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ON COMPATIBLE REGULARIZING DATA FOR SECOND ORDER HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS

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1. Introduciton.

It is well known that a necessary condition to solve an initial-boundary value problem in a proper domain of $\boldsymbol{R}^n$ is that the data of the problem satisfy, at the boundary of the domain, compatibility conditions of a certain order, which generally depends on the regularity assumed of the data, and required of the solution. In many situations, one is led to consider approximations of the solutions, obtained by solving problems with more regular data; thus, one needs to construct more regular data that not only approximate the given ones, but also satisfy compatibility conditions of higher order. A typecal example occurs when, in order to prove the existence of a solution to the original problem by means of energy methods, one first establishes the required energy estimates on more regular solutions (which it is possible to differentiate), and then resorts to a density argument. This is, for instance, the method followed by Ikawa, [8], and Shibata, [16], for linear hyperbolic equations of second order with Neumann type boundary conditions, and by Dan, [6], for a linear coupled hyperbolic-parabolic system, again with Neumann boundary conditions. A similar situation was considered by Rauch and Massey, [15], while proving the regularity of solutions to a linear first order hyperbolic system, under general boundary conditions.

More recently, Beirao DaVeiga ([1,2,3,4,5]) presents and develops a general method to prove the strong continuous dependence with respect to the data of solutions to nonlinear hyperbolic problems, including the nonlinear Neumann problems considered by Shibata-Kikuchi, [18], and Shibata-Nakamura, [19], as well as several systems of nonlinear fluid dynamics; in particular, the model nonlinear Neumann problem

$$(1.1)\qquad \begin{aligned} &u_{tt}-\operatorname{div}A(\nabla u)=f(x,t) && \text{in } \Omega\times]0,T[\\ &u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) && \text{in } \Omega\\ &\nu\cdot A(\nabla u)+b(u)=\phi(x,t) && \text{in } \partial\Omega\times]0,T[\end{aligned}$$

with $\Omega \subset \boldsymbol{R}^n$ bounded open domain with smooth boundary $\partial\Omega$ and (small) $T>0$ is considered in [3]. DaVeiga's method is partially based on a regularizing procedure for the linearized equations, and the required compatible regularizing data are constructed by adapting the procedure of Rauch and Massey; this essentially consists in approximating all data in a standard way, and then modifying one of the regularized initial values so as to satisfy the compatibility conditions (this is also the method followed by Ikawa and Shibata). As another application of his method, in [4] DaVeiga shows the strong convergence, under certain conditions, of solutions of the compressible Euler equations to that of the incompressible ones; in a similar way, we can consider, as in [11], the convergence of solutions to the dissipative quasilinear hyperbolic equation

$$\varepsilon u_{tt} + u_t - \sum_{i,j=1}^{n} a_{ij}(u, \nabla u)\partial_i \partial_j u = f(x,t) \tag{1.2}$$

to the solution of the corresponding limit parabolic equation

$$u_t - \sum_{i,j=1}^{n} a_{ij}(u, \nabla u)\partial_i \partial_j u = f(x,t) \tag{1.3}$$

as $\varepsilon \downarrow 0$: in the case $\Omega = \boldsymbol{R}^n$, weak convergence is proved in [11], and strong convergence in [12].

All these results on the nonlinear problem are obtained by the usual method of linearizing, and constructing compatible regularizing data for the linearized equations; thus, the problem of such construction under minimal regularity assumptions on the coefficients is of fundamental importance. In this respect, the mentioned results of Ikawa, Rauch and Massey, Shibata and Dan are not optimal, in that the regularity of the coefficients makes their results unsuitable for direct application to the corresponding nonlinear problems; in fact, DaVeiga explicitly shows in [1] and [5] how the procedure of Rauch and Massey can be suitably adapted, so as to be applicable to the nonlinear problem.

With these motivations in mind, in this note we present a simple, direct and selfcontained method to construct compatible regularizing data in the two model cases of a linear second order hyperbolic equation, with Dirichlet or Neumann boundary conditions, under the minimal regularity assumptions of the coefficients that are sufficient for applications to the corresponding nonlinear problems. More precisely, we consider the problems

$$\begin{aligned}
&u_{tt} - \sum_{i,j=1}^{n} a_{ij}(x,t)\partial_i \partial_j u = f(x,t) && \text{in } \Omega \times]0,T[\\
&u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) && \text{in } \Omega \\
&u(x,t) = 0 && \text{in } \partial\Omega \times]0,T[
\end{aligned} \tag{1.4}$$

and

$$
(1.5)\qquad
\begin{aligned}
&u_{tt}-\sum_{i,j=1}^{n} a_{ij}(x,t)\partial_i\partial_j u=f(x,t) && \text{in } \Omega\times]0,T[\\
&u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) && \text{in } \Omega\\
&\sum_{i,j=1}^{n} \nu_j a_{ij}(x,t)\partial_i u(x,t)=\phi(x,t) && \text{in } \partial\Omega\times]0,T[
\end{aligned}
$$

where ν is the outward unit normal to $\partial\Omega$; as an application, we shall show how our method can be used to give a direct, simplified proof of DaVeiga's result on the strong well-posedness of the quasilinear Neumann problem (1.1). The Dirichlet problem could be treated in a similar way (for an alternative, simpler method, see also [13]). Finally, we would like to remark that our method should be sufficiently general to also apply to parabolic equations; for instance, following a similar technique, in [10] we prove the strong well-posedness in the large for Sobolev solutions of (1.3) with Dirichlet boundary conditions. Although we have not checked the details, we believe it should also apply to first order systems, as considered by Rauch and Massey, [15], thereby recovering the improved results of DaVeiga, [1].

