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ON THE CONVERGENCE RATES FOR SOLUTIONS OF SOME CHEMICAL INTERFACIAL REACTION PROBLEMS

MASATO IIDA, YOSHIO YAMADA AND SHOJI YOTSUTANI

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1. Introduction

In this paper, we treat some diffusion equations with a nonlinear system of boundary conditions, which appear in chemical engineering. Our concern is to investigate the asymptotic behavior of solutions to the following initial boundary value problem in $\bar{I}\times(0,\infty)$:

$$
\text{(P)}\qquad
\begin{cases}
a(x)\dfrac{\partial u}{\partial z}=\dfrac{\partial^2 u}{\partial x^2}, \quad b(x)\dfrac{\partial v}{\partial z}=\dfrac{\partial^2 v}{\partial x^2} \qquad \text{for} \quad (x,z)\in I\times(0,\infty);\\[2mm]
\dfrac{\partial u}{\partial x}(0,z)=R_1(u(0,z),v(0,z)), \qquad \dfrac{\partial v}{\partial x}(0,z)=R_2(u(0,z),v(0,z)),\\[2mm]
\dfrac{\partial u}{\partial x}(1,z)=0, \quad \dfrac{\partial v}{\partial x}(1,z)=0 \qquad \text{for} \quad z\in(0,\infty);\\[2mm]
u(x,0)=\phi_1(x), \quad v(x,0)=\phi_2(x) \qquad \text{for} \quad x\in I.
\end{cases}
$$

Here I and $\bar{I}$ denote $(0,1)$ and $[0,1]$, respectively; $a(x)$ and $b(x)$ are given functions satisfying

$$
\text{(A)}\qquad
\begin{cases}
a\in C^\infty(\bar{I}), \quad b\in C^\infty(\bar{I}),\\
a(x)>0, \quad b(x)>0 \quad \text{for} \quad x\in[0,1),\\
a(1)=b(1)=0;
\end{cases}
$$

$\phi_i(x)$ $(i=1,2)$ are nonnegative initial data; $R_i(u,v)=k_iR_0(u,v)$ $(i=1,2)$, where k_i $(i=1,2)$ are positive constants and

$$R_0(u,v)=u^m v^n$$

with positive integers m and n.

The problem (P) was proposed by Kawano and Nakashio [5] to describe some chemical models in which reactions are taking place only on the interface between two liquid phases flowing concurrently in contact. In their model, u and v represent the concentrations of the chemical substances in consideration; $a(x)$ and $b(x)$ are given by

$$a(x)=a_0(1-x^2), \quad b(x)=b_0(1-x^2)$$

with some positive constants a_0, b_0. For the derivation of the model see [5].

We are interested in the asymptotic behavior of solutions to (P) as $z\to\infty$ from the mathematical viewpoint. In the study of their asymptotic behavior the difficulty is how to deal with the nonlinear coupled boundary conditions, while it is relatively easy to show the global existence of nonnegative solutions owing to the monotonicity of $R_i(u,v)$ (cf. Yamada and Yotsutani [8]). Shinomiya [7] has first succeeded in showing the uniform convergence of solutions to the corresponding equilibrium by finding a nice Lyapunov function. The authors [4] have improved his results and derived the uniform convergence of the solutions together with all their derivatives as $z\to\infty$ by establishing a method to construct infinitely many Lyapunov functions systematically. On the other hand, Nagasawa [6] has obtained a partial answer to the rates of convergence by using a weighted L^p-norm.

In the present paper, we will complete Nagasawa's results, i.e., we will give the rates of convergence for

$$\begin{cases} \|u(\cdot,z)-u_\infty\|_\infty, & \left\|\dfrac{\partial^k u}{\partial z^k}(\cdot,z)\right\|_\infty, \quad \left\|\dfrac{\partial^k u}{\partial x\partial z^{k-1}}(\cdot,z)\right\|_\infty, \\ \|v(\cdot,z)-v_\infty\|_\infty, & \left\|\dfrac{\partial^k v}{\partial z^k}(\cdot,z)\right\|_\infty, \quad \left\|\dfrac{\partial^k v}{\partial x\partial z^{k-1}}(\cdot,z)\right\|_\infty \end{cases}$$

$(k=1,2,3,\cdots)$, where (u_∞,v_∞) is the equilibrium corresponding to $(u(\cdot,z),v(\cdot,z))$. The remarkable point here is that (u_∞,v_∞) depends on the initial data. The equilibrium changes its essential character according to the sign of

$$E:=\frac{1}{k_1}\|\phi_1 a\|_1-\frac{1}{k_2}\|\phi_2 b\|_1,$$

where $\|\cdot\|_1$ denotes $L^1(I)$-norm (see Proposition 2.1). Because of this feature, the rates of the convergence also vary depending on the sign of E. This fact makes our analysis complicated.

The organization of this paper is as follows. §2 contains our main theorem. In §3 we will give lemmas which are useful throughout the paper. In §4 we will summarize some fundamental properties of solutions to (P). §§5, 6 and 7 are devoted to the proof of our main theorem.

NOTATION

We will use the following notation throughout this paper. For $U=(u,v)$, we abbreviate $R_i(u,v)$ to $R_i(U)$ $(i=0,1,2)$. For any vector-valued function $U=U(x,z)=(u(x,z),v(x,z))$, its derivatives are denoted by

$$D_x^i D_z^j U=(D_x^i D_z^j u, D_x^i D_z^j v)=\left(\frac{\partial^{i+j}u}{\partial x^i \partial z^j}, \frac{\partial^{i+j}v}{\partial x^i \partial z^j}\right).$$

For any vector-valued function $U=U(x)=(u(x),v(x))$ on I, we use the following norms:

$$\|U\|=(\|u\|^2+\|v\|^2)^{1/2}=\left\{\int_I (u^2+v^2)dx\right\}^{1/2},$$

$$\|U\|_\infty=\operatorname*{ess\,sup}_{x\in I}|U(x)|=\operatorname*{ess\,sup}_{x\in I}\{u(x)^2+v(x)^2\}^{1/2},$$

$$\|u\|_{2;a}=\left\{\int_I u^2 a dx\right\}^{1/2}, \quad \|v\|_{2;b}=\left\{\int_I v^2 b dx\right\}^{1/2},$$

$$\|U\|_{2;}=\{\|u\|_{2;a}^2+\|v\|_{2;b}^2\}^{1/2}.$$

2. Main result

For convenience, we recall some results in [4] which will be fundamental for the subsequent arguments.

Proposition 2.1. *Suppose that* (A) *holds and that* $\phi=(\phi_1,\phi_2)$ *satisfies*

$$\phi\in L^\infty(I)^2, \quad \phi_i\geq 0 \text{ in } I \ (i=1,2).$$

Then there exists a unique solution $U=(u,v)\in C^\infty(\bar{I}\times(0,\infty))^2$ *of* (P) *which satisfies* $\lim_{z\to 0}\|U(\cdot,z)-\phi\|=0$. *Furthermore,*

(i) $0\leq u(x,z)\leq\|\phi_1\|_\infty$, $0\leq v(x,z)\leq\|\phi_2\|_\infty$ *in* $\bar{I}\times[0,\infty)$,

(ii) *U satisfies the "mass conservation" law, i.e.,*

$$\frac{1}{k_1}\|ua\|_1-\frac{1}{k_2}\|vb\|_1=E, \quad z\in[0,\infty),$$

where

$$E=\frac{1}{k_1}\|\phi_1 a\|_1-\frac{1}{k_2}\|\phi_2 b\|_1,$$

(iii) *if* $\phi_1 \not\equiv 0$ $(\phi_2 \not\equiv 0)$ *in* I, *then* $u>0$ (*resp.* $v>0$) *in* $\bar{I}\times(0,\infty)$.

