Ichinose, W.
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A NOTE ON THE EXISTENCE AND $\hbar$-DEPENDENCY OF THE SOLUTION OF EQUATIONS IN QUANTUM MECHANICS

Dedicated to Professor Hiroki Tanabe for his 60th birthday

WATARU ICHINOSE

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0. Introduction

In the present paper we study the Schrödinger equation, the Dirac equation, the equation describing the motion of relativistic spinless particles, and somewhat more general equation:

$$i\hbar\frac{\partial u}{\partial t}(t)=K(t)u(t) \quad \text{on } [0,T] \ (x\in R^n), \quad u(0)=u^{(0)}. \tag{0.1}$$

Here $0<\hbar\leq 1$ is the Planck constant, $u(t)={}^t(u_1(t),\cdots,u_N(t))$, and $K(t)=(K_{jl}(t,\frac{X+X}{2},\hbar D_X);{}^{j\downarrow}_{l\rightarrow}1,2,\cdots,N)$ where $K_{jl}(t,\frac{X+X}{2},\hbar D_X)$ is a pseudo-differential operator with the Weyl symbol $k_{jl}(t,x,\xi)$ defined by

$$\iint e^{i(x-x')\cdot\xi}k_{jl}(t,\frac{x+x'}{2},\hbar\xi)f(x')dx'đ\xi \qquad (x\cdot\xi=\textstyle\sum_{j=1}^n x_j\xi_j,\ i=\sqrt{-1},\ đ\xi=(2\pi)^{-n}d\xi).$$

Throughout the present paper we assume that $k_{jl}(t,x,\xi);{}^{j\downarrow}_{l\rightarrow}1,2,\cdots,N)$ is a Hermitian matrix and that $k_{jl}(t,x,\xi)$ is a continuous function on $[0,T]\times R^{2n}$ and C^∞-differentiable with respect to (x,ξ).

We use the following function spaces : $L^2=L^2(R^n)$ is the space of all square integrable functions with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$; Let a, b, and s be non-negative constants. We define the weighted Sobolev space $B^s_{a,b}(\hbar)$ by $B^s_{a,b}(\hbar)=\{f\in L^2;\ \|f\|_{B^s_{a,b}(\hbar)}\equiv\|\langle\cdot\rangle^{as}f\|+\|\langle\hbar\cdot\rangle^{bs}\hat{f}\|<\infty\}$ and denote its dual space by $B^{-s}_{a,b}(\hbar)$ with norm $\|f\|_{B^{-s}_{a,b}(\hbar)}$, where $<x>=(1+|x|^2)^{1/2}$ and $\hat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)dx$.

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Let F be a Banach space with norm $\|f\|_F$. We denote by F^N the direct product space of N copies of F with norm $\|(f_1,\cdots,f_N)\|_{F^N}\equiv(\Sigma_{j=1}^N\|f_j\|_F^2)^{1/2}$ and by $\mathscr{E}_t^j([0,T];F)$ $(j=0,1,\cdots)$ the space of all F-valued j times continuously differentiable functions in $[0,T]$.

At first consider the Schrödinger equation. Let $N=1$ and set $k(t,x,\xi)=k_{11}(t,x,\xi)$ in (0.1). Assume the following:

(0.2) For any multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$ and $\beta=(\beta_1,\cdots,\beta_n)$ with $|\alpha+\beta|\equiv\sum_{j=1}^n(\alpha_j+\beta_j)\geq 2$ there exists a constant $C_{\alpha,\beta}$ such that $|k_{(\beta)}^{(\alpha)}(t,x,\xi)|\leq C_{\alpha,\beta}$ on $[0,T]\times R^{2n}$,

where $k_{(\beta)}^{(\alpha)}(t,x,\xi)=\partial_\xi^\alpha(\frac{1}{i}\partial_x)^\beta k(t,x,\xi)\equiv\left(\frac{\partial}{\partial\xi_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial\xi_n}\right)^{\alpha_n}\left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\beta_1}\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\beta_n}k(t,x,\xi)$.

Then we get the existence and uniqueness of the unitary solution on L^2 of (0.1) and the regularity property on this solution stated just below from results in [6], [12], and [18]. For any $u^{(0)}\in B_{1,1}^s(\hbar)$ $(s\geq 0)$ the equation (0.1) admits a unique solution $u_\hbar(t)\in\mathscr{E}_t^0([0,T];B_{1,1}^s(\hbar))$ and this $u_\hbar(t)$ satisfies

$$(0.3)\qquad\begin{cases}\|u_\hbar(t)\|=\|u^{(0)}\| & \text{on } [0,T],\\ \|u_\hbar(t)\|_{B_{1,1}^s(\hbar)}\leq C_s(T)\|u^{(0)}\|_{B_{1,1}^s(\hbar)} & \text{on } [0,T],\end{cases}$$

where $C_s(T)$ is independent of $0<\hbar\leq 1$. The first equality in (0.3) implies that the solution has the unitarity on L^2 and the second one guarantees for many observables G in quantum mechanics and initial data $u^{(0)}$ that the expectation value $(Gu_\hbar(t),u_\hbar(t))$ exists for all $t\in[0,T]$. Then $\hbar$-independence of $C_s(T)$ is also important for the study on the classical limit $\lim_{\hbar\to 0}(Gu_\hbar(t),u_\hbar(t))$. For example see [21] for it. But the assumption (0.2) is too restrictive. In fact this result is applicable to the Schrödinger equation with an external electro-magnetic field

$$(0.4)\qquad i\hbar\frac{\partial u}{\partial t}(t)=\sum_{j=1}^n\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-a_j(t,x)\right)\circ\left(\frac{\hbar}{i}\frac{\partial}{\partial x_j}-a_j(t,x)\right)u+V(t,x)u$$

only if all real valued $a_j(t,x)$ are polynomials of degree one in x, where "$\cdot\circ\cdot$" denotes the product of operators. We note that the right-hand side in (0.4) can be written as $K(t,\frac{X+X}{2},\hbar D_x)u$ with the Weyl symbol $k(t,x,\xi)=\sum_{j=1}^n(\xi_j-a_j(t,x))^2+V(t,x)$.

One of our purposes in the present paper is to show that we can obtain the same result as in the above under a more general assumption (A) than (0.2):

(A) For any α and β with $|\alpha+\beta|\geq 1$ there exists a constant $C_{\alpha,\beta}$ such that $|k_{(\beta)}^{(\alpha)}(t,x,\xi)|\leq C_{\alpha,\beta}(1+|x|+|\xi|)$ on $[0,T]\times R^{2n}$.

Then our result is applicable to (0.4) if $\partial_x^\beta a_j(t,x)$ $(j=1,2,\cdots,n)$ are bounded on $[0,T]\times R^n$ for all $|\beta|\neq 0$ and we have $|\partial_x^\beta V(t,x)|\leq C_\beta\langle x\rangle$ for all $|\beta|\neq 0$. So, for example, we can apply it to (0.4) with periodic vector potential $(a_1(t,x),\cdots,a_n(t,x))$ in x.

Next we consider the Dirac equation and the equation describing the motion of relativistic spinless particles. Assume the following in general :

(B) We have $|k_{jl(\beta)}^{(\alpha)}(t,x,\xi)|\leq C_{\alpha,\beta}\langle x\rangle$ on $[0,T]\times R^{2n}$ $(j,l=1,2,\cdots,N)$ for all $|\alpha|\neq 0$ and β. In addition, there exists a constant $M\geq 1$ such that $|k_{jl(\beta)}(t,x,\xi)|\leq C_\beta(\langle x\rangle^M+\langle\xi\rangle)$ for all β.

Then we can also get the existence and uniqueness of the unitary solution on $(L^2)^N$ of (0.1) and the regularity property on it. To show this is our second purpose.