2. Notations and results.

Let $\Omega\subset \boldsymbol{R}^n$ be a bounded domain, with a smooth boundary $\partial\Omega$, whose outward unit normal we denote by ν, and $T>0$; we consider in $Q=\Omega\times]0,T[$ the linear hyperbolic initial-boundary value problems (1.4) and (1.5) for $u=u(x,t)\in\boldsymbol{R}^n$, with $u_t=\partial u/\partial t$ and $\partial_j u=\partial u/\partial x_j$. Following the theory developed by Kato in [9], we consider solutions with values in Sobolev spaces; thus, given $m\in\boldsymbol{N}$, we set $H^m=H^m(\Omega)$, $H^m_*=H^m\cap H^1_0$ if $m\geq 2$, and note $\|\cdot\|_m$ the norm in H^m, $\|\cdot\|$ the norm in $H^0=L^2$. Similarly, we set $\tilde{H}^r=H^r(\partial\Omega)$ and note $\ll\cdot\gg_r$ its norm. We also set

$$
X^m=\bigcap_{j=0}^{m} C^j([0,T];H^{m-j}),\qquad X^m_*=\bigcap_{j=0}^{m} C^j([0,T];H^{m-j}_*),
$$

$$
\tilde{X}_r=\bigcap_{j=0}^{[r]} C^j([0,T];\tilde{H}^{r-j}_*),
$$

and endow these spaces with their canonical norms, noted $|||\cdot|||$, $\lll\cdot\ggg_r$: that is, we set

$$
|||u|||^2_m \doteq \max_{0\leq t\leq T}\sum_{j=0}^{m}\|\partial_t^j u(t)\|^2_{m-j},
$$

$$
\lll u\ggg^2_r \doteq \max_{0\leq t\leq T}\sum_{j=0}^{[r]}\ll\partial_t^j u(t)\gg^2_{r-j}.
$$

We assume that the coefficents $a_{ij}\in X^s$, with integer $s\geq[\frac{n}{2}]+2$, so that in particular $a_{ij}\in C^1(\bar{Q})$; that they are symmetric and strongly elliptic, i.e. $a_{ij}=a_{ji}$ and

$$\exists\alpha>0\ \forall(x,t)\in\bar{Q},\forall\xi\in\boldsymbol{R}^n,\ \sum_{i,j=1}^{n}a_{ij}(x,t)\xi^i\xi^j\geq\alpha|\xi|^2. \tag{2.1}$$

In the sequel, we shall assume $\alpha=1$, and abbreviate

$$\sum_{i,j=1}^{n}a_{ij}(x,t)\partial_i\partial_j u\doteq a(x,t)\partial^2 u;$$

also, whenever it makes sense, we define

$$u^2(x)\doteq f(x,0)+a(x,0)\partial^2 u_0(x). \tag{2.2}$$

To describe the compatibility conditions on the data, necessary to solve the Dirichlet problem (1.4), we introduce the spaces

$$\mathscr{D}_1=\{(u_0,u_1,f)|u_0\in H^2_*,\ u_1\in H^1_0,\ f\in \mathrm{Lip}([0,T];L^2)\},$$
$$\mathscr{D}_2=\{(u_0,u_1,f)|u_0\in H^3_*,\ u_1\in H^2_*,\ u_2\in H^1_0,\ f\in X^1,\ f_t\in \mathrm{Lip}([0,T];L^2)\},$$

and we say that $\{u_0,u_1,f\}$ satisfy the compatibility conditions of order one [two] if $\{u_0,u_1,f\}\in\mathscr{D}_1$ [$\mathscr{D}_2$]; note that, indeed, $u_2\in H^1$ if $\{u_0,u_1,f\}\in\mathscr{D}_2$, while if $\{u_0,u_1,f\}\in\mathscr{D}_1$, then in general $u_2\in L^2$ only, so it does not necessarily have a trace on $\partial\Omega$. Analogously, for the Neumann problem (1.5), we say that the data $\{u_0,u_1,f,g\}$ satisfy the first or second order compatibility conditions if they belong respectively to the spaces

$$\begin{aligned}\mathscr{N}_1=\{(u_0,u_1,f,g)|u_0\in H^2,\ u_1\in H^1,\ f\in \mathrm{Lip}([0,T];L^2),&\\ g\in \mathrm{Lip}([0,T];\tilde{H}^{\frac{1}{2}}),\ a^{\nu}(x,0)\partial u_0(x)=g(x)\}&\\ \mathscr{N}_2=\{(u_0,u_1,f,g)|u_0\in H^3,\ u_1\in H^2,\ f\in X^1,\ f_t\in \mathrm{Lip}([0,T];L^2),&\\ g\in \tilde{X}_{\frac{3}{2}},\ g_t\in \mathrm{Lip}([0,T];\tilde{H}^{\frac{1}{2}}),&\\ a^{\nu}(x,0)\partial u_0(x)=g(x,0),&\\ a^{\nu}(x,0)\partial u_1(x)+a^{\nu}_t(x,0)\partial u_0(x)=g_t(x,0)\},&\end{aligned}$$

where we have abbreviated

$$\sum_{i,j=1}^{n}\nu_j a_{ij}(x,t)\partial_i u\doteq a^{\nu}(x,t)\partial u;$$

again, note that if $\{u_0,u_1,f,g\}\in\mathcal{N}_2$, then $u_2\in H^1$ and the two conditions at the boundary make sense in $\tilde{H}^{\frac{1}{2}}$, while if $\{u_0,u_1,f,g\}\in\mathcal{N}_1$, then in general $u_2\in L^2(\Omega)$ only, and only the first condition makes sense in $\tilde{H}^{\frac{1}{2}}$.

It is well known then that, under the stated assumptions on the coefficients, to solve the Dirichlet and Neumann problems (1.4) and (1.5) it is necessary and sufficient that the data satisfy the compatibility conditions; in fact, setting $D^2 \doteq \{\partial_t^j\partial_x^\alpha; j+|\alpha|\le 2\}$, we have the following theorem.

Theorem 1. *Let $i=1,2$. Given any $\{u_0,u_1,f\}\in\mathcal{D}_i$, there exists a unique $u\in X^{i+1}$, solution of* (1.4). *Moreover, there exists $M>0$, depending on T and $|||a|||_s$, such that $\forall t\in[0,T]$*

$$\|D^2u(t)\|^2\le M\{\|D^2u(0)\|^2+\|f(.,0)\|^2+\int_0^T\|f_t\|^2\}. \tag{2.3}$$

Proof. See e.g. Kato, [9] (II, 10.1), and Ikawa, [7] (Prop. 2.6).