In the study of asymptotic properties for (P) as $z\to\infty$, (ii) of Propostion 2.1 plays an important role. As a limit problem associated with (P), we consider the following algebraic problem for $U_\infty=(u_\infty,v_\infty)\in \boldsymbol{R}^2$.

$$(\mathrm{P}_\infty)\qquad \begin{cases} u_\infty\geq 0, \quad v_\infty\geq 0,\\ R_0(U_\infty)=0,\\ \dfrac{\|a\|_1}{k_1}u_\infty-\dfrac{\|b\|_1}{k_2}v_\infty=E. \end{cases}$$

Clearly, (P_∞) has a unique solution $U_\infty=(u_\infty,v_\infty)$ with

$$u_\infty=\frac{k_1E}{\|a\|_1} \text{ and } v_\infty=0 \qquad \text{if } E>0,$$

$$u_\infty=v_\infty=0 \qquad \text{if } E=0,$$

$$u_\infty=0 \textit{ and } v_\infty=\frac{k_2|E|}{\|b\|_1} \qquad \text{if } E<0.$$

By constructing infinitely many Lyapunov functions, we have obtained the following results on the asymptotic behavior.

Proposition 2.2. *Suppose that* (A) *holds. Then*

$$\begin{cases} \lim\limits_{z\to\infty}\|U(\cdot,z)-U_\infty\|_\infty=0,\\ \lim\limits_{z\to\infty}\|D_z^iD_x^jU\|_\infty=0 \end{cases}$$

for all nonnegative integers i, j *with* $(i,j)\neq(0,0)$.

For the proofs of Propositions 2.1 and 2.2, see [8] and [4]. In the present paper we investigate the rates of convergence in Proposition 2.2. Our goal is to show the following theorem.

Main Theorem. *In addition to* (A), *assume* $\phi_1\not\equiv 0\,(\geq 0)$ *and* $\phi_2\not\equiv 0\,(\geq 0)$. *Then*

$$\begin{cases} \|u(\cdot,z)-u_\infty\|_\infty=O(\rho_u(z)), \qquad \|v(\cdot,z)-v_\infty\|_\infty=O(\rho_v(z)),\\ \|D_xD_z^{k-1}u\|_\infty=O\left(\dfrac{d^k}{dz^k}\rho_u(z)\right), \qquad \|D_xD_z^{k-1}v\|_\infty=O\left(\dfrac{d^k}{dz^k}\rho_v(z)\right), \end{cases}$$

$$\left\{\ \|D_z^k u\|_\infty = O\left(\frac{d^k}{dz^k}\rho_u(z)\right), \qquad \|D_z^k v\|_\infty = O\left(\frac{d^k}{dz^k}\rho_v(z)\right)\right.$$

as $z \to \infty$ for every positive integer k. Here $\rho_u(z)$, $\rho_v(z)$ are defined in the following way depending on E and reaction exponents m, n:

Case (I) ———— $E>0$, $n=1$,

$$\rho_u(z)=\exp(-\tilde{\lambda}_0 z), \quad \rho_v(z)=\exp(-\lambda_0 z);$$

Case (II) ———— $E>0$, $n>1$,

$$\rho_u(z)=\rho_v(z)=z^{-\alpha} \quad \textit{with } \alpha=\frac{1}{n-1};$$

Case (III) ———— $E=0$,

$$\rho_u(z)=\rho_v(z)=z^{-\beta} \quad \textit{with } \beta=\frac{1}{m+n-1};$$

Case (IV) ———— $E<0$, $m>1$,

$$\rho_u(z)=\rho_v(z)=z^{-\gamma} \quad \textit{with } \gamma=\frac{1}{m-1};$$

Case (V) ———— $E<0$, $m=1$,

$$\rho_u(z)=\exp(-\mu_0 z), \quad \rho_v(z)=\exp(-\tilde{\mu}_0 z),$$

where λ_0, $\tilde{\lambda}_0$, μ_0 and $\tilde{\mu}_0$ are appropriate positive constants.

REMARK 2.1. The constants λ_0, $\tilde{\lambda}_0$ are characterized in the following way. Let $E>0$ and $n=1$. Consider the eigenvalue problem

$$\text{(2.1)} \qquad \begin{cases} -D_x^2 f = \lambda a(x) f, & -D_x^2 g = \lambda b(x) g, \qquad x \in I, \\ D_x f(0) = k_1(u_\infty)^m g(0), & D_x g(0) = k_2(u_\infty)^m g(0), \\ D_x f(1) = 0, & D_x g(1) = 0, \end{cases}$$

which is corresponding to the linearization of (P) around U_∞:

$$\text{(LP)} \qquad \begin{cases} a(x) D_z u = D_x^2 u, \quad b(x) D_z v = D_x^2 v \quad \text{for} \quad (x,z) \in I \times (0,\infty); \\ D_x u(0,z) = k_1(u_\infty)^m v(0,z), \qquad D_x v(0,z) = k_2(u_\infty)^m v(0,z), \\ D_x u(1,z) = 0, \qquad D_x v(1,z) = 0 \quad \text{for} \quad z \in (0,\infty). \end{cases}$$

It is easy to see that the set of eigenvalues for (2.1) coincides with the union of the eigenvalues for

$$\text{(2.2)} \qquad \begin{cases} -D_x^2 g = \lambda b(x) g & \text{in } I, \\ D_x g(0) = k_2(u_\infty)^m g(0), & D_x g(1) = 0 \end{cases}$$

and

$$\text{(2.3)} \qquad \begin{cases} -D_x^2 f = \lambda^* a(x) f & \text{in } I, \\ D_x f(0) = D_x f(1) = 0. \end{cases}$$

The constant λ_0 is the least eigenvalue for (2.2) and can be characterized as

$$\text{(2.4)} \qquad \lambda_0 := \inf\left\{\frac{\|D_x g\|^2 + k_2(u_\infty)^m g(0)^2}{\|g\|_{2;b}^2};\ g \in H^1(0,1), g \not\equiv 0\right\} > 0.$$

The constant $\tilde{\lambda}_0$ is given by

$$\tilde{\lambda}_0 = \begin{cases} \min\{\lambda_0, \lambda_0^*\} & \text{if } \lambda_0 \neq \lambda_0^* \\ \lambda_0 - \varepsilon & \text{if } \lambda_0 = \lambda_0^*. \end{cases}$$

where ε is an arbitrarily small positive number and λ_0^* is the least positive eigenvalue for (2.3). Observe that λ_0^* is characterized as

$$\text{(2.5)} \qquad \lambda_0^* := \inf\left\{\frac{\|D_x f\|^2}{\|f\|_{2;a}^2};\ f \in H^1(0,1),\ f \not\equiv 0,\ \int_I f a dx = 0\right\} > 0.$$

In particular, $\min\{\lambda_0, \lambda_0^*\}$ is the least positive eigenvalue for (2.1).

