscr{F}}^0_L=\{u\in\mathscr{F}_L|\ \iota^{-1}(u)(x)=0 \quad \text{for } x\in V_1\},$$

$$\tilde{\mathscr{E}}^0_L(u,v)=\mathscr{E}_L(u,v) \quad \text{for } u,v\in\tilde{\mathscr{F}}^0_L.$$

Lemma 6.3. *The following hold*
(i) $\mathscr{F}\subset\tilde{\mathscr{F}}$, *and* $\mathscr{E}(u,u)=\tilde{\mathscr{E}}(u,u)$ *for* $u\in\mathscr{F}$.
(ii) $\mathscr{F}_L\subset\tilde{\mathscr{F}}_L$, *and* $\mathscr{E}_L(u,u)=\tilde{\mathscr{E}}_L(u,u)$ *for* $u\in\mathscr{F}_L$.

Proof. For any $u\in\mathscr{F}$, by the definition of ι and ι_L, we can verify

$$u\circ F_i=\iota(\iota^{-1}(u)\circ F_i)\in\mathscr{F} \quad \text{for } i\in I_s.$$

$$u\circ F_j=\iota_L(\iota^{-1}(u)\circ F_j)\in\mathscr{F}_L \quad \text{for } j\in I_b.$$

Thus, for any $u,v\in\mathscr{F}$, Lemma 3.11 gives

$$\begin{aligned}\mathscr{E}(u,v)&=\sum_{i\in I}\mathscr{E}_{K_i}(\iota^{-1}(u),\iota^{-1}(v))\\&=\sum_{i\in I_s}\bar{r}_i^{-1}\mathscr{E}(\iota^{-1}(u)\circ F_i,\iota^{-1}(v)\circ F_i)+\sum_{j\in I_b}\bar{r}_j^{-1}\mathscr{E}_L(\iota^{-1}(u)\circ F_j,\iota^{-1}(v)\circ F_j)\\&=\sum_{i\in I_s}\bar{r}_i^{-1}\mathscr{E}(u\circ F_i,v\circ F_i)+\sum_{j\in I_b}\bar{r}_j^{-1}\mathscr{E}_L(u\circ F_j,v\circ F_j)=\tilde{\mathscr{E}}(u,v).\end{aligned}$$

Hence, we obtain the first assertion. We can show the second assertion in the same way. □

We can see the following proposition in the same argument as in Section 4.

Proposition 6.4.
(i) $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$, $(\mathscr{E}^0,\mathscr{F}^0)$ *and* $(\tilde{\mathscr{E}}^0,\tilde{\mathscr{F}}^0)$ *are Dirichlet forms on* $L^2(K,\mu)$.
(ii) $(\mathscr{E}_L,\mathscr{F}_L)$, $(\tilde{\mathscr{E}}_L,\tilde{\mathscr{F}}_L)$, $(\mathscr{E}^0_L,\mathscr{F}^0_L)$ *and* $(\tilde{\mathscr{E}}^0_L,\tilde{\mathscr{F}}^0_L)$ *are Dirichlet forms on* $L^2(L,\mu)$.

Now, let $\rho(x)$ be a counting function of eigenvalues of $(\mathscr{E},\mathscr{F})$ less than x:

(6.7) $$\rho(x)=\#\{\lambda|\lambda \text{ is an eigenvalue of } (\mathscr{E},\mathscr{F}),\lambda\leq x\}.$$

Substituting $(\tilde{\mathscr{E}},\tilde{\mathscr{F}})$ for $(\mathscr{E},\mathscr{F})$ in (6.7), we define $\tilde{\rho}(x)$. So ρ^0, $\tilde{\rho}^0$, ρ_L, $\tilde{\rho}_L$, ρ^0_L, $\tilde{\rho}^0_L$ are defined in a similar way.

