

L[∞] BOUNDEDNESS OF NONLINEAR EIGENFUNCTION UNDER SINGULAR VARIATION OF DOMAINS

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1. Introduction.

Let M be a bounded domain in R^3 with smooth boundary ∂M . Let \tilde{w} be a point in M . We remove ball B_ε of radius ε with the center \tilde{w} from M and we get $M_\varepsilon = M \setminus \bar{B}_\varepsilon$.

Fix $p \in (1, 2)$. We fix $k > 0$. We put

$$(1.1) \quad \lambda(\varepsilon) = \inf_{u \in X} \left(\int_{M_\varepsilon} |\nabla u|^2 dx + k \int_{\partial B_\varepsilon} u^2 d\sigma_x \right),$$

where $X = \{u \in H^1(M_\varepsilon), u = 0 \text{ on } \partial M \text{ and } u \geq 0 \text{ in } M_\varepsilon, \|u\|_{p+1, \varepsilon} = 1\}$. Here $\|u\|_{L^q(M_\varepsilon)} = \|u\|_{q, \varepsilon}$. We see that there exists at least one solution v_ε of the above problem which attains (1.1)_ε. We know that v_ε satisfies $-\Delta v_\varepsilon(x) = \lambda(\varepsilon)v_\varepsilon(x)^p$ in M_ε , $v_\varepsilon(x) = 0$ on ∂M and $kv_\varepsilon(x) + (\partial/\partial v_x)v_\varepsilon(x) = 0$ on ∂B_ε . Here $\partial/\partial v_x$ denotes the derivative along the exterior normal vector with respect to M_ε .

Let S_ε denote the set of positive function u_ε which attains the minimum of (1.1).

Main result of this paper is the following

Theorem 1. *Fix $p \in (1, 2)$. Then, there exists a constant C independent of ε such that*

$$\sup_{u_\varepsilon \in S_\varepsilon} \sup_{x \in M_\varepsilon} |v_\varepsilon(x)| < C.$$

Related topics are discussed in Lin [1], Ozawa [4].

2. Preliminary Lemmas.

We have the following Lemma 2.1.

Lemma 2.1. *Consider the following equations.*

$$(2.1) \quad \Delta v_\varepsilon(x) = 0 \quad x \in M \setminus \bar{B}_\varepsilon$$

$$(2.2) \quad v_\varepsilon(x) = 0 \quad x \in \partial M$$

$$(2.3) \quad k v_\varepsilon(x) + (\partial v_\varepsilon / \partial \nu_x) = \alpha(\omega) \quad x = w + \varepsilon \omega \in \partial B_\varepsilon.$$

Here $\omega \in S^2$. Then, the solution of (2.1), (2.2), (2.3) satisfies

$$(2.4) \quad |v_\varepsilon(x)| \leq C \varepsilon^2 r^{-1} \|\alpha\|_{C^{(\sigma/2)}(S^2)}$$

for any $\sigma > 1$. Here C may depend on σ but independent of ε . Here $r = |x - \tilde{w}|$.

Proof. Let Δ_{S^2} denote the Laplace-Beltrami operator on S^2 . It is well known that $-n(n+1)$ is an eigenvalue of Δ_{S^2} whose multiplicity is $(2n+1)$. We can write it explicitly by using the Legendre polynomial but we do not do it. We write complete orthonormal basis of $L^2(S^2)$ consisting of eigenfunction by $\{\varphi_j(\omega)\}_{j=1}^\infty$. If $\Delta \varphi_j(\omega) = -n(n+1)\varphi_j(\omega)$, then we write j as $j \in (n)$. Thus, $\#\{j; j \in (n)\} = 2n+1$.

First we want to construct a solution of (2.1), (2.3).

We put

$$v^*(x) = \sum_{n=0}^\infty r^{-(n+1)} \sum_{j \in (n)} c_{j,n} \varphi_j(\omega).$$

Then,

$$\Delta v^*(x) = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_\varepsilon.$$

We expand α by $\varphi_j(\omega)$ as

$$\alpha(\omega) = \sum_{n=0}^\infty \sum_{j \in (n)} a_{j,n} \varphi_j(\omega).$$

By the equality (2.3), we have

$$(k c_{j,n} \varepsilon^{-(n+1)} + (n+1) c_{j,n} \varepsilon^{-(n+2)}) = a_{j,n}.$$

Then,

$$c_{j,n} = a_{j,n} \varepsilon^{n+1} (k + (n+1) \varepsilon^{-1})^{-1}.$$

Therefore,

$$|v^*(x)| \leq \varepsilon \sum_{n=0}^\infty \sum_{j \in (n)} a_{j,n} (\varepsilon/r)^{n+1} (\varepsilon k + (n+1))^{-1} \varphi_j(\omega).$$

We notice that we have

$$(2.5) \quad |\varphi_j(\omega)| \leq C(n+1)^{1/2}$$

for $j \in (n)$ by the property of Legendre and associated Legendre polynomial. See Mizohata [2, p.312].