Theorem 2. *Let $i=1,2$. Given any $\{u_0,u_1,f,g\}\in\mathcal{N}_i$, there exists a unique $u\in X^{i+1}$, solution of* (1.5). *Moreover, there exists $M>0$, depending on T and $|||a|||_s$, such that $\forall t\in[0,T]$*

$$\|D^2u(t)\|^2\le M\{\|D^2u(0)\|^2+\|f(.,0)\|^2+\int_0^T\|f_t\|^2+\ll g(.,0\gg_{\frac{1}{2}}^2\}. \tag{2.4}$$

Proof. See Shibata, [16](sct. 4). We recall that the second order estimate (2.4) is a consequence of the first order estimate

$$\|u(t)\|^2+\|Du(t)\|^2+\int_0^t\ll u_t\gg_{\frac{1}{2}}^2\le M\{\|u_0\|_1^2+\|u_1\|_0^2+\int_0^t\|f\|^2+\int_0^t\ll g\gg_{\frac{1}{2}}^2\}, \tag{2.5}$$

where $D\doteq\{\partial_i,\partial_t\}$; estimate (2.5) was first established for solutions of (1.5) by Miyatake in [14]; (2.4) follows then by regularization, differentiation in t, and ellipticity. □

We now address the following question for the Dirichlet problem: given $\{u_0,u_1,f\}\in\mathcal{D}_1$, construct a sequence $\{u_0^\lambda,u_1^\lambda,f^\lambda\}\in\mathcal{D}_2$ such that, as $\lambda\downarrow 0$,

$$\|u_i^\lambda-u_i\|_{2-i}\to 0,\qquad i=0,1,2 \tag{2.6}$$

$$\|f^\lambda(.,0)-f(.,0)\|\to 0,\qquad \int_0^T\|f_t^\lambda-f_t\|^2\to 0, \tag{2.7}$$

and, calling u^λ the solution to (1.4) corresponding to $\{u_0^\lambda, u_1^\lambda, f^\lambda\}$ (so that $u^\lambda \in X^3$),

$$|||u^\lambda - u||| \to 0. \tag{2.8}$$

Similarly, for the Neumann problem, given $\{u_0, u_1, f, g\} \in \mathcal{N}_1$, we want to construct a sequence of data $\{(u_0^\lambda, u_1^\lambda, f^\lambda, g^\lambda\} \in \mathcal{N}_2$ such that, as $\lambda \downarrow 0$, (2.6) and (2.7) hold, together with

$$\ll g^\lambda(.,0) - g(.,0) \gg_{\frac{1}{2}} \to 0, \qquad \int_0^T \ll g_t^\lambda - g_t \gg_{\frac{1}{2}}^2 \to 0, \tag{2.9}$$

and, with the analogous meaning of $u^\lambda \in X^3$, (2.8) as well.

Our goal is to show that the stated assumptions on the coefficients a_{ij} are sufficient to construct such compatible regularizing data; in fact, we claim:

Theorem 3. *Given any $\{u_0, u_1, f\} \in \mathcal{D}_1$, there exists a sequence $\{u_0^\lambda, u_1^\lambda, f^\lambda\} \in \mathcal{D}_2$, such that* (2.6), (2.7) *and* (2.8) *hold.*

Theorem 4. *Given any $\{u_0, u_1, f, g\} \in \mathcal{N}_1$, there exists a sequence $\{u_0^\lambda, u_1^\lambda, f^\lambda, g^\lambda\} \in \mathcal{N}_2$, such that* (2.6), (2.7), (2.8) *and* (2.9) *hold.*

As we mentioned, the regularity $a_{ij} \in X^s$ assumed of the coefficients is precisely the minimal one sufficient for applications to nonlinear problems; actually, since we only need that $a_{ij} \in C^1(\bar{Q})$ (while Ikawa, [8], and Shibata,[16], assumed $a_{ij} \in C^2(\bar{Q})$), it would be sufficient to assume that $a_{ij} \in C^0([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$. As an application, we shall indicate how to merge our method with DaVeiga's one to prove the strong well-posedness of the nonlinear Neumann problem (1.1) (with, for simplicity, $b \equiv 0$); thus, under the usual assumptions on A, which we shall recall in section 5, we claim (compare to [3], Theorem 1.1):

Theorem 5. *Let $f, g \in Z^s(T) \doteq X^{s-1} \cap C^s([0,T]; L^2)$; $\phi, \psi \in \tilde{Z}_s(T) \doteq \tilde{X}_{s-\frac{1}{2}} \cap C^s([0,T]; \tilde{H}^{\frac{1}{2}})$; $u_0, v_0 \in H^{s+1}$; $u_1, v_1 \in H^s$ be such that $\{u_0, u_1, f, \phi\}$ and $\{v_0, v_1, g, \psi\}$ satisfy the compatibility conditions of order s (to be recalled explicitly in section 5). Let $u, v \in X^{s+1}(T')$ $(T' \in]0,T])$ be the local solutions to* (1.1), *corresponding to such data. Then, given any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|||f-g|||_s^2 + \lll \phi - \psi \ggg_{s-\frac{1}{2}}^2 + \|u_0 - v_0\|_{s+1}^2 + \|u_1 - v_1\|_s^2 \le \delta^2$, then $|||u-v|||_{s+1} \le \varepsilon$.*

(We recall that local in time solvability in X^{s+1} for (1.1) under the stated assumptions on the data is established by Shibata- Kikuchi in [18]; see section 5).