Lemma 6.5. *For any* $x>0$, *the following hold*

(6.8) $$\tilde{\rho}(x)\geq\rho(x)\geq\rho^0(x)\geq\tilde{\rho}^0(x).$$

$$\tilde{\rho}_L(x) \geq \rho_L(x) \geq \rho_L^0(x) \geq \tilde{\rho}_L^0(x). \tag{6.9}$$

$$\tilde{\rho}(x) = 2\rho\left(\frac{x}{c(\boldsymbol{r})}\right) + 2\rho_L\left(\frac{x}{c(\boldsymbol{r})}\right). \tag{6.10}$$

$$\tilde{\rho}^0(x) = 2\rho^0\left(\frac{x}{c(\boldsymbol{r})}\right) + 2\rho_L^0\left(\frac{x}{c(\boldsymbol{r})}\right). \tag{6.11}$$

$$\tilde{\rho}_L(x) = 2\rho_L\left(\frac{x}{c(\boldsymbol{r})}\right). \tag{6.12}$$

$$\tilde{\rho}_L^0(x) = 2\rho_L^0\left(\frac{x}{c(\boldsymbol{r})}\right). \tag{6.13}$$

Proof. From the definition, it is clear that (6.8) and (6.9) hold.
Let u be an eigenfunction of $(\tilde{\mathscr{E}}, \tilde{\mathscr{F}})$ with eigenvalue λ:

$$\mathscr{E}(u,v) = \lambda(u,v)_{L^2(K,\mu)} \qquad \text{for any } v \in \mathscr{F}. \tag{6.14}$$

By the self-similarity of μ, we have

$$\begin{aligned}(u,v)_{L^2(K,\mu)} &= \sum_{i\in I_s}(u,v)_{L^2(K_i,\mu_{K_i})} \\ &= \sum_{j\in I_s} c(\boldsymbol{r})^{-1}\bar{r}_i^{-1}(u_i,v_i)_{L^2(K,\mu)} + \sum_{j\in I_b} c(\boldsymbol{r})^{-1} r_j^{-1}(u_j,v_j)_{L^2(L,\mu_L)}.\end{aligned} \tag{6.15}$$

Thus we have

$$\begin{aligned}&\sum_{i\in I_s}\bar{r}_i^{-1}\mathscr{E}(u_i,v_i) + \sum_{j\in I_b} r_j^{-1}\mathscr{E}_L(u_j,v_j) \\ &= \frac{\lambda}{c(\boldsymbol{r})}\sum_{i\in I_s}\bar{r}_i^{-1}(u_i,v_i)_{L^2(K,\mu)} + \frac{\lambda}{c(\boldsymbol{r})}\sum_{j\in I_b} r_j^{-1}(u_j,v_j)_{L^2(L,\mu_L)}.\end{aligned} \tag{6.16}$$

Since $v \in \tilde{\mathscr{F}}$ is arbitrary, this implies (6.10). By a similar argument, we have (6.11), (6.12) and (6.13). □

Theorem 6.6. *Let* $d_s = \frac{2\log 2}{\log c(\boldsymbol{r})}$. *Then*

$$0 < \varliminf_{x\to\infty} \frac{\rho(x)}{x^{\frac{d_s}{2}}\log x} \leq \varlimsup_{x\to\infty} \frac{\rho(x)}{x^{\frac{d_s}{2}}\log x} < \infty. \tag{6.17}$$

Proof. For any $n \in N$, by Lemma 6.5, we have

$$(6.18)\qquad 2^n\rho\left(\frac{x}{c(r)^n}\right)+n2^n\rho_L\left(\frac{x}{c(r)^n}\right)\geq\rho(x)\geq\rho^0(x)\geq 2^n\rho^0\left(\frac{x}{c(r)^n}\right)+n2^n\rho_L^0\left(\frac{x}{c(r)^n}\right).$$