The constants μ_0, $\tilde{\mu}_0$ are characterized in a similar way:

$$\mu_0 := \inf\left\{\frac{\|D_x f\|^2 + k_1(v_\infty)^n f(0)^2}{\|f\|_{2;a}^2};\ f \in H^1(0,1),\ f \not\equiv 0\right\} > 0,$$

$$\mu_0^* := \inf\left\{\frac{\|D_x g\|^2}{\|g\|_{2;b}^2};\ g \in H^1(0,1),\ g \not\equiv 0,\ \int_I g b dx = 0\right\} > 0,$$

$$\tilde{\mu}_0 := \begin{cases} \min\{\mu_0, \mu_0^*\} & \text{if } \mu_0 \neq \mu_0^*, \\ \mu_0 - \varepsilon & \text{if } \mu_0 = \mu_0^*, \end{cases}$$

where ε is an arbitrarily small positive number.

REMARK 2.2. In Remark 2.1 We can expect neither that $\lambda_0 = \lambda_0^*$ implies $\tilde{\lambda}_0 = \lambda_0$ nor that $\mu_0 = \mu_0^*$ implies $\tilde{\mu}_0 = \mu_0$. This fact is recently proved by Iida and Ninomiya in [2] by using an argument on invariant manifolds. Moreover we see from [2] that the rates of convergence in Main Theorem are optimal.

REMARK 2.3. Recently, Hoshino and Yamada [1] have investigated a mathematical model for chemical reactions in a bounded domain. Using a different method from ours, they have obtained similar results to ours.

In the sequel, we will use C or $C_1, C_2, \cdots$ to denote various positive canstants. For simplicity, the same C sometimes denotes several different constants if there is no confusion.

3. Preliminaries

We begin with an imbedding lemma of the Sobolev type.

Lemma 3.1. *Suppose that $\rho(x) \in C(\bar{I})$ satisfies $\rho(x) \geq 0$ in $\bar{I}$ and $\rho(x) > 0$ in I. For any $\delta > 0$, there exists a positive constant C_δ depending only on ρ and δ such that*

$$\|w\|_\infty \leq \delta \|D_x w\| + C_\delta \|w\|_{2;\rho} \qquad \text{for all } w \in H^1(I).$$

Proof. See (3.2) of [4]. ∎

We give several lemmas on differential inequalities which are very useful to derive the rates of convergence for solutions from various energy estimates.

Lemma 3.2. *For a positive integer m, let $\{p_k(z)\}_{0 \leq k \leq m}$ be a sequence of nonnegative functions of class $C^1([\hat{z}, \infty))$ and let $\{q_k(z)\}_{0 \leq k \leq m}$, $\{\rho_k(z)\}_{0 \leq k \leq m}$ be sequences of nonnegative functions of class $C([\hat{z}, \infty))$. Suppose that*

$$\begin{cases} \bar{\rho} := \sup\{\rho_k(z);\ 0 \leq k \leq m,\ z \geq \hat{z}\} < 1, \\ \dfrac{dp_k}{dz} + q_k \leq \rho_k \displaystyle\sum_{j=0}^{k} q_j, & z \in [\hat{z}, \infty), \\ \lambda p_k \leq q_k, & z \in [\hat{z}, \infty) \end{cases}$$

for $k = 0, 1, \cdots, m$, where λ is a positive constant. Then

$$p_k(z) = O(\exp(-\tilde{\lambda} z)) \quad \text{as } z \to \infty \ (k = 0, 1, \cdots, m)$$

with a suitable constant $\tilde{\lambda} \in (0, \lambda)$.

Moreover, if $\rho_k(z) \in L^1(\hat{z}, \infty)$ $(k = 0, 1, \cdots, m)$, then $\tilde{\lambda}$ can be replaced by λ.

Proof. See Lemmas 3.2 and 3.3 of Iida, Yamada, Yanagida and Yotsutani [3]. ∎

Lemma 3.3. *Let $p(z)$ and $q(z)$ be nonnegative functions of class $C^1([\hat{z}, \infty))$ and $C([\hat{z}, \infty))$, respectively. Suppose that*

$$\begin{cases} \dfrac{dp}{dz}+q\leq\eta\exp\left(-\dfrac{\mu z}{2}\right)q^{1/2}, & z\in[\hat{z},\infty), \\ \lambda p\leq q, & z\in[\hat{z},\infty) \end{cases}$$

with some positive constants λ, μ *and* η. *Then*

$$p(z)=O(\exp(-\bar{\lambda}z)) \text{ as } z\to\infty,$$

where

$$\bar{\lambda}=\begin{cases} \min\{\lambda,\mu\} & \text{if } \lambda\neq\mu, \\ \lambda-\varepsilon & \text{if } \lambda=\mu, \end{cases}$$

and ε (>0) *is an arbitrarily small number.*

Proof. We divide the proof into three cases ($\lambda>\mu$, $\lambda<\mu$, $\lambda=\mu$).

(i) Consider the case where $\lambda>\mu$. Choose a sufficiently small $\delta>0$ such that $\lambda-\delta>\mu$. Since the right-hand side of the first inequality in the assumptions can be estimated as

$$\eta\exp\left(-\frac{\mu z}{2}\right)q^{1/2}\leq\frac{\delta}{\lambda}q+C\exp(-\mu z),$$

we have

$$\frac{dp}{dz}+(\lambda-\delta)\frac{q}{\lambda}\leq C\exp(-\mu z).$$

Thus, by the second inequality in the assumptions, we get

$$\frac{dp}{dz}+(\lambda-\delta)p\leq C\exp(-\mu z), \qquad z\in[\hat{z},\infty),$$

from which we can easily derive

$$p(z)=O(\exp(-\mu z)) \qquad \text{as } z\to\infty.$$

(ii) Consider the case where $\lambda<\mu$. Take a new small $\delta(>0)$ such that $\lambda<\mu-\delta$. We may assume $\hat{z}>0$ without loss of generality. Observing that

$$\eta\exp\left(-\frac{\mu z}{2}\right)q^{1/2}\leq\exp(-\delta z)q+C\exp(-\{\mu-\delta\}z),$$

we have

$$\frac{dp}{dz}+\{1-\exp(-\delta z)\}\lambda p\leq\frac{dp}{dz}+\{1-\exp(-\delta z)\}q$$
$$\leq C\exp(-\{\mu-\delta\}z)$$

for $z\in[\hat{z},\infty)$. Multiplication by

$$\exp\left(\lambda\int_{\hat{z}}^{z}\{1-\exp(-\delta\zeta)\}d\zeta\right)$$

and integration from $\hat{z}$ to z eventually yield

$$p(z)=O(\exp(-\lambda z))\qquad \text{as } z\to\infty.$$

(iii) Consider the case where $\lambda=\mu$. Let $\delta>1$ and assume $\hat{z}>0$ again. In view of

$$\eta\exp\left(-\frac{\mu z}{2}\right)q^{1/2}\leq(1+z)^{-\delta}q+C(1+z)^{\delta}\exp(-\mu z),$$

it is easy to derive

$$\frac{dp}{dz}+\{1-(1+z)^{-\delta}\}\lambda p\leq C(1+z)^{\delta}\exp(-\lambda z),\quad z\in[\hat{z},\infty).$$

