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MODULES WITH EVERY SUBGENERATED MODULE LIFTING

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It was shown in Dung-Smith [2] that, for a module M, every module in $\sigma[M]$ is extending (CS module) if and only if every module in $\sigma[M]$ is a direct sum of indecomposable modules of length 2 or, equivalently, every module in $\sigma[M]$ is a direct sum of M-injective module and a semisimple module. Here we charcterize these modules by the fact that every module in $\sigma[M]$ is lifting or, equivalently, decompose as a direct sum of a semisimple module and a projective module in $\sigma[M]$. They are also determined by the functor ring of $\sigma[M]$ being a QF-2 ring with Jacobson radical square zero.

As a corollary we obtain a result of Vanaja-Purav [8]: All (left) R-modules are lifting if and only if R is a generalizad uniserial ring with Jacobson radical aquare zero.

1. Preliminaries

Let R denote an associative ring with unit, R-Mod the category of unital left R-modules, and M a left R-module. We call M locally artinian, noetherian, of finite length every finitely generated submodule of M has the corresponding property. The natation $K \ll M$ means that K is a small (superfluous) submodule of M.

By $\sigma[M]$ we denote the full subcategory of R-Mod whose objects are submodules of M-generated modules.

For any R-module N, $E(N)$ will denote the injective hull of N in R-Mod. For $N \in \sigma[M]$, $\hat{N}$ is the injective hull of N in $\sigma[M]$. $\hat{N}$ is also called the M-injective hull of N and is isomorphic to the trace of M in $E(N)$.

$N \in \sigma[M]$ is injective in $\sigma[M]$ if and only if N is M-injective hull.

Proposition 1.1 (Functor ring). *Denote by* $\{U_\lambda\}_\Lambda$ *a representing set of all finitely generated modules in* $\sigma[M]$ *and* $U = \oplus_\Lambda U_\lambda$.

$T := \hat{End}(U_R) = \{f \in End_R(U) | (U_\lambda)f = 0$ *almost every where*$\}$ *is called the funtor ring of* $\sigma[M]$. *T has no unit but has enough idempotents. The following hold:*

(1) *T is left perfect if and only if every module in* $\sigma[M]$ *is a direct sum of finitely generated modules. In this case M is called pure semisimple* ([10], 53.4]).

(2) *Assume M is locally of finite length. Then T is semiperfect* ([10], 51.7).

(3) *Assume for every primitive idempotent* $e \in T$, *Te is finitely cogenerated. Then M is locally artinian* ([10], 52.1).

A ring T with enough idempotents is called semiperfect if every simple T-modules has projective covers (see [10], 49.10). T is said to be a left (right) QF-2 ring if it is a semiperfect and, for every primitive idempotent $e \in T$, Te (resp. eT) has a simple essential socle (e.g., [3], section 4).

Theorem 1.2. *For an R-module M with functor ring T the following are equivalent:*

(a) *For some* $k \in N$, *every module in* $\sigma[M]$ *is a direct sum of uniserial modules of length* $\leq k$;

(b) *T is a left and right QF-2 ring and Jac(T) is nilpotent.*

Proof. Consider a representing set $\{U_\lambda\}_\Lambda$ of all finitely generated modules in $\sigma[M]$, $U = \oplus_\Lambda U_\lambda$ and $T = \hat{End}_R(U)$.

(a)$\Rightarrow$(b) By condition (a), U is a direst sum of indecomosable modules of bounded length. Hence, by the Haraba-Sai Lemma (e.g., [10], 54.1), T is semiperfect and $Jac(T)$ is nilpotent.

Since M is locally of finite length, we know from [10], 53.5 that U_T is T-injective. Now we can use the conclusions (a)$\Rightarrow$(b)$\Rightarrow$(c) of [10], 55.15 to derive that T is left and right QF-2.

(b)$\Rightarrow$(a) Assume T is a left and right QF-2 ring and $Jac(T)^n = 0$, for some $n \in N$. Then M is pure semisimple and locally artinian (see 1.1) and hence locally of finite length. With the proof of (c)$\Rightarrow$(a) in [10], 55.15 we see that indecomposable modules in $\sigma[M]$ are uniserial.

It remain to show that for every uniserial module $N \in \sigma[M]$, length $N \leq n$. Assume N has composition series

$$0 \neq N_1 \subset \cdots \subset N_n \subset N_{n+1} = N.$$

From this we obtain a sequence of n morphisms in $Jac(T)$,

$$N_n \to N \to N/N_1 \to \cdots \to N/N_{n-1},$$

whose product is not zero, contradicting $Jac(T)^n = 0$.