Give a sufficient large positive number x_0 which satisfies $\rho_L^0(x_0)>0$, then (6.18) implies

$$\frac{\rho(c(r)^n x_0)}{n2^n}\geq\frac{\rho^0(x_0)}{n}+\rho_L^0(x_0)\geq\rho_L^0(x_0)>0$$

and

$$\frac{\rho(c(r)^n x_0)}{n2^n}\leq\frac{\rho(x_0)}{n}+\rho_L(x_0)\leq\rho(x_0)+\rho_L(x_0)<\infty.$$

Thus, there are positive constants c_1, c_2 such that

$$(6.19)\qquad 0<c_1\leq\frac{\rho(c(r)^n x_0)}{n2^n}\leq c_2<\infty.$$

For $x\geq x_0$, there is $n\in N$ such that $c(r)^n x_0\leq x<c(r)^{n+1}x_0$. So, there are positive constants c_3, c_4 such that

$$(6.20)\qquad n2^n\leq c_3x^{\frac{d_s}{2}}(\log x+c_4)\leq(n+1)2^{n+1}.$$

And, obviously $\rho(c(r)^n x_0)\leq\rho(x)\leq\rho(c(r)^{n+1}x_0)$. This implies our assertion. □

7. An extension

So far (except Section 5), we have taken the self-similar measure μ as the basicmeasure for the Dirichlet form $(\mathscr{E},Dom(\mathscr{E}))$. In this section, we introduce more general self-similar measure κ instead of μ, and examine how much the argument in Section 4 and Section 6 works.

Let s_1, s_2, s_3, s_4 be positive numbers with $s_1+s_2=1$ and $s_3+s_4=1$. Let $\tau>1$ and let $\bar{s}_i$ equal τs_i for $i=1,2$, and equal s_i for $i=3,4$. Moreover, we assume the following

Condition C. $s_1s_2=s_3s_4$: *i.e.*, $s_1=s_3$ *or* s_4.

Hereafter, we shall assume Condition C instead of Condition A. Let $c(s)=(s_1s_2)^{-1}=(s_3s_4)^{-1}$. Now, let us define a probability κ measure on K as follows: for any non-negative integer k, with $k<m$

$$(7.1)\qquad \kappa(K_\omega)=\begin{cases}(1-\tau^{-1})c(s)^{-m}\bar{s}_\omega^{-1} & \text{if } \omega\in I_s^kI_b^{m-k}\\ c(s)^{-m}\bar{s}_\omega^{-1} & \text{if } \omega\in I_s^m.\end{cases}$$

Then, we can prove the following proposition in a similar way as Proposition 3.16.

Proposition 7.1. *κ_m converges weakly to κ, where κ_m is a probability measure on V_m defined by*

(7.2)
$$\kappa_m(x)=\frac{\sum_{\substack{\eta\in W_m\setminus I_s^m\\ x\in F_\eta(V_0)}}\bar{s}_\eta^{-1}+\frac{\tau}{\tau-1}\sum_{\substack{\eta\in I_s^m\\ x\in F_\eta(V_0)}}\bar{s}_\eta^{-1}}{2\sum_{\omega\in W_m\setminus I_s^m}\bar{s}_\omega^{-1}+2\frac{\tau}{\tau-1}\sum_{\omega\in I_s^m}\bar{s}_\omega^{-1}}\quad \textit{for all } x\in V_m.$$

Let us assume the following condition in addition to Condition C.

Condition D. $\tau\geq t(>1)$.

Let $\gamma_i=c(s)\bar{s}_i\bar{r}_i^{-1}$ for $i\in I$, and let $\gamma=\min\{\gamma_i\colon i=1,2,3,4\}$. Then $\gamma>1$. The following lemma is proved analogously to Lemma 4.6.

