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KdV POLYNOMIALS AND Λ-OPERATOR

MAYUMI OHMIYA

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1. Introduction

The purpose of the present paper is to clarify certain algebraic properties of the spectrum of the second order ordinary differential operator

$$H(u) = -\partial^2 + u(x),$$

where $u(x)$ is a meromorphic function defined in a region of the complex plane and $\partial = ' = d/dx$. The integro-differential operator

$$\Lambda(u) = \partial^{-1} \cdot \left(\frac{1}{2} u'(x) + u(x)\partial - \frac{1}{4}\partial^3 \right)$$

plays crucial role in our approach, where $A \cdot B$ denotes the product of the operators A and B. The operator $\Lambda(u)$ is usually called the *Λ-operator* or the *recursion operator*. The Λ-operator generates the infinite sequence of differential polynomials as follows; put $Z_0(u) = 1$ and define functions $Z_n(u)$, $n \in \boldsymbol{N}$ by the recurrence relation $Z_n(u) = \Lambda(u) Z_{n-1}(u)$, $n \in \boldsymbol{N}$. Then it turns out that $Z_n(u)$ are the differential polynomials in $u, u', \cdots, u^{(2n-2)}$ with constant coefficients. We call the differential polynomials $Z_n(u)$, $n \in \boldsymbol{Z}_+$ the KdV *polynomials.*

Now, let $V(u)$ be the vector space over the complex number field $\boldsymbol{C}$ spanned by $Z_n(u)$, $n \in \boldsymbol{Z}_+$, then $\Lambda(u) \in \mathrm{End}(V(u))$, i.e. $\Lambda(u)$ can be regarded as the operator in $V(u)$. If $V(u)$ is finite dimensional then the principal part of the problem concerned with $H(u)$ can be reduced to consideration of certain algebraic properties of $\Lambda(u) \in \mathrm{End}(V(u))$. We want to call this method the *Λ-algorithm.* The main purpose of the present paper is to investigate the spectrum of $H(u)$ by the Λ-algorithm.

On the other hand, the present work is deeply related to the algebraic theory of the Darboux transformation. Those problems were discussed in [18]. See also [17].

The contents of this paper are as follows. In § 2, the precise definitions of the Λ-operator and the KdV polynomials are given. In § 3, the expansion theorem for the KdV polynomials is obtained. In § 4, the notion of Λ-rank is introduced. In § 5, the spectrum $\Gamma(u)$ of the opertor $H(u)$ is defined and certain class of eigenfunctions of $H(u)$ are exactly constructed by using the Λ-operator. In § 6, the problem related to the classical theorem of Ince is discussed. In § 7, the trace formulae of McKean-Trubowitz type are proved by Λ-algorithm.

A part of the present paper is announced in [16].

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2. KdV polynomials

Let $\mathscr{A}$ be a differential algebra over the complex number field $\boldsymbol{C}$ of polynomials in infinite formal symbols u_ν, $\nu \in \boldsymbol{Z}_+ = \boldsymbol{N} \cup \{0\}$ with the derivation $\delta = \Sigma_{\nu=0}^{\infty} u_{\nu+1}\partial/\partial u_\nu$. We denote its subalgebra of polynomials without constants by $\mathscr{A}_0$. Put $\hat{\mathscr{A}}_0 = \delta\mathscr{A}_0$ then one can define the inverse δ^{-1} of the derivation $\delta: \mathscr{A}_0 \to \hat{\mathscr{A}}_0$. On the other hand, put

$$\hat{K} = \frac{1}{2}u_1 + u_0\delta - \frac{1}{4}\delta^3,$$

then it is known that $\hat{K}\cdot(\delta^{-1}\cdot\hat{K})^{n-1}1$ belong to $\hat{\mathscr{A}}_0$ for all $n \in \boldsymbol{N}$ (cf. [20] or [15, p. 621 Lemma 3.1]). Hence the set $\{\hat{\Lambda}^n 1 | n \in \boldsymbol{N}\}$ is well defined as the orbit in $\mathscr{A}_0$, where $\hat{\Lambda} = \delta^{-1}\cdot\hat{K}$. Since $\hat{\Lambda}^n 1$ are the polynomials in $u_0, u_1, \cdots, u_{2n-2}$, $n \in \boldsymbol{N}$ (cf. [20] or [15]), we denote them by $P_n(u_0, u_1, \cdots, u_{2n-2})$;

$$\hat{\Lambda}^n 1 = P_n(u_0, u_1, \cdots, u_{2n-2}).$$

On the other hand, let $u(x)$ be a meromorphic function of the one complex variable x. Let $\mathscr{A}(u)$ be the differential algebra of differential polynomials in $u(x)$. Now let us identify the derivatives $u^{(\nu)} = \partial^\nu u(x) \in \mathscr{A}(u)$, $\nu \in \boldsymbol{Z}_+$ and the differential operator ∂ with the variables $u_\nu, \nu \in \boldsymbol{Z}_+$ and the derivation δ respectively. By this identification, we can define the subalgebras $\mathscr{A}_0(u)$ and $\hat{\mathscr{A}}_0(u)$ correponding to $\mathscr{A}_0$ and $\hat{\mathscr{A}}_0$ respectively. Then one can define the operator $\partial^{-1}: \hat{\mathscr{A}}_0(u) \to \mathscr{A}_0(u)$ by identifying with the operator $\delta^{-1}: \hat{\mathscr{A}}_0 \to \mathscr{A}_0$. The operators $\hat{K}$ and $\hat{\Lambda}$ are identified with the third order differential operator

$$K(u) = \frac{1}{2}u'(x) + u(x)\partial - \frac{1}{4}\partial^3$$

and $\Lambda(u) = \partial^{-1}\cdot K(u)$ respectively. Moreover put

$$Z_n(u(x)) = P_n(u(x), u'(x), \cdots, u^{(2n-2)}(x)), \quad n \in \boldsymbol{N}$$

and $Z_0(u(x)) \equiv 1$, which are called the KdV polynomials. We also use the differential polynomials $X_n(u(x)) = \partial Z_n(u(x))$. The KdV polynomials $Z_n(u)$, $n \in \boldsymbol{Z}_+$ are represented by the recurrence relation

$$Z_n(u) = \Lambda(u) Z_{n-1}(u), \quad n \in \boldsymbol{N}$$

with $Z_0(u)=1$. At the same time, they are represented by the commutator representation of Lax type

$$(1) \qquad Z_n(u)=\frac{1}{2}\partial^{-1}[A_n(u),H(u)],$$

where

$$(2) \qquad A_n(u)=\sum_{j=0}^{n}(Z_j(u)\partial-\frac{1}{2}X_j(u))\cdot H(u)^{n-j}$$

and $[A,B]=A\cdot B-B\cdot A$ (cf. [10, p.220, Lemma 12.3.1] or [20, p.4]).