3. Peoof of Theorem 3.

We start with the following technical result:

Lemma 1. *$H^\infty \cap H_0^1$ is dense in H_*^2.*

Proof. Given $u \in H_*^2$, *let* $\{v^\delta\}$ be a sequence from H^∞, such that

$$\|v^\delta - u\|_2 \to 0. \tag{3.1}$$

To correct v^δ at $\partial\Omega$ so that its trace is zero, we define w^δ to be the solution of the elliptic problem

$$\begin{aligned} &-\Delta w^\delta + \lambda w^\delta = 0 && \text{in } \Omega, \\ &w^\delta = v^\delta && \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Indeed, (3.2) has, for sufficiently large $\lambda > 0$ (which can be chosen independently of δ), a unique solution $w^\delta \in H^\infty$. Then $u^\delta \doteq v^\delta - w^\delta \in H^\infty \cap H_0^1$; and since $u^\delta - u$ solves the problem

$$\begin{aligned} &-\Delta(u^\delta - u) + \lambda(u^\delta - u) = -\Delta(v^\delta - u) + \lambda(v^\delta - u) && \text{in } \Omega, \\ &u^\delta - u = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.3}$$

it satisfies the elliptic estimate

$$\|u^\delta - u\|_2 \le c\{\|u^\delta - u\| + \|(\Delta - \lambda)(v^\delta - u)\|\} \le c\{\|u^\delta - u\| + \|v^\delta - u\|_2\}. \tag{3.4}$$

From (3.3), using (3.1), we see that $\|u^\delta - u\| \to 0$ as $\delta \downarrow 0$; thus, from (3.4), using again (3.1), we deduce that $\|u^\delta - u\|_2 \to 0$. □

Since $C_0^\infty(\Omega)$ is dense in both L^2 and H_0^1, by Lemma 1 we can choose sequences $\{u_0^\lambda\}$, $\{u_1^\lambda\}$ and $\{u_2^\lambda\}$ from $H^\infty \cap H_0^1$ such that (2.6) holds as $\lambda \downarrow 0$ (again, note that u_2 is defined, by (2.2), in L^2). Following Ikawa, [8] (sct. 3), by mollification in t we can also construct a sequence h^λ from $C^\infty([0,T];H^\infty)$ such that

$$\int_0^T \|h_t^\lambda - f_t\|^2 \to 0 \tag{3.5}$$

as $\lambda \downarrow 0$, holds. Define

$$l^\lambda(x) \doteq h^\lambda(x,0) - u_2^\lambda(x) + a(x,0)\partial^2 u_0^\lambda(x): \tag{3.6}$$

then, since $a(.,0) \in H^s$ and $s > \frac{n}{2} + 1$, $l^\lambda \in H^1$. Define further

(3.7) $$f^\lambda(x,t) \doteq h^\lambda(x,t) - l^\lambda(x) = h^\lambda(x,t) - h^\lambda(x,0) + u_2^\lambda(x) - a(x,0)\partial^2 u_0^\lambda(x):$$

then $f^\lambda \in C^\infty([0,T];H^1)$ and

(3.8) $$u_2^\lambda(x) = f^\lambda(x,0) + a(x,0)\partial^2 u_0^\lambda(x),$$

so $\{u_0^\lambda, u_1^\lambda, f^\lambda\} \in \mathscr{D}_2$. Since $f_t^\lambda \equiv h_t^\lambda$, (3.5) implies the second part of (2.7); moreover, from (3.8) and (2.2) we have

$$f^\lambda(x,0) - f(x,0) = u_2^\lambda(x) - u_2(x) - a(x,0)\partial^2(u_0^\lambda(x) - u_0(x)),$$

and therefore

$$\|f^\lambda(.,0) - f(.,0)\| \leq \|u_2^\lambda - u_2\| + \|a(.,0)\|_{s-1}\|\partial^2(u_0^\lambda - u_0)\|,$$

whence the first part of (2.7) follows, by means of (2.6). Finally, (2.8) is a consequence of estimate (2.3), applied to the difference $u^\lambda - u$, with f replaced by $f^\lambda - f$, and (2.6), (2.7). □

4. Proof of Theorem 4.

We now turn to the Neumann problem (1.5), and follow a similar procedure. At first, we choose sequences $\{u_0^\lambda\}$, $\{u_1^\lambda\}$ from $C^\infty([0,T];H^\infty)$ such that (2.6) and (2.7) hold, and a sequence $\{h^\lambda\}$ from $C^\infty([0,T];\tilde{H}^\infty)$ such that

(4.1) $$\ll h^\lambda(.,0) - g(.,0) \gg_{\frac{1}{2}} \to 0, \qquad \int_0^T \ll h_t^\lambda - g_t \gg_{\frac{1}{2}}^2 \to 0$$

as $\lambda \downarrow 0$, holds. Define, for $x \in \partial\Omega$,

(4.2) $$l_0^\lambda(x) \doteq a^\nu(x,0)\partial u_0^\lambda(x) - h^\lambda(x,0),$$

(4.3) $$l_1^\lambda(x) \doteq a^\nu(x,0)\partial u_1^\lambda(x) + a_t^\nu(x,0)\partial u_0^\lambda(x) + h_t^\lambda(x,0):$$

then, since $a(.,0) \in H^s$ and $a_t(.,0) \in H^{s-1}$, l_0^λ and l_1^λ are in $H^{\frac{3}{2}}$, at least if $n \geq 2$. (If $n = 1$, $\partial\Omega$ reduces to two points, say $\{x_1, x_2\}$; g is in fact a pair of functions $\{g(x_1,.), g(x_2,.)\}$, and we choose sequences $\{h_1^\lambda\}, \{h_2^\lambda\} \in C^\infty([0,T])$, approximating g in the sense that

$$|h_i^\lambda(0) - g(x_i,0)| \to 0, \qquad \int_0^T |h_i^\lambda(t) - g(x_i,t)|^2 dt \to 0$$

as $\lambda \downarrow 0$, for $i = 1,2$). Consider now a function $\rho \in C_0^\infty(\boldsymbol{R})$, such that $\rho(r) \equiv 1$ in a neighborhood of $r = 0$, and set

(4.4)
$$R_\lambda \doteq \frac{1}{\lambda}(1 + \ll l_1^\lambda \gg_{\frac{1}{2}}^2),$$

$$g^\lambda(x,t) \doteq h^\lambda(x,t) + l_0^\lambda(x) + t\rho(R_\lambda t) l_1^\lambda(x):$$

then, clearly, $g^\lambda \in C^\infty([0,T]; \tilde{H}^{\frac{3}{2}})$, and

(4.5)
$$g_t^\lambda(x,t) = h_t^\lambda(x,t) + [\rho(R_\lambda t) + (R_\lambda t)\rho'(R_\lambda t)] l_1^\lambda(x).$$