By the Schwarz inequality we have

$$|v_\varepsilon^*| \leq C\varepsilon^2 r^{-1} \left(\sum_{n=0}^\infty (n+1)^\sigma \sum_{j \in (n)} a_{j,n}^2 \right)^{1/2} \times \left(\sum_{n=0}^\infty (n+1)^{-\sigma} (2n+1)n(n+1)^{-2} \right)^{1/2}.$$

If we take $\sigma > 1$, then

$$(2.6) \quad |v^*(x)| \leq C\varepsilon^2 r^{-1} \left(\sum_{n=0}^\infty (n+1)^\sigma \sum_{j \in (n)} a_{j,n}^2 \right)^{1/2} \leq C\varepsilon^2 r^{-1} \|\alpha\|_{H^{\sigma/2}(S^2)}.$$

Here we note that $\|\alpha\|_{H^2(S^2)}^2$ is equivalent to $\left(\sum_{n=0}^\infty (n+1)^4 \sum_{j \in (n)} a_{j,n}^2 \right)$, since the eigenvalue of $-\Delta_{S^2}$ is $n(n+1)$. We used representation of norm of fractional Sobolev space. Notice that $C^{\sigma'/2}(S^2) \subset H^{\sigma/2}(S^2)$ for $\sigma < \sigma'$.

Now $v^*(x)$ does not satisfy $v^*(x) = 0$ on ∂M .

By the same procedure as in the repeated construction of the function $v_\varepsilon^{(n)}$ in Proposition 1 of Ozawa [3] we proved Lemma 2.1.

The following Lemm is very useful.

Lemma 2.3. *There exists an extension operator $E: H^1(M_\varepsilon) \ni u \rightarrow Eu = \tilde{u} \in H^1(M)$ satisfying the followings;*

$$(2.9) \quad \tilde{u}(x) = u(x) \quad \text{a.e. } M_\varepsilon$$

holds for any $u \in H^1(M_\varepsilon)$,

$$(2.10) \quad \|\tilde{u}\|_{L^s(M)} \leq C\|u\|_{s,\varepsilon} \quad (1 \leq s \leq \infty)$$

holds for any $u \in H^1(M_\varepsilon) \cap L^s(M_\varepsilon)$,

$$(2.11) \quad \|\tilde{u}\|_{H^1(M)} \leq C\|u\|_{H^1(M_\varepsilon)} + \varepsilon^{((s-2)3/(2s)-1)} \|u\|_{s,\varepsilon}$$

for any $u \in H^1(M_\varepsilon) \cap L^s(M_\varepsilon)$ with $2 \leq s < \infty$,

$$\|\tilde{u}\|_{H^1(M)} \leq C\|u\|_{H^1(M_\varepsilon)} + C\varepsilon^{(1/2)} \|u\|_{\infty,\varepsilon}$$

holds for any $u \in H^1(M_\varepsilon) \cap L^\infty(M_\varepsilon)$.

Proof. Without loss of generality, we may assume that $\tilde{w}=0$. We take an arbitrary $u \in H^1(M_\varepsilon)$ and put

$$\begin{aligned} \tilde{u}(x) &= u(x) & x \in M_\varepsilon \\ &= u(\varepsilon^2 x |x|^{-2}) \eta_\varepsilon(x) & x \in B_\varepsilon, \end{aligned}$$

where $\eta_\varepsilon \in C^\infty(\mathbb{R}^3)$ satisfies $0 \leq \eta_\varepsilon \leq 1$, $\eta_\varepsilon = 1$ on $\mathbb{R}^3 \setminus \bar{B}_{\varepsilon/2}$, $\eta_\varepsilon = 0$ on $B_{\varepsilon/4}$ and $|\nabla \eta_\varepsilon| \leq 8\varepsilon^{-1}$. Notice that both $\eta_\varepsilon(\varepsilon^2 x |x|^{-2})$ and $(\nabla \eta_\varepsilon)(\varepsilon^2 x |x|^{-2})$ vanish on $\mathbb{R}^3 \setminus B_{4\varepsilon}$. Then, by using the Kelvin transformation of co-ordinates, $y = \varepsilon^2 x |x|^{-2}$, we have

