Nagatomo, K. and Koga, Y.
Osaka J. Math.
32 (1995), 451–465

# $q$-DIFFERENCE ANALOGUE OF THE EULER-POISSON-DARBOUX EQUATION AND ITS LAPLACE SEQUENCE

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(Received October 19, 1993)

## 1. Introduction and main result

The Euler-Poisson-Darboux (EPD) equation is the second order hyperbolic equation

$$\bar{L}(\beta,\beta')u=\left\{\partial_x\partial_y-\frac{\beta-\beta'}{x-y}\partial_x+\frac{\beta(\beta'-1)}{(x-y)^2}\right\}u=0$$

which appears in various areas of mathematics and physics such as theory of surfaces [2], propagation of sounds [1] and collidings of gravitational waves [3], etc. By the conjugate transform of the differential operator $\bar{L}(\beta,\beta')$ with $(x-y)^{-\beta}$, we have the operator

$$(x-y)^{-\beta}\bar{L}(\beta,\beta')(x-y)^{\beta}=\bar{E}(\beta,\beta')=\partial_x\partial_y-\frac{\beta'}{x-y}\partial_x+\frac{\beta}{x-y}\partial_y.$$

In this note we consider a q-difference analogue of the operator

$$E(\beta,\beta')=(x-y)\bar{E}(\beta,\beta')=(x-y)\partial_x\partial_y-\beta'\partial_x+\beta\partial_y \tag{1}$$

and demonstrate that q-deformation of $E(\beta,\beta')$ is the q-difference operator (see section 2)

$$E_q(\beta,\beta')=[\theta_x+\beta]_q[\partial_y]_q-[\theta_y+\beta']_q[\partial_x]_q. \tag{2}$$

The EPD equation has very interesting properties, for example, Miller's symmetry, Laplace sequence and the relation to Toda molecule equation, etc. (see [2] and [6]). First we consider a q-deformation of Miller's symmetry explained below. Let $V(\beta,\beta')$ be the space of solutions of the differential equation $E(\beta,\beta')u=0$. Then $V(\beta,\beta')$ is invariant under the action of $SL(2,C)$ defined by

[1]This work was partially supported by Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.

$$u(x,y)\mapsto(bx+d)^{-\beta}(by+d)^{-\beta'}u\left(\frac{ax+c}{bx+d},\frac{ay+c}{by+d}\right),\ \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,C)$$

and hence infinitesimal generators of this symmetry are

$$E=-x^2\partial_x-y^2\partial_y-\beta x-\beta' y,$$
$$H=\partial_x+\partial_y,$$
$$F=2x\partial_x+2y\partial_y+\beta+\beta'.$$

We call this Miller's symmetry. Indeed, this Lie algebra is isomorphic to Lie algebra $sl(2,C)$. It shall be shown that its q-deformation is quantum group $U_q(sl(2,C))$ with generators

$$e=-\{q^{-\theta_x}y[\theta_y+\beta']_q+q^{\theta_y}x[\theta_x+\beta]_q\},$$
$$f=q^{-\theta_x-\beta}y^{-1}[\theta_y]_q+q^{\theta_y+\beta'}x^{-1}[\theta_x]_q,$$
$$q^h=q^{2\theta_x+2\theta_y+\beta+\beta'}.$$

If the parameter q tends to unit, obviously we get Miller's symmetry.

**Theorem 1.1.** *The difference operators e, f and $q^h$ are symmetries of the q-difference EPD equation and are generators of the quantum group $U_q(sl(2,C))$.*

REMARK 1.1. *This kind of representation of quuantum group can be seen in* [4] *and* [5].

The second aim of our research is to find a q-deformation of the so-called Laplace sequence. We give a brief explanation of the Laplace sequence for the EPD equation. Let us consider a family of differential operators parametrized by an integer n

$$E_n(\beta,\beta')=(x-y)\partial_x\partial_y-(\beta'+n)\partial_x+(\beta-n)\partial_y. \tag{3}$$

This is a typical example of Laplace sequence for the second order hyperbolic equation with two independent variables (also see [2], [6]). Define two operators $H_n$ and $B_n$ by

$$H_n=(x-y)\partial_y-(\beta'+n),\quad B_n=(x-y)\partial_x+(\beta-n).$$

Then we have

$$H_{n+1}E_n=E_{n+1}H_n,\quad B_{n-1}E_n=E_{n-1}B_n$$

for any integer n. These equations mean that if $u_n$ is a solution of the equation

$E_n(\beta,\beta')u=0$, then $u_{n+1}=H_n u_n$ or $u_{n-1}=B_n u_n$ is a solution of the equation $E_{n+1}u=0$ or $E_{n-1}u=0$, respectively. Therefore we may think that $H_n$ and $B_n$ are a kind of increasing or decreasing operators. We shall show that q-analogues of $H_n$ and $B_n$ are

$$H_{q,n}=-q^{-\theta_x}[\theta_y+\beta'+n]_q+q^{-\theta_x-(\beta-n-1)}xy^{-1}[\theta_y]_q,$$
$$B_{q,n}=q^{\theta_y}[\theta_x+\beta-n]_q-q^{\theta_y+(\beta'+n-1)}yx^{-1}[\theta_x]_q.$$