Thus, from (4.2) and (4.3) we have that, since $\rho(0)=1$,

(4.6)
$$g^\lambda(x,0) = h^\lambda(x,0) + l_0^\lambda(x) = a^\nu(x,0)\partial u_0^\lambda(x),$$

(4.7)
$$g_t^\lambda(x,0) = h_t^\lambda(x,0) + l_1^\lambda(x) = a^\nu(x,0)\partial u_1^\lambda(x) + a_t^\nu(x,0)\partial u_0^\lambda(x).$$

Since $\{u_0, u_1, f, g\} \in \mathcal{N}_1$, we have that $g(x,0) = a^\nu(x,0)\partial u_0(x)$, and therefore, by (4.6),

$$\ll g^\lambda(\cdot,0) - g(\cdot,0) \gg_{\frac{1}{2}} \le c \sum_{i,j=1}^n \|a_{ij}(\cdot,0)\partial_j(u_0^\lambda - u_0)\|_1 \le c \sum_{i,j=1}^n \|a_{ij}(\cdot,0)\|_{s-1} \|u_0^\lambda - u_0\|_2;$$

thus, the first part of (2.9) follows from (2.6). Next, from (4.5) we have that

$$g_t^\lambda(x,t) - g_t(x,t) = h_t^\lambda(x,t) - g_t(x,t) + \chi(R_\lambda t) l_1^\lambda(x),$$

where $\chi(t) = \rho(t) + t\rho'(t)$; thus,

(4.8)
$$\int_0^T \ll g_t^\lambda - g_t \gg_{\frac{1}{2}}^2 \le \int_0^T \ll h_t^\lambda - g_t \gg_{\frac{1}{2}}^2 + 2 \ll l_1^\lambda \gg_{\frac{1}{2}}^2 \int_0^T |\chi(R_\lambda t)|^2 dt.$$

Now, by our choice (4.4) of R_λ,

$$\int_0^T |\chi(R_\lambda t)|^2 dt \le \frac{1}{R_\lambda} \int_0^\infty |\chi(s)|^2 ds \le c\lambda(1 + \ll l_1^\lambda \gg_{\frac{1}{2}}^2)^{-1},$$

and therefore the second part of (2.9) follows from (4.8) and the first part of (4.1). Finally, (2.8) follows from (2.6),(2.7) and (2.9), by means of estimate (2.4), with u, f and g replaced by $u^\lambda - u$, $f^\lambda - f$ and $g^\lambda - g$. □

5. Proof of Theorem 5.

5.1. We assume that $A: \mathcal{U} \subseteq \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a smooth function with symmetric derivatives, that is, setting $a_{ij} \doteq \partial_j A_i$, $a_{ij} = a_{ji}$, and that (compare to (2.1))

(UE)
$$\exists \mu > 0 | \forall p \in \mathcal{U},\ \forall q \in \boldsymbol{R}^n, \quad \sum_{i,j=1}^n a_{ij}(p) q^i q^j \ge \mu |q|^2.$$

To describe the natural compatibility conditions on the data at $\partial\Omega$ for $t=0$, we formally compute by recursion from (1.1) the functions

$$u_k \doteq (\partial_u^k u)(0) \qquad \text{for } 0 \le k \le s+1$$

with their usual interpretation: i.e., for instance,

$$u_2(x) = f(x,0) + \operatorname{div} A(\nabla u_0(x)),$$
$$u_3(x) = f_t(x,0) + \operatorname{div} [A'(\nabla u_0(x)) \cdot \nabla u_1(x)],$$

etc.; note that our assumptions on u_0, u_1 and f guarantee that $u_k \in H^{s+1-k}$ for $0 \le k \le s+1$, and moreover, setting $\#\tilde{u}_0\#_{s+1}^2 \doteq \Sigma_{k=0}^{s+1} \|u_k\|_{s+1-k}^2$,

$$\#\tilde{u}_0\#_{s+1}^2 \le c\{\|u_0\|_{s+1}^2 + \|u_1\|_s^2\}, \tag{5.1}$$

for suitable $c>0$ depending on $\|u_0\|_{s+1}$ and $\|u_1\|_s$. Also, we can define on $\partial\Omega$ the functions, that we formally denote by

$$a_{ij}^k(x) \doteq \partial_t^k(a_{ij}(\nabla u(x,t)))|_{t=0},$$

in the usual recursive way, that is for instnce

$$a_{ij}^0(x = a_{ij}(\nabla u_0(x)),$$
$$a_{ij}^1(x) = a'_{ij}(\nabla u_0(x)) \cdot \nabla u_1(x),$$

etc.; again, we have that $a_{ij}^k \in \tilde{H}^{s-k-\frac{1}{2}}$.

Accordingly, we say that the data $\{u_0, u_1, f, \phi\}$ satisfy the compatibility conditions of order s if the following conditions hold on $\partial\Omega$:

$$\begin{aligned} &\nu \cdot A(\nabla u_0) = \phi(.,0), \\ &\sum_{r=1}^{k} \sum_{i,j=1}^{n} \nu_j \binom{k}{r} a_{ij}^r \partial_i u_{k-r} = \partial_t^k \phi(.,0) \qquad \text{for } 1 \le k \le s. \end{aligned} \tag{5.2}$$

5.2. We now sketch the main ideas of the proof of Theorem 5. At first, we recall from Shibata- Kikuchi, [18], that problem (1.1) has a unique local solution $u \in X^{s+1}(T')$, for some $T' \in]0,T]$, corresponding to $\{u_0, u_1, f, \phi\}$ satisfying the compatibility conditions of order s. Similarly, there is a unique $v \in X^{s+1}(T'')$, $T'' \in]0,T]$, corresponding to $\{v_0, v_1, g, \psi\}$ satisfying the same compatibility conditions; in the sequel, we rename T by $\min(T, T', T'')$.