$$\begin{aligned} \int_{B_\varepsilon} |\tilde{u}(x)|^s dx &\leq \int_{\mathbb{R}^3 \setminus \bar{B}_\varepsilon} |u(y)|^s \eta_\varepsilon(\varepsilon^2 y |y|^{-2})^s (\varepsilon |y|^{-1})^6 dy \\ &\leq \int_{M_\varepsilon} |u(y)|^s dy \quad (1 \leq s < \infty), \end{aligned}$$

and

$$\begin{aligned} \int_{B_\varepsilon} |\nabla u(x)|^2 dx &\leq C \int_{B_\varepsilon} |u(\varepsilon^2 x |x|^{-2})|^2 |(\nabla \eta_\varepsilon)(x)|^2 dx \\ &\quad + C \int_{B_\varepsilon} (\varepsilon |x|^{-1})^4 |(\nabla u)(\varepsilon^2 x |x|^{-2})|^2 \eta_\varepsilon(x)^2 dx \\ &\leq C\varepsilon^4 \int_{M_\varepsilon} |u(y)|^2 |y|^{-6} dy + C \int_{M_\varepsilon} |\nabla u|^2 dy. \end{aligned}$$

By Hölder's inequality, we see that

$$\int_{M_\varepsilon} |u(y)|^2 |y|^{-6} dy \leq \begin{cases} C\varepsilon^{-(1+(2/s)3)} \|u\|_{L^s(M_\varepsilon)}^2 & (2 \leq s < \infty) \\ C\varepsilon^{-3} \|u\|_{L^\infty(M_\varepsilon)}^2 & (s = \infty). \end{cases}$$

Thus, we get Lemma 2.3.

3. The Green function.

Let $G(x, y)$ be the Green function of the Laplacian in M under the Dirichlet condition on ∂M . We introduce the following kernel $p_\varepsilon(x, y)$.

$$p_\varepsilon(x, y) = G(x, y) + g(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y) + h(\varepsilon)\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle,$$

where

$$\langle \nabla_w u(\tilde{w}), \nabla_w v(\tilde{w}) \rangle = \sum_{n=1}^3 \frac{\partial u}{\partial w_n} \frac{\partial v}{\partial w_n} \Big|_{w=\tilde{w}}$$

when $w=(w_1, w_2, w_3)$ is an orthonormal frame of R^3 .

Here we put

$$g(\varepsilon) = -(\gamma + (4\pi\varepsilon)^{-1} + k^{-1}(4\pi)^{-1}\varepsilon^{-2})^{-1},$$

where

$$\gamma = \lim_{x \rightarrow \tilde{w}} (G(x, \tilde{w}) - (4\pi)^{-1}|x - \tilde{w}|^{-1})$$

and

$$h(\varepsilon) = k^{-1}/((4\pi)^{-1}\varepsilon^{-2} + k^{-1}(2\pi)^{-1}\varepsilon^{-3}).$$

We put

$$G(x, y) - (4\pi)^{-1}|x - y|^{-1} = S(x, y).$$

Then, $S(x, y) \in C^\infty(M \times M)$. We have the following:

$$\begin{aligned} & \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \\ &= (4\pi)^{-1}\varepsilon^{-2} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \\ & \frac{\partial}{\partial x_1} \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \\ &= -(2\pi)^{-1}\varepsilon^{-3} \frac{\partial}{\partial w_1} G(\tilde{w}, y) + \frac{\partial}{\partial x_1} \langle \nabla_w S(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \end{aligned}$$

for $x=(\varepsilon, 0, 0)$.

Then,

$$\begin{aligned} & kp_\varepsilon(x, y) + \frac{\partial}{\partial v_x} p_\varepsilon(x, y) \\ &= kG(x, y) + kg(\varepsilon)G(x, \tilde{w})G(\tilde{w}, y) + kh(\varepsilon)\langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \\ & \quad - \frac{\partial}{\partial x_1} G(x, y) - g(\varepsilon) \left(\frac{\partial}{\partial x_1} G(x, \tilde{w}) \right) G(\tilde{w}, y) \\ & \quad - h(\varepsilon) \frac{\partial}{\partial x_1} \langle \nabla_w G(x, \tilde{w}), \nabla_w G(\tilde{w}, y) \rangle \\ &= kG(x, y) - kG(\tilde{w}, y) - g(\varepsilon) \frac{\partial}{\partial x_1} S(x, \tilde{w})G(\tilde{w}, y) \end{aligned}$$