3. Expansion theorem

In this section, we consider the expansion theorem for the KdV polynomial. First we have the following.

Lemma 1. *For any $\lambda\in \mathbf{C}$ and $m\in \mathbf{N}$, $Z_m(u(x)+\lambda)$ belongs to $V(u)$, i.e., there exist $\alpha_{mv}(\lambda), v=0,1,\cdots,m$ such that*

$$Z_m(u(x)+\lambda)=\sum_{v=0}^{m}\alpha_{mv}(\lambda)Z_v(u(x)).$$

The coefficients $\alpha_{mv}(\lambda)$ satisfy the recurrence formulae

$$(3) \qquad \alpha_{m+1\,v}(\lambda)=\begin{cases}\alpha_{mm}(\lambda) & \text{for } v=m+1\\ \alpha_{mv-1}(\lambda)+\lambda\alpha_{mv}(\lambda) & \text{for } v=1,2,\cdots,m\end{cases}$$

with $\alpha_{00}(\lambda)=1$.

Proof. First assume that

$$Z_l(u(x)+\lambda)=\sum_{v=0}^{l}\alpha_{lv}(\lambda)Z_v(u(x)).$$

are valid for any $l\leq m$. Actually this is true for $m=1$. Differentiating both sides of the above, we have

$$X_l(u(x)+\lambda)=\sum_{v=1}^{l}\alpha_{lv}(\lambda)X_v(u(x)).$$

Hence, by (2), one has

$$
\begin{aligned}
&A_m(u(x)+\lambda)\\
&\quad=\sum_{l=0}^{m}(Z_l(u(x)+\lambda)\partial-\frac{1}{2}X_l(u(x)+\lambda))\cdot H(u(x)+\lambda)^{m-l}\\
&\quad=\sum_{l=0}^{m}\sum_{\nu=0}^{l}\alpha_{l\nu}(\lambda)(Z_\nu(u(x))\partial-\frac{1}{2}X_\nu(u(x)))\cdot H(u(x)+\lambda)^{m-l}.
\end{aligned}
$$

Let $f(x)$ be a nontrivial solution of the equation

$$
H(u+\lambda)f(x)=-f''(x)+(u(x)+\lambda)f(x)=0, \tag{4}
$$

then, by (1), one verifies

$$
\begin{aligned}
X_{m+1}(u(x)+\lambda)f(x)&=\frac{1}{2}[A_m(u+\lambda),H(u+\lambda)]f(x) \\
&=-\frac{1}{2}H(u+\lambda)A_m(u+\lambda)f(x)
\end{aligned} \tag{5}
$$

and

$$
A_m(u+\lambda)f(x)=\sum_{\nu=0}^{m}\alpha_{m\nu}(\lambda)(Z_\nu(u(x))\partial-\frac{1}{2}X_\nu(u(x)))f(x). \tag{6}
$$

Combining (5) and (6), one has

$$
\begin{aligned}
&X_{m+1}(u(x)+\lambda)f(x)\\
&\quad=-\frac{1}{2}H(u+\lambda)\sum_{\nu=0}^{m}\alpha_{m\nu}(\lambda)(Z_\nu(u(x))f'(x)-\frac{1}{2}X_\nu(u(x))f(x)).
\end{aligned}
$$

Calculate the right hand side of the above and eliminate f'' and f''' by (4) and

$$
f'''(x)=u'(x)f(x)+(u(x)+\lambda)f'(x),
$$

then we have immediately

$$
\begin{aligned}
&X_{m+1}(u(x)+\lambda)f(x)\\
&\quad=\sum_{\nu=0}^{m}\alpha_{m\nu}(\lambda)(K(u)Z_\nu(u(x))+\lambda\partial Z_\nu(u(x)))f(x)\\
&\quad=\alpha_{mm}(\lambda)X_{m+1}(u(x))f(x)+\sum_{\nu=1}^{m}(\alpha_{m\nu-1}(\lambda)+\lambda\alpha_{m\nu}(\lambda)X_\nu(u(x)))f(x).
\end{aligned}
$$

This implies that there exist $\alpha_{m+1\nu}(\lambda),\nu=1,2,\cdots,m+1$ such that

$$X_{m+1}(u(x)+\lambda)=\sum_{\nu=1}^{m+1}\alpha_{m+1\nu}(\lambda)X_\nu(u(x))$$

and

$$\alpha_{m+1\nu}(\lambda)=\begin{cases}\alpha_{mm}(\lambda) & \text{for} \quad \nu=m+1\\ \alpha_{m\nu-1}(\lambda)+\lambda\alpha_{m\nu}(\lambda) & \text{for} \quad \nu=1,2,\cdots,m.\end{cases}$$

This completes the proof.

Note that we can not determine $\alpha_{m0}(\lambda)$ by the recurrence formulae (3). To determine them, we prove the following.

Lemma 2. *The differential polynomial $Z_m(u(x))$ contains $\beta_m u(x)^m$ as its term, while the remainder terms of $Z_m(u(x))$ contain derivatives of $u(x)$ as their variables, where*

$$\beta_m=\frac{(2m)!}{2^{2m}(m!)^2}.$$

Proof. We prove this by induction on m. First, note that the assertion holds for $m=1$ because $Z_1(u(x))=\frac{1}{2}u(x)$. Assume now that the assertion is correct for $m-1$. Put

$$Y_m(u)=Z_{m-1}(u)-\beta_{m-1}u^{m-1},$$

then each term of $Y_m(u)$ contains derivatives of $u(x)$ as its variables. By direct calculation, we have

$$\begin{aligned}Z_m(u)&=\beta_{m-1}\Lambda(u)u^{m-1}+\Lambda(u)Y_m(u)\\&=\frac{2m-1}{2m}\beta_{m-1}u^m-\frac{1}{4}\beta_{m-1}(m-1)(m-2)u^{m-3}u'^2\\&\quad-\frac{1}{4}\beta_{m-1}(m-1)u^{m-2}u''+\Lambda(u)Y_m(u).\end{aligned}$$

Note that

$$\beta_m=\frac{2m-1}{2m}\beta_{m-1}$$

holds. Therefore it suffices to show that each term of $\Lambda(u)Y_m(u)$ contains the derivatives of $u(x)$ as their variable. Conversely assume that at least one of terms of $\Lambda(u)Y_m(u)$ contains no derivatives of $u(x)$ as its variables. Let l be the lowest degree of such terms. This implies that $K(u)Y_m(u)$ contains the term of the form $l\beta u^{l-1}u'$. On the other hand, we have

$$K(u)Y_m(u)=\frac{1}{2}Y_m(u)u'+Y_m(u)'u-\frac{1}{4}Y_m(u)'''.$$

Hence one can see that $Y(u)$ contains the term $2l\beta u^{l-1}$. This is contradiction. Therefore this completes the proof.