The method of proving our results is much different from and much easier than that in [6], [12], and [18], where the theory of Fourier or oscillatory integral operators was used. In the present paper we will use the theory of pseudo-differential operators with basic weight function which will be studied in Section 2. Then our proof becomes analogous to that of the similar result on hyperbolic equations in [13] and [20]. We will state our results in Section 1. Their proof will be given in Section 3.

1. Results

We state the main theorem.

Theorem. *(i) Suppose that all* $k_{jl}(t,x,\xi)$ *satisfy the assumption (A). Then for any* $u^{(0)}\in B_{1,1}^s(\hbar)^N$ $(-\infty<s<\infty)$ *there exists a unique solution* $u_\hbar(t)\in\mathscr{E}_t^0([0,T];B_{1,1}^s(\hbar)^N)\cap\mathscr{E}_t^1([0,T];B_{1,1}^{s-2}(\hbar)^N)$ *of* (0.1). *In addition, there exits a constant* $C_s(T)$ *independent of* $0<\hbar\leq 1$ *such that*

$$\|u_\hbar(t)\|_{B_{1,1}^s(\hbar)^N}\leq C_s(T)\|u^{(0)}\|_{B_{1,1}^s(\hbar)^N}\quad\text{on }[0,T]. \tag{1.1}$$

In particular, when $s=0$, *we have*

$$\|u_\hbar(t)\|_{(L^2)^N}=\|u^{(0)}\|_{(L^2)^N}\quad\text{on }[0,T]. \tag{1.2}$$

(ii) Suppose the assumption (B). Then for any $u^{(0)}\in B_{M,1}^s(\hbar)^N$ $(-\infty<s<\infty)$ *there exists a unique solution* $u_\hbar(t)\in\mathscr{E}_t^0([0,T];B_{M,1}^s(\hbar)^N)\cap\mathscr{E}_t^1([0,T];B_{M,1}^{s-1}(\hbar)^N)$ *of* (0.1). *In addition,* (1.1) *where* $B_{1,1}^s(\hbar)^N$ *is replaced by* $B_{M,1}^s(\hbar)^N$ *holds and* (1.2) *also does when* $s=0$.

EXAMPLE 1.1. Consider the Dirac equation

$$i\hbar\frac{\partial u}{\partial t}(t)=\sum_{j=1}^n\alpha_j\Big(\frac{\hbar}{i}\frac{\partial}{\partial x_j}\Big)u+\gamma u+V(t,x)u,$$

where $u={}^t(u_1,\cdots,u_N)$ and α_j, γ, and $V(t,x)$ are N by N Hermitian matrices. α_j and γ are constant ones. Suppose that all elements $V_{jl}(t,x)$ of $V(t,x)$ are continuous on $[0,T]\times R^n$ and C^∞-differentiable with respect to x and that there exists a constant $M\geq 1$ satisfying

$$|\partial_x^\beta V_{jl}(t,x)|\leq C_\beta\langle x\rangle^M \tag{1.3}$$

for all β. Then we can apply (ii) of Theorem to this Dirac equation.

EXAMPLE 1.2. Let

$$k(t,x,\xi)=\{1+\textstyle\sum_{j=1}^n(\xi_j-a_j(t,x))^2\}^{1/2}+V(t,x),$$

where $a_j(t,x)$ and $V(t,x)$ are real valued continuous functions on $[0,T]\times R^n$ and C^∞-differentiable with respect to x. Then the equation (0.1) where $N=1$ and $k_{11}(t,x,\xi)=k(t,x,\xi)$ describes the motion of relativistic spinless particles in an external electro-magnetic field (e.g. [4], [7], and [9]). Suppose that $|\partial_x^\beta a_j(t,x)|$ are bounded by $C_\beta\log\langle x\rangle$ on $[0,T]\times R^n$ for all $|\beta|\neq 0$ and that $V(t,x)$ satisfies (1.3) for all β. Then we can apply (ii) of Theorem.

Let $\mathcal{S}$ be the space of rapidly decreasing functions and $\mathcal{S}'$ its dual space. Suppose that $(k_{jl}(x,\xi);{}^{j}_{l}{=}1,2,\cdots,N)$ is independent of t and satisfies the assumption in (i) or (ii) of Theorem. Then $K(\frac{X+X}{2},\hbar D_x)$ is a continuous operator on $\mathcal{S}^N$ and can be extended uniquely to a continuous one on $(\mathcal{S}')^N$, defining $K(\frac{X+X}{2},\hbar D_x)f\in(\mathcal{S}')^N$ for $f\in(\mathcal{S}')^N$ by

$$(K(\frac{X+X}{2},\hbar D_x)f,g)=(f,K(\frac{X+X}{2},\hbar D_x)g)\qquad(g\in\mathcal{S}^N)$$

(e.g. Chapter XVIII in [8]). Then we get the following :

Corollary. *Under the assumption above we denote* $K(\frac{X+X}{2},\hbar D_x)$ with domain $\mathcal{S}^N$ and that with domain $\{f\in(L^2)^N;K(\frac{X+X}{2},\hbar D_x)f\in(L^2)^N\}$ *by* K_0 *and* K, *respectively. Then* K_0 *is essentially self-adjoint on* $(L^2)^N$ *and its self-adjoint extension is* K.

Proof. It follows from the definition of $K(\frac{X+X}{2},\hbar D_X)f\in(\mathcal{S}')^N$ for $f\in(\mathcal{S}')^N$ that we can easily have $K_0^*=K$. So if we can prove that K_0 is essentially self-adjoint, the proof of Corollary can be completed. For we have $K=K_0^*=(\overline{K_0})^*=\overline{K_0}$ from Theorem VIII.1 in [16]. $\overline{K_0}$ denotes the closure of K_0.

We follow the proof of the Stone theorem in [16]. Suppose that there is an $f\in D(K_0^*)$ so that $K_0^*f=-if$. Let g be an arbitrary function in $\mathscr{S}^N$ and $u_\hbar(t)$ the solution of (0.1) with $u^{(0)}=g$. We write $u_\hbar(t)$ as $U(t)g$. Then $U(t)g\in\mathscr{E}_t^1([0,T]);\mathscr{S}^N)$ follows from Thorem, because we can easily prove that $\bigcap_{-\infty<s<\infty} B_{a,b}^s(\hbar)=\mathscr{S}$ for $a>0$ and $b>0$. Hence we have $\frac{d}{dt}(U(t)g,f)=(-\frac{i}{\hbar}K_0U(t)g,f)=-\frac{i}{\hbar}(U(t)g,K_0^*f)=\frac{1}{\hbar}(U(t)g,f)$ and so $(U(t)g,f)=e^{t/\hbar}(g,f)$. The equality (1.2) implies that $U(t)$ is unitary on $(L^2)^N$. So both sides must be bouded, which implies $(g,f)=0$. Hence we have $f=0$. A similar proof shows that $K_0^*f=if$ can have no non-zero solutions. Thus K_0 is essentially self-adjoint. Q.E.D.

REMARK 1.1. In [22] the equation (0.4) was studied. There a similar result to (i) of Theorem was obtained (Teorem 3 in [22]). See also [23], where the problem was studied under a different situation. The assumption imposed in both papers is more restrictive than (A). For example their assumption can not be satisfied by (0.4) with periodic vector potential in x.

REMARK 1.2. We can get from Corollary the result on the self-adjontness of the Schrödinger operator, the Dirac one, and the one describing the motion of relativistic spinless particles. As to the essential self-adjointness, more general results have been obtained. See [2], [3], [5], [9], [10], [11], and their references.