$$\begin{aligned}
 & -\frac{\partial}{\partial x_1}G(x,y) + \frac{\partial}{\partial w_1}G(\tilde{w},y) + kh(\varepsilon)\langle \nabla_w S(x,\tilde{w}), \nabla_w G(\tilde{w},y) \rangle \\
 & -h(\varepsilon)\frac{\partial}{\partial x_1}\langle \nabla_w S(x,\tilde{w}), \nabla_w G(\tilde{w},y) \rangle + O(\varepsilon)g(\varepsilon)G(\tilde{w},y)
 \end{aligned}$$

for $x=(\varepsilon,0,0)$.

Let $G_\varepsilon(x,y)$ denote the Green function of Δ in M_ε satisfying $G_\varepsilon(x,y)=0$ on ∂M and $kG_\varepsilon(x,y) + \frac{\partial}{\partial v_x}G_\varepsilon(x,y)=0$ on ∂B_ε .

Let Q_ε and P_ε be the operator defined by

$$\begin{aligned}
 Q_\varepsilon f(x) &= \int_{M_\varepsilon} G_\varepsilon(x,y)f(y)dy \\
 P_\varepsilon f(x) &= \int_{M_\varepsilon} p_\varepsilon(x,y)f(y)dy.
 \end{aligned}$$

We put $Q_\varepsilon f(x)=P_\varepsilon f(x)$. Then, it satisfies

$$\begin{aligned}
 \Delta Q_\varepsilon f(x) &= 0 & x \in M_\varepsilon \\
 Q_\varepsilon f(x) &= 0 & x \in \partial M \\
 kQ_\varepsilon f(x) + \frac{\partial}{\partial v_x}Q_\varepsilon f(x) &= kP_\varepsilon f(x) + \frac{\partial}{\partial v_x}P_\varepsilon f(x) & x \in \partial B_\varepsilon.
 \end{aligned}$$

We know that

$$\begin{aligned}
 (3.1) \quad & kP_\varepsilon f(x) + \frac{\partial}{\partial v_x}P_\varepsilon f(x) \\
 &= k(Gf)(x) - (Gf)(\tilde{w}) - \frac{\partial}{\partial x_1}Gf(x) + \frac{\partial}{\partial w_1}Gf(\tilde{w}) \\
 & \quad + O(1)g(\varepsilon)Gf(\tilde{w}) + O(h(\varepsilon))\frac{\partial}{\partial x_1}\langle \nabla_w S(x,\tilde{w}), \nabla_w Gf(\tilde{w}) \rangle \\
 & \quad + O(h(\varepsilon))\langle \nabla_w S(x,\tilde{w}), \nabla_w Gf(\tilde{w}) \rangle.
 \end{aligned}$$

4. Proof of Theorem 1.

We have the following decomposition of u_ε .

$$(4.1) \quad u_\varepsilon = -\lambda(\varepsilon)Q_\varepsilon u_\varepsilon^p + \lambda(\varepsilon)P_\varepsilon u_\varepsilon^p.$$

Lemma 4.1. Fix $p \in (1, 2)$. Then,

$$(4.2) \quad \sup_{x \in M_\varepsilon} |P_\varepsilon u_\varepsilon^p| \leq C(1 + \varepsilon \|u_\varepsilon\|_{\infty, \varepsilon}^\tau)$$

holds for any $\tau < 1$ which is close enough to 1.

Proof. Recall that

$$P_\varepsilon u_\varepsilon^p(x) = G\hat{u}_\varepsilon^p(x) + g(\varepsilon)G(x, \tilde{w})G\hat{u}_\varepsilon^p(\tilde{w}) + h(\varepsilon)\langle \nabla_w G(x, \tilde{w}), \nabla_w G\hat{u}_\varepsilon^p(\tilde{w}) \rangle.$$

Here $\hat{u}_\varepsilon^p(x)$ is the extension of u_ε^p to M which is zero outside M_ε .

We have $|G\hat{u}_\varepsilon^p(\tilde{w})| \leq C\|u_\varepsilon^p\|_{t, \varepsilon} \leq C'\|u_\varepsilon\|_{t, p, \varepsilon}^p$ for any $t > (3/2)$. Notice that we can take $t > (3/2)$ as close as $3/2$ so that $tp < p + 1$ for $p \in (1, 2)$. Therefore, $|G\hat{u}_\varepsilon^p(w)|$ is bounded.