Since $Z_\nu(0)\equiv 0$ for $\nu\geq 1$ and $Z_0(0)\equiv 1$, one verifies readily

$$Z_m(\lambda)=\sum_{\nu=0}^{m}\alpha_\nu^{(m)}(\lambda)Z_\nu(0)=\alpha_{m0}(\lambda).$$

On the other hand, by lemma 2, we have

$$Z_m(\lambda)=\beta_m\lambda^m=\frac{(2m)!}{2^{2m}(m!)^2}\lambda^m.$$

This implies

$$\alpha_{m0}(\lambda)=\frac{(2m)!}{2^{2m}(m!)^2}\lambda^m.$$

Calculating the recurrence formulae (3) with the above expression for $\alpha_{m0}(\lambda)$, we have the following.

Theorem 3. *Define* $\alpha_\nu^{(n)}, \nu=0,1,2,\cdots,n$ *by the recurrence formulae*

$$\alpha_\nu^{(n)}=\begin{cases}1 & \text{for } \nu=n\\ \alpha_{\nu-1}^{(n-1)}+\alpha_\nu^{(n-1)} & \text{for } \nu=1,2,\cdots,n-1\\ \dfrac{(2n)!}{2^{2n}(n!)^2} & \text{for } \nu=0\end{cases}$$

with $\alpha_\nu^{(0)}=1$. *Then*

$$Z_n(u(x)+\lambda)=\sum_{\nu=0}^{n}\alpha_\nu^{(n)}Z_\nu(u(x))\lambda^{n-\nu} \tag{7}$$

holds for any $\lambda\in \boldsymbol{C}$.

Next we consider certain arithmetic properties of the coefficients $\alpha_\nu^{(n)}$.

Proposition 4. *The binomial coefficients* $\alpha_\nu^{(n)}, \nu=0,1,\cdots,n$ *satisfy the following relations*;

$$\sum_{\nu=1}^{n}(-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n)}=1, \tag{8}$$

$$\sum_{\nu=0}^{n}(-1)^{\nu}\alpha_0^{(\nu)}\alpha_\nu^{(n)}=0. \tag{9}$$

Proof. Suppose $n\geq 1$. Since $Z_n(0)=0$, by Theorem 3, we have

$$Z_n(1-1)=\sum_{\nu=0}^{n}(-1)^{n-\nu}\alpha_\nu^{(n)}Z_\nu(1)=0$$

On the other hand, by (7), one verifies

$$\sum_{\nu=0}^{n}(-1)^{n-\nu}\alpha_\nu^{(n)}Z_\nu(1)=(-1)^n\sum_{\nu=0}^{n}(-1)^{\nu}\alpha_0^{(\nu)}\alpha_\nu^{(n)}.$$

Hence (9) follows. Next we prove (8) by induction on n. Since $\alpha_0^{(0)}\alpha_1^{(1)}=1$, (8) holds for $n=1$. Assume that

$$\sum_{\nu=1}^{n-1}(-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n-1)}=1$$

holds. Then we have

$$\begin{aligned}
&\sum_{\nu=1}^{n}(-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n)}\\
&\quad=\sum_{\nu=1}^{n-1}(-1)^{\nu-1}\alpha_0^{(\nu-1)}(\alpha_{\nu-1}^{(n-1)}+\alpha_\nu^{(n-1)})+(-1)^{n-1}\alpha_0^{(n-1)}\alpha_n^{(n)}\\
&\quad=\sum_{\nu=1}^{n}(-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_{\nu-1}^{(n-1)}+\sum_{\nu=1}^{n-1}(-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n-1)},
\end{aligned}$$

since $\alpha_n^{(n)}=\alpha_{n-1}^{(n-1)}=1$. The first term of the above vanishes by (9) and the second term coincides with 1 by the assumption. This completes the proof.

3. Λ-rank

In this section we introduce the notion of Λ-rank. Let $V(u)$ be the vector space over $\boldsymbol{C}$ spanned by the infinite sequence of the KdV polynomials $Z_m(u), m\in \boldsymbol{Z}_+$, i.e.,

$$V(u)=\bigcup_{m\in \boldsymbol{Z}_+} \boldsymbol{C}Z_m(u).$$

If $V(u)$ is finite diminsional, then we say that the Λ-rank of the meromorphic function $u(x)$ is finite and define $\operatorname{rank}_\Lambda u(x)$ by

$$\operatorname{rank}_\Lambda u(x)=\dim_{\boldsymbol{C}} V(u)-1.$$

First we have the following.

Lemma 5. *If* $n=\operatorname{rank}_\Lambda u(x)<\infty$ *then* $V(u)$ *is spanned by* $Z_\nu(u)$, $\nu=0,1,\cdots,n$, *i.e.,* $V(u)=\oplus_{\nu=0}^n \boldsymbol{C}Z_\nu(u)$.

Proof. Since $Z_0(u)\neq 0$ and $V(u)$ is finite dimensional, there exists $m\in\boldsymbol{N}$ such that $Z_0(u),Z_1(u),\cdots,Z_m(u)$ are linearly independent and $Z_0(u),Z_1(u),\cdots,Z_m(u),Z_{m+1}(u)$ are linearly dependent. Hence there exist $c_\nu,\nu=0,1,\cdots,m$ such that

$$Z_{m+1}(u)=\sum_{\nu=0}^m c_\nu Z_\nu(u).$$

Then, operating with $\Lambda(u)$ on both sides of the above, we have

$$\begin{aligned} Z_{m+2}(u)&=\Lambda(u)Z_{m+1}(u)\\ &=\sum_{\nu=0}^m c_\nu\Lambda(u)Z_\nu(u)\\ &=c_mZ_{m+1}(u)+\sum_{\nu=0}^{m-1} c_\nu Z_{\nu+1}(u)\\ &=c_m\sum_{\nu=0}^m c_\nu Z_\nu(u)+\sum_{\nu=0}^{m-1} c_\nu Z_{\nu+1}(u)\\ &=\sum_{\nu=1}^m (c_{\nu-1}+c_mc_\nu)Z_\nu(u)+c_mc_0Z_0(u).\end{aligned}$$

Similarly to the above, one verifies that $Z_{m+\nu}(u)$ can be expressed as the linear combination of $Z_0(u),Z_1(u),\cdots,Z_m(u)$ for any $\nu\in\boldsymbol{N}$. This implies $\dim V(u)=m+1$. Hence $m=n$ follows. This completes the proof.