Lemma 7.2. *Let u be a continuous function on K. Then*

(7.3)
$$\int_K|u(b_x^m)-u(x)|^2\kappa(dx)\leq\gamma^{-m}\mathscr{E}(u,u).$$

Proof. For any $\omega\in W_m$, let $U_n(\omega)=F_\omega(L)\cap(V_{m+n}\setminus V_m)$. Then

(7.4)
$$\kappa_{m+n}(U_n(\omega))\leq c(s)^{-m}\bar{s}_\omega^{-1}.$$

By (4.14), we see that

(7.5)
$$\begin{aligned}\sum_{x\in U_n(\omega)}|u(b_x^m)-u(x)|^2\kappa_{m+n}(x)&\leq c(s)^{-m}\bar{s}_\omega^{-1}\bar{r}_\omega\mathscr{E}^{m+n}_{F_\omega(L)}(u,u)\\ &\leq\gamma^{-m}\mathscr{E}^{m+n}_{F_\omega(L)}(u,u).\end{aligned}$$

For any $\omega\in I_s^m$ and $h\geq 1$, let $L_\omega^h=K_\omega\cap\left(\cup_{\eta\in I_s^{m+h}}F_\eta(L)\right)$ and let $U_n^h(\omega)=L_\omega^h\cap(V_{m+n}\setminus V_m)$. Then, we see

(7.6)
$$\kappa_{m+n}(U_n^h(\omega))\leq c(s)^{-m}\tau^{-h}\bar{s}_\omega^{-1}.$$

(7.7)
$$|u(b_x^m)-u(x)|^2\leq\bar{r}_\omega t^h\mathscr{E}^{m+n}_{L_\omega^h}(u,u).$$

From Condition D, we have

(7.8)
$$\sum_{x\in U_n^h}|u(b_x^m)-u(x)|^2\kappa_{m+n}(x)\leq c(s)^{-m}\bar{s}_\omega^{-1}\bar{r}_\omega\left(\frac{t}{\tau}\right)^h\mathscr{E}^{m+n}_{L_\omega^h}(u,u)$$

$$\leq \gamma^{-m}\mathscr{E}^{m+n}_{L^h_\omega}(u,u).$$

The result now follows from Proposition 7.1. □

Using the same argument with Corollary 4.7 and Proposition 4.8, there exists a injective map from $Dom(\mathscr{E})$ to $L^2(K,\kappa)$. We denote the image of $Dom(\mathscr{E})$ by $\mathscr{G}$. Then we see the following theorem analogously to Theorem 4.9.

Theorem 7.3. *$(\mathscr{E},\mathscr{G})$ is a regular local Dirichlet form on $L^2(K,\kappa)$.*

In virtue of Theorem 7.3, we can determine the spectral dimension of $(\mathscr{E},\mathscr{G})$ with respect to κ.

DEFINITION 7.4. Let $d_s^{(s)}$ and $d_s^{(b)}$ be the solutions of the following equalities;

$$\sum_{i\in I_s}\left(\frac{1}{\gamma_i}\right)^{\frac{d_s^{(s)}}{2}}=1, \qquad \sum_{j\in I_b}\left(\frac{1}{\gamma_j}\right)^{\frac{d_s^{(b)}}{2}}=1.$$

Let $d_s=d_s^{(s)}\vee d_s^{(b)}$, and let $a_i=(1/\gamma_i)^{\frac{d_s^{(s)}}{2}}$ for $i=1,2,3,4$. Following Kigami-Lapidus [8], we shall define a collection of words with various length $\Lambda_n\subset\cup_{m\geq 1}W_m$ by

$$\Lambda_n=\{\omega=\omega_1\omega_2\cdots\omega_m\,|\,a_{\omega_1}a_{\omega_2}\cdots a_{\omega_{m-1}}>a^n\geq a_\omega\},$$

where $a=\min\{a_i\,|\,i=1,2,3,4\}$.