These q-difference operators are found by quantizing some solution of the EPD equation. The EPD equation has a formal solution

$$\varphi(\lambda,\mu;\beta,\beta';x,y)=\sum_{n\in Z}\frac{[\lambda+\beta;n][\mu-n+1;n]}{[\mu-n+\beta';n][\lambda+1;n]}x^{\lambda+n}y^{\mu-n}$$

where $[\alpha;n]=\Gamma(\alpha+n)/\Gamma(\alpha)$ and $\Gamma(\alpha)$ is the gamma function. We may think that its q-deformation is

$$\varphi_q(\lambda,\mu;\beta,\beta';x,y)=\sum_{n\in Z}\frac{[\lambda+\beta;n]_q[\mu-n+1;n]_q}{[\mu-n+\beta';n]_q[\lambda+1;n]_q}x^{\lambda+n}y^{\mu-n}$$

where $[\alpha;n]_q=\Gamma_q(\alpha+n)/\Gamma_q(\alpha)$ and $\Gamma_q(\alpha)$ is the basic gamma function (see section 2). We use the notations $\varphi_q^{\lambda}$ and $\varphi_{q,\lambda}$ to denote contiguous functions of $\varphi_q$, such as $\varphi_q^{\lambda}=\varphi_q(\lambda+1,\mu;\beta,\beta';x,y)$ and $\varphi_{q,\lambda}=\varphi_q(\lambda-1,\mu;\beta,\beta';x,y)$, etc. To describe the action of $e$, $f$, $q^{\pm h}$, $H_{q,n}$ and $B_{q,n}$ in a simple form, it is convenient to introduce the function

$$\Phi_q=\frac{\Gamma_q(\lambda+\beta)\Gamma_q(\mu+\beta')}{\Gamma_q(\lambda)\Gamma_q(\mu+1)}\varphi_q.$$

By using this function we can get the next expression of the action of $U_q(sl(2,C))$ and Laplace sequence

$$e\Phi_q=-[\lambda+\mu+1]_q\Phi_q^{\lambda},$$
$$f\Phi_q=[\lambda+\mu+\beta+\beta'-1]_q\Phi_{q,\mu},$$
$$q^h\Phi_q=q^{2(\lambda+\mu)+\beta+\beta'}\Phi_q,$$
$$H_{q,0}\Phi_q=-[\beta-1]_q\Phi_{q,\beta}^{\beta'},$$
$$B_{q,0}\Phi_q=[\beta'-1]_q\Phi_{q,\beta'}.$$

Finally we give the explanation of the organization of this paper. In the next section, we introduce and fix our notations appeared in the q-analogue calcules. In section 3, we define a q-difference analogue of the EPD equation and give a proof of theorem 1.1 and we shall find its q-Laplace sequence $H_{q,n}$ and $B_{q,n}$ in section 4. The classical results about the EPD equation are stated in Appendix A. A part of the proof of Theorem 1.1 is given in Appendix B. Finally we express the Casimir operator of $U_q(sl(2,C))$ by means of the operator $E_{q,0}$ in Appendix C.

## 2. q-difference calculus

In this section, a few elementary results involving basic differentiation are obtained. For any number $A$, we define basic number $[A]_q$ by the relation

$$[A]_q=\frac{q^A-q^{-A}}{q-q^{-1}}$$

where q may be real or complex. Then we can easily verify the formula

$$\begin{aligned}[A+B]_q&=q^A[B]_q+q^{-B}[A]_q\\&=q^{-A}[B]_q+q^B[A]_q\end{aligned} \tag{4}$$

and

$$[A+1]_q[B+1]_q-[A]_q[B]_q=[A+B+1]_q. \tag{5}$$

In the following sections, we need q-difference operator (q-differentiation or basic differentiation). First we introduce q-shift operator $T$ by

$$(Tf)(x)=f(qx),$$

then q-difference operator $[\partial]_q$ is defined by

$$\begin{aligned}([\partial]_q f)(x)&=\frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})x}\\&=\frac{1}{x}\left(\frac{T-T^{-1}}{q-q^{-1}}f\right)(x).\end{aligned}$$

Further we need q-difference Euler operator $[\theta]_q$

$$[\theta]_q=\frac{T-T^{-1}}{q-q^{-1}}.$$

Because of this definiton, we may identify $T$ and $q^\theta$, namely, $q^\theta\overset{\text{def}}{=}T$. One of the important properties of the operator $[\theta]_q$ is that it behaves just as the ordinary Euler differential, i.e.

$$[\theta]_q x^n=[n]_q x^n$$

We shall often use the following relations

$$x^n q^{-\theta} = q^{-\theta+n} x^n, \quad x^n q^{\theta} = q^{\theta-n} x^n, \quad x^n[\theta+\alpha]_q = [\theta+\alpha-n]_q x^n,$$

where these all relations are considered as operators. Finally we define basic gamma function by

$$\Gamma_q(x) = q^{(x^2-3x)/2} \frac{(q^2)_\infty}{(q^{2x})_\infty} (1-q^2)^{1-x}, \quad (a)_\infty = \prod_{j=0}^{\infty}(1-q^j a).$$

For this basic gamma function we have fundamental difference relation

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

## 3. q-difference analogue of the EPD equation

Let us prove that the q-deformed function $\varphi_q$ satisfies Eq.(2). From the difference relation of the basic gamma function and the expression of the q-deformed function $\varphi_q$

$$\varphi_q(\lambda,\mu;\beta,\beta';x,y) = \sum_{n\in Z} \frac{[\lambda+\beta;n]_q[\mu-n+1;n]_q}{[\mu-n+\beta';n]_q[\lambda+1;n]_q} x^{\lambda+n} y^{\mu-n},$$

we can get the following contiguous relations of $\varphi_q$.