2. Pseudo-differential operators with basic weight function

For constants $a\geq 0$ and $b\geq 0$ we set

$$\omega(x,\xi;a,b)=\langle x\rangle^a+\langle\xi\rangle^b. \tag{2.1}$$

Let $p(x,\xi)$ be a C^∞-function in Hörmander's symbol class $S(\omega(x,\xi;a,b)^m,\ |dx|^2+|d\xi|^2)$ $(-\infty<m<\infty)$. That is, for any α and β there exists a constant $C_{\alpha,\beta}$ such that

$$|p_{(\beta)}^{(\alpha)}(x,\xi)|\leq C_{\alpha,\beta}\,\omega(x,\xi;a,b)^m \quad \text{on } R^{2n}. \tag{2.2}$$

Then the pseudo-differential operator $P(X,D_X)$ with symbol $\sigma(P(X,D_X))=p(x,\xi)$ is defined by

$$P(X,D_X)f(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{f}(\xi)đ\xi$$

for $f(x)\in\mathscr{S}$. It is easy to see that $P(X,D_X)$ is a continuous operator on $\mathscr{S}$ and can be extended uniquely to a continuous one on $\mathscr{S}'$ ([8]). We write $S(\omega(x,\xi;a,b)^m,\ |dx|^2+|d\xi|^2)$ as $S(\omega(x,\xi;a,b)^m)$ and call $\omega(x,\xi;a,b)$ a basic weight function, following

[13], [14], and [19].

Set

$$|p|_l^{(m)} = \max_{|\alpha+\beta|\le l} \sup_{x,\xi} \{|p_{(\beta)}^{(\alpha)}(x,\xi)|\omega(x,\xi;a,b)^{-m}\} \quad (l=0,1,2,\cdots). \tag{2.3}$$

Following the proof of Theorem 2.5 of Chapter 2 in [13] (c.f. Theorem 18.5.4 in [8]), we can easily have

Lemma 2.1. *Let* $p_j(x,\xi) \in S(\omega(x,\xi;a,b)^{m_j})$ $(j=1,2)$ *and set*

$$q(x,\xi;\hbar) = Os-\iint e^{-iy\cdot\eta} p_1(x,\xi+\hbar\eta) p_2(x+y,\xi) dy đ\eta, \tag{2.4}$$

where $Os-\iint \cdots dyđ\eta$ *means the oscillatory integral* (*Chapter* 1 *of* [13]). *Then for any* $l_1 = 0,1,2,\cdots$ *there exist an integer* $l_2 \ge 0$ *and a constant* C_{l_1} *independent of* $0<\hbar\le 1$ *such that*

$$|q|_{l_1}^{(m_1+m_2)} \le C_{l_1} |p_1|_{l_2}^{(m_1)} |p_2|_{l_2}^{(m_2)}. \tag{2.5}$$

So $\{q(x,\xi;\hbar)\}_{0<\hbar\le 1}$ *makes a bounded set in* $S(\omega^{m_1+m_2})$. *We also have*

$$Q(X,\hbar D_x;\hbar) = P_1(X,\hbar D_x) \circ P_2(X,\hbar D_x) \quad \text{on } \mathscr{S}. \tag{2.6}$$

Lemma 2.2. *Let* $\{p_\varepsilon(x,\xi)\}_{0<\varepsilon\le 1}$ *be a bounded set in* $S(1)$ *such that*

$$\lim_{\varepsilon\to 0} p_\varepsilon^{(\alpha)}(x,\xi) = 0 \quad \textit{pointwisely on } R^{2n}$$

for all $|\alpha| \le 2n$. *Then we have for all* $f \in L^2$

$$\lim_{\varepsilon\to 0} \|P_\varepsilon(X,D_x)f\| = 0.$$

Proof. Let $f \in \mathscr{S}$ and l an integer such that $\frac{n}{4} < l \le n$. Then, integrating by parts, we have

$$\begin{aligned} P_\varepsilon(X,D_x)f(x) &= \int e^{ix\cdot\xi} p_\varepsilon(x,\xi)\hat{f}(\xi) đ\xi \\ &= \langle x\rangle^{-2l} \int e^{ix\cdot\xi}(1-\Delta_\xi)^l \{p_\varepsilon(x,\xi)\hat{f}(\xi)\} đ\xi \\ &\equiv \langle x\rangle^{-2l} g_\varepsilon(x). \end{aligned}$$

Using $f \in \mathscr{S}$ and the assumptions on $\{p_\varepsilon(x,\xi)\}_{0<\varepsilon\le 1}$, we see that $g_\varepsilon(x)$ are uniformly bounded on R^n in $0<\varepsilon\le 1$ and that

$$\lim_{\varepsilon\to 0} g_\varepsilon(x) = 0 \quad \text{pointwisely on } R^n.$$

Here we used the Lebesgue dominated convergence theorem. Applying the

Lebesgue theorem again, we obtain $\lim_{\varepsilon\to 0}\|P_\varepsilon(X,D_x)f\|=0$.

Let $f\in L^2$. We have assumed that $\{p_\varepsilon(x,\xi)\}_{0<\varepsilon\le 1}$ is bounded in $S(1)$. So, applying the Calderón-Vaillancourt theorem in [1], for any $\eta>0$ we can determine a $v\in\mathscr{S}$ independent of ε such that

$$\|P_\varepsilon(X,D_x)(v-f)\|<\eta$$

for all ε. Then we have

$$\overline{\lim}_{\varepsilon\to 0}\|P_\varepsilon f\|\le\eta$$

because of $\lim_{\varepsilon\to 0}\|P_\varepsilon v\|=0$. Hence we can complete the proof. Q.E.D.

We set for $s\ge 0$

$$\gamma_s(x,\xi;a,b)=\omega(x,\xi;a,b)^s=(\langle x\rangle^a+\langle\xi\rangle^b)^s. \tag{2.7}$$

Then there exist constants $C_{\alpha,\beta}$ such that

$$|\gamma_{s(\beta)}^{(\alpha)}(x,\xi;a,b)|\le\begin{cases} C_{\alpha,\beta}\omega(x,\xi)^{s-1}\langle x\rangle^{a-1} \\ \qquad \text{for all } \alpha \text{ and } |\beta|\neq 0, \\ C_{\alpha,\beta}\omega(x,\xi)^{s-1}\langle\xi\rangle^{b-1} \\ \qquad \text{for all } |\alpha|\neq 0 \text{ and } \beta. \end{cases} \tag{2.8}$$

Lemma 2.3. *Let $s\ge 0$. Then we have:*

(i) There exist a constant $\mu(s)=\mu(s;a,b)\ge 1$ independent of $0<\hbar\le 1$ and a bounded set $\{w_s(x,\xi;\hbar)\}_{0<\hbar\le 1}=\{w_s(x,\xi;\hbar,a,b)\}_{0<\hbar\le 1}$ in $S(\omega(x,\xi;a,b)^{-s})$ such that $W_s(x,\hbar D_x;\hbar)=\{\mu(s)+\Gamma_s(x,\hbar D_x)\}^{-1}$ on $\mathscr{S}$.