Suppose $n=\operatorname{rank}_\Lambda u(x)<\infty$ then, by lemma 1, there uniquely exist $a_\nu(u),\nu=0,1,\cdots,n$ such that

$$Z_{n+1}(u(x))=\sum_{\nu=0}^n \alpha_\nu(u)Z_\nu(u(x)). \tag{10}$$

We call $a_\nu(u), \nu=0,1,\cdots,n$ the *Λ-characteristic coefficients* of $u(x)$. Moreover we call the monic polynomial of degree $n+1$

$$\Omega(\lambda;u)=\lambda^{n+1}-\sum_{\nu=0}^{n}a_\nu(u)\lambda^\nu$$

the *Λ-characteristic polynomial.* By Theorem 3, we have readily the following.

Proposition 6. *For any* $\lambda\in \boldsymbol{C}$

$$\operatorname{rank}_\Lambda(u(x)-\lambda)=\operatorname{rank}_\Lambda u(x)$$

holds.

Hence, if $n=\operatorname{rank}_\Lambda u(x)<\infty$ then there exist $a_\nu(\lambda;u), \nu=0,1,\cdots,n$ such that

$$Z_{n+1}(u(x)-\lambda)=\sum_{\nu=0}^{n}a_\nu(\lambda;u)Z_\nu(u(x)-\lambda).$$

Of course, $a_\nu(0;u)=a_\nu(u)$ holds for any $\nu=0,1,\cdots,n$. More precisely, we have the following.

Lemma 7. *If* $n=\operatorname{rank}_\Lambda u(x)<\infty$ *then* $a_\nu(\lambda;u), \nu=0,1,\cdots,n$ *are the polynomials in* λ *of degree* $n-\nu+1$;

$$a_\nu(\lambda;u)=-\alpha_\nu^{(n+1)}\lambda^{n-\nu+1}+\sum_{j=\nu}^{n}\alpha_\nu^{(j)}a_j(u)\lambda^{j-\nu}.$$

Proof. By Therem 3, we have

$$\begin{aligned}Z_{n+1}(u(x))&=Z_{n+1}((u(x)-\lambda)+\lambda)\\&=Z_{n+1}(u(x)-\lambda)+\sum_{\nu=0}^{n}\alpha_\nu^{(n+1)}Z_\nu(u(x)-\lambda)\lambda^{n-\nu+1}\\&=\sum_{\nu=0}^{n}(a_\nu(\lambda;u)+\alpha_\nu^{(n+1)}\lambda^{n-\nu+1})Z_\nu(u(x)-\lambda).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}Z_{n+1}(u(x))&=\sum_{j=0}^{n}a_j(u)Z_j((u(x)-\lambda)+\lambda)\\&=\sum_{j=0}^{n}a_j(u)\sum_{\nu=0}^{j}\alpha_\nu^{(j)}Z_\nu(u(x)-\lambda)\lambda^{j-\nu}\end{aligned}$$

$$= \sum_{\nu=0}^{n}\left(\sum_{j=\nu}^{n}\alpha_{\nu}^{(j)}a_j(u)\lambda^{j-\nu}\right)Z_\nu(u(x)-\lambda).$$

This implies that

$$a_\nu(\lambda;u)+\alpha_\nu^{(n+1)}\lambda^{n-\nu+1}=\sum_{j=\nu}^{n}\alpha_\nu^{(j)}a_j(u)\lambda^{j-\nu}$$

are valid for $\nu=0,1,\cdots,n$. This completes the proof.

4. Construction of eigenfunctions

In this section we construct special class of exact solutions of the eigenvalue problem

(11) $$(H(u)-\lambda)f(x)=0, \qquad \lambda\in \boldsymbol{C}$$

by the Λ-algorithm when Λ-rank of $u(x)$ is finite.

Suppose $n=\mathrm{rank}_\Lambda u(x)<\infty$ and put

(12) $$F(x;\lambda)=Z_n(u(x)-\lambda)-\sum_{\nu=1}^{n}a_\nu(\lambda;u)Z_{\nu-1}(u(x)-\lambda).$$

Then, since $\mathrm{rank}_\Lambda(u(x)-\lambda)=n$, $F(x;\lambda)$ is not identically zero for any $\lambda\in\boldsymbol{C}$. One verifies

$$\begin{aligned}&\Lambda(u(x)-\lambda)F(x;\lambda)\\&=\partial^{-1}\cdot\left(\frac{1}{2}u'(x)+(u(x)-\lambda)\partial-\frac{1}{4}\partial^3\right)F(x;\lambda)\\&=a_0(\lambda;u).\end{aligned}$$

Hence

$$\begin{aligned}&K(u(x)-\lambda)F(x;\lambda)\\&=\frac{1}{2}u'(x)F(x;\lambda)+(u(x)-\lambda)F_x(x;\lambda)-\frac{1}{4}F_{xxx}(x;\lambda)\\&=0\end{aligned}$$

follows. Suppose that $u(x)$ is holomorphic at $x=a$. Let $f_j(x;\lambda), j=1,2$ be the fundamental system of solutions of (11) such that

$$\begin{pmatrix} f_1(a;\lambda) & f_2(a;\lambda)\\ f_1'(a;\lambda) & f_2'(a;\lambda)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Then, by [19, p.23, Theorem 7], there exist $\alpha_j(\lambda), j=1,2,3$ such that

$$F(x;\lambda)=\alpha_1(\lambda)f_1(x;\lambda)^2+\alpha_2(\lambda)f_1(x;\lambda)f_2(x;\lambda)+\alpha_3(\lambda)f_2(x;\lambda)^2,$$

that is, $F(x;\lambda)$ can be represented as the quadratic form with the variables $f_j(x;\lambda)$, $j=1,2$. We have the following.

Lemma 8. *The coefficients $\alpha_j(\lambda), j=1,2,3$ are the polynomials in λ expressed as*

$$\alpha_j(\lambda)=\begin{cases} F(a;\lambda) & \text{for } j=1 \\ F_x(a;\lambda) & \text{for } j=2 \\ \frac{1}{2}F_{xx}(a;\lambda)-(u(a)-\lambda)F(a;\lambda) & \text{for } j=3. \end{cases}$$

Proof. Note that $f_1(a;\lambda)=f_2'(a;\lambda)=1$ and $f_1'(a;\lambda)=f_2(a;\lambda)=0$. Then, by direct calculation, one verifies

$$F(a;\lambda)=\alpha_1(\lambda)$$

$$F_x(a;\lambda)=\alpha_2(\lambda)$$

and

$$F_{xx}(a;\lambda)=2\alpha_1(\lambda)(u(a)-\lambda)+2\alpha_3(\lambda).$$

By lemma 7, $F(a;\lambda)$, $F_x(a;\lambda)$ and $F_{xx}(a;\lambda)$ are polynomials in λ. This completes the proof.