**Proposition 3.1.** *The function $\varphi_q$ has the following contiguous relations:*

1. $x^{-1}[\theta_x]_q \varphi_q = [\lambda]_q \varphi^{\beta}_{q,\lambda}, \qquad y^{-1}[\theta_y]_q \varphi_q = [\mu]_q \varphi^{\beta'}_{q,\mu}.$
2. $[\theta_x+\beta]_q \varphi_q = [\lambda+\beta]_q \varphi^{\beta}_q, \qquad [\theta_y+\beta']_q \varphi_q = [\mu+\beta']_q \varphi^{\beta'}_q.$

By using these contiguous relations, we have

$$[\theta_x+\beta]_q[\partial_y]_q\varphi_q = [\mu]_q[\lambda+\beta]_q \varphi^{\beta,\beta'}_{q,\mu},$$

$$[\theta_y+\beta']_q[\partial_x]_q\varphi_q = [\lambda]_q[\mu+\beta']_q \varphi^{\beta,\beta'}_{q,\lambda},$$

and further we can easily verify

$$[\mu]_q[\lambda+\beta]_q \varphi^{\beta,\beta'}_{q,\mu} = [\lambda]_q[\mu+\beta']_q \varphi^{\beta,\beta'}_{q,\lambda},$$

by direct calculation. Hence we have proved

$$[\theta_x+\beta]_q[\partial_y]_q\varphi_q = [\theta_y+\beta']_q[\partial_x]_q\varphi_q,$$

which we call the q-difference EPD equation.

Now we will prove that the algebra generated by three q-defference operators

$$e=-\{q^{-\theta_x}y[\theta_y+\beta']_q+q^{\theta_y}x[\theta_x+\beta]_q\},$$
$$f=q^{-\theta_x-\beta}y^{-1}[\theta_y]_q+q^{\theta_y+\beta'}x^{-1}[\theta_x]_q,$$
$$q^h=q^{2\theta_x+2\theta_y+\beta+\beta'}$$

is a q-deformation of Miller's symmetry. First we show the next proposition.

**Proposition 3.2.** *Let $E_q(\beta,\beta')$ be the q-difference EPD operator defined by Eq.(2), then operators e, f and $q^h$ satisfy the following relations*:

1. $E_q(\beta,\beta')e=-\{q^{-\theta_x-1}[\theta_y+\beta']_q y+q^{\theta_y+1}[\theta_x+\beta]_q x\}E_q(\beta,\beta')$.
2. $E_q(\beta,\beta')f=fE_q(\beta,\beta')$.
3. $E_q(\beta,\beta')q^{\pm h}=q^{\pm 2}q^{\pm h}E_q(\beta,\beta')$.

From this proposition we immediately have the next corollary.

**Corollary 3.1.** *The q-difference operators e, f and $q^{\pm h}$ are symmetries of the q-difference EPD equation.*

Proof of Proposition 3.2. Let us prove the first relation. From the difinition of the difference operator $e$, we have

$$\begin{aligned}&E_q(\beta,\beta')e\\&=[\theta_x+\beta']_q x^{-1}[\theta_x]_q q^{-\theta_x}y[\theta_y+\beta']_q+[\theta_y+\beta']_q x^{-1}[\theta_x]_q q^{\theta_y}x[\theta_x+\beta]_q\\&\quad-[\theta_x+\beta]_q y^{-1}[\theta_y]_q q^{-\theta_x}y[\theta_y+\beta']_q-[\theta_x+\beta]_q y^{-1}[\theta_y]_q q^{\theta_y}x[\theta_x+\beta]_q.\end{aligned}$$

By using the following relation

$$x^{-1}q^{-\theta_x}=q^{-\theta_x-1}x^{-1},\quad x^{-1}q^{\theta_x}=q^{\theta_x+1}x^{-1},\quad x^{-1}[\theta_x]_q x=[\theta_x+1]_q,$$

we see

$$\begin{aligned}&E_q(\beta,\beta')e\\&=q^{-\theta_x-1}[\theta_y+\beta']_q y[\partial_x]_q[\theta_y+\beta']_q+q^{\theta_y}[\theta_x+\beta]_q[\theta_x+1]_q[\theta_y+\beta']_q\\&\quad-q^{-\theta_x}[\theta_y+\beta']_q[\theta_y+1]_q[\theta_x+\beta]_q-q^{\theta_y+1}[\theta_x+\beta]_q x[\partial_y]_q[\theta_x+\beta]_q.\end{aligned}$$