(ii) *$\{w_1(x,\xi;\hbar,as,bs)\}_{0<\hbar\le 1}$ is bounded in $S(\omega(x,\xi;a,b)^{-s})$.*

Proof. Let $\tau>0$ and $\kappa\ge 0$ be constants. By direct calculations we get for $\mu>0$

$$\max_{1\le\theta}\frac{\theta^{\kappa-1/\tau}}{\mu+\theta^\kappa}\le\begin{cases}\mu^{-1}, \text{when } 0\le\tau\kappa\le 1, \\ \dfrac{(\tau\kappa-1)^{1-1/\tau\kappa}}{\tau\kappa}\mu^{-1/\tau\kappa}, \quad \text{when } \tau\kappa>1 \text{ and } \mu\ge\dfrac{1}{\tau\kappa-1}.\end{cases} \tag{2.9}$$

We set for $\mu\ge 1$

$$p_\mu(x,\xi)=\{\mu+\gamma_s(x,\xi;a,b)\}^{-1}. \tag{2.10}$$

Then we have from (2.8)

$$(2.11)\qquad |p^{(\alpha)}_{\mu(\beta)}(x,\xi)| \le \begin{cases} C'_{\alpha,\beta}\omega(x,\xi)^{s-1}\langle x\rangle^{a-1}p_\mu(x,\xi)^2 \\ \quad \text{for all } \alpha \text{ and } |\beta|\neq 0, \\ C'_{\alpha,\beta}\omega(x,\xi)^{s-1}\langle \xi\rangle^{b-1}p_\mu(x,\xi)^2 \\ \quad \text{for all } |\alpha|\neq 0 \text{ and } \beta. \end{cases}$$

We note that the above constants $C'_{\alpha,\beta}$ are independent of $\mu\ge 1$.

Set

$$q_\mu(x,\xi;\hbar)=Os-\iint e^{-iy\cdot\eta}p_\mu(x,\xi+\hbar\eta)\{\mu+\gamma_s(x+y,\xi)\}dy đ\eta.$$

Since $p_\mu(x,\xi)\in S(\omega^{-s})$, we see by Lemma 2.1 that $\{q_\mu(x,\xi;\hbar)\}_{0<\hbar\le 1}$ is bounded in $S(1)$ and that

$$(2.12)\qquad Q_\mu(X,\hbar D_x;\hbar)=P_\mu(X,\hbar D_x)\circ\{\mu+\Gamma_s(X,\hbar D_x)\}.$$

We write

$$(2.13)\qquad \begin{aligned} q_\mu(x,\xi;\hbar)&=1+\hbar\textstyle\int_0^1 d\theta\Sigma_{|\alpha|=1}Os-\iint e^{-iy\cdot\eta}p^{(\alpha)}_\mu(x,\xi+\theta\hbar\eta)\gamma_{s(\alpha)}(x+y,\xi)dy đ\eta \\ &\equiv 1+\hbar r_\mu(x,\xi;\hbar). \end{aligned}$$

Integrating by parts, for even integers l_1 and l_2 we have

$$\begin{aligned} |r_\mu(x,\xi;\hbar)| &\le \textstyle\int_0^1 d\theta\Sigma_{|\alpha|=1}|\iint e^{-iy\cdot\eta}\langle y\rangle^{-l_1}(1-\Delta_\eta)^{l_1/2}\{\langle\eta\rangle^{-l_2} \\ &\qquad (1-\Delta_y)^{l_2/2}p^{(\alpha)}_\mu(x,\xi+\theta\hbar\eta)\gamma_{s(\alpha)}(x+y,\xi)\}dy đ\eta| \\ &\le C_1\textstyle\int_0^1 d\theta\Sigma_{|\alpha|=1}\Sigma_{|\alpha'|\le l_1,|\beta'|\le l_2}\iint\langle y\rangle^{-l_1}\langle\eta\rangle^{-l_2} \\ &\qquad |p^{(\alpha+\alpha')}_\mu(x,\xi+\theta\hbar\eta)\gamma_{s(\alpha+\beta')}(x+y,\xi)|dy đ\eta \end{aligned}$$

for a constant C_1 independent of $\hbar$ and $\mu\ge 1$. Using (2.8) and (2.11), we get

$$\begin{aligned} |r_\mu| &\le C'_1\textstyle\int_0^1 d\theta\iint\langle y\rangle^{-l_1}\langle\eta\rangle^{-l_2}\omega(x,\xi+\theta\hbar\eta)^{s-1}\langle\xi+\theta\hbar\eta\rangle^{b-1} \\ &\qquad p_\mu(x,\xi+\theta\hbar\eta)^2\omega(x+y,\xi)^{s-1}\langle x+y\rangle^{a-1}dy đ\eta. \end{aligned}$$

We can easily see $(\sqrt{2}\langle y\rangle)^{-1}\langle x\rangle\le\langle x+y\rangle\le\sqrt{2}\langle y\rangle\langle x\rangle$ and so

$$\begin{cases} (\sqrt{2}\langle y\rangle)^{-a}\omega(x,\xi;a,b)\le\omega(x+y,\xi;a,b)\le(\sqrt{2}\langle y\rangle)^{a}\omega(x,\xi;a,b), \\ (\sqrt{2}\langle \eta\rangle)^{-b}\omega(x,\xi;a,b)\le\omega(x,\xi+\eta;a,b)\le(\sqrt{2}\langle \eta\rangle)^{b}\omega(x,\xi;a,b). \end{cases}$$

Hence we have

$$\begin{aligned} |r_\mu| &\le C_2\iint\langle y\rangle^{-l_1+a|s-1|+|a-1|}\langle\eta\rangle^{-l_2+b|s-1|+|b-1|+2bs}dy đ\eta \\ &\qquad \times\omega(x,\xi)^{s-1}\langle\xi\rangle^{b-1}p_\mu(x,\xi)^2\omega(x,\xi)^{s-1}\langle x\rangle^{a-1} \end{aligned}$$

$$\leq C_2'\left(\frac{\omega(x,\xi)^{s-1}\langle x\rangle^{a-1}}{\mu+\omega(x,\xi)^s}\right)\left(\frac{\omega(x,\xi)^{s-1}\langle\xi\rangle^{b-1}}{\mu+\omega(x,\xi)^s}\right),$$

taking l_1 and l_2 large. Here we note

$$\frac{\omega(x,\xi;a,b)^{s-1}\langle x\rangle^{a-1}}{\mu+\omega(x,\xi;a,b)^s}\leq\begin{cases}\dfrac{\omega(x,\xi)^{s-1}}{\mu+\omega(x,\xi)^s} & (0\leq a\leq 1),\\[2ex] \dfrac{\omega(x,\xi)^{s-1/a}}{\mu+\omega(x,\xi)^s} & (1<a).\end{cases}$$

For it is clear when $0\leq a\leq 1$. When $1<a$, it can be proved from $\omega(x,\xi)^{1-1/a}=\{\omega(x,\xi)^{1/a}\}^{a-1}\geq\langle x\rangle^{a-1}$. Apply (2.9) as $\theta=\omega(x,\xi)$ to the above. Then we can determine constants $\zeta=\zeta(a,b,s)>0$ and $\mu^*=\mu^*(a,b,s)\geq 1$ independently of $\hbar$ so that

$$|r_\mu(x,\xi;\hbar)|\leq C_3(\frac{1}{\mu})^\zeta$$

for all $\mu\geq\mu^*$. In the same way we get the following. For any α and β there exists a constant $C''_{\alpha,\beta}$ independent of $\hbar$ such that

$$(2.14)\qquad |r^{(\alpha)}_{\mu(\beta)}(x,\xi;\hbar)|\leq C''_{\alpha,\beta}(\frac{1}{\mu})^\zeta\quad(\mu\geq\mu^*).$$

Now, noting (2.14), we obtain the following result from Theorem I.1 in Appendix of [13] and its proof. We can determine a $\mu(s;a,b)\geq 1$ and a bounded set $\{z_s(x,\xi;\hbar)\}_{0<\hbar\leq 1}$ in $S(1)$ such that

$$(2.15)\qquad Z_s(X,\hbar D_x;\hbar)=\{I+\hbar R_{\mu(s)}(X,\hbar D_x;\hbar)\}^{-1}\quad\text{on }\mathscr{S}.$$

Consequently we get from (2.12) and (2.13)

$$(2.16)\qquad Z_s(X,\hbar D_x;\hbar)\circ P_{\mu(s)}(X,\hbar D_x)\circ\{\mu(s)+\Gamma_s(X,\hbar D_x)\}=\text{Identity}.$$

Thus, using Lemma 2.1, we can determine a bounded set $\{w_s(x,\xi;\hbar)\}_{0<\hbar\leq 1}$ in $S(\omega^{-s})$ such that $W_s(X,\hbar D_x;\hbar)$ is the left inverse operator of $\mu(s)+\Gamma_s(X,\hbar D_x)$. In the same way we can determine the right inverse operator. Since the right inverse operator and the left one is equal, we could complete the proof of (i).