2. Lifting modules

An R-module M is called extending of CS module if every submodule is essential in a direct summand of M.

M is said to be lifting if every submodule $K \subset M$ lies above a direct summand, i.e., there is a direct summand $X \subset M$ with $X \subset K$ and $K/X \ll M/X$. For characterizations of this condition refer to [10], 41.11 and 41.12.

A familly $\{N_\lambda\}_\Lambda$ of independent submodules of M is said to be a local direct smmand of M if finite (direct) sum of N_λ's is a direct summand in M, and we say it is a direct summand if $\oplus_\Lambda N_\lambda$ is a direct summand in M (see [4], Definition 2.15).

A module is called continuous if it is extending and direct injective. In particular, self-injective modules are continuous.

Recall two results about these modules :

Lemma 2.1. *Let M be an R-module.*

(1) *Assume every local direct summand of M is a direct summand. Then M is a direct sum of indecomposable submodules.*

(2) *Assume M is lifting and continuous. Then every local direct summand of M is a direct summand.*

Proof. (1) See [5], Lemma 2.4 or [4], Theorem 2.17.

(2) This is shown in [5], Lemma 2.5.

A ring R is called a left H-ring if every injecitve module is R-*Mod* is lifting. Some of the characterizations of H-rings (see [5], Theorem 1) can be extended to modules. For this we need the

DEFINITION. A module $K \in \sigma[M]$ is said to be small in $\sigma[M]$ if it is small submodule in its M-injective hull, i.e., $K \ll \hat{K}$.

Theorem 2.2. *For any R-module M, the following are equivalent:*

(a) *Every injective module in $\sigma[M]$ is lifting :*

(b) *M is locally noetherian and every non-small module in $\sigma[M]$ contains an M-injective submodule;*

(c) *Every module in $\sigma[M]$ is a direct sum of an M-injective module and a small module.*

Proof. (a)$\Rightarrow$(b) By 2.1, every injective module in $\sigma[M]$ is a direct sum of indecomposable submodules. This implies that M is locally noetherian (see [10], 27.5).

Assume N is not small in its M-injective hull $\hat{N}$. Since $\hat{N}$ is lifting there is a direct summand $X \subset \hat{N}$ with $X \subset N$ and $N/X \ll \hat{N}/X$. By assumption, X is not zero.

(b)$\Rightarrow$(a) Referring to [10], 27.3, apply the proof of Proposition 2.7 in [5].

(a)$\Rightarrow$(c) Consider $N \in \sigma[M]$ with M-injective hull N. Since $\hat{N}$ is lifting, by [10], 41.11, a direct summand $X \subset \hat{N}$ is contained in N and $N = X + Y$ with $Y \ll \hat{N}$. This implies that Y is small in $\sigma[M]$.

(c) $\Rightarrow$ (a) With respect to [10], 41.11, this is obvious.

It was pointed out in Osofsky [6], Lemma B (also in the proof (1) $\Rightarrow$ (3) of Vanaja-Purav, Proposition 2.13) that, for a uniserial module M with composition series $0 \neq V \subset U \subset M$, $M \oplus U/V$ is not an extending module. For the same situation we observe:

Lemma 2.3. *Assume M is a uniserial module with composition series $0 \neq V \subset U \subset M$. Then the module $M \oplus U/V$ is not lifting.*

Proof. Assume $M \oplus U/V$ is lifting. Then, by Theorem 1 in [1], U/V is M-projective. However, the diagram

$$\begin{array}{c} U/V \\ \downarrow \\ M \rightarrow M/V \rightarrow 0 \end{array}$$

can not be extended commutatively by any $h: U/V \rightarrow M$, since the image of such a morphisem always is contained in V.

The main purpose of this note is to prove:

Theorem 2.4. *For any R-module M the following are equivalent*:

(a) *Every module in $\sigma[M]$ is lifting*;

(b) *every module in $\sigma[M]$ is direct sum of a semisimple module and a projective module in $\sigma[M]$*;

(c) *every module in $\sigma[M]$ is direct sum of modules of lenth ≤ 2*

(d) *T is left and right OF-2 ring and $Jac(T)^2 = 0$.*

If this conditions hold, there is a projective generator in $\sigma[M]$ and all indecomposable modules of length ≤ 2 are M-projective.