Further by applying the addition foumula Eq. (4)

$$[\theta_x+1]_q=q[\theta_x]_q+q^{-\theta_x}=q^{-1}[\theta_x]_q+q^{\theta_x},$$

in the second and third terms of the above equation, we get

$$E_q(\beta,\beta')e$$
$$=q^{-\theta_x-1}[\theta_y+\beta']_q y[\partial_x]_q[\theta_y+\beta']_q+q^{\theta_y+1}[\theta_x+\beta]_q[\theta_x]_q[\theta_y+\beta']_q$$
$$-q^{-\theta_x-1}[\theta_y+\beta']_q[\theta_y]_q[\theta_x+\beta]_q-q^{\theta_y+1}[\theta_x+\beta]_q x[\partial_y]_q[\theta_x+\beta]_q.$$

Therefore we have

$$E_q(\beta,\beta')e=-\{q^{-\theta_x-1}[\theta_y+\beta']_q y+q^{\theta_y+1}[\theta_x+\beta]_q x\}E_q(\beta,\beta').$$

The second relation is proved just above by using the relation

$$x^{-1}q^{-\theta_x-\beta}=q^{-\theta_x-1-\beta}x^{-1},\quad x^{-1}q^{\theta_x+\beta}=q^{\theta_x+\beta+1}x^{-1},$$
$$[\theta_x+\beta]_q x^{-1}=x^{-1}[\theta_x+\beta-1]_q,$$

and the addition formula

$$[\theta_x-1+\beta]_q=q^{-1}[\theta_x+\beta]_q-q^{-\theta_x-\beta}=q[\theta_x+\beta]_q-q^{\theta_x+\beta}.$$

Finally we prove the third relation. By the definition of $q^h$ and the formula

$$x^{-1}q^{2\theta_x}=q^2q^{2\theta_x}x^{-1},\quad y^{-1}q^{2\theta_y}=q^2q^{2\theta_y}y^{-1},$$

we get

$$E_q(\beta,\beta')q^h=q^2q^hE_q(\beta,\beta').$$

q.e.d

Thus we have proved the first statement of Theorem 1. A proof of the second statement, that three operators $e,f$ and $q^{\pm h}$ are generators of $U_q(sl(2,C))$, namely,

$$q^heq^{-h}=q^2e,$$
$$q^hfq^{-h}=q^{-2}f,$$
$$[e,f]=\frac{q^h-q^{-h}}{q-q^{-1}}.$$

is given in appendix B.

In the following we give a kind of representation of $U_q(sl(2,C))$ on the space of contiguous functions of $\varphi_q$.

**Proposition 3.3.** *The q-difference operators e, f and $q^{\pm h}$ act on the space of contiguous functions of $\varphi_q$ as follows*:

$$e\varphi_q=-\frac{[\lambda+\beta]_q[\lambda+\mu+1]_q}{[\lambda+1]_q}\varphi_q^{\lambda},$$

$$f\varphi_q=\frac{[\mu]_q[\lambda+\mu+\beta+\beta'-1]_q}{[\mu+\beta'-1]_q}\varphi_{q,\mu},$$

$$q^h\varphi_q=q^{2(\lambda+\mu)+\beta+\beta'}\varphi_q,\quad q^{-h}\varphi_q=q^{-2(\lambda+\mu)-\beta-\beta'}\varphi_q.$$

Proof. By the definition of $\varphi_q$, we get

$$\begin{aligned}-e\varphi_q&=\sum_{n\in Z}q^{-\lambda-n}[\mu-n+\beta']_q\frac{[\mu-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[\mu-n+\beta';n]_q}x^\lambda y^{\mu+1}t^n\\&+\sum_{n\in Z}q^{\mu-n}[\lambda+n+\beta]_q\frac{[\mu-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n\\&=I_1+I_2,\end{aligned}$$

where we put $t=x/y$. Hence by replacing $n$ by $n+1$ in the first term $I_1$, we have

$$\begin{aligned}I_1&=\sum_{n\in Z}q^{-\lambda-n-1}[\mu-n-1+\beta']_q\times\frac{[\mu-n;n+1]_q[\lambda+\beta;n+1]_q}{[\lambda+1;n+1]_q[\mu-n-1+\beta';n+1]_q}x^{\lambda+1}y^\mu t^n\\&=\frac{[\lambda+\beta]_q}{[\lambda+1]_q}\sum_{n\in Z}q^{-(\lambda+1)-n}[\mu-n]_q\times\frac{[\mu-n+1;n]_q[\lambda+1+\beta;n]_q}{[\lambda+2;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n.\end{aligned}$$

On the other hand,

$$\begin{aligned}I_2=&\frac{[\lambda+\beta]_q}{[\lambda+1]_q}\sum_{n\in Z}q^{\mu-n}[\lambda+n+1]_q\\&\times\frac{[\mu-n+1;n]_q[\lambda+1+\beta;n]_q}{[\lambda+2;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n.\end{aligned}$$