We can differentiate (1.1) r times in t, for $1 \le r \le s-1$, and find that $\partial_t^r u \in X^{s+1-r}$

is solution of the problem

$$
\begin{aligned}
&(\partial_t^r u)_{tt} - \alpha(x,t)\partial^2(\partial_t^r u) = F^r(x,t) && \text{in } \Omega\times]0,T[\\
(5.3)\qquad &(\partial_t^r u)(x,0) = u_r(x), \quad (\partial_t^r u)_t(x,0) = u_{r+1}(x) && \text{in } \Omega \\
&\alpha^\nu(x,t)\partial(\partial_t^r u) = \Phi^r(x,t) && \text{in } \partial\Omega\times]0,T[
\end{aligned}
$$

with $\alpha = \{\alpha_{ij}\} \doteq \{a_{ij}(\nabla u)\}$, $F^r = \partial_t^r f + F_1^r$, $\Phi^r = \partial_t^r\phi - \Phi_1^r$, $\Phi_1^1 = 0$, and

$$
F_1^r \doteq \sum_{k=1}^r \binom{r}{k} \sum_{i,j=1}^n (\partial_t^k \alpha_{ij})(\partial_t^{r-k}\partial_i\partial_j u) \qquad \text{for } r \geq 1,
$$

$$
\Phi_1^r \doteq \sum_{k=1}^r \binom{r-1}{k} \sum_{i,j=1}^n \nu_j \partial_t^k(\alpha_{ij})(\partial_t^{r-k}\partial_j u) \qquad \text{for } r \geq 2.
$$

We have a similar set of equations satisfied, for $0 \leq r \leq s-1$, by $\partial_t^r v \in X^{s+1-r}$, with coefficients $\beta = \{\beta_{ij}\} \doteq \{a_{ij}(\nabla v)\}$, and G^r, Ψ^r defined analogously to F^r, Φ^r; since $u, v \in X^{s+1}$, we have that $\alpha, \beta \in X^s$ and, as it is not difficult to verify, $F^r, G^r \in C^1([0,T]; L^2)$ and $\Phi^r, \Psi^r \in C^1([0,T]; \tilde{H}^{\frac{1}{2}})$ at least.

We now proceed first to establish the Lipschits type estimates for lower order norms in H^m, $1 \leq m \leq s$,

$$
(5.4)\qquad |||u-v|||_m^2 \leq \gamma(|||f-g|||_s^2 + \langle\langle\langle \phi-\psi \rangle\rangle\rangle_{s-\frac{1}{2}}^2 + \|u_0 - v_0\|_{s+1}^2 + \|u_1 - v_1\|_s^2),
$$

with γ depending on T, $|||u|||_{s+1}$ and $|||v|||_{s+1}$. The difference $\partial_t^r u - \partial_t^r v$ satisfies the equations

$$
\begin{aligned}
&(\partial_t^r u - \partial_t^r v)_{tt} - \alpha\partial^2(\partial_t^r u - \partial_t^r v) = F^r - G^r + (\alpha-\beta)\partial^2\partial_t^r v, \\
(5.5.a)\qquad &(\partial_t^r u - \partial_t^r v)(0) = u_r - v_r, \quad (\partial_t^r u - \partial_t^r v)_t(0) = u_{r+1} - v_{r+1}, \\
&\alpha^\nu\partial(\partial_t^r u - \partial_t^r v) = \Phi^r - \Psi^r + (\alpha^\nu - \beta^\nu)\partial^2\partial_t^r v,
\end{aligned}
$$

for $r \geq 1$ and

$$
\begin{aligned}
&(u-v)_{tt} - \sum_{i,j=1}^n \partial_i(\alpha'_{ij}\partial_j(u-v)) = f - g, \\
(5.5.b)\qquad &(u-v)(0) = u_0 - v_0, \quad (u-v)_t(0) = u_1 - v_1, \\
&\sum_{i,j=1}^n \nu_i \alpha'_{ij}\partial_j(u-v) = \phi - \psi,
\end{aligned}
$$

for $r=0$ where $\alpha'_{ij} = \int_0^1 a_{ij}(\nabla(v + \theta(u-v)))d\theta$, and therefore, by Shibata, [17] (Thm.

3.8), it satisfies, for $0 \le r \le s-2$, the elliptic estimate

$$
\begin{aligned}
\|\partial_t^r u-\partial_t^r v\|_{s-r} \le & \|\partial_t^r f-\partial_t^r g\|_{s-r-2}+d_r\|F_1^r-G_1^r\|_{s-r-2} \\
& +d_r\|(\alpha-\beta)\partial^2(\partial_t^r v)\|_{s-r-2}+\ll \partial_t^r\phi-\partial_t^r\psi \gg_{s-r-3/2} \\
& +d_r\ll \Phi_1^r-\Psi_1^r \gg_{s-r-3/2}+d_r\ll (\alpha^\nu-\beta^\nu)\partial(\partial_t^r v) \gg_{s-r-3/2} \\
& +\|\partial_t^{r+2}u-\partial_t^{r+2}v\|_{s-r-2} \equiv A_1+\cdots+A_7,
\end{aligned}
\tag{5.6}
$$

with M depending on $|||u|||_{s+1}$ via the coefficients $\{\alpha_{ij}\}$ and $\{\alpha'_{ij}\}$, where $d_r=1$ for $r\ge 1$ and $=0$ for $r=0$. Following again Shibata, [17] (sct. 4), by means of Sobolev's product estimates and classical calculus inequalities we can estimate A_2 and A_5 in terms of $\Sigma_{k=0}^s\|\partial_t^k u-\partial_t^k v\|_{s-k-1}$, with quantities that depend at most on $|||u|||_{s+1}$ and $|||v|||_{s+1}$; the same is true for A_3 and A_6, keeping in mind that $\partial_t^k v\in C([0,T];H^{s+1-k})$. Thus, summing all inequalities (5.6) for $0\le r\le s-1$, and using the interpolation inequality