Multiply each side of the inequality by

$$\exp\left(\lambda\int_{\hat{z}}^{z}\{1-(1+\zeta)^{-\delta}\}d\zeta\right),$$

and integrate from $\hat{z}$ to z. After some calculations, we find that

$$p(z)=O(z^{\delta+1}\exp(-\lambda z))\qquad \text{as } z\to\infty,$$

which implies

$$p(z)=O(\exp(-\{\lambda-\varepsilon\}z))\qquad \text{as } z\to\infty,$$

where we can make $\varepsilon>0$ arbitrarily small. ■

Lemma 3.4. *Let $p(z)$ be a nonnegative function of class $C^1([\hat{z},\infty))$. Suppose that*

$$\frac{dp}{dz}+\eta p^{\omega}\leq 0,\qquad z\in([\hat{z},\infty),$$

where $\eta\in(0,\infty)$ and $\omega\in(1,\infty)$ are constants. Then

$$p(z)=O(z^{-1/(\omega-1)})\qquad \text{as } z\to\infty.$$

Proof. It is sufficient to multiply both sides of the given inequality by $p(z)^{-\omega}$ and integrate from $\hat{z}$ to z. ■

Lemma 3.5. *Let $p_1(z)$, $p_2(z)$ be nonnegative functions of class $C^1([\hat{z},\infty))$ and let $q_0(z)$, $q_1(z)$, $q_2(z)$ be nonnegative functions of class $C([\hat{z},\infty))$. Let $\rho(z)$ be a positive function of class $C([\hat{z},\infty))$ such that*

$$\rho(z)=O(z^{-\theta}) \qquad \text{as } z\to\infty,$$

where θ is a positive constant. Suppose that $p_i(z)$ $(i=1,2)$, $q_j(z)$ $(j=0,1,2)$ and $\rho(z)$ satisfy

$$\begin{cases} \dfrac{dq_0}{dz}+p_1\leq\eta\rho+\varepsilon q_1, & z\in[\hat{z},\infty), \\ \dfrac{dp_1}{dz}+q_1\leq\kappa(\rho+p_1), & z\in[\hat{z},\infty), \\ \dfrac{dq_1}{dz}+p_2\leq\eta(\rho+p_1+q_1)+\varepsilon q_2, & z\in[\hat{z},\infty), \\ \dfrac{dp_2}{dz}+q_2\leq\kappa(\rho+p_1+q_1+p_2), & z\in[\hat{z},\infty), \\ q_0\leq\kappa p_1, & z\in[\hat{z},\infty), \end{cases}$$

where ε, κ and η are positive canstants with $\varepsilon<\kappa^{-1}$. Then

$$q_0(z)+p_1(z)+q_1(z)+p_2(z)=O(z^{-\theta}) \qquad \text{as } z\to\infty.$$

Proof. Since $\varepsilon<\kappa^{-1}$, there exist positive constants σ_1, σ_2 and σ_3 such that

$$\begin{cases} \eta\sigma_2+\kappa\sigma_3<\min\{1-\kappa\sigma_1,\sigma_1-\varepsilon\}, \\ \kappa\sigma_3<\sigma_2, \\ \varepsilon\sigma_2<\sigma_3. \end{cases}$$

It is easy to see from the inequalities in the assumption that

$$\begin{aligned} &\frac{d}{dz}(q_0+\sigma_1p_1+\sigma_2q_1+\sigma_3p_2)+p_1+\sigma_1q_1+\sigma_2p_2+\sigma_3q_2 \\ &\leq(\eta+\kappa\sigma_1+\eta\sigma_2+\kappa\sigma_3)\rho+(\kappa\sigma_1+\eta\sigma_2+\kappa\sigma_3)p_1 \\ &\quad+(\varepsilon+\eta\sigma_2+\kappa\sigma_3)q_1+\kappa\sigma_3p_2+\varepsilon\sigma_2q_2, \qquad z\in[\hat{z},\infty). \end{aligned}$$

Hence we have

$$\frac{d}{dz}(q_0+\sigma_1 p_1+\sigma_2 q_1+\sigma_3 p_2)$$
$$+\frac{1}{C}(p_1+q_1+p_2+q_2)\le C\rho, \qquad z\in[\hat{z},\infty).$$

Moreover, since

$$\psi(z):=q_0(z)+\sigma_1 p_1(z)+\sigma_2 q_1(z)+\sigma_3 p_2(z)$$
$$\le C\{p_1(z)+q_1(z)+p_2(z)\},$$

we get

$$\frac{d\psi}{dz}+\frac{1}{C}\psi\le C\rho, \qquad z\in(\hat{z},\infty).$$

Therefore, with the aid of $\rho(z)=O(z^{-\theta})$ as $z\to\infty$, we obtain

$$\psi(z)=O(z^{-\theta}) \qquad \text{as } z\to\infty,$$

which completes the proof. ■

4. Fundamental properties of solutions

In the sequel, we sometimes write $u_*=u-u_\infty$ ($v_*=v-v_\infty$). We give three types of fundamental identities which will later yield various useful estimates.

Lemma 4.1. *Let $U=(u,v)$ be the solution of* (P). *The first component u satisfies*

$$\text{(4.1)}\qquad \begin{cases} \dfrac{1}{2}\dfrac{d}{dz}\|u_*\|_{2;a}^2=-\|D_x u\|^2-u_*(0,z)R_1(U(0,z)), \\[2ex] \dfrac{1}{2}\dfrac{d}{dz}\|D_z^k u\|_{2;a}^2=-\|D_x D_z^k u\|^2-D_z^k u(0,z)\dfrac{d^k}{dz^k}R_1(U(0,z)), \end{cases}$$

$$\text{(4.2)}\qquad \frac{1}{2}\frac{d}{dz}\|D_z^{k-1}u-D_z^{k-1}\bar{u}\|_{2;a}^2 = -\|D_x D_z^{k-1}u\|^2-\{D_z^{k-1}u(0,z)-D_z^{k-1}\bar{u}(z)\}\frac{d^{k-1}}{dz^{k-1}}R_1(U(0,z)),$$

$$\text{(4.3)}\qquad \frac{1}{2}\frac{d}{dz}\|D_x D_z^{k-1}u\|^2=-\|D_z^k u\|_{2;a}^2-D_z^k u(0,z)\frac{d^{k-1}}{dz^{k-1}}R_1(U(0,z))$$

for $z\in(0,\infty)$ and every $k=1,2,3,\cdots$. Here

$$\bar{u}(z)=\frac{1}{\|a\|_1}\int_I ua\,dx.$$

Similar equalities hold for v *with some modification.*

Proof. It is easy to see the first equality of (4.1) from (P). The second equality of (4.1) and (4.3) are obtained from the boundary value problems for $D_z^k U$ and $D_z^{k-1} U$, respectively, which are obtained by differentiation of (P) with respect to z.

We will show (4.2) with $k=1$. We have

$$\frac{1}{2}\frac{d}{dz}\|u-\bar{u}\|_{2;a}^2=\int_I (u-\bar{u})(D_z u-\frac{d\bar{u}}{dz})a\,dx$$
$$=\int_I (u-\bar{u})D_x^2 u\,dx-\frac{d\bar{u}}{dz}\int_I (u-\bar{u})a\,dx.$$

Noting the fact

$$\int_I (u-\bar{u})a\,dx=0$$

and integrating by parts, we get (4.2).

The same argument for $D_z^{k-1}u$ instead of u yields (4.2) with $k\geq 2$. ■

The following identities are a key property of solutions.

Lemma 4.2. *The solution* $U=(u,v)$ *of* (P) *satisfies*

$$\frac{1}{k_1}\int_I u_*(x,z)a(x)dx=\frac{1}{k_2}\int_I v_*(x,z)b(x)dx,$$
$$\frac{1}{k_1}\int_I D_z^k u(x,z)a(x)dx=\frac{1}{k_2}\int_I D_z^k v(x,z)b(x)dx$$

for $z\in(0,\infty)$ *and* $k=1,2,3,\cdots$.