It follows from (i) that $\{w_1(x,\xi;\hbar,as,bs)\}_{0<\hbar\leq 1}$ is bounded in $S(\omega(x,\xi;as,bs)^{-1})$. We have $2^{-s}(\langle x\rangle^a+\langle\xi\rangle^b)^s\leq <x>^{as}+\langle\xi\rangle^{bs}\leq 2(<x>^a+\langle\xi\rangle^b)^s$ $(s\geq 0,\ x,\xi\in R^n)$, which can be derived from $2^{-s}(1+\theta)^s\leq 1+\theta^s\leq 2(1+\theta)^s$ $(s\geq 0,\ 0\leq\theta\leq 1)$. So we can easily prove (ii).

Q.E.D.

Lemma 2.4. *Let $B^s_{a,b}(\hbar)$ and $B^{-s}_{a,b}(\hbar)$ $(0 \leq s < \infty)$ be the weighted Sobolev space and its dual space defined in Introduction, respectively. Let $\gamma_s(x,\xi)=\gamma_s(x,\xi;a,b)$ and $\mu(s)=\mu(s;a,b)$ be the symbol and the constant in Lemma 2.3. Then we have:*

(i) $f(x) \in B^s_{a,b}(\hbar)$ $(s \geq 0)$ is equivalent to $\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f \in L^2$. Moreover, there exists a constant $C_{sB}>0$ independent of $0<\hbar \leq 1$ such that

$$C_{sB}^{-1}\|\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f\| \leq \|f\|_{B^s_{a,b}(\hbar)} \leq C_{sB}\|\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f\| \tag{2.17}$$

for all $f \in B^s_{a,b}(\hbar)$.

(ii) We can identify $B^{-s}_{a,b}(\hbar)$ with the space $\{f \in \mathscr{S}'; \{\mu(s)+\Gamma_s(X,\hbar D_x)\}^{-1}f \in L^2\}$. Moreover, there exists a constant $C'_{sB}>0$ independent of $0<\hbar \leq 1$ such that

$$C'^{-1}_{sB}\|\{\mu(s)+\Gamma_s(X,\hbar D_x)\}^{-1}f\| \leq \|f\|_{B^{-s}_{a,b}(\hbar)} \leq C'_{sB}\|\{\mu(s)+\Gamma_s(X,\hbar D_x)\}^{-1}f\| \tag{2.18}$$

for all $f \in B^{-s}_{a,b}(\hbar)$.

Proof. (i) We see from Lemmas 2.1 and 2.3 that the set of symbols $\{\sigma(\langle x\rangle^{as}\circ\{\mu(s)+\Gamma_s(X,\hbar D_x)\}^{-1})\}_{0<\hbar\leq 1}$ is bounded in $S(1)$. So, applying the Calderón-Vaillancourt theorem, we get

$$\begin{aligned}&\|\langle\cdot\rangle^{as}f\| = \|\langle\cdot\rangle^{as}\circ\{\mu(s)+\Gamma_s(X,\hbar D_x)\}^{-1}\circ\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f\|\\ &\leq C_1\|\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f\|\end{aligned}$$

for all f such that $\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f \in L^2$. In the same way we obtain the second inequality of (2.17) for all f such that $\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f \in L^2$.

Conversely suppose $f \in B^s_{a,b}(\hbar)$. We write $\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f = \{\mu(s)+\Gamma_s\}\circ\{\mu(1;as,bs)+\Gamma_1(X,\hbar D_x;as,bs)\}^{-1}\circ\{\mu(1;as,bs)+\Gamma_1(X,\hbar D_x;as,bs)\}f$. Using (ii) of Lemma 2.3, we see from Lemma 2.1 that $\{\sigma(\{\mu(s)+\Gamma_s\}\circ\{\mu(1;as,bs)+\Gamma_1(X,\hbar D_x;as,bs)\}^{-1})\}_{0<\hbar\leq 1}$ is bounded in $S(1)$. So, applying the Calderón-Vaillancourt theorem, we have

$$\|\{\mu(s)+\Gamma_s(X,\hbar D_x)\}f\| \leq C_2(\|\langle\cdot\rangle^{as}f\| + \|\langle\hbar D_x\rangle^{bs}f\|).$$

Thus we could prove (2.17).

(ii) Set $\nu=\mu(1;2as,2bs)$. We define an inner product in $B^s_{a,b}(\hbar)$ by

$$(f,g)_{B^s_{a,b}(\hbar)} = \nu(f,g) + (\langle\cdot\rangle^{as}f, \langle\cdot\rangle^{as}g) + (\langle\hbar D_x\rangle^{bs}f, \langle\hbar D_x\rangle^{bs}g)$$

and denote its norm by $\|g\|'_{B^s_{a,b}(\hbar)}$. Then we have

$$\|g\|_{B^s_{a,b}(\hbar)} \leq \|g\|'_{B^s_{a,b}(\hbar)} \leq (\nu+1)^{1/2}\|g\|_{B^s_{a,b}(\hbar)}. \tag{2.19}$$

Let $f \in B^{-s}_{a,b}(\hbar)$, which is a linear functional: $B^s_{a,b}(\hbar) \ni g \to f(g)$. Then the Riesz theorem shows that there exists a unique $\tilde{f} \in B^s_{a,b}(\hbar)$ satisfying

$$(g,\tilde{f})_{B^s_{a,b}(\hbar)} = f(g) \tag{2.20}$$

for all $g \in B^s_{a,b}(\hbar)$ and that we have

$$\|\tilde{f}\|'_{B^s_{a,b}(\hbar)} = \|f\|_{B^{-s}_{a,b}(\hbar)}. \tag{2.21}$$

If $g \in \mathscr{S}$, we have

$$f(g) = (g,\tilde{f})_{B^s_{a,b}(\hbar)} = (g, (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})\tilde{f}).$$

It is easy to see that the mapping: $B^{-s}_{a,b}(\hbar) \ni f \to (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})\tilde{f} \in \{(\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})v\,;\, v \in B^s_{a,b}(\hbar)\}$ is one to one and onto, because $\mathscr{S}$ is dense in $B^s_{a,b}(\hbar)$. So we can identify f with

$$(\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})\tilde{f}. \tag{2.22}$$

Under this identification, noting $\nu = \mu(1;2as,2bs)$, we see by Lemma 2.3 that $f \in B^{-s}_{a,b}(\hbar)$ is equivalent to $(\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})^{-1} f\ (=\tilde{f}) \in B^s_{a,b}(\hbar)$.

Suppose $f \in B^{-s}_{a,b}(\hbar)$. We write $\{\mu(s) + \Gamma_s(X,\hbar D_x)\}^{-1} f = [\{\mu(s)+\Gamma_s\}^{-1} \circ (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs}) \circ \{\mu(s)+\Gamma_s\}^{-1}] \circ \{\mu(s)+\Gamma_s\} \circ (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})^{-1} f$. Then as in the proof of (2.17) we get

$$\begin{aligned} &\|\{\mu(s) + \Gamma_s(X,\hbar D_x)\}^{-1} f\| \\ &\leq C_3 \|\{\mu(s) + \Gamma_s(X,\hbar D_x)\} \circ (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})^{-1} f\| \\ &= C_3 \|\{\mu(s) + \Gamma_s(X,\hbar D_x)\}\tilde{f}\| \end{aligned}$$

and so from (2.17)

$$\|\{\mu(s) + \Gamma_s(X,\hbar D_x)\}^{-1} f\| \leq C_3' \|\tilde{f}\|_{B^s_{a,b}(\hbar)}.$$

Hence we obtain the first inequality of (2.18) by (2.19) and (2.21).