Theorem 7.5.
(i) *If $d_s^{(s)}=d_s^{(b)}$, then*

$$0<\varliminf_{x\to\infty}\frac{\rho(x)}{x^{\frac{d_s}{2}}\log x}\leq\varlimsup_{x\to\infty}\frac{\rho(x)}{x^{\frac{d_s}{2}}\log x}<\infty, \tag{7.9}$$

where $\rho(x)$ is a counting function of eigenvalues of $(\mathscr{E},\mathscr{G})$ less than x, see (6.7).
(ii) *If $d_s^{(s)}\neq d_s^{(b)}$, then*

$$0<\varliminf_{x\to\infty}\frac{\rho(x)}{x^{\frac{d_s}{2}}}\leq\varlimsup_{x\to\infty}\frac{\rho(x)}{x^{\frac{d_s}{2}}}<\infty. \tag{7.10}$$

To prove the theorem above, we need to show the following lemmas.

Lemma 7.6. *Suppose that $d_s^{(s)}\leq d_s^{(b)}$. Let*

$$\Lambda_n^k=\{\omega\in\Lambda_n\,|\,\omega\text{ is in the }k\text{-th bridge: i.e., }\omega\in I_s^kI_b^{|\omega|-k}\},$$

where $|\omega|$ is the length of the word ω. Then

(7.11) $$\sum_{\omega\in\Lambda_n^k} a_\omega \le (a_1+a_2)^k.$$

In particular, equality holds if $k=0,1,\cdots,n$.

Proof. Notice that for any $m\ge 1$ and any $k=0,1,\cdots,m$,

(7.12) $$\sum_{\omega\in I_s^k I_b^{m-k}} a_\omega = (a_1+a_2)^k (a_3+a_4)^{m-k}.$$

Let $M_n=\max\{|\omega|:\omega\in\Lambda_n\}$, and let

$$[\Lambda_n^k]=\coprod_{\omega\in\Lambda_n^k}\{\omega\xi\,|\,\xi\in I_b^{M_n-|\omega|}\}.$$

Then we can see that the natural projection f from $[\Lambda_n^k]$ to $I_s^k I_b^{M_n-k}$ is injective. If $d_s^{(s)}\le d_s^{(b)}$, then $a_1+a_2<1$ and $a_3+a_4=1$. Hence, (7.12) gives

(7.13) $$\sum_{\omega\in\Lambda_n^k} a_\omega = \sum_{\omega'\in[\Lambda_n^k]} a_{\omega'} \le \sum_{\eta\in I_s^k I_b^{M_n-k}} a_\eta = (a_1+a_2)^k.$$

If $k=0,1,\cdots,n$, then for any $\eta\in I_s^k$,

(7.14) $$a_\eta \ge (a_1\wedge a_2)^k \ge (a_1\wedge a_2)^n \ge a^n.$$

This implies natural projection f is surjective. Thus, in case of $k=0,1,\cdots,n$, equality holds in (7.11). □

Lemma 7.7. *Suppose that* $d_s^{(s)}>d_s^{(b)}$. *Let* $\Lambda_n^s=\Lambda_n\cap(\cup_{m\ge 1}I_s^m)$ *and let*

$$\tilde\Lambda_n^l=\{\omega\in\Lambda_n\,|\,\omega\in I_s^{|\omega|-l}I_b^l\}.$$

Then

(7.15) $$\sum_{\omega\in\Lambda_n^s} a_\omega = 1.$$

And there is a constant $c_1>0$ *such that*

(7.16) $$\sum_{\omega\in\tilde\Lambda_n^l} a_\omega \le c_1 (a_3+a_4)^l.$$

Proof. Notice that if $d_s^{(s)}>d_s^{(b)}$, then $a_1+a_2=1$ and $a_3+a_4<1$. Let

$$[\Lambda_n^s]=\coprod_{\omega\in\Lambda_n^s}\{\omega\xi|\xi\in I_s^{M_n-|\omega|}\}.$$