Proof. Clearly, the first identity follows from (ii) of Proposition 2.1. It is sufficient to apply D_z^k to its both sides to complete the proof. ■

Finally we give some a priori estimates for solutions.

Lemms 4.3. *Let* $U=(u,v)$ *be the solution of* (P). *Then*

$$\begin{cases} \|D_x D_z^{k-1}u\|_\infty\leq\|a\|_1^{1/2}\|D_z^k u\|_{2;a}, \\ \|D_x D_z^{k-1}v\|_\infty\leq\|b\|_1^{1/2}\|D_z^k v\|_{2;b} \end{cases}$$

hold true for $z \in (0, \infty)$ *and* $k = 1, 2, 3, \cdots$.

Proof. By virtue of the boundary condition at $x = 1$ of (P), we have

$$\begin{aligned} D_x u(x,z) &= D_x u(1,z) - \int_x^1 D_x^2 u(\xi, z) d\xi \\ &= -\int_x^1 D_z u(\xi, z) a(\xi) d\xi \end{aligned}$$

for $x \in I$. Thus, with the aid of Schwarz' inequality, we get

$$\|D_x u\|_\infty \le \|a\|_1^{1/2} \|D_z u\|_{2;a}.$$

For $k \ge 2$, we have only to use the boundary value problem which is obtained by differentiating (P) $k-1$ times with respect to z. ■

Lemma 4.4. *Let* $U = (u, v)$ *be the solution of* (P). *Then*

$$\begin{cases} \|u_*\|_\infty \le K(\|u_*\|_{2;a} + \|D_z u\|_{2;a}), \\ \|v_*\|_\infty \le K(\|v_*\|_{2;b} + \|D_z v\|_{2;b}), \\ \|D_z^k u\|_\infty \le K(\|D_z^k u\|_{2;a} + \|D_z^{k+1} u\|_{2;a}), \\ \|D_z^k v\|_\infty \le K(\|D_z^k v\|_{2;b} + \|D_z^{k+1} v\|_{2;b}) \end{cases}$$

hold true for $z \in (0, \infty)$ *and* $k = 1, 2, 3, \cdots$. *Here K is a positive constant independent of z and k.*

Proof. The assertion follows from Lemmas 3.1 and 4.3. ■

5. Rates of convergence Cases ... (I) and (V)

In this section we will prove Main Theorem in Case (I), i.e., $E > 0$ and $n = 1$. The proof for Case (V) is the same. Note that

$$u_\infty = \frac{k_1}{\|a\|_1} E > 0.$$

Lemma 5.1. *The solution* $U = (u, v)$ *of* (P) *satisfies*

$$(5.1) \quad \begin{cases} \dfrac{1}{2}\dfrac{d}{dz}\|v\|_{2;b}^2 + \|D_x v\|^2 + k_2 (u_\infty)^m |v(0,z)|^2 \le L_0 |u_*(0,z)| |v(0,z)|^2, \\ \dfrac{1}{2}\dfrac{d}{dz}\|D_z^k v\|_{2;b}^2 + \|D_x D_z^k v\|^2 + k_2 (u_\infty)^m |D_z^k v(0,z)|^2 \\ \qquad \le L_k \left\{ |u_*(0,z)| + \sum_{l=1}^{k} |D_z^l u(0,z)| \right\} \sum_{j=0}^{k} |D_z^j v(0,z)|^2, \end{cases}$$

$$\frac{1}{2}\frac{d}{dz}\|D_z^{k-1}u-D_z^{k-1}\bar{u}\|_{2;a}^2+\|D_xD_z^{k-1}u\|^2\leq L_k\|D_xD_z^{k-1}u\|\sum_{j=0}^{k-1}|D_z^jv(0,z)|, \tag{5.2}$$

$$\begin{cases} \|u_*\|_{2;a}\leq L(\|v\|_\infty+\|u-\bar{u}\|_{2;a}), \\ \|D_z^ku\|_{2;a}\leq L(\|D_z^kv\|_\infty+\|D_z^ku-D_z^k\bar{u}\|_{2;a}) \end{cases} \tag{5.3}$$

for $z\in[1,\infty)$ and $k=1,2,3,\cdots$, where L and $L_0, L_1,\cdots$, are positive constants independent of z.

Proof. Observing that

$$R_i(U(0,z))=k_i(u_\infty)^m v(0,z)+k_i v(0,z)\sum_{l=1}^{m}\binom{m}{l}(u_\infty)^{m-l}u_*(0,z)^l,$$

and $\lim_{z\to\infty}D_z^ju_*(0,z)=0$, we obtain (5.1) from (4.1) (use similar identities for v). Similarly, we can derive (5.2) from (4.2) by using a version of Poincaré's inequality:

$$|D_z^ku(0,z)-D_z^k\bar{u}(z)|\leq\|D_xD_z^ku\|.$$

To obtain the first inequality of (5.3), we have only to use the identity

$$\begin{aligned} u_*(x,z)&=\frac{1}{\|a\|_1}\int_I u_*a\,dx+u_*(x,z)-\bar{u}_*(z) \\ &=\frac{k_1}{k_2\|a\|_1}\int_I vb\,dx+u(x,z)-\bar{u}(z), \end{aligned}$$

which is a corollary of Lemma 4.2. The second inequality of (5.3) is similarly verified. ■

Proof of Main Theorem ... Case (I). The proof is carried out by dividing it into several steps:

Step 1 $\|D_z^kv\|_\infty=O(\exp(-\nu z))$ as $z\to\infty$ for $k\geq0$, where ν is a positive constant;

Step 2 $\|D_z^ku_*\|_\infty=O(\exp(-\tilde{\nu}z))$ as $z\to\infty$ for $k\geq0$, where $\tilde{\nu}$ is a positive constant;

Step 3 $\|D_z^kD_x^jv\|_\infty=O(\exp(-\lambda_0z))$ as $z\to\infty$ for $k\geq0$ and $j\geq0$;

Step 4 $\|D_z^kD_x^ju_*\|_\infty=O(\exp(-\tilde{\lambda}_0z))$ as $z\to\infty$ for $k\geq0$ and $j\geq0$.

Step 1. By (2.4), v satisfies

$$\lambda_0 \|D_z^k v\|_{2;b}^2 \le \|D_x D_z^k v\|^2 + k_2(u_\infty)^m |D_z^k v(0,z)|^2 \tag{5.4}$$

for $z \in (0,\infty)$ and $k=0,1,2,\cdots$. Since

$$\begin{cases} \lim_{z\to\infty} u_*(0,z)=0, \\ \lim_{z\to\infty} D_z^k u(0,z)=0 \qquad (k=1,2,\cdots), \end{cases}$$

we can apply Lemma 3.2 to (5.1); so that

$$\|D_z^k v\|_{2;b} = O(\exp(-\nu z)) \quad \text{as } z\to\infty \qquad (k=0,1,2,\cdots),$$

where $\nu \in (0,\lambda_0)$ is a constant independent of z. Thus, making use of Lemma 4.4, we can also show

$$\|D_z^k v\|_\infty = O(\exp(-\nu z)) \quad \text{as } z\to\infty \qquad (k=0,1,2,\cdots). \tag{5.5}$$