$$
\|\partial_t^k u-\partial_t^k v\|_{s-k-1}\le c\|\partial_t^k u-\partial_t^k v\|_{s-k}^{1-a}\|\partial_t^k u-\partial_t^k v\|_0^a,
$$

with $a=\frac{1}{s-k}$, and adding the two extra terms for $r=s-1$ and $r=s$, we deduce the estimate

$$
\begin{aligned}
\sum_{k=0}^{s}\|\partial_t^k u-\partial_t^k v\|_{s-k}^2 \le M\{ & \sum_{k=0}^{s-2}\|\partial_t^k f-\partial_t^k g\|_{s-k-2}^2 \\
& +\sum_{k=0}^{s-2}\ll \partial_t^k\phi-\partial_t^k\psi \gg_{s-r-\frac{3}{2}}^2+\sum_{k=0}^{s}\|\partial_t^k u-\partial_t^k v\|^2+\|\partial_t^{s-1}u-\partial_t^{s-1}v\|_1^2\}.
\end{aligned}
\tag{5.7}
$$

The last two terms of (5.7) can be estimated by means of the first order energy estimate (2.5) applied to (5.5) for $0\le r\le s-1$; note, however, that when $r=s-1$ we can estimate $\|F_1^{s-1}-G_1^{s-1}\|$, etc., only in terms of $\Sigma_{k=1}^s\|\partial_t^k u-\partial_t^k v\|_{s-k}$. Still, we obtain from (5.7) that

$$
\begin{aligned}
\sum_{k=0}^{s}\|\partial_t^k u-\partial_t^k v\|_{s-k}^2 \le M\Bigg\{ & \sum_{k=0}^{s-2}\|\partial_t^k f-\partial_t^k g\|_{s-k-2}^2+\sum_{k=0}^{s-2}\ll \partial_t^k\phi-\partial_t^k\psi \gg_{s-k-\frac{3}{2}}^2 \\
& +\sum_{k=0}^{s}\|u_k-v_k\|_1^2+\sum_{k=0}^{s}\|u_{k+1}-v_{k+1}\|^2 \\
& +\sum_{k=0}^{s-1}\int_0^T\|\partial_t^k f-\partial_t^k g\|_{s-k-1}^2+\sum_{k=0}^{s-1}\int_0^T\ll \partial_t^k\phi-\partial_t^k\psi \gg_{s-k-\frac{3}{2}}^2 \\
& +\sum_{k=0}^{s}\int_0^t\|\partial_t^k u-\partial_t^k v\|_{s-k}^2\Bigg\},
\end{aligned}
$$

from which, recalling (5.1),(5.4) follows by Gronwall's inequality.

5.3. We now consider the highest order norm $|||u-v|||_{s+1}$. Exactly in the same way we established (5.7), we obtain from (5.5) for $0 \le r \le s-1$ the estimate

$$\sum_{k=0}^{s+1} \|\partial_t^k u-\partial_t^k v\|_{s+1-k}^2 \le M\Bigg\{\sum_{k=0}^{s-1} \|\partial_t^k f-\partial_t^k g\|_{s-k-1}^2 + \sum_{k=0}^{s-1} \ll \partial_t^k \phi-\partial_t^k \psi \gg_{s-r-\frac{1}{2}}^2 + \sum_{k=0}^{s} \|\partial_t^k u-\partial_t^k v\|_{s-k}^2 + \|D^2\partial_t^{s-1}u-D^2\partial_t^{s-1}v\|^2\Bigg\}; \tag{5.8}$$

thus, because of the lower order estimates (5.4), to estimate $|||u-v|||_{s+1}$ it is sufficient to estimate $\|D^2 w-D^2 z\|$, where $w \doteq \partial_t^{s-1}u$ and $z \doteq \partial_t^{s-1}v \in X^2$. This we cannot do directly, because of the loss of regularity (that is, we cannot consider (5.5) for $r=s$); rather, we remark that from the boundary conditions in (5.3) for $r=s-1$ we deduce that, since $u_{s-1} \in H^2$ and $u_s \in H^1$, setting $F \doteq F^{s-1}$, $\Phi \doteq \Phi^{s-1}$, we have that $\{u_{s-1}, u_s, F, \Phi\} \in \mathcal{N}_1$. Thus, we can apply Theorem 4 to (5.3), and construct a sequence $\{u_{s-1}^\lambda, u_s^\lambda, F^\lambda, \Phi^\lambda\}$ from approximating $\{u_{s-1}, u_s, F, \Phi\}$ in $\mathcal{N}_1$, and such that (from (2.8))

$$|||w-w^\lambda|||_2 \to 0 \tag{5.9}$$

as $\lambda \downarrow 0$. Applying (2.4) to the difference $z-w^\lambda$ we obtain

$$\begin{aligned}\|D^2 z(t)-D^2 w^\lambda(t)\|^2 \le M\{&\|D^2 z(0)-D^2 w^\lambda(0)\|^2 \\ &+\|G(0)-F^\lambda(0)\|^2+\|\tilde{B}^\lambda(0)\|^2 \\ &+\ll \Psi(0)-\Phi^\lambda(0)\gg_{\frac{1}{2}}^2+\ll B^\lambda(0)\gg_{\frac{1}{2}}^2 \\ &+\int_0^T \|G_t-F_t^\lambda\|^2+\int_0^T \|B_t^\lambda\|^2 \\ &+\int_0^T \ll \Psi_t-\Phi_t^\lambda\gg_{\frac{1}{2}}^2+\int_0^T \ll \tilde{B}_t^\lambda\gg_{\frac{1}{2}}^2 \\ &\equiv C_1+\cdots+C_9,\end{aligned} \tag{5.10}$$

where $B^\lambda \doteq (\beta-\alpha)\partial^2 w^\lambda$ and $\tilde{B}^\lambda \doteq (\alpha^\nu-\beta^\nu)\partial w^\lambda$. We split

$$C_1 \le 2M\{\|D^2 z(0)-D^2 w(0)\|^2+\|D^2 w(0)-D^2 w^\lambda(0)\|^2\}, \tag{5.11}$$

and similarly for the terms C_2, C_4, C_6, C_8; we note that

$$\|D^2z(0)-D^2w(0)\|^2=\|v_{s-1}-u_{s-1}\|_2^2+\|v_s-u_s\|_1^2 \tag{5.12}$$
$$+\|v_{s+1}-u_{s+1}\|^2\leq \#\tilde{u}_0-\tilde{v}_0\#_{s+1}^2.$$