Let $\Delta(\lambda,u)=\alpha_2(\lambda)^2-4\alpha_1(\lambda)\alpha_3(\lambda)$ be the discriminant of the quadratic form $F(x;\lambda)$. Then, by lemma 8, we have immediately

$$\Delta(\lambda,u)=F_x(a;\lambda)^2-2F(a;\lambda)F_{xx}(a;\lambda)+4(u(a)-\lambda)F(a;\lambda)^2. \tag{13}$$

Hence $\Delta(\lambda,u)$ is the polynomial in λ. To investigate it more precisely, we have the following.

Lemma 9. *$F(x;\lambda)$ is the monic polynomial of degree n in λ for any x.*

Proof. By Theorem 3 and lemma 7, we have

$$F(x;\lambda)=\sum_{j=0}^{n}(-1)^{n-j}\alpha_j^{(n)}Z_j(u(x))\lambda^{n-j}$$
$$+\sum_{\nu=1}^{n}\alpha_\nu^{(n+1)}\lambda^{n-\nu+1}\sum_{k=0}^{\nu-1}(-1)^{\nu-k-1}\alpha_k^{(\nu-1)}Z_k(u(x))\lambda^{\nu-k-1}$$

$$+\text{lower terms}$$
$$=\sum_{\nu=1}^{n+1}(-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n+1)}\lambda^n+\text{lower terms}.$$

The assertion immediately follows from the formula (8).

Hence we have the following.

Corollary 10. *The discriminant $\Delta(\lambda;u)$ is the polynomial of degree $2n+1$ in λ;*

$$\Delta(\lambda;u)=-4\lambda^{2n+1}+\text{lower terms}.$$

Therefore, if we put

$$\Gamma(u)=\{\lambda\in\mathbf{C}|\Delta(\lambda;u)=0\},$$

then $\#\Gamma(u)\leq 2n+1$ follows, where # denotes cardinality of the set. Moreover, since

$$\frac{\partial}{\partial a}\Delta(\lambda;u)=8F(a;\lambda)\Big(\frac{1}{2}u'(a)F(a;\lambda)$$
$$+(u(a)-\lambda)F_x(a;\lambda)-\frac{1}{4}F_{xxx}(a;\lambda)\Big)=0,$$

$\Delta(\lambda;u)$ and $\Gamma(u)$ are independent of choice of the holomorphic point $x=a$ of $u(x)$. In the case of Hill's operator, $\Gamma(u)$ corresponds to its periodic spectrum (cf. [12]). Hence we call $\Gamma(u)$ the *Λ-spectrum.*

Now suppose $\lambda_j\in\Gamma(u)$ then there exist the constants $\beta_{ij}, i=1,2, j=0,1,\cdots,2n$ such that

$$F(x;\lambda_j)=(\beta_{1j}f_1(x;\lambda_j)+\beta_{2j}f_2(x;\lambda_j))^2.$$

Thus we proved the following.

Theorem 11. *Suppose $n=\text{rank}_\Lambda u(x)<\infty$. Then the Λ-spectrum $\Gamma(u)$ is uniquely defined for $u(x)$ and $\#\Gamma(u)\leq 2n+1$ holds. Moreover, if $\lambda_j\in\Gamma(u), j=0,1,\cdots,2n$ then*

$$g_j(x)=\sqrt{F(x;\lambda_j)},\qquad j=0,1,\cdots,2n$$

are the corresponding eigenfunctions of the eigenvalue problem (11).

Such an algorithm to construct eigenfunctions as above has been already developed by several authors from somewhat different point of view. See e.g.

[12, §6, pp. 235–236].

On the other hand, it is known that $Z_i(u)\partial Z_j(u)\in\hat{\mathscr{A}}_0(u)$ hold for any $i,j\in \boldsymbol{Z}_+$ (cf. [6, p. 168, Proposition 12.1.12]). Hence there exist $I_{ij}(u)=\partial^{-1}(Z_i(u)\partial Z_j(u))\in \mathscr{A}_0(u), i,j\in \boldsymbol{N}$. Put

$$J_k(u(x))=I_{n+1k}(u(x))-\sum_{\nu=0}^{n}a_\nu(u)I_{\nu k}(u(x)), \qquad k=1,2,\cdots,n$$

then they are the nontrivial first integrals of the $2n$ - 2 th order ordinary differential equation (10), i.e., $\partial J_k(u)\equiv 0$. Hence there exist the constants c_k such that $J_k(u)\equiv c_k, k=1,2,\cdots,n$. Using these relations, one can reduce the expression of $F(x;\lambda_j),\lambda_j\in\Gamma(u)$ as the differential polynomials. Here we refer [6] for the Hamiltonian method in the study of the differential equation (10). See also [2] and [21].

5. Ince's theorem

Let $\mathscr{P}(x)$ be the Weierstrass elliptic function with the real primitive period $\omega_1=\pi$ and the imaginary primitive period ω_3. Put $p(x)=\mathscr{P}(x+\frac{1}{2}\omega_3)$, $x\in\boldsymbol{R}$. Ince [9] proved that if $n\in\boldsymbol{Z}_+$ then the differential operator $H(n(n+1)p(x))$ in the class of functions of period 2π has $2n+1$ simple eigenvalues $\lambda_0<\lambda_1<\cdots<\lambda_{2n}$. See also [1] and [11]. Hence, by the results of soliton theory (cf. [7, p. 84] or [12, p. 234]), $\mathrm{rank}_\Lambda n(n+1)\mathscr{P}(x)=n$ follows. The purpose of this section is to prove the above fact within the framework of Λ-algorithm.