Step 2. We see from (2.5) that

$$\lambda_0^* \|D_z^{k-1}u - D_z^{k-1}\bar{u}\|_{2;a}^2 \le \|D_x D_z^{k-1} u\|^2$$

for $z \in (0,\infty)$ and $k=1,2,3,\cdots$. Hence, by virtue of (5.2) and (5.5) it follows from Lemma 3.3 that

$$\|D_z^{k-1}u - D_z^{k-1}\bar{u}\|_{2;a} = O(\exp(-\tilde{\nu} z)) \quad \text{as } z\to\infty \qquad (k=1,2,3,\cdots),$$

where $\tilde{\nu} \in (0,\min\{\lambda_0^*,\nu\}]$ is a constant independent of z. Therefore, with use of (5.3) and (5.5), we have

$$\begin{cases} \|u_*\|_{2;a} = O(\exp(-\tilde{\nu} z)) \quad \text{as } z\to\infty, \\ \|D_z^k u\|_{2;a} = O(\exp(-\tilde{\nu} z)) \quad \text{as } z\to\infty \qquad (k=1,2,3,\cdots); \end{cases}$$

so that Lemma 4.4 yields

$$\begin{cases} \|u_*\|_\infty = O(\exp(-\tilde{\nu} z)) \quad \text{as } z\to\infty, \\ \|D_z^k u\|_\infty = O(\exp(-\tilde{\nu} z)) \quad \text{as } z\to\infty \qquad (k=1,2,3,\cdots). \end{cases} \tag{5.6}$$

Step 3. In view of (5.6), we invoke the latter half of Lemma 3.2. Then it follows from (5.1) and (5.4) that

$$\|D_z^k v\|_{2;b} = O(\exp(-\lambda_0 z)) \quad \text{as } z\to\infty \qquad (k=0,1,2,\cdots),$$

which, together with Lemma 4.4, imply

$$\|D_z^k v\|_\infty = O(\exp(-\lambda_0 z)) \quad \text{as } z \to \infty \qquad (k=0,1,2,\cdots). \tag{5.7}$$

Using Lemma 4.3, we get

$$\|D_z^k D_x v\|_\infty = O(\exp(-\lambda_0 z)) \quad \text{as } z \to \infty \qquad (k=0,1,2,\cdots).$$

Moreover, since

$$\|D_z^k D_x^j v\|_\infty = \|D_x^{j-2}(D_z^k D_x^2 v)\|_\infty = \|D_x^{j-2}(bD_z^{k+1}v)\|_\infty \le C_j \sum_{l=0}^{j-2} \|D_z^{k+1} D_x^l v\|_\infty,$$

we can inductively derive

$$\|D_z^k D_x^j v\|_\infty = O(\exp(-\lambda_0 z)) \quad \text{as } z \to \infty \qquad (k \ge 0,\ j \ge 0).$$

Step 4. Repeat the argument in Step 2 with (5.5) replaced by (5.7). We get

$$\|D_z^k u_*\|_\infty = \begin{cases} O(\exp(-\min\{\lambda_0, \lambda_0^*\} z)) & \text{if } \lambda_0 \ne \lambda_0^*, \\ O(\exp(-\{\lambda_0 - \varepsilon\} z)) & \text{if } \lambda_0 = \lambda_0^* \end{cases}$$

as $z \to \infty$ $(k=0,1,2,\cdots)$, where $\varepsilon > 0$ is an arbitrarily small number. Thus we obtain

$$\|D_z^k D_x^j u_*\|_\infty = \begin{cases} O(\exp(-\min\{\lambda_0, \lambda_0^*\} z)) & \text{if } \lambda_0 \ne \lambda_0^*, \\ O(\exp(-\{\lambda_0 - \varepsilon\} z)) & \text{if } \lambda_0 = \lambda_0^* \end{cases}$$

as $z \to \infty$ $(k \ge 0,\ j \ge 0)$ in the same manner as in Step 3. ■

6. Rates of convergence ... Cases (II) and (IV)

In this section we will prove Main Theorem in Case (II) where $E>0$ and $n>1$. The discussion for Case (IV) is quite the same.

We will divide the proof into four steps and establish the following asymptotic properties for $U=(u,v)$ in each step:

$$\|v(\cdot,z)\|_\infty = O(z^{-\alpha}) \quad \text{as } z \to \infty, \tag{6.1}$$

$$\|D_x U(\cdot,z)\|_\infty + \|D_z U(\cdot,z)\|_\infty = O(z^{-\alpha-1}) \quad \text{as } z \to \infty, \tag{6.2}$$

$$\|D_x D_z^{k-1} U(\cdot,z)\|_\infty + \|D_z^k U(\cdot,z)\|_\infty = O(z^{-\alpha-k}) \quad \text{for } k \ge 2 \text{ as } z \to \infty, \tag{6.3}$$

$$\|u_*(\cdot,z)\|_\infty = O(z^{-\alpha}) \quad \text{as } z \to \infty, \tag{6.4}$$

where $\alpha = 1/(n-1)$.

In Steps 1–4 we will use Lemmas 6.1–6.6, whose proofs will be given at the end of this section.

Step 1. The following lemmas give essential estimates.

Lemma 6.1. *Let $U=(u,v)$ satisfy* (P). *Then*

$$\frac{d}{dz}\|v\|_{2;b}^2+\|D_xv\|^2+\frac{1}{M}v(0,z)^{n+1}\leq 0, \tag{6.5}$$

$$\frac{d}{dz}\|D_zv\|_{2;b}^2+\|D_xD_zv\|^2\leq M\{\|D_zv\|_{2;b}^2+v(0,z)^{n+1}\} \tag{6.6}$$

for $z\in[1,\infty)$, where M is a positive constant independent of z.

Lemma 6.2. *Let $U=(u,v)$ satisfy* (P). *For any positive number ε there exists a positive constant M_ε such that*

$$\frac{d}{dz}\|D_xv\|^2+\|D_zv\|_{2;b}^2\leq\varepsilon\|D_xD_zv\|^2+M_\varepsilon v(0,z)^{n+1} \tag{6.7}$$

for $z\in(0,\infty)$.

Let $\varepsilon\in(0,1/M)$. Combining (6.5), (6.6) and (6.7), we can deduce

$$\frac{d}{dz}(\|v\|_{2;b}^2+C_1\|D_xv\|^2+C_2\|D_zv\|_{2;b}^2)$$
$$+C_3\{v(0,z)^{n+1}+\|D_xv\|^2+\|D_zv\|_{2;b}^2\}\leq 0.$$

Since $n+1>2$, the boundedness of $\|D_xv\|$ and $\|D_zv\|_{2;b}$ for $z\in[1,\infty)$ leads us to

$$\|D_xv\|^{n+1}\leq C\|D_xv\|^2,\qquad z\in[1,\infty),$$
$$\|D_zv\|_{2;b}^{n+1}\leq C\|D_zv\|_{2;b}^2,\qquad z\in[1,\infty).$$

Thus, with the aid of Lemma 3.4, we have

$$\|v\|_{2;b}+\|D_xv\|=O(z^{-\alpha})\qquad \text{as } z\to\infty,$$

which, together with Lemma 3.1, yields (6.1).

Step 2. Observe that (4.1) and (4.3), combined with (6.1), yield the following estimates (actually we will prove them later).

Lemma 6.3. *Let $U=(u,v)$ satisfy* (P). *Then*

$$\frac{d}{dz}\|D_zU\|_{2;}^2+\|D_xD_zU\|^2\leq M_1\|D_zU\|_{2;}^2,$$

$$\frac{d}{dz}\|D_z^2U\|_{2;}^2+\|D_xD_z^2U\|\le M_1(\|D_zU\|_{2;}^2+\|D_xD_zU\|^2+\|D_z^2U\|_{2;}^2)$$

for $z\in[1,\infty)$, *where* M_1 *is a positive constant independent of* z.