Therefore we get

$$\begin{aligned}e\varphi_q&=-I_1-I_2\\&=-\frac{[\lambda+\beta]_q}{[\lambda+1]_q}\sum_{n\in Z}\{q^{-(\lambda+1)-n}[\mu-n]_q+q^{\mu-n}[\lambda+n+1]_q\}\\&\quad\times\frac{[\mu-n+1;n]_q[\lambda+1+\beta;n]_q}{[\lambda+2;n]_q[\mu-n+\beta';n]_q}x^{\lambda+1}y^\mu t^n\\&=-\frac{[\lambda+\beta]_q[\lambda+\mu+1]_q}{[\lambda+1]_q}\varphi_q^\lambda,\end{aligned}$$

where we use the addition formula Eq. (4). Similarly as above, we have

$$f\varphi_q=\frac{[\mu]_q}{[\mu-1+\beta']_q}\sum_{n\in Z}\{q^{-\lambda-n-\beta}[\mu-n-1+\beta']_q+q^{(\mu-1)-n+\beta'}[\lambda+\beta+n]_q\}$$
$$\times\frac{[(\mu-1)-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[(\mu-1)-n+\beta';n]_q}x^{\lambda}y^{\mu-1}t^n$$
$$=\frac{[\mu]_q}{[\mu-1+\beta']_q}\sum_{n\in Z}[\lambda+\mu+\beta+\beta'-1]_q$$
$$\times\frac{[(\mu-1)-n+1;n]_q[\lambda+\beta;n]_q}{[\lambda+1;n]_q[(\mu-1)-n+\beta';n]_q}x^{\lambda}y^{\mu-1}t^n$$
$$=\frac{[\mu]_q[\lambda+\mu+\beta+\beta'-1]_q}{[\mu-1+\beta']_q}\varphi_{q,\mu}.$$

The last statement is easily proved by direct calculation. q.e.d

By using the funciton $\Phi_q$, we get a simple expression of the action of operators $e$, $f$ and $q^h$.

**Corollary 3.2.** *The action of operators $e$, $f$ and $q^h$ on the function $\Phi_q$ is*

$$e\Phi_q=-[\lambda+\mu+1]_q\Phi_q^{\lambda},$$
$$f\Phi_q=[\lambda+\mu+\beta+\beta'-1]_q\Phi_{q,\mu},$$
$$q^h\Phi_q=q^{2(\lambda+\mu)+\beta+\beta'}\Phi_q.$$

## 4. q-Laplace sequence

Here we consider a family of the difference operators

$$E_{q,n}(\beta,\beta')=[\theta_x+\beta-n]_q[\partial_y]_q-[\theta_y+\beta'+n]_q[\partial_x]_q,\qquad n\in \boldsymbol{Z} \tag{6}$$

which may be thought as a q-difference analogue of the operator $E_n$ defined by Eq. (2). Our purpose is to find a kind of increasing or decreasing operators. Let us denote two types of q-difference operators $H_{q,n}$ and $B_{q,n}$ by

$$H_{q,n}=-q^{-\theta_x}[\theta_y+\beta'+n]_q+q^{-\theta_x-(\beta-n-1)}xy^{-1}[\theta_y]_q,$$
$$B_{q,n}=q^{\theta_y}[\theta_x+\beta-n]_q-q^{\theta_y+(\beta'+n-1)}yx^{-1}[\theta_x]_q.$$

Then the next theorem can be proved by direct calculation.

**Theorem 4.1.**

1. $H_{q,n+1}E_{q,n}=qE_{q,n+1}H_{q,n}$

$$2.\quad B_{q,n-1}E_{q,n}=q^{-1}E_{q,n-1}B_{q,n}$$

Proof. By replacing $\beta$ and $\beta'$ by $\beta+n$ or $\beta'-n$, it is enough to prove when $n=0$. From the difiniton, we see

$$\begin{aligned}
&H_{q,1}E_{q,0}\\
&=-q^{-\theta_x}[\theta_y+\beta'+1]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q+q^{-\theta_x}[\theta_y+\beta'+1]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q\\
&\quad+q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q-q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q.
\end{aligned}$$

By using the relations

$$[\theta_x+\beta]_q=q[\theta_x+\beta-1]_q+q^{-\theta_x-\beta+1},\quad [\theta_x]_q=q^{-1}[\theta_x+1]_q-q^{-\theta_x-1},$$