Suppose $\mathrm{rank}_\Lambda u(x)=1$ and $k\in\boldsymbol{C}\backslash\{0\}$. Put $u_k=u_k(x)=ku(x)$. Since

$$Z_1(u)=\frac{1}{2}u, \qquad Z_2(u)=\frac{1}{8}(3u^2-u''),$$

one verifies

$$u_k''=\frac{3}{k}u_k^2-4a_1(u)u_k-8a_0(u)k, \tag{14}$$

where $a_0(u)$ and $a_1(u)$ are the Λ-characteristic coefficients of $u(x)$. This also implies

$$(u_k')^2=\frac{2}{k}u_k^3-4a_1(u)u_k^2-16a_0(u)ku_k+c, \tag{15}$$

where c is a constant. By (14) and (15), one can eliminate the derivatives $u_k^{(s)}(x), s\geq 2$ and $(u_k'(x))^{2l}, l\geq 1$ from the differential polynomial $Z_m(u_k)$. Thus we have

$$Z_m(u_k)=P_m(u_k)+Q_m(u_k)u_k', \tag{16}$$

where $P_m(u_k)$ and $Q_m(u_k)$ are the polynomials in u_k. On the other hand, one verifies

$$\partial Z_m(u_k)=\frac{\partial}{\partial u_k}P_m(u_k)u_k'+\frac{\partial}{\partial u_k}Q_m(u_k)(u_k')^2+Q_m(u_k)u_k''.$$

Now put

$$R_m(u_k)=\left(\frac{2}{k}u_k^3-4a_1(u)u_k^2-16a_0(u)ku_k+c\right)\frac{\partial}{\partial u_k}Q_m(u_k)$$
$$+\left(\frac{3}{k}u_k^2-4a_1(u)u_k-8ka_0(u)\right)Q_m(u_k)$$

and

$$S_m(u_k)=\frac{\partial}{\partial u_k}P_m(u_k),$$

which are the polynomials in u_k. Then, by (14) and (15), one verifies

$$\partial Z_m(u_k)=R_m(u_k)+S_m(u_k)u_k'.$$

Here we show the following.

Lemma 12. *$P_m(u_k)$ is the polynomial of degree m in u_k;*

$$P_m(u_k)=\sum_{j=0}^{m}P_{mj}(k)u_k^j.$$

The leading coefficient $p_{mm}(k)$ satisfies the recurrence relation

$$p_{m+1\,m+1}(k)=\frac{(2m+1)(2k-m(m+1))}{4k(m+1)}p_{mm}(k). \tag{17}$$

Moreover

$$Q_m(u_k)=R_m(u_k)=0$$

holds.

Proof. We prove this by induction on m. The assertion is obviously correct for $m=1$. Assume that the assertion is correct for m. Operate with $\Lambda(u)$ on both sides of (16). Then we have

$$Z_{m+1}(u_k)=\Lambda(u_k)(P_m(u_k)+Q_m(u_k)u_k')=\sum_{j=0}^{m}\frac{2j+1}{2(j+1)}p_{mj}(k)u_k^{j+1}$$
$$-\frac{1}{4}\sum_{j=2}^{m}j(j-1)p_{mj}(k)u_k^{j-2}(u_k')^2-\frac{1}{4}\sum_{j=1}^{m}ju_k^{j-1}u_k''. \tag{18}$$

Eliminate u_k'' and $(u_k')^2$ by (14) and (15) respectively. Then $Z_{m+1}(u_k)$ turns out to be the polynomial of degree $m+1$ in u_k, that is,

$$Z_{m+1}(u_k)=P_{m+1}(u_k)$$

and $Q_{m+1}(u_k)=0$. Moreover one verifies (17) by calculating the coefficient of u_k^{m+1} in (18). This completes the proof.

Finally we prove the following.

Theorem 13. *If* $\mathrm{rank}_A u(x)=1$ *then*

$$\mathrm{rank}_A \frac{n(n+1)}{2} u(x)=n$$

holds for any $n\in \boldsymbol{N}$.

Proof. For brevity, we use the notation $v_n=\frac{n(n+1)}{2}u$ in this proof. By lemma 12, we have

$$Z_{n+1}(v_n)=\sum_{j=0}^{n+1} p_{n+1j}\left(\frac{n(n+1)}{2}\right)v_n^j. \tag{19}$$

Moreover, $p_{n+1n+1}(\frac{n(n+1)}{2})=0$ follows from (17). On the other hand, let us consider the system of $n+1$ linear algebraic equations

$$\sum_{i=j}^{n} p_{ij}\left(\frac{n(n+1)}{2}\right)b_i=p_{n+1j}\left(\frac{n(n+1)}{2}\right), \qquad j=0,1,\cdots,n \tag{20}$$

for the $n+1$ unknowns $b_0,b_1,\cdots,b_n$. The coefficient matrix of the system of linear equations (20) is the upper triangle matrix with the diagonal elements $p_{mm}(\frac{n(n+1)}{2})$, $m=0,1,\cdots,n$. By induction based on the recurrence formula (17), one easily verifies

$$p_{mm}\left(\frac{n(n+1)}{2}\right)=\frac{(2m)!}{2^{2m}n^m(n+1)^m(m!)^2}\prod_{j=1}^{m}(n+j)(n-j+1).$$

Hence $p_{mm}(\frac{n(n+1)}{2})\neq 0$, $m=0,1,\cdots,n$ follows. Thus (20) is uniquely solvable. Let $b_0,b_1,\cdots,b_n$ be the unique solutions of (20). Then, by (19) and (20), one has

$$\begin{aligned}\sum_{i=0}^{n} b_i Z_i(v_n) &= \sum_{i=0}^{n} b_i \sum_{j=0}^{i} p_{ij}\left(\frac{n(n+1)}{2}\right)v_n^i \\ &= \sum_{j=0}^{n}\left(\sum_{i=j}^{n} p_{ij}\left(\frac{n(n+1)}{2}\right)b_i\right)v_n^i\end{aligned}$$

$$= \sum_{j=0}^{n} p_{n+1j}\left(\frac{n(n+1)}{2}\right) v_n^j = Z_{n+1}(v_n)$$

This implies $\mathrm{rank}_{\Lambda} v_n \leq n$. On the other hand, suppose that

$$\sum_{\nu=0}^{n} c_\nu Z_\nu(v_n) = 0$$

are valid for some $c_0, c_1, \cdots, c_n$. Then, similarly to the above, one verifies that $c_0 = c_1 = \cdots = c_n = 0$ hold. Thus we proved $\mathrm{rank}_{\Lambda} v_n = n$. This completes the proof.

Here we briefly mention about the function of Λ-rank 1. Let $\mathrm{rank}_{\Lambda} u(x) = 1$ then there exist the Λ-characteristic coefficients $a_\nu(u), \nu = 0, 1$ such that

$$Z_2(u(x)) = a_1(u) Z_1(u(x)) + a_0(u) Z_0(u(x)).$$

We have

$$u'' - 3u^2 + 4a_1(u)u + 8a_0(u) = 0$$

This equation has the following three type solutions; the rational function $2\lambda^2(\lambda x + a)^{-2} + b$, the trigonometric function $2\lambda^2 \sin^{-2}(\lambda x + a) + b$, and the elliptic function $2\lambda^2 \mathscr{P}(\lambda x + a) + b$. Therefore we have the following.