Lemma 6.4. *Let* ε *be any positive number and let* $U=(u,v)$ *satisfy* (P). *Then there exists a positive constant* $M_{\varepsilon,1}$ *such that*

$$\frac{d}{dz}\|D_xU\|^2+\|D_zU\|_{2;}^2\le\varepsilon\|D_xD_zU\|^2+M_{\varepsilon,1}z^{-2\alpha-2},$$

$$\frac{d}{dz}\|D_xD_zU\|^2+\|D_z^2U\|_{2;}^2\le\varepsilon\|D_xD_z^2U\|^2+M_{\varepsilon,1}(\|D_zU\|_{2;}^2+\|D_xD_zU\|^2)$$

for $z\in[1,\infty)$.

We see from Lemma 4.3 that

$$\|D_xU\|^2\le C\|D_zU\|_{2;}^2$$

for $z\in[1,\infty)$. Hence Lemmas 6.3 and 6.4 enable us to apply Lemma 3.5 to get

$$\|D_zU\|_{2;}+\|D_xD_zU\|=O(z^{-\alpha-1})\qquad \text{as } z\to\infty.$$

By using Lemmas 3.1 and 4.3 again, we obtain (6.2).

Step 3. Let k be a positive integer with $k\ge 2$. We need the following estimates which correspond to Lemmas 6.3 and 6.4 in Step 2.

Lemma 6.5. *Let* $U=(u,v)$ *satisfy* (P). *Suppose that*

$$\|D_z^jU(\cdot,z)\|_\infty=O(z^{-\alpha-j})\qquad \text{as } z\to\infty$$

for $j=1,2,\cdots,k$-1. *Then*

$$\frac{d}{dz}\|D_z^kU\|_{2;}^2+\|D_xD_z^kU\|^2\le M_k(z^{-2\alpha-2k-2}+\|D_z^kU\|_{2;}^2),$$

$$\frac{d}{dz}\|D_z^{k+1}U\|_{2;}^2+\|D_xD_z^{k+1}U\|^2$$
$$\le M_k(z^{-2\alpha-2k-4}+\|D_z^kU\|_{2;}^2+\|D_xD_z^kU\|^2+\|D_z^{k+1}U\|_{2;}^2)$$

for $z\in[1,\infty)$, *where* M_k *is a positive constant.*

Lemma 6.6. *Let ε be any positive number. Under the assumption of Lemma 6.5, there exists a positive constant $M_{\varepsilon,k}$ such that*

$$\frac{d}{dz}\|D_xD_z^{k-1}U\|^2+\|D_z^kU\|_{2;}^2\leq\varepsilon\|D_xD_z^kU\|^2+M_{\varepsilon,k}z^{-2\alpha-2k},$$

$$\frac{d}{dz}\|D_xD_z^kU\|^2+\|D_z^{k+1}U\|_{2;}^2$$

$$\leq\varepsilon\|D_xD_z^{k+1}U\|^2+M_{\varepsilon,k}(z^{-2\alpha-2k-2}+\|D_z^kU\|_{2;}^2+\|D_xD_z^kU\|^2)$$

for $z\in[1,\infty)$.

Recall that we have already verified the assumption of Lemmas 6.5 for $k=2$. In a similar manner to Step 2, combining Lemma 3.5 with Lemmas 6.5 and 6.6 ($k=2$) we can show

$$\|D_xD_zU\|_\infty+\|D_z^2U\|_\infty=O(z^{-\alpha-2})\qquad\text{as } z\to\infty.$$

By repeating this argument, we can inductively derive

$$\|D_xD_z^{k-1}U\|_\infty+\|D_z^kU\|_\infty=O(z^{-\alpha-k})\qquad\text{as } z\to\infty$$

for $k=3,4,5,\cdots$. Thus (6.3) has been proved.

Step 4. Since it follows from Lemma 4.2 with $k=0$ that

$$\|u^*\|_\infty\leq\|u-\bar{u}\|_\infty+\left|\frac{1}{\|a\|_1}\int_I u_*a\,dx\right|$$

$$\leq C(\|D_xu\|_\infty+\|v\|_\infty),$$

we get (6.4) from (6.1) and (6.2). This completes the proof. ∎

Now we will give the proofs of Lemmas. We can derive Lemmas 6.1 and 6.2 directly from Lemma 4.1.

Proofs of Lemmas 6.1 and 6.2. Since $\lim_{z\to\infty}u(0,z)=u_\infty>0$ and $u(0,z)>0$ for $z\in(0,\infty)$, we see that

$$v(0,z)R_2(U(0,z))\geq Cv(0,z)^{n+1},\qquad z\in[1,\infty),$$

so that (6.5) follows from the equality for v corresponding to the first one of (4.1).

Recalling $n+1>2$, we see from the boundedness of u, v and D_zu that

$$\begin{aligned}
&\left|D_z v(0,z)\frac{d}{dz}R_2(U(0,z))\right| \\
&\quad \le \left|D_z v(0,z)\frac{\partial R_2}{\partial u}(U(0,z))D_z u(0,z)\right| + \left|\frac{\partial R_2}{\partial v}(U(0,z))\right||D_z v(0,z)|^2 \\
&\quad \le C\{|D_z v(0,z)|v(0,z)^n + |D_z v(0,z)|^2\} \\
&\quad \le C\{|D_z v(0,z)|^2 + v(0,z)^{n+1}\} \\
&\quad \le \frac{1}{2}\|D_x D_z v\|^2 + C\{\|D_z v\|_{2;b}^2 + v(0,z)^{n+1}\}
\end{aligned}$$

for $z\in[1,\infty)$. In the last inequality we have used Lemma 3.1. Then (6.6) comes from (4.1) with $k=1$.

Observe that

$$|R_2(U(0,z))|^2 \le Cv(0,z)^{n+1};$$

it is easy to get (6.7) from Lemma 3.1 and (4.3) with $k=1$. ∎

Proofs of Lemmas 6.3 and 6.4. We see from (6.1) that

$$(6.8) \qquad |R_0(U(0,z))| \le Cz^{-\alpha-1}, \qquad z\in[1,\infty).$$

Moreover, by the boundedness of U and $D_z U$,

$$(6.9) \qquad \left|\frac{d}{dz}R_0(U(0,z))\right| \le C|D_z U(0,z)|, \qquad z\in[1,\infty),$$

$$(6.10) \qquad \left|\frac{d^2}{dz^2}R_0(U(0,z))\right| \le C\{|D_z U(0,z)| + |D_z^2 U(0,z)|\}, \qquad z\in[1,\infty).$$

In the right-hand sides of (6.9) and (6.10), use Lemma 3.1 making $\delta>0$ sufficiently small. Then we get Lemma 6.3 from (4.1) with $k=1,2$.