at the first and the second terms, we have

$$\begin{aligned}
&H_{q,1}E_{q,0}\\
&=-q^{-\theta_x+1}[\theta_y+\beta'+1]_q[\theta_x+\beta-1]_q y^{-1}[\theta_y]_q-q^{-\theta_x}[\theta_y+\beta'+1]_q q^{-\theta_x-\beta+1}y^{-1}[\theta_y]_q\\
&\quad+q^{-\theta_x}[\theta_y+\beta'+1]_q[\theta_x+\beta']_q x^{-1}[\theta_x]_q+q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q\\
&\quad-q^{-\theta_x-(\beta-1)}xy^{-1}[\theta_y]_q[\theta_y+\beta']_q x^{-1}[\theta_x+1]_q\\
&\qquad\qquad\qquad\qquad+q^{-\theta_x-(\beta-2)}xy^{-1}[\theta_y]_q[\theta_y+\beta']_q x^{-1}q^{-\theta_x-1}\\
&=-q[\theta_x+\beta-1]_q y^{-1}[\theta_y]_q q^{-\theta_x}[\theta_y+\beta']_q-q^{-2\theta_x-(\beta-1)}y^{-1}[\theta_y+\beta']_q[\theta_y]_q\\
&\quad+q[\theta_y+\beta'+1]_q x^{-1}[\theta_x]q^{-\theta_x}[\theta_y+\beta']_q+q[\theta_x+\beta-1]_q y^{-1}[\theta_y]_q q^{-\theta_x-(\beta-1)}xy^{-1}[\theta_y]_q\\
&\quad-q[\theta_y+\beta'+1]_q x^{-1}[\theta_x]_q q^{-\theta_x-(\beta-1)}xy^{-1}[\theta_y]_q+q^{-2\theta_x-(\beta-1)}y^{-1}[\theta_y]_q[\theta_y+\beta']_q\\
&=qE_{q,1}H_{q,0}.
\end{aligned}$$

Thus the first statement is proved. We will show the second statement.

$$\begin{aligned}
&B_{q,-1}E_{q,0}\\
&=q^{\theta_y}[\theta_x+\beta+1]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q-q^{\theta_y}[\theta_x+\beta+1]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q\\
&\quad-q^{\theta_y+(\beta'-2)}yx^{-1}[\theta_x]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q+q^{\theta_y+(\beta'-2)}yx^{-1}[\theta_x]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q.
\end{aligned}$$

Substituting

$$[\theta_y+\beta']_q=q^{-1}[\theta_y+\beta'-1]_q+q^{\theta_y+\beta'-1},\quad [\theta_y]_q=q[\theta_y+1]_q-q^{\theta_y+1}$$

into the second and third terms, we have

$$\begin{aligned}
&B_{q,-1}E_{q,0}\\
&=q^{\theta_y}[\theta_x+\beta+1]_q[\theta_x+\beta]_q y^{-1}[\theta_y]_q-q^{\theta_y-1}[\theta_x+\beta+1]_q[\theta_y+\beta'-1]_q x^{-1}[\theta_x]_q
\end{aligned}$$

$$-q^{\theta_y}[\theta_x+\beta+1]_q q^{\theta_y+\beta'-1}x^{-1}[\theta_x]_q-q^{\theta_y+(\beta'-1)}yx^{-1}[\theta_x]_q[\theta_x+\beta]_q y^{-1}[\theta_y+1]_q$$
$$+q^{\theta_y+(\beta'-2)}yx^{-1}[\theta_x]_q[\theta_x+\beta]_q y^{-1}q^{\theta_y+1}+q^{\theta_y+(\beta'-2)}xy^{-1}[\theta_x]_q[\theta_y+\beta']_q x^{-1}[\theta_x]_q$$
$$=q^{-1}[\theta_x+\beta+1]_q y^{-1}[\theta_y]_q q^{\theta_y}[\theta_x+\beta]_q-q^{-1}[\theta_y+\beta'-1]_q x^{-1}[\theta_x]_q q^{\theta_y}[\theta_x+\beta]_q$$
$$-q^{2\theta_y+(\beta'-1)}x^{-1}[\theta_x+\beta]_q[\theta_x]_q-q^{-1}[\theta_x+\beta+1]_q y^{-1}[\theta_y]_q q^{\theta_y+(\beta'-1)}yx^{-1}[\theta_x]_q$$
$$+q^{2\theta_y+(\beta'-1)}x^{-1}[\theta_x]_q[\theta_x+\beta]_q+q^{-1}[\theta_y+\beta'-1]_q x^{-1}[\theta_x]_q q^{\theta_y+(\beta'-1)}yx^{-1}[\theta_x]_q$$
$$=q^{-1}E_{q,-1}B_{q,0}.$$
q.e.d

REMARK 4.1. *The above theorem implies that if $u_n$ is a solution of the equation $E_{q,n}u_n=0$, then $u_{n+1}=H_{q,n}u_n$ or $u_{n-1}=B_{q,n}u_n$ is a solution of $E_{q,n+1}u=0$ or $E_{q,n-1}u=0$, respectively.*

We have more infomation about the action of $H_{q,n}$ and $B_{q,n}$.

**Proposition 4.1.** *The action of operators $H_{q,0}$ and $B_{q,0}$ on the space of contiguous functions of $\varphi_q$ is*

$$H_{q,0}\varphi_q=-\frac{[\mu+\beta']_q[\beta-1]_q}{[\lambda+\beta-1]_q}\varphi^{\beta'}_{q,\beta},\quad B_{q,0}\varphi_q=\frac{[\lambda+\beta]_q[\beta'-1]_q}{[\mu+\beta'-1]_q}\varphi^{\beta}_{q,\beta'}. \tag{7}$$