Conversely suppose $\{\mu(s) + \Gamma_s(X,\hbar D_x)\}^{-1} f \in L^2$. We write $\{\mu(s) + \Gamma_s(X,\hbar D_x)\} \circ (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})^{-1} f = [\{\mu(s)+\Gamma_s\} \circ (\nu + \langle X\rangle^{2as} + \langle \hbar D_x\rangle^{2bs})^{-1} \circ \{\mu(s)+\Gamma_s\}] \circ \{\mu(s)+\Gamma_s\}^{-1} f$. Then as in the proof of the first inequality we can prove the second one of (2.18). We see from the above arguments that $f \in B^{-s}_{a,b}(\hbar)$ is equivalent to $\{\mu(s) + \Gamma_s(X,\hbar D_x)\}^{-1} f \in L^2$. Thus we could complete the proof of (ii). Q.E.D.

The following lemma can be easily shown from Lemmas 2.1, 2.3, 2.4, and the

Calderón-Vaillancourt theorem.

Lemma 2.5. *Let $p(x,\xi)\in S(\omega(x,\xi;a,b)^m)$ $(-\infty<m<\infty)$. Then $P(X,\hbar D_x)$ $(0<\hbar\leq 1)$ is a bounded operator from $B^s_{a,b}(\hbar)$ to $B^{s-m}_{a,b}(\hbar)$ for all $-\infty<s<\infty$. Moreover, its operator norm is bounded in $0<\hbar\leq 1$.*

3. Proof of Theorem

Our proof of Theorem is analogous to that of the similar result on hyperbolic equations in [13] and [20]. We will give the proof when $N=1$. General result can be proved similarly.

Let $N=1$ and set $k(t,x,\xi)=k_{11}(t,x,\xi)$. Then it follows from our assumption that $k(t,x,\xi)$ is real valued. Using $\mu(s;a,b)$ and $\gamma_s(x,\xi;a,b)$ in Section 2, we define

$$\lambda_s(x,\xi;a,b)=\mu(s;a,b)+\gamma_s(x,\xi;a,b). \tag{3.1}$$

Let $\chi(\theta)$ be a real valued C^∞-function on R^1 with compact support such that $\chi(0)=1$. Then we have the following:

Lemma 3.1. *Suppose that $k(x,\xi)$ satisfies the assumption (A) or (B). When (A) is assumed, we define $\chi_\varepsilon(x,\xi)$ by $\chi(\varepsilon(\langle x\rangle+\langle\xi\rangle))$ for each $\varepsilon>0$. When (B) is done, we do $\chi_\varepsilon(x,\xi)$ by $\chi(\varepsilon(\langle x\rangle^M+\langle\xi\rangle))$. Here $M\geq 1$ is the constant appearing in (B). In both cases we set*

$$k_\varepsilon(x,\xi)=\chi_\varepsilon(x,\xi)k(x,\xi). \tag{3.2}$$

Let $s\geq 0$. Then there exists a bounded set $\{q_{s\varepsilon}(x,\xi;\hbar)\}_{0<\varepsilon,\hbar\leq 1}$ in $S(1)$ such that

$$Q_{s\varepsilon}(X,\hbar D_x;\hbar)=\frac{1}{\hbar}[\Lambda_s(X,\hbar D_x),K_\varepsilon(\frac{X+X}{2},\hbar D_x)]\circ\Lambda_s(X,\hbar D_x)^{-1}.$$

Here $\Lambda_s(X,\hbar D_x)$ denotes $\Lambda_s(X,\hbar D_x;1,1)$ when (A) is assumed and $\Lambda_s(X,\hbar D_x;M,1)$ when (B) is done, respectively.

Proof. At first we will prove this lemma under the assumption (A). Let $\omega=\omega(x,\xi;1,1)$. It follows from (A) that $k(x,\xi)\in S(\omega^2)$. It is easy to see that $\chi_\varepsilon(x,\xi)\in\bigcap_{-\infty<m<\infty}S(\omega^m)$ for each $0<\varepsilon\leq 1$ and that $\{\chi_\varepsilon(x,\xi)\}_{0<\varepsilon\leq 1}$ is a bounded set in $S(1)$. In addition, if $|\alpha+\beta|=1$, then $\{\chi^{(\alpha)}_{\varepsilon(\beta)}\}_{0<\varepsilon\leq 1}$ is bounded in $S(\omega^{-1})$. In fact we can prove it by using $\dfrac{\partial}{\partial z_j}\chi_\varepsilon(x,\xi)=\varepsilon(\langle x\rangle+\langle\xi\rangle)\dfrac{d\chi}{d\theta}(\varepsilon(\langle x\rangle+\langle\xi\rangle))\dfrac{\partial(\langle x\rangle+\langle\xi\rangle)}{\partial z_j}$ $\times\left(\dfrac{1}{\langle x\rangle+\langle\xi\rangle}\right)$ where $z=x$ and ξ. Consequently we have from Lemma 2.1 together with (A)

$$(3.3)\quad \begin{cases} \text{(i)} \quad k_\varepsilon(x,\xi)\in S(1) \quad \text{for each } 0<\varepsilon\leq 1, \\ \text{(ii)} \quad \{k_\varepsilon(x,\xi)\}_{0<\varepsilon\leq 1} \text{ is bounded in } S(\omega^2), \\ \text{(iii)} \quad \text{if } |\alpha+\beta|=1,\ \{k^{(\alpha)}_{\varepsilon(\beta)}(x,\xi)\}_{0<\varepsilon\leq 1} \text{ is bounded in } S(\omega). \end{cases}$$

We define $k_{\varepsilon L}(x,\xi;\hbar)$ by

$$(3.4)\qquad k_{\varepsilon L}(x,\xi;\hbar)=\mathrm{Os}-\iint e^{-iy\cdot\eta}k_\varepsilon(x+\frac{y}{2},\xi+\hbar\eta)dy đ\eta.$$

Then

$$(3.5)\qquad K_{\varepsilon L}(x,\hbar D_x;\hbar)=K_\varepsilon(\frac{X+X}{2},\hbar D_x)$$

holds and we have from (3.3)

$$(3.6)\quad \begin{cases} \text{(i)} \quad k_{\varepsilon L}(x,\xi;\hbar)\in S(1) \quad \text{for each } 0<\varepsilon,\ \hbar\leq 1, \\ \text{(ii)} \quad \{k_{\varepsilon L}\}_{0<\varepsilon,\hbar\leq 1} \quad \text{is bounded in } S(\omega^2), \\ \text{(iii)} \quad \text{if } |\alpha+\beta|=1,\ \{k^{(\alpha)}_{\varepsilon L(\beta)}\}_{0<\varepsilon,\hbar\leq 1} \text{ is bounded in } S(\omega). \end{cases}$$

Their proofs are analogous to that of Lemma 2.1.