Next, we decompose $F_t-G_t=\partial_t^{s-1}g-\partial_t^{s-1}f+\partial_tG^{s-1}-\partial_tF^{s-1}$, and analogously for $\Phi_t-\Psi_t$; carrying out the differentiation and adding and subtracting similar terms, proceeding as in the estimate of A_2 and A_5 it is not difficult to obtain the estimate

$$\|\partial_tG^{s-1}-\partial_tF^{s-1}\|^2+\ll\partial_t\Psi_1-\partial_t\Phi_1\gg_{\frac{1}{2}}^2\leq M\sum_{k=0}^{s-1}\|\partial_t^ku-\partial_t^kv\|_{s+1-k}^2. \tag{5.13}$$

Exactly in the same way, we also obtain

$$\|G(0)-F(0)\|^2+\ll\Psi(0)-\Phi(0)\gg_{\frac{1}{2}}^2 \tag{5.14}$$
$$\leq\|\partial_t^{s-1}g(0)-\partial_t^{s-1}f(0)\|^2+\ll\partial_t^{s-1}\psi(0)-\partial_t^{s-1}\phi(0)\gg_{\frac{1}{2}}^2$$
$$+M\sum_{k=0}^{s}\|\partial_t^ku(0)-\partial_t^kv(0)\|_{s-k}^2;$$

finally, recalling that $\|\partial^2w^\lambda(.,t)\|$ is uniformly bounded with respect to λ (and t), we have

$$\|B_t^\lambda\|^2+\ll\tilde{B}_t^\lambda\gg_{\frac{1}{2}}^2\leq M(1+\|\partial^2w_t^\lambda\|)\sum_{k=0}^{s}\|\partial_t^ku-\partial_t^kv\|_{s-k}^2, \tag{5.15}$$

$$\|B^\lambda(0)\|^2+\ll\tilde{B}^\lambda(0)\gg_{\frac{1}{2}}^2\leq M\#\tilde{u}_0-\tilde{v}_0\#_{s+1}^2. \tag{5.16}$$

Replacing (5.11), $\cdots$ (5.16) into (5.10) we obtain

$$\|D^2z(t)-D^2w(t)\|^2\leq 2\|D^2w(t)-D^2w^\lambda(t)\|^2+2\|D^2z(t)-D^2w^\lambda(t)\|^2$$
$$\leq M\Big\{\|D^2w(t)-D^2w^\lambda(t)\|^2+\#\tilde{u}_0-\tilde{v}_0\#_{s+1}^2$$
$$+\|D^2w(0)-D^2w^\lambda(0)\|^2+|||f-g|||_s^2+⟪\phi-\psi⟫_{\frac{1}{2}}^2 \tag{5.17}$$
$$+\|F(0)-F^\lambda(0)\|^2+\ll\Phi(0)-\Phi^\lambda(0)\gg_{\frac{1}{2}}^2$$
$$+\int_0^T\|F_t-F_t^\lambda\|^2+\int_0^T\ll\Phi_t-\Phi_t^\lambda\gg_{\frac{1}{2}}^2$$
$$+\int_0^t(1+\|\partial^2w_t^\lambda\|^2)\sum_{k=0}^{s}\|\partial_t^ku-\partial_t^kv\|_{s-k}^2\Big\}$$
$$\equiv D_1+\cdots+D_{10}.$$

Recalling (5.9), we have that

$$D_1+D_3+D_6+D_7+D_8+D_9\le k(\lambda),$$

with $k(\lambda)\downarrow 0$ as $\lambda\downarrow 0$; thus, from (5.17) we obtain

$$\|D^2z(t)-D^2w(t)\|^2\le k(\lambda)+M\{\#\tilde{u}_0-\tilde{v}_0\#^2_{s+1}+|||f-g|||^2_s + \langle\!\langle\!\langle \phi-\psi\rangle\!\rangle\!\rangle^2_{s-\frac{1}{2}}+\int_0^t(1+\|\partial^2w^\lambda_t\|^2)\sum_{k=0}^{s}\|\partial^k_tu-\partial^k_tv\|^2_{s-k}\}.$$

Adding this to (5.8), and recalling the lower order estimates (5.4), we obtain that

$$|||u-v|||^2_{s+1}\le k(\lambda)+M\{\#\tilde{u}_0-\tilde{v}_0\#^2_{s+1}+|||f-g|||^2_s + \langle\!\langle\!\langle \phi-\psi\rangle\!\rangle\!\rangle^2_{s-\frac{1}{2}}+(1+r(\lambda))T|||u-v|||^2_s\}, \tag{5.18}$$

where the function $r(\lambda)=\max_{0\le t\le T}\|\partial^2w^\lambda_t(t)\|^2$ is not necessarily bounded as $\lambda\downarrow 0$.

By (5.4) with $m=s$, we deduce from (5.18) that

$$|||u-v|||^2_{s+1}\le k(\lambda)+M(1+(1+r(\lambda))T)\{|||f-g|||^2_s + \langle\!\langle\!\langle \phi-\psi\rangle\!\rangle\!\rangle^2_{s-\frac{1}{2}}+\|u_0-v_0\|^2_{s+1}+\|u_1-v_1\|^2_s\}; \tag{5.19}$$

therefore, to conclude the proof of Theorem 5, given $\varepsilon>0$ it is sufficient to first fix $\lambda=\bar{\lambda}$ so that $k(\bar{\lambda})\le\frac{1}{2}\varepsilon^2$, and then to choose $\delta>0$ so that $M\delta^2(1+(1+r(\bar{\lambda}))T)\le\frac{1}{2}\varepsilon^2$: from (5.19) with $\lambda=\bar{\lambda}$ we deduce that $|||u-v|||_{s+1}\le\varepsilon$. □ □

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