Corollary 14. *The following are valid;*

$$\mathrm{rank}_{\Lambda}\left(\frac{n(n+1)\lambda^2}{(\lambda x + a)^2} + b\right) = n,$$

$$\mathrm{rank}_{\Lambda}\left(\frac{n(n+1)\lambda^2}{\sin^2(\lambda x + a)} + b\right) = n,$$

$$\mathrm{rank}_{\Lambda}(n(n+1)\lambda^2 \mathscr{P}(\lambda x + a) + b) = n.$$

7. McKean-Trubowitz type trace formula

Let $q(x), -\infty < x < \infty$ be a real smooth function of period 1, then the spectrum of Hill's operator $-\partial^2 + q(x)$ in the class of functions of period 2 is a discrete series

$$-\infty < \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots < \lambda_{2i-1} \leq \lambda_{2i} < \cdots.$$

Let $f_\nu(x), \nu \in \boldsymbol{Z}_+$ be corresponding normalized eigenfunctions. In [13], McKean and Trubowitz proved that there exist $\varepsilon_\nu \in \boldsymbol{R}, \nu \in \boldsymbol{Z}_+$ such that

$$\sum_{\nu=0}^{\infty} \varepsilon_\nu f_\nu(x)^2 = 1, \tag{21}$$

where $\varepsilon_0>0$ and $\varepsilon_\nu \geq 0$ with equality if and only if $\lambda_{2\nu}=\lambda_{2\nu-1}$. See also [12], [4] and [5]. In this section, we want to understand the above trace formula (21) of McKean-Trubowitz type from the viewpoint of Λ-algorithm.

Suppose $n=\mathrm{rank}_\Lambda u(x)<\infty$ and define $F(x;\lambda)$ by (12). Then we have the following.

Theorem 15. *For any $\lambda \in C$,*

$$\Lambda(u)F(x;\lambda)=\lambda F(x;\lambda)-\Omega(\lambda;u)$$

holds, where $\Omega(\lambda;u)$ is the Λ-characteristic polynomial.

Proof. Put

$$P(\lambda)=(\lambda-\Lambda(u))F(x;\lambda)$$

then one has

$$\begin{aligned}&\frac{\partial}{\partial x}((\lambda-\Lambda(u))F(x;\lambda))\\&=\lambda F_x(x;\lambda)-\left(\frac{1}{2}u'(x)F(x;\lambda)+u(x)F_x(x;\lambda)-\frac{1}{4}F_{xxx}(x;\lambda)\right)\\&=-K(u-\lambda)F(x;\lambda)=0.\end{aligned}$$

This implies that $P(\lambda)$ is the polynomial with constant coefficients. On the other hand, since $(\lambda-\Lambda(u))F(x;\lambda)$ can be expressed as the linear combination of $Z_0(u),Z_1(u),\cdots,Z_{n+1}(u)$, there exist the polynomials $p_j(\lambda), j=0,1,\cdots,n+1$ in λ with constant coefficients such that

$$P(\lambda)=\sum_{j=0}^{n+1}p_j(\lambda)Z_j(u).$$

Since

$$\sum_{j=0}^{n+1}p_j(\lambda)Z_j(u)=\sum_{j=0}^{n}(p_j(\lambda)+a_j(u)p_{n+1}(\lambda))Z_j(u)$$

and $Z_0(u),\cdots,Z_n(u)$ are linearly independent, we have

$$P(\lambda)=p_0(\lambda)+a_0(u)p_{n+1}(\lambda).$$

By Theorem 3, one verifies

$$
\begin{aligned}
&(\lambda - A(u))F(x;\lambda)\\
&= \sum_{j=0}^{n} (-1)^{n-j}\alpha_j^{(n)} Z_j(u)\lambda^{n-j+1}\\
&\quad - \sum_{\nu=1}^{n} a_\nu(\lambda;u) \sum_{j=0}^{\nu-1} (-1)^{\nu-j-1}\alpha_j^{(\nu-1)} Z_j(u)\lambda^{\nu-j}\\
&\quad - \sum_{j=0}^{n} (-1)^{n-j}\alpha_j^{(n)} Z_{j+1}(u)\lambda^{n-j}\\
&\quad + \sum_{\nu=1}^{n} a_\nu(\lambda;u) \sum_{j=0}^{\nu-1} (-1)^{\nu-j-1}\alpha_j^{(\nu-1)} Z_{j+1}(u)\lambda^{\nu-j-1}.
\end{aligned}
$$

Therefore we have

$$
p_0(\lambda) = (-1)^n \alpha_0^{(n)} \lambda^{n+1} - \sum_{\nu=1}^{n} (-1)^{\nu-1} a_\nu(\lambda;u)\alpha_0^{(\nu-1)}\lambda^\nu
$$

and

$$
p_{n+1}(\lambda) = -\alpha_n^{(n)} = -1.
$$

On the other hand, by lemma 7 and Proposition 4, we have

$$
\begin{aligned}
p_0(\lambda) &= (-1)^n \alpha_0^{(n)} \lambda^{n+1}\\
&\quad - \sum_{\nu=1}^{n} (-1)^{\nu-1}\alpha_0^{(\nu-1)} \Big(-\alpha_\nu^{(n+1)}\lambda^{n-\nu+1} + \sum_{j=\nu}^{n} \alpha_\nu^{(j)} a_j(u)\lambda^{j-\nu}\Big)\lambda^\nu\\
&= \Big((-1)^n\alpha_0^{(n)} + \sum_{\nu=1}^{n} (-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n+1)}\Big)\lambda^{n+1}\\
&\quad - \sum_{\nu=1}^{n} (-1)^{\nu-1}\alpha_0^{(\nu-1)} \sum_{j=\nu}^{n} \alpha_\nu^{(j)} a_j(u)\lambda^j\\
&= \left(\sum_{\nu=1}^{n+1} (-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(n+1)}\right)\lambda^{n+1} - \sum_{j=1}^{n}\left(\sum_{\nu=1}^{j} (-1)^{\nu-1}\alpha_0^{(\nu-1)}\alpha_\nu^{(j)}\right) a_j(u)\lambda^j\\
&= \lambda^{n+1} - \sum_{j=1}^{n} a_j(u)\lambda_j
\end{aligned}
$$

Hence we have

$$
p_0(\lambda) + a_0(u)p_{n+1}(\lambda) = \lambda^{n+1} - \sum_{j=0}^{n} a_j(u)\lambda^j = \Omega(\lambda;u).
$$

This completes the proof.