We see from (4.3) that

$$\frac{1}{2}\frac{d}{dz}\|D_x D_z^{k-1}U\|^2 + \|D_z^k U\|_{2;}^2 \le \eta|D_z^k U(0,z)|^2 + C_\eta\left|\frac{d^{k-1}}{dz^{k-1}}R_0(U(0,z))\right|^2$$

for $z\in(0,\infty)$. Here η is an arbitrary positive number and C_η is a corresponding positive constant which is independent of z. Taking a sufficiently small $\eta>0$ for ε, we can show Lemma 6.4 with the aid of (6.8), (6.9) and Lemma 3.1. ∎

Proofs of Lemmas 6.5 and 6.6. In place of (6.8), (6.9) and (6.10), use the following fundamental Lemma. Then the same argument as the proofs of Lemmas 6.3 and 6.4 completes the proofs. ∎

Lemma 6.7. *Let $U=(u,v)$ be the solution of* (P) *and let k be a positive integer with $k \geq 2$. Suppose that*

$$\|v(\cdot,z)\|_\infty = O(z^{-\alpha}) \qquad \text{as } z \to \infty$$

and

$$\|D_z^j U(\cdot,z)\|_\infty = O(z^{-\alpha-j}) \qquad \text{as } z \to \infty$$

for $j=1,2,\cdots,k$-1. Then there exists a positive constant M_k' (independent of z) which satisfies

$$\begin{cases} \left|\dfrac{d^{k-1}}{dz^{k-1}} R_0(U(0,z))\right| \leq M_k' z^{-\alpha-k}, \\ \left|\dfrac{d^k}{dz^k} R_0(U(0,z))\right| \leq M_k'\{z^{-\alpha-k-1} + |D_z^k U(0,z)|\}, \\ \left|\dfrac{d^{k+1}}{dz^{k+1}} R_0(U(0,z))\right| \leq M_k'\{z^{-\alpha-k-2} + |D_z^k U(0,z)| + |D_z^{k+1} U(0,z)|\} \end{cases}$$

for $z \in [1,\infty)$.

7. Rates of convergence ... Case (III)

We will prove Main Theorem in the case $E=0$, where $u_\infty = v_\infty = 0$. The following Lemmas correspond to Lemmas 6.1 and 6.2; so that their proofs can be accomplished with use of Lemmas 3.1 and 4.1.

Lemma 7.1. *Let $U=(u,v)$ satisfy* (P). *Then*

$$\frac{1}{2}\frac{d}{dz}\|U\|_{2;}^2 + \|D_x U\|^2 + \{k_1 u(0,z) + k_2 v(0,z)\} R_0(U(0,z)) = 0, \tag{7.1}$$

$$\frac{d}{dz}\|D_z U\|_{2;}^2 + \|D_x D_z U\|^2 \leq N \|D_z U\|_{2;}^2 \tag{7.2}$$

for $z \in (0,\infty)$, where N is a positive constant independent of z.

Lemma 7.2. *Let $U=(u,v)$ satisfy* (P). *For any positive number ε there exists a positive constant N_ε such that*

$$\frac{d}{dz}\|D_x U\|^2 + \|D_z U\|_{2;}^2 \le \varepsilon \|D_x D_z U\|^2 + N_\varepsilon R_0(U(0,z))^2 \tag{7.3}$$

for $z \in (0, \infty)$.

Since u and v are nonnegative and uniformly bounded, we have

$$0 \le R_0(U(0,z)) \le C\{k_1 u(0,z) + k_2 v(0,z)\}$$

for $z \in [0, \infty)$. Therefore, it follows from (7.1), (7.2) and (7.3) that

$$\begin{aligned}&\frac{d}{dz}\{\|U\|_{2;}^2 + C_1\|D_x U\|^2 + C_2\|D_z U\|_{2;}^2\}\\ &\quad + C_3\{\|D_x U\|^2 + \|D_z U\|_{2;}^2 + (k_1 u(0,z) + k_2 v(0,z))R_0(U(0,z))\} \le 0\end{aligned} \tag{7.4}$$

with some positive constants C_1, C_2 and C_3. Here we use the following lemma.

Lemma 7.3. *The solution* $U=(u,v)$ *of* (P) *satisfies*

$$\|U(\cdot,z)\|_{2;}^{m+n+1} \le N^*[\|D_x U(\cdot,z)\|^2 + \{k_1 u(0,z) + k_2 v(0,z)\}R_0(U(0,z))],$$

for $z \in [1, \infty)$ *with some* $N^* > 0$.

We will continue the proof of main theorem. It follows from (7.4) and Lemma 7.3 that the assumption of Lemma 3.4 is satisfied with $p(z) = \|U\|_{2;}^2 + C_1\|D_x U\|^2 + C_2\|D_z U\|_{2;}^2$ and $\omega = (m+n+1)/2$. Thus

$$\|U(\cdot,z)\|_\infty = O(z^{-\beta}) \qquad \text{as } z \to \infty$$

with $\beta = 1/(m+n-1)$.

In order to proceed the proof, we observe that Lemmas 6.3–6.6 remain valid with α replaced by β. So it is sufficient to follow the argument of Steps 2 and 3 in section 6. ■

Proof of Lemma 7.3. Using Lemma 4.2, we have

$$\begin{aligned}u(x,z) &= u(x,z) - \bar{u}(z) + \frac{k_1}{k_2\|a\|_1}\int_I v(x,z)b(x)dx\\ &= u(x,z) - \bar{u}(z) + \frac{k_1}{k_2\|a\|_1}\int_I \{v(x,z) - v(0,z)\}b(x)dx + \frac{k_1\|b\|_1}{k_2\|a\|_1}v(0,z).\end{aligned}$$

The right-hand side is bounded by

$$C\{\|D_xU\|+v(0,z)\}.$$

Hence we can show

$$|U(x,z)|\leq C\{\|D_xU\|+v(0,z)\} \tag{7.5}$$

with the aid of

$$v(x,z)\leq v(0,z)+\|D_xv\|.$$

Similarly,

$$|U(x,z)|\leq C\{\|D_xU\|+u(0,z)\}. \tag{7.6}$$

Setting $x=0$ in (7.5) and (7.6) we get

$$\begin{aligned}
&|U(0,z)|^{m+n+1}\\
&\quad\leq C_1\{\|D_xU\|+u(0,z)\}^{m+1}\{\|D_xU\|+v(0,z)\}^n\\
&\quad\leq C_2\{\|D_xU\|^{m+1}+u(0,z)^{m+1}\}\{\|D_xU\|^n+v(0,z)^n\}\\
&\quad=C_2\{\|D_xU\|^{m+n+1}+u(0,z)R_0(U(0,z))+u(0,z)^{m+1}\|D_xU\|^n+v(0,z)^n\|D_xU\|^{m+1}\}\\
&\quad\leq C_3\{\|D_xU\|^{m+n+1}+u(0,z)R_1(U(0,z))\}+\frac{1}{2}|U(0,z)|^{m+n+1}.
\end{aligned}$$

We have used Young's inequality to derive the last inequality. Recalling $m+n+1>2$ and the boundedness of $\|D_xU(\cdot,z)\|_\infty$ for $z\in[1,\infty)$, we obtain

$$|U(0,z)|^{m+n+1}\leq C\{\|D_xU\|^2+u(0,z)R_1(U(0,z))\}, \qquad z\in[1,\infty).$$

We also see from similar calculations that

$$|U(0,z)|^{m+n+1}\leq C\{\|D_xU\|^2+v(0,z)R_2(U(0,z))\}, \qquad z\in[1,\infty).$$

Consequently making use of these two estimates we can obtain the assertion from (7.5) and (7.6). ■

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M. Iida
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Oh-Okayama, Meguro, Tokyo 152 Japan

Y. Yamada
Department of Mathematics
Waseda University
3–4–1 Ohkubo, Shinjuku, Tokyo 169
Japan

S. Yotsutani
Department of Applied Mathematics and Informatics
Ryukoku University
Seta, Ohtsu 520–21
Japan