Then we can see that the natural projection from $[\Lambda_n^s]$ to $I_s^{M_n}$ is bijective. If $d_s^{(s)} \geq d_s^{(b)}$, then $a_1 + a_2 = 1$. Hence, (7.12) gives

$$\sum_{\omega \in \Lambda_n^s} a_\omega = \sum_{\omega' \in [\Lambda_n^s]} a_{\omega'} = \sum_{\eta \in I_s^{M_n}} a_\eta = 1. \tag{7.17}$$

Thus we have (7.15). For any $\omega \in \tilde{\Lambda}_n^l$, we write $\omega = \omega^{(1)}\omega^{(2)}$, where $\omega^{(1)} \in I_s^{|\omega| - l}$ and $\omega^{(2)} \in I_b^l$. Let

$$[\tilde{\Lambda}_n^l] = \coprod_{\omega \in \tilde{\Lambda}_n^l} \{\omega^{(1)} \xi \omega^{(2)} \mid \xi \in I_s^{M_n - |\omega|}\}.$$

We denote the natural projection from $[\tilde{\Lambda}_n^l]$ to $I_b^{M_n - l} I_b^l$ by g. From the definition of a, it is obvious that there is a constant $c_1 \in \boldsymbol{N}$ such that

$$(a_1 \vee a_2)^{c_1 - 1} > a \geq (a_1 \vee a_2)^{c_1}$$

Then we can verify that

$$\sharp\{g^{-1}(\eta)\} \leq c_1 \qquad \text{for } \eta \in I_s^{M_n - l} I_b^l. \tag{7.18}$$

Hence, (7.12) gives

$$\sum_{\omega \in \tilde{\Lambda}_n^l} a_\omega = \sum_{\omega' \in [\tilde{\Lambda}_n^l]} a_{\omega'} \leq c_1 \sum_{\eta \in I_s^{M_n - l} I_b^l} a_\eta = c_1 (a_3 + a_4)^l. \tag{7.19}$$

□

Lemma 7.8.

(i) *If* $d_s^{(s)} = d_s^{(b)}$, *then*

$$n + 1 \leq \sum_{\omega \in \Lambda_n} a_\omega \leq c_1 n + 1, \tag{7.20}$$

where c_1 *is the same constant as in* Lemma 7.7.

(ii) *If* $d_s^{(s)} \neq d_s^{(b)}$, *then there are positive constants* c_2 *and* c_3 *such that*

$$0 < c_2 \leq \sum_{\omega \in \Lambda_n} a_\omega \leq c_3 < \infty. \tag{7.21}$$

Proof. Suppose that $d_s^{(s)} = d_s^{(b)}$. Let $M_n^s = \max\{|\omega| : \omega \in \Lambda_n^s\}$. Notice that if $k > M_n^s$, then $\Lambda_n^k = \emptyset$. By the definition of c_1, we have $M_n^s \leq c_1 n$. Since $a_1 + a_2 = 1$, Lemma 7.6 gives

$$n+1=\sum_{k=0}^{n}\sum_{\omega\in\Lambda_n^k}a_\omega\le\sum_{\omega\in\Lambda_n}a_\omega$$

$$=\sum_{k=0}^{M_n^s}\sum_{\omega\in\Lambda_n^k}a_\omega\le M_n^s+1\le c_1n+1.$$

Hence, we have (7.20).

Suppose that $d_s^{(s)}<d_s^{(b)}$. Lemma 7.6 gives

$$1=\sum_{\omega\in\Lambda_n^0}a_\omega\le\sum_{\omega\in\Lambda_n}a_\omega=\sum_{k=0}^{M_n^s}\sum_{\omega\in\Lambda_n^k}a_\omega\le\sum_{k=0}^{M_n^s}(a_1+a_2)^k.$$