One easily verifies that

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_0(u) \\ 1 & 0 & \cdots & 0 & a_1(u) \\ 0 & 1 & \cdots & 0 & a_2(u) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n(u) \end{pmatrix}$$

is the matrix of $A(u) \in \mathrm{End}(V(u))$ relative to the basis $Z_0(u), \cdots, Z_n(u)$ of $V(u)$. Hence

$$\det(\lambda - A(u)) = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & -a_0(u) \\ -1 & \lambda & 0 & \cdots & 0 & -a_1(u) \\ 0 & -1 & \lambda & \cdots & 0 & -a_2(u) \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & \lambda & -a_{n-1}(u) \\ 0 & 0 & \cdots & \cdots & -1 & \lambda - a_n(u) \end{vmatrix}$$

$$= \Omega(\lambda; u)$$

follows. Hence if we put

$$\Gamma_0(u) = \{\lambda | \Omega(\lambda; u) = 0\},$$

then we have the following.

Corollary 16. *$\Gamma_0(u)$ is the set of eigenvalues of $A(u) \in \mathrm{End}(V(u))$. Moreover $F(x; \mu_j)$ are the eigenvectors of $A(u)$ corresponding to the eigenvalues $\mu_j \in \Gamma_0(u)$, $j = 0, 1, \cdots, n$ respectively.*

Hence, if $n = \mathrm{rank}_A u(x) < \infty$ and $\sharp\Gamma_0(u) = n+1$ then $V(u)$ is spanned by $F(x; \mu_j), j = 0, 1, \cdots, n$;

$$V(u) = \bigoplus_{j=0}^{n} CF(x; \mu_j).$$

By lemma 9, $F(x; \lambda)$ is the polynomial of degree n in λ for each x. Hence if $\sharp\Gamma_0(u) = n+1$ then, by Lagrange's interpolation formula, we have

$$F(x; \lambda) = \sum_{j=0}^{n} \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{\lambda - \mu_i}{\mu_j - \mu_i} F(x; \mu_j). \tag{23}$$

Operate with $\Lambda(u)$ on both sides of (23) then, by Theorem 16, one has immediately

$$-\Omega(\lambda;u)+\lambda F(x;\lambda)=\sum_{j=0}^{n}\mu_j\prod_{\substack{i=0\\ i\neq j}}^{n}\frac{\lambda-\mu_i}{\mu_j-\mu_i}F(x;\mu_j).$$

Therefore, we have

$$-\prod_{j=0}^{n}(\lambda-\mu_j)+\sum_{j=0}^{n}\lambda\prod_{\substack{i=0\\ i\neq j}}^{n}\frac{\lambda-\mu_i}{\mu_j-\mu_i}F(x;\mu_j)=\sum_{j=0}^{n}\mu_j\prod_{\substack{i=0\\ i\neq j}}^{n}\frac{\lambda-\mu_i}{\mu_j-\mu_i}F(x;\mu_j).$$

Thus we proved the following.

Proposition 17. *If* $n=\mathrm{rank}_\Lambda u(x)<\infty$ *and* $\sharp\Gamma_0(u)=n+1$ *then the formula*

$$\sum_{j=0}^{n}\varepsilon_j^{(n)}F(x;\mu_j)=1 \tag{24}$$

holds, where $\Gamma_0(u)=\{\mu_0,\mu_1,\cdots,\mu_n\}$ *and*

$$\varepsilon_j^{(n)}=\prod_{\substack{i=0\\ i\neq j}}^{n}\frac{1}{\mu_j-\mu_i}. \tag{25}$$

Furthermore, by operating with $\Lambda(u)^m$ both sides of (24), one has

$$\sum_{j=0}^{n}\mu_j^m\varepsilon_j^{(n)}F(x;\mu_j)=Z_m(u(x)),\qquad m\in \boldsymbol{Z}_+.$$

Next suppose that $F(x;\mu_j)$ has at least one zero $x=a_j$ of second order for each $j=0,1,\cdots,n$, i.e.

$$F(a_j;\mu_j)=F_x(a_j;\mu_j)=0,\qquad j=0,1,\cdots,n.$$

Then, by (13), $\Delta(\mu_j;u)=0$ are valid for any $j=0,1,\cdots,n$. Hence $\Gamma_0(u)\subset\Gamma(u)$ holds in this case. Therefore, by Theorem 11, we proved the following.

Theorem 18. *Suppose that* $n=\mathrm{rank}_\Lambda u(x)<\infty$ *and* $\sharp\Gamma_0(u)=n+1$. *Moreover assume that* $F(x;\mu_j), j=0,1,\cdots,n$ *have at least one zero of second order respectively. Then*

$$\phi_j(x)=\sqrt{\varepsilon_j^{(n)}F(x;\mu_j)},\qquad j=0,1,\cdots,n$$

are the corresponding eigenfunctions of eigenvalues μ_j *of the eigenvalue problem* (11) *and the following trace formulae are valid for all* $m \in \boldsymbol{Z}_+$;

$$\sum_{j=0}^{n} \mu_j^m \phi_j(x)^2 = Z_m(u(x)), \tag{26}$$

$$\sum_{j=0}^{n} \mu_j^m \phi_j'(x)^2 = \frac{1}{2}(\partial^2 - [u\partial, \partial^{-1}])Z_m(u(x)), \tag{27}$$

where $\varepsilon_j^{(n)}, j=0,1,\cdots,n$ *are defined by* (25). *The right hand side of* (27) *is the differential polynomial in* $u(x)$. *Particularly*

$$\sum_{j=0}^{n} \phi_j'(x)^2 = -\frac{1}{2}u(x) \tag{28}$$

holds.

Proof. It suffices to prove (27). Differentiate twice both sides of (26), then we have

$$2\sum_{j=0}^{n} \mu_j^m \phi_j'(x)^2 + 2\sum_{j=0}^{n} \mu_j^m \phi_j(x)\phi_j''(x) = \partial^2 Z_m(u(x)).$$

Eliminate $\phi_j''(x)$ by $\phi_j''(x) = (u(x) - \mu_j)\phi_j(x)$ from the above then one easily verifies (27) by direct calculation. Moreover, by [6, p. 168, Proposition 12.1.12], it turns out that the right hand side of (27) belongs to $\mathscr{A}_0(u)$. The formula (28) follows immediately from (27). This completes the proof.

It is well known that the trace formulae of McKean-Trubowitz type (26) and (27) have many applications. Particularly, they play fundamental roles in many geometric theories of Hill's operator. See [3], [8], [13] and [14].

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Department of Mathematical Sciences
Faculty of Integrated Arts and Sciences
Tokushima University
Minamijohsanjima-cho 1–1
Tokushima 770
Japan