We have

$$\begin{aligned} \varepsilon |\nabla_w G\hat{u}_\varepsilon^p(\tilde{w})| &\leq C\varepsilon \|u_\varepsilon\|_{\infty, \varepsilon}^\tau \left(\int_{M_\varepsilon} u_\varepsilon^{p-\tau}(y) |y - \tilde{w}|^{-2} dy \right) \\ &\leq C'\varepsilon \|u_\varepsilon\|_{\infty, \varepsilon}^\tau \|u_\varepsilon\|_{p+1, \varepsilon}^{(p-\tau)/(p+1)} \left(\int_{M_\varepsilon} |y - w|^{-2r} dy \right)^{1/r} \end{aligned}$$

where $r = (p + 1)/(1 + \tau)$. Thus, it does not exceed $\varepsilon \|u_\varepsilon\|_{\infty, \varepsilon}^\tau$ since $2r < 3$, if we take $p \in (1, 2)$ and $\tau < 1$ as close as 1.

Lemma 4.2. Fix $p \in (1, 2)$. Then,

$$(4.3) \quad |Q_\varepsilon u_\varepsilon^p| \leq C\varepsilon^{1-p+(\mu/q)} \|u_\varepsilon\|_{\infty, \varepsilon}^{(\mu/q)}$$

holds for $q > 6, q > \mu$.

Proof. By Lemma 2.1 we see that

$$(4.4) \quad |Q_\varepsilon f(x)| \leq C\varepsilon^2 r^{-1} \|G\hat{f}\|_{C^{1+(\sigma/2)}(M)}$$

for $\sigma > 1$ by using Lemma 2.1 and (3.1). Here \hat{f} is the extension of f to M which is zero outside M_ε .

We have

$$\|G\hat{f}\|_{C^{1+(\sigma/2)}(M)} \leq C\|f\|_{q, \varepsilon}$$

for $q > 3/(1 - (\sigma/2))$. We take $\sigma > 1$ as close as 1.

Then, we have

$$(4.5) \quad |Q_\varepsilon f(x)| \leq C\varepsilon \|f\|_{q,\varepsilon}$$

for $q > 6, x \in M_\varepsilon$.

We have

$$\varepsilon \|u_\varepsilon^p\|_{q,\varepsilon} \leq \varepsilon \|u_\varepsilon\|_{\infty,\varepsilon}^{(\mu/q)} \left(\int_{M_\varepsilon} |u_\varepsilon|^{pq-\mu} dx \right)^{1/q}.$$

We take $\mu < q$ as close as q and $q > 6$ as close as 6. Then, $pq - \mu < 6$ for $p \in (1, 2)$.

On the other hand we have

$$\|u_\varepsilon\|_{pq-\mu,\varepsilon} \leq \|\tilde{u}_\varepsilon\|_{L^{pq-\mu}(M)} \leq C \|\tilde{u}_\varepsilon\|_{H^1(M)}$$

by the Sobolev embedding theorem using $pq - \mu < 6$. Here \tilde{u}_ε is an extension of u_ε in Lemma 2.3 which is different from \hat{u}_ε . It should be noted, since $\lambda(\varepsilon)$ is defined as an infimum of a functional so that it is easy to see that $\limsup_{\varepsilon \rightarrow 0} \lambda(\varepsilon) < \infty$. Therefore,

$$\|\tilde{u}_\varepsilon\|_{H^1(M)} \leq \|u_\varepsilon\|_{H^1(M_\varepsilon)} + C\varepsilon^{-1} \|u_\varepsilon\|_{2,\varepsilon}$$

by taking $s=2$ in Lemma 2.3. Since u_ε is a minimizer of (1.1) and

$$\|u_\varepsilon\|_{2,\varepsilon} \leq C$$

we see that

$$\|\tilde{u}_\varepsilon\|_{H^1(M)} \leq C\varepsilon^{-1}.$$

Therefore,

$$\|u_\varepsilon\|_{pq-\mu,\varepsilon} \leq C''\varepsilon^{-1}.$$

Therefore, we get the desired result.

Proof of Theorem 1. We put $\xi = \|u_\varepsilon\|_{\infty,\varepsilon}$. By Lemma 4.1 and 4.2 we get

$$\xi \leq C'(1 + \varepsilon\xi^\tau + \varepsilon^{1-p+(\mu/q)}\xi^{\mu/q}).$$

Since $p \in (1, 2)$, we can take $\mu < q$ as close as q so that $1 - p + (\mu/q) > 0$. We take $\tau < 1$. Therefore, $\xi \leq C''$. We get the desired result.

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