Since $a_1+a_2<1$, we have (7.21) in the case of $d_s^{(s)}<d_s^{(b)}$

Suppose that $d_s^{(s)}>d_s^{(b)}$. Notice that $\tilde{\Lambda}_n^0=\Lambda_n^s$. Let $M_n^b=\max\{|\omega|:\omega\in\Lambda_n^b\}$, where $\Lambda_n^b=\Lambda_n\backslash\Lambda_n^s$. We see that if $l>M_n^b$, then $\tilde{\Lambda}_n^l=\emptyset$. Thus Lemma 7.7 gives

$$1=\sum_{\omega\in\Lambda_n^s}a_\omega\le\sum_{\omega\in\Lambda_n}a_\omega=\sum_{l=0}^{M_n^b}\sum_{\omega\in\tilde{\Lambda}_n^l}a_\omega\le c_1\sum_{l=0}^{M_n^b}(a_3+a_4)^l.$$

Since $a_3+a_4<1$, we have (7.21) in the case of $d_s^{(s)}>d_s^{(b)}$. □

Proof of Theorem 7.5. As in Section 6, we can define the Dirichet form $(\tilde{\mathscr{E}},\tilde{\mathscr{G}})$ on $L^2(K,\kappa)$. The counting function of eigenvalues of $(\tilde{\mathscr{E}},\tilde{\mathscr{G}})$ is denoted by $\tilde{\rho}$. Similarly, we define $(\mathscr{E}^0,\mathscr{G}^0),\rho^0,\cdots,(\tilde{\mathscr{E}}_L^0,\tilde{\mathscr{G}}_L^0),\tilde{\rho}_L^0$. Then, Lemma 6.4 can be easily extended to the present case. For example, (6.8) and (6.9) are obvious. The assertion (6.10) is replaced by

$$\tilde{\rho}(x)=\sum_{i\in I_s}\rho\left(\frac{x}{\gamma_i}\right)+\sum_{j\in I_b}\rho_L\left(\frac{x}{\gamma_j}\right).$$

The counterparts of (6.11), (6.12), (6.13) will be obvious. Using the results above, we can verify

$$\sum_{\omega\in\Lambda_n^s}\rho\left(\frac{x}{\gamma_\omega}\right)+\sum_{\omega\in\Lambda_n^b}\rho_L\left(\frac{x}{\gamma_\omega}\right)\ge\rho(x) \tag{7.22}$$

$$\ge\rho^0(x)\ge\sum_{\omega\in\Lambda_n^s}\rho^0\left(\frac{x}{\gamma_\omega}\right)+\sum_{\omega\in\Lambda_n^b}\rho_L^0\left(\frac{x}{\gamma_\omega}\right).$$

By the definition of Λ_n, we see that

$$a^{\frac{2n}{d_s}}\ge\frac{1}{\gamma_\omega}>a^{\frac{2(n+1)}{d_s}}\qquad\text{for }\omega\in\Lambda_n. \tag{7.23}$$

Hence,

$$\sharp(\Lambda_n^s)\rho(\alpha^{\frac{2n}{d_s}}x)+\sharp(\Lambda_n^b)\rho_L(a^{\frac{2n}{d_s}}x)\geq\rho(x)$$

$$\geq\rho^0(x)\geq\sharp(\Lambda_n^s)\rho^0(a^{\frac{2(n+1)}{d_s}}x)+\sharp(\Lambda_n^b)\rho_L^0(a^{\frac{2(n+1)}{d_s}}x). \tag{7.24}$$

This implies that there exists $x_0>0$ and constants c_4, c_5 such that

$$0<c_4\leq\frac{\rho(a^{-\frac{2n}{d_s}}x_0)}{\sharp(\Lambda_n)}\leq c_5<\infty. \tag{7.25}$$

From (7.23)

$$a^{-n}\sum_{\omega\in\Lambda_n}a_\omega\leq\sharp(\Lambda_n)<a^{-(n+1)}\sum_{\omega\in\Lambda_n}a_\omega. \tag{7.26}$$

This, combined with Lemma 7.8 and (7.25), implies our assertion. □

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