Proof. By the definition of $\varphi_q$, we get

$$H_{q,0}\varphi_q=-\sum_{n\in Z}q^{-\lambda-n}[\mu-n+\beta']_q\frac{[\lambda+\beta;n]_q[\mu-n+1;n]_q}{[\mu-n+\beta';n]_q[\lambda+1;n]_q}x^\lambda y^\mu t^n$$
$$+\sum_{n\in Z}q^{-\lambda-n-(\beta-1)}[\mu-n+1]_q\frac{[\lambda+\beta;n-1]_q[\mu-n+2;n-1]_q}{[\mu-n+1+\beta';n-1]_q[\lambda+1;n-1]_q}x^\lambda y^\mu t^n$$
$$=\frac{[\mu+\beta']_q}{[\lambda+\beta-1]_q}\sum_{n\in Z}\{-q^{-\lambda-n}[\lambda+n+\beta-1]_q+q^{-\lambda-n-(\beta-1)}[\lambda+n]_q\}$$
$$\times\frac{[\lambda+(\beta-1);n]_q[\mu-n+1;n]_q}{[\mu-n+(\beta'+1);n]_q[\lambda+1;n]_q}x^\lambda y^\mu t^n$$
$$=-\frac{[\mu+\beta']_q[\beta-1]_q}{[\lambda+\beta-1]_q}\varphi^{\beta'}_{q,\beta}.$$

Here we used the addition formula

$$-q^{-\lambda-n}[\lambda+n+\beta-1]_q+q^{-\lambda-n-(\beta-1)}[\lambda+n]_q=-[\beta-1]_q.$$

The second statement is proved just above by using addition formula

$$q^{\mu-n}[\mu-n+(\beta'-1)]_q-q^{\mu-n+(\beta'-1)}[\mu-n]_q=[\beta'-1]_q$$

as follows:

$$B_{q,0}\varphi_q=\frac{[\lambda+\beta]_q}{[\mu+(\beta'-1)]_q}\sum_{n\in Z}\{q^{\mu-n}[\mu-n+(\beta'-1)]_q-q^{\mu-n+(\beta'-1)}[\mu-n]_q\}$$
$$\times\frac{[\lambda+(\beta+1);n]_q[\mu-n+1;n]_q}{[\mu-n+(\beta'-1);n]_q[\lambda+1;n]_q}x^\lambda y^\mu t^n$$
$$=\frac{[\lambda+\beta]_q[\beta'-1]_q}{[\mu+(\beta'-1)]_q}\varphi^{\beta}_{q,\beta'}.$$

q.e.d

**Remark 4.2.** *The action of* $H_{q,0}$ *and* $B_{q,0}$ *on* $\Phi_q$ *is*

$$H_{q,0}\Phi_q=-[\beta-1]_q\Phi^{\beta'}_{q,\beta},\quad B_{q,0}\Phi_q=[\beta'-1]_q\Phi^{\beta}_{q,\beta'}.$$

ACKNOWLEDGEMENT

This work is strongly influenced by Lectures at Osaka University given by Prof. K. Okamoto 1987 and Prof. E. Horikawa 1992 to whom we would like to express our special thanks. And we also heartly express our indebtedness to Prof. N. Ikeda, Prof. M. Ikawa and Prof. E. Date who gave us great encouragement.

## A. The Euler-Poisson-Darboux Equation

Let us consider some analytic properties of the equation

$$E(\beta,\beta')u=\{(x-y)\partial_x\partial_y-\beta'\partial_x+\beta\partial_y\}u=0 \tag{8}$$

We would like to find a solution of the form

$$u=x^\lambda y^\mu\varphi\left(\frac{x}{y}\right)$$

where $\lambda$ and $\mu$ are complex parameters. By substituting this expression into Eq. (8) we have

$$t^2(1-t)\varphi''(t)+t\{(\mu-\lambda-1-\beta)t-(\mu-\lambda-1+\beta')\}\varphi'(t)$$
$$+\{(\lambda+\beta)\mu t-\lambda(\mu+\beta')\}\varphi(t)=0$$

Especially in the case of $\lambda=0$ this equation is reduced to Gauss's hypergeometric euqation

$$t(1-t)\varphi''(t)+t\{(\mu-1-\beta)t-(\mu-1+\beta')\}\varphi'(t)+\beta\mu\varphi(t)=0$$

Hence Eq. (8) have special solutions related to hypergeometric series. For example, we have a solution

$$u(x,y)=y^{\mu}F\left(\mu,-\beta,1-\mu-\beta';\frac{y}{x}\right),$$

where

$$F(a,b,c;t)=\sum_{n=0}^{\infty}\frac{[a;n][b;n]}{[c;n][1;n]}t^n,\quad [a;n]=\Gamma(a+n)/\Gamma(a),$$

is Gauss's hypergeometric series. Hence by using the action of $SL(2,C)$, we obtain Appell's formula

$$u(x,y)=(bx+d)^{-\beta}(by+d)^{-\beta'}(ay+c)^{\mu}(by+d)^{-\mu}F(\mu,-\beta,1-\mu+\beta';\sigma)$$

$$\sigma=\frac{(bx+d)(ay+c)}{(ax+c)(by+d)}.$$

**B. A proof of Theorem 1**

Here we will prove that three operators $e$, $f$ and $q^h$ are generators of the quantum group $U_q(sl(2,C))$. Namely, let us prove Serre's relations

$$q^heq^{-h}=q^2e,\quad q^hfq^{-h}=q^{-2}f,\quad [e,f]=\frac{q^h-q^{-h}}{q-q^{-1}}$$

which characterize $U_q(sl(2,C))$. From the definition, we see

$$q^heq^{-h}=-q^{2\theta_x+2\theta_y+\beta+\beta'}q^{-\theta_x}y[\theta_y+\beta']_q q^{-2\theta_x-2\theta_y-\beta-\beta'}$$
$$-q^{2\theta_x+2\theta_y+\beta+\beta'}q^{\theta_y}x[\theta_x+\beta]_q q^{-2\theta_x-2\theta_y-\beta-\beta'}$$