Set

$$(3.7)\ p_{s\varepsilon}(x,\xi;\hbar)=\frac{1}{\hbar}Os-\iint e^{-iy\cdot\eta}\{\lambda_s(x,\xi+\hbar\eta)k_{\varepsilon L}(x+y,\xi)-k_{\varepsilon L}(x,\xi+\hbar\eta)\lambda_s(x+y,\xi)\}dy đ\eta.$$

Then we have $P_{s\varepsilon}(X,\hbar D_x;\hbar)=\frac{1}{\hbar}[\Lambda_s(X,\hbar D_x), K_{\varepsilon L}(X,\hbar D_x)]$ from Lemma 2.1 and so

$$(3.8)\qquad P_{s\varepsilon}(X,\hbar D_x;\hbar)=\frac{1}{\hbar}[\Lambda_s(X,\hbar D_x), K_\varepsilon(\frac{X+X}{2},\hbar D_x)]$$

by (3.5). Its symbol $p_{s\varepsilon}(x,\xi;\hbar)$ can be written as

$$\Sigma_{|\alpha|=1}\int_0^1 d\theta Os-\iint e^{-iy\cdot\eta}\{\lambda^{(\alpha)}_s(x,\xi+\theta\hbar\eta)k_{\varepsilon L(\alpha)}(x+y,\xi)-k^{(\alpha)}_{\varepsilon L}(x,\xi+\theta\hbar\eta)\lambda_{s(\alpha)}(x+y,\xi)\}dy đ\eta.$$

Using (2.8) with $a=b=1$, we get $\lambda^{(\alpha)}_{s(\beta)}(x,\xi)\in S(\omega^{s-1})$ for $|\alpha+\beta|=1$. Consequently, applying Lemma 2.1, we see by (3.6) that $\{p_{s\varepsilon}\}_{0<\varepsilon,\hbar\leq 1}$ is bounded in $S(\omega^s)$. Hence we can complete the proof from Lemmas 2.1 and 2.3.

The proof under the assumption (B) can be given similarly as follows. Let $\omega=\omega(x,\xi;M,1)$. It follows from (B) that $k(x,\xi)\in S(\omega)$. We see that $\chi_\varepsilon(x,\xi)\in \bigcap_{-\infty<m<\infty} S(\omega^m)$ for each $0<\varepsilon\leq 1$ and that $\{\chi_\varepsilon\}_{0<\varepsilon\leq 1}$ and $\{\chi_\varepsilon^{(\alpha)}\}_{0<\varepsilon\leq 1}$ with $|\alpha|=1$ are bounded in $S(1)$ and $S(\omega^{-1})$, respectively. Consequently we have together with (B)

$$(3.9)\quad \begin{cases} \text{(i)}\ \ k_\varepsilon(x,\xi)\in S(1) \quad \text{for each } 0<\varepsilon\leq 1, \\ \text{(ii)}\ \ \{k_\varepsilon(x,\xi)\}_{0<\varepsilon\leq 1} \text{ is bounded in } S(\omega), \\ \text{(iii)}\ \ \text{for any } |\alpha|\neq 0 \text{ and } \beta \text{ there exists a constant } C_{\alpha,\beta} \text{ independent of } \varepsilon \\ \quad \text{such that } |k_{\varepsilon(\beta)}^{(\alpha)}(x,\xi)|\leq C_{\alpha,\beta}\langle x\rangle. \end{cases}$$

Define $k_{\varepsilon L}(x,\xi;\hbar)$ and $p_{s\varepsilon}(x,\xi;\hbar)$ by (3.4) and (3.7), respectively. Then we also get (3.5),(3.8), and

$$(3.10)\quad \begin{cases} \text{(i)}\ \ k_{\varepsilon L}(x,\xi;\hbar)\in S(1) \quad \text{for each } 0<\varepsilon,\ \hbar\leq 1, \\ \text{(ii)}\ \ \{k_{\varepsilon L}\}_{0<\varepsilon,\hbar\leq 1} \text{ is bounded in } S(\omega), \\ \text{(iii)}\ \ \text{for any } |\alpha|\neq 0 \text{ and } \beta \text{ there exists another constant } C_{\alpha,\beta} \text{ independent} \\ \quad \text{of } \varepsilon,\ \hbar \text{ such that } |k_{\varepsilon L(\beta)}^{(\alpha)}(x,\xi)|\leq C_{\alpha,\beta}\langle x\rangle. \end{cases}$$

Now, using (2.8) with $a=M$ and $b=1$, we see that $\lambda_s^{(\alpha)}(x,\xi)\in S(\omega^{s-1})$ for $|\alpha|=1$ and that $|\lambda_{s(\beta)}^{(\alpha)}(x,\xi)|\leq C'_{\alpha,\beta}\omega(x,\xi)^{s-1}\langle x\rangle^{M-1}$ for all α and $|\beta|\neq 0$. Hence, applying Lemma 2.1, we can prove together with (3.10) that $\{p_{s\varepsilon}\}_{0<\varepsilon,\hbar\leq 1}$ is bounded in $S(\omega^s)$. Thus we can complete the proof by Lemmas 2.1 and 2.3. Q.E.D.

Proof of Theorem. We will prove Theorem only under the assumption (A). The proof under the assumption (B) can be given similarly. We write a solution $u_\hbar(t)$ as $u(t)$, omitting $\hbar$, and $B_{1,1}^s(\hbar)$ as $B^s(\hbar)$. Let $\omega=\omega(x,\xi;1,1)$ and $\lambda_d=\lambda_d(x,\xi;1,1)$ $(d\geq 0)$. We will decompose the proof into three steps.

1st step. Let $k_\varepsilon(t,x,\xi)=\chi(\varepsilon(\langle x\rangle+\langle\xi\rangle))k(t,x,\xi)$ $(0<\varepsilon\leq 1)$ and write $K_\varepsilon(t,\frac{X+X'}{2},\hbar D_x)$ as $K_\varepsilon(t)$. Throughout this step we suppose $u^{(0)}\in B^{s+2}(\hbar)$ $(-\infty<s<\infty)$ and $f(t)\in \mathscr{E}_t^0([0,T];B^{s+2}(\hbar))$. We consider the equation

$$(3.11)\qquad i\hbar\frac{\partial u}{\partial t}(t)=K_\varepsilon(t)u(t)+i\hbar f(t) \text{ on } [0,T],\ u(0)=u^{(0)}.$$

Using (3.5) and (3.6), we see by Lemma 2.5 that $K_\varepsilon(t)$ is a bounded operator on $B^d(\hbar)$ for any $-\infty<d<\infty$. The equation (3.11) is equivalent to

$$(3.11)' \qquad i\hbar u(t) = i\hbar u^{(0)} + \int_0^t \{K_\varepsilon(\theta)u(\theta) + i\hbar f(\theta)\} d\theta.$$

So we can find a solution $u_\varepsilon(t) \in \mathscr{E}_t^1([0,T]; B^{s+2}(\hbar))$ of (3.11) by the succesive iteration for each fixed $0<\varepsilon, \hbar \leq 1$.

At first suppose $0 \leq d \leq s+2$. We denote by Re$(\cdot)$ its real part. Since $K_\varepsilon(t)$ is symmetric on L^2, we can easily have

$$\begin{aligned}
\frac{d}{dt}\|\Lambda_d(X,\hbar D_x)u_\varepsilon(t)\|^2 &= 2\mathrm{Re}(\Lambda_d(X,\hbar D_x)\partial_t u_\varepsilon(t), \Lambda_d u_\varepsilon(t)) \\
&= -2\mathrm{Re}\, i(\frac{1}{\hbar}\Lambda_d \circ K_\varepsilon(t)u_\varepsilon, \Lambda_d u_\varepsilon) + 2\mathrm{Re}(\Lambda_d f, \Lambda_d u_\varepsilon) \\
&= -2\mathrm{Re}\, i(\frac{1}{\hbar}[\Lambda_d, K_\varepsilon]u_\varepsilon, \Lambda_d u_\varepsilon) - 2\mathrm{Re}\, i(\frac{1}{\hbar}K_\varepsilon \circ \Lambda_d u_\varepsilon, \Lambda_d u_\varepsilon) + 2\mathrm{Re}(\Lambda_d f, \Lambda_d u_\varepsilon) \\
&= -2\mathrm{Re}\, i(\frac{1}{\hbar}[\Lambda_d, K_\varepsilon] \circ \Lambda_d^{-1} \circ \Lambda_d u_\varepsilon, \Lambda_d u_\varepsilon) + 2\mathrm{Re}(\Lambda_d f, \Lambda_d u_\varepsilon).
\end{aligned}$$

Applying Lemma 3.1, we get

$$\|\Lambda_d(X,\hbar D_x)u_\varepsilon(t)\| \leq \tilde{C}_d(T)(\|\Lambda_d(X,\hbar D_x)u^{(0)}\| + \int_0^t \|\Lambda_d(X,\hbar D_x)f(\theta)\| d\theta)$$

for a constant $\tilde{C}_d(T)$ independent of $\varepsilon, \hbar$. Consequently, using Lemma 2.4, we obtain

$$\|u_\varepsilon(t)\|_{B^d(\hbar)} \leq \tilde{C}_d(T)(\|u^{(0)}\|_{B^d(\hbar)} + \int_0^t \|f(\theta)\|_{B^d(\hbar)} d\theta)$$

for another constant $\tilde{C}_d(T)$.