Proof. (a) $\Rightarrow$ (d) Assume every module in $\sigma[M]$ is lifting. Then by Theorem 2.2, M is locally noetherian. It is easy to see that finitely generated uniform lifting module are local modules, i.e., their factor modules are indecomposable.

Consider an indecomposable injective module $Q \in \sigma[M]$. Then for any finitely generated submodule $K \subset Q$, $K/Rad(K)$ is simple and hence Q is uniserial (see [10], 55.1). In particular, every uniform module in $\sigma[M]$ is uniserial of lenght ≤ 2 (by Lemma 2.3). So the M-injective hull $\hat{M}$ of M is a direct sum of modules of length ≤ 2 and hence $\hat{M}$ (and M) is locally of finite length. This implies that every finitely generated module in $\sigma[M]$ is a direct sum of indecomposable module (of length ≤ 2).

Denote by $\{U_\lambda\}_\Lambda$ a representing set of all finitely generated modules in $\sigma[M]$ and $U=\oplus_\Lambda U_\lambda$. By the Harada-Sai Lemma, the functor ring $T:=\hat{End}_R(U)$ has the properties that $T/Jac(T)$ is semisimple and $Jac(T)$ is nilpotent.

In particular, M is pure-semisimple, i.e., every module in $\sigma[M]$ is a direct sum of finitely generated modules and these are direct sums of uniserial submodules of length ≤ 2. Now the assertion follows from Theorem 1.2.

Since T is ritht perfect, there exists a projective generator in $\sigma[M]$ by [10], 51.13.

Consider an indecomposable module N of length 2. This is a factor module of a supplemented projective module in $\sigma[M]$ and hence has a projective cover P (see [10], 42.1), which again is indecomposable and hence of length ≤ 2. This implies $P=N$, i.e., N is M-projective.

(c) $\Rightarrow$ (d) This is clear by Theorem 1.2.

(c) $\Rightarrow$ (a) Consider any module $N=\oplus_\Lambda N_\alpha$ in $\sigma[M]$, with N_α uniserial of length ≤ 2. By Theorem 1 in [1], N is lifting if and only if $\{N_\alpha\}_\Lambda$ is locally semi-T-nilpotent and N_α is almost N_β projective for any $\alpha \neq \beta$ in Λ.

The first condition is satisfied by the Harada-Sai Lemma (see [10], 54.1]. Any N_α of length 2 is projective in $\sigma[M]$ (as noted above) and hence is almost K-projective for any $K \subset \sigma[M]$.

Assume N_α has length 1 and consider any diagram with exact line

$$\begin{array}{ccc} & N_\alpha & \\ & \downarrow^{f} & \\ N_\beta & \xrightarrow{p} L & \to 0, \end{array}$$

with length $N_\beta \leq 2$. If p is not an isomorphism and $f \neq 0$, there exists an epimorphism $g: N_\beta \to N_\alpha$ with $p=gf$. From this we see that N_α is almost N_β-projective and N is lifting.

(c) $\Rightarrow$ (b) It is clear from the above that modules of length 2 are M-projective. Recall that finitely generated M-projecitve modules are projective in $\sigma[M]$. From this the assertion is obvious.

(b) $\Rightarrow$ (c) Consider a finitely generated $N \in \sigma[M]$. Then any factor module of N is a direct sum of a projective module and a noetherian module and hence N is noetherian by [7], section 3. This imlies that M is locally noetherian.

Now let $K \in \sigma[M]$ be any indecomposable M-injective module. Assume K is not semisimple. Then it is projective in $\sigma[M]$. Since $End_R(K)$ is local. K is a local module, i.e., every factor module is indecomposable (see [10], 19.7) and hence simple. From this we deduce that k has length ≤ 2.

Since every M-injective module in $\sigma[M]$ is a direct sum of indecomsables, the assertions follows.

From Theorem 2.4 together with Theorem 11 in Dung-Smith [2] we obtain a characterization of rings with all modules lifting which extends Proposition 2.13 in Vanaja-Purvav [8] :

Corollary 2.5. *For any ring R the following are equivalent*:

(a) *Every left R-module is lifting*;

(b) *Every left R-module is extending*;

(c) *Every left R-module is a direct sum of a semisimple module and a projective module*;

(d) *Every left R-module is a direct sum of modules of length* ≤ 2;

(e) *R is a generalized uniserial ring with* $Jac(J)^2 = 0$

It follows from (e) *that the conditions* (a)-(d) *are left right symmetric.*

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