By using the relations $xq^{-2\theta_x}=q^{-2\theta_x+2}x$ and $yq^{-2\theta_y}=q^{-2\theta_y+2}y$, we obtain

$$q^heq^{-h}=-\{q^{-\theta_x+2}y[\theta_y+\beta']_q+q^{\theta_y+2}x[\theta_x+\beta]_q\}=q^2e$$

and just as the same above we can show $q^hfq^{-h}=q^{-2}f$.

Now we prove the relation

$$[e,f]=\frac{q^h-q^{-h}}{q-q^{-1}}.$$

From the definition of $e$ and $f$, we have

$$
\begin{aligned}
&[e,f]\\
&=-[q^{\theta_y}x[\theta_x+\beta]_q, q^{\theta_y+\beta'}x^{-1}[\theta_x]_q]-[q^{\theta_y}x[\theta_x+\beta]_q, q^{-\theta_x-\beta}y^{-1}[\theta_y]_q]\\
&\quad-[q^{-\theta_x}y[\theta_y+\beta']_q, q^{\theta_y+\beta'}x^{-1}[\theta_x]_q]-[q^{-\theta_x}y[\theta_y+\beta']_q, q^{-\theta x-\beta}y^{-1}[\theta_y]_q]\\
&=-C_1-C_2-C_3-C_4.
\end{aligned}
$$

Now we calculate each term $C_i$ $i=1,2,3,4$. We have

$$
\begin{aligned}
C_1&=q^{2\theta_y+\beta'}[x[\theta_x+\beta]_q, x^{-1}[\theta_x]_q]\\
&=q^{2\theta_y+\beta'}\{x[\theta_x+\beta]_q x^{-1}[\theta_x]_q-x^{-1}[\theta_x]_q x[\theta_x+\beta]_q\}\\
&=q^{2\theta_y+\beta'}\{[\theta_x+\beta-1]_q[\theta_x]_q-[\theta_x+1]_q[\theta_x+\beta]_q\}\\
&=-q^{2\theta_y+\beta'}[2\theta_x+\beta]_q,
\end{aligned}
$$

where we use Eq. (5). The second term is

$$
\begin{aligned}
C_2&=q^{\theta_y}x[\theta_x+\beta]_q q^{-\theta_x-\beta}y^{-1}[\theta_y]_q\\
&\quad-q^{-\theta_x-\beta}y^{-1}[\theta_y]_q q^{\theta_y}x[\theta_x+\beta]_q\\
&=q^{\theta_y-\theta_x-\beta+1}x[\theta_x+\beta]_q y^{-1}[\theta_y]_q\\
&\quad-q^{-\theta_x-\beta+\theta_y+1}y^{-1}[\theta_y]_q x[\theta_x+\beta]_q\\
&=0.
\end{aligned}
$$

Similary just above, we obtain $C_3=0$. Finally

$$
\begin{aligned}
C_4&=q^{-2\theta_x-\beta}[y[\theta_y+\beta']_q, y^{-1}[\theta_y]_q]\\
&=q^{-2\theta_x-\beta}\{y[\theta_y+\beta']_q y^{-1}[\theta_y]_q-y^{-1}[\theta_y]_q y[\theta_y+\beta']_q\}\\
&=q^{-2\theta_x-\beta}\{[\theta_y+\beta'-1]_q[\theta_y]_q-[\theta_y+1]_q[\theta_y+\beta']_q\}\\
&=-q^{-2\theta_x-\beta}[2\theta_y+\beta']_q,
\end{aligned}
$$

where we use the addition formula Eq. (5). Hence we have

$$
\begin{aligned}
[e,f]&=q^{2\theta_y+\beta'}[2\theta_x+\beta]_q+q^{-2\theta_x-\beta}[2\theta_y+\beta']_q\\
&=[2\theta_x+\beta+2\theta_y+\beta']_q\\
&=\frac{q^h-q^{-h}}{q-q^{-1}}.
\end{aligned}
$$

### C. Casimir operator

Here we express Casimir operator by means of the operator $E_{q,0}$. It is well known that the Casimir element $C$ of $U_q(sl(2,C))$ is

$$C = \frac{q^{-1} \cdot q^h - 2 + q \cdot q^{-h}}{(q - q^{-1})^2} + ef.$$

In our case, by the direct calculation, we have

$$C = -q^{\theta_y - \theta_x}(q^{-\beta+1}x - q^{\beta'-1}y)E_{q,0} + \left[\frac{\beta + \beta' - 1}{2}\right]_q^2.$$

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