Next let $d<0$ and $d \leq s+2$. Then we have

$$\begin{aligned}
\frac{d}{dt}\|\Lambda_{-d}(X,\hbar D_x)^{-1}u_\varepsilon(t)\|^2 &= -2\mathrm{Re}\, i(\frac{1}{\hbar}\Lambda_{-d}^{-1} \circ K_\varepsilon u_\varepsilon, \Lambda_{-d}^{-1}u_\varepsilon) + 2\mathrm{Re}(\Lambda_{-d}^{-1}f, \Lambda_{-d}^{-1}u_\varepsilon) \\
&= -2\mathrm{Re}\, i(\frac{1}{\hbar}\Lambda_{-d}^{-1} \circ [K_\varepsilon, \Lambda_{-d}] \circ \Lambda_{-d}^{-1}u_\varepsilon, \Lambda_{-d}^{-1}u_\varepsilon) + 2\mathrm{Re}(\Lambda_{-d}^{-1}f, \Lambda_{-d}^{-1}u_\varepsilon).
\end{aligned}$$

Hence we can also get the similar inequality as in the case $0 \leq d \leq s+2$. Thus for any $-\infty < d \leq s+2$ there exists a constant $C_d(T)$ independent of $0<\varepsilon, \hbar \leq 1$ such that

$$(3.12) \qquad \|u_\varepsilon(t)\|_{B^d(\hbar)} \leq C_d(T)(\|u^{(0)}\|_{B^d(\hbar)} + \int_0^t \|f(\theta)\|_{B^d(\hbar)} d\theta).$$

We see by (3.12) that for each fixed $\hbar$ $\{u_\varepsilon(t)\}_{0<\varepsilon\leq 1}$ is uniformly bounded as a family of functions from $[0,T]$ to $B^{s+2}(\hbar)$. Moreover, using (3.5) and (ii) of (3.6), we see from this and (3.11)' that $\{u_\varepsilon(t)\}_{0<\varepsilon\leq 1}$ is equi-continuous as a family of functions from $[0,T]$ to $B^s(\hbar)$. See page 29 in [16] about the terminology above. The compactness of the embedding map from $B^2(\hbar)$ into L^2 follows from the Rellich criterion (e.g. Theorem XIII. 65 on page 247 of [17]). So we can prove from Lemmas 2.4 and 2.5 that the embedding map from $B^{s+2}(\hbar)$ into $B^s(\hbar)$ is compact. Hence we can apply the Ascoli-Arzelà theorem to $\{u_\varepsilon(t)\}_{0<\varepsilon\leq 1}$. Consequently there exist a sequence $\{\varepsilon_j\}_{j=1}^\infty$ tending to zero and a $u(t)\in\mathscr{E}_t^0([0,T];B^s(\hbar))$ so that

$$\lim_{j\to\infty}u_{\varepsilon_j}(t)=u(t)\quad\text{in }\mathscr{E}_t^0([0,T];B^s(\hbar)). \tag{3.13}$$

Here the norm of $g(t)\in\mathscr{E}_t^0([0,T];B^s(\hbar))$ is $\max_{0\leq t\leq T}\|g(t)\|_{B^s(\hbar)}$.

Since $u_\varepsilon(t)$ is a solution of (3.11), we have

$$i\hbar u_{\varepsilon_j}(t)=i\hbar u^{(0)}+\int_0^t K_{\varepsilon_j}(\theta)u(\theta)d\theta+i\hbar\int_0^t f(\theta)d\theta+\int_0^t K_{\varepsilon_j}(\theta)\{u_{\varepsilon_j}(\theta)-u(\theta)\}d\theta.$$

Let j tend to infinity. Then, using (3.5),(ii) of (3.6), and (3.13), we can prove from Lemma 2.5 that the last term in the right-hand side tends to zero in $B^{s-2}(\hbar)$. Moreover, noting (3.4) and (3.5), we can see by the analogous arguments used in the proof of Lemma 2.1 that we can apply Lemma 2.2 to the second term. Then the second term tends to $\int_0^t K(\theta)u(\theta)d\theta$ in $B^{s-2}(\hbar)$. Hence $u(t)$ defined by (3.13) belongs to $\mathscr{E}_t^1([0,T];B^{s-2}(\hbar))$ and satisfies

$$i\hbar\frac{\partial u}{\partial t}(t)=K(t)u(t)+i\hbar f(t)\quad\text{on }[0,T],\ u(0)=u^{(0)}. \tag{3.14}$$

Thus we could find a solution $u(t)\in\mathscr{E}_t^0([0,T];B^s(\hbar))\cap\mathscr{E}_t^1([0,T];B^{s-2}(\hbar))$ of (3.14). In addition, we have

$$\|u(t)\|_{B^s(\hbar)}\leq C_s(T)(\|u^{(0)}\|_{B^s(\hbar)}+\int_0^t\|f(\theta)\|_{B^s(\hbar)}d\theta) \tag{3.15}$$

from (3.12) and (3.13).

2nd step. In this step we will prove that the solution of (3.14) is unique in $\mathscr{E}_t^0([0,T];B^d(\hbar))\cap\mathscr{E}_t^1([0,T];B^{d-2}(\hbar))$ for any $-\infty<d<\infty$.

Let $g(t)$ be an arbitrary function in $\mathscr{E}_t^0([0,T];B^{|d|+4}(\hbar))$ and consider

$$i\hbar\frac{\partial v}{\partial t}(t)=K(t)v(t)+i\hbar g(t)\quad\text{on }[0,T],\ v(T)=0.$$

Then we can find a solution $v(t)\in\mathscr{E}_t^0([0,T];B^{|d|+2}(\hbar))\cap\mathscr{E}_t^1([0,T];B^{|d|}(\hbar))$ from the arguments in the 1st step. Let $u(t)\in\mathscr{E}_t^0([0,T];B^d(\hbar))\cap\mathscr{E}_t^1([0,T];B^{d-2}(\hbar))$ be a solution of (3.14) to $u^{(0)}=0$ and $f(t)=0$. Then we have

$$0=\int_0^T(i\hbar\frac{\partial u}{\partial t}(\theta)-K(\theta)u(\theta), v(\theta))d\theta$$

$$=i\hbar\int_0^T(\frac{\partial u}{\partial t}(\theta), v(\theta))d\theta-\int_0^T(K(\theta)u(\theta), v(\theta))d\theta$$

$$=\int_0^T(u(\theta), i\hbar\frac{\partial v}{\partial t}(\theta)-K(\theta)v(\theta))d\theta$$

and hence

$$0=\int_0^T(u(\theta), g(\theta))d\theta.$$

This shows that $u(t)$ vanishes on $[0,T]$.

3rd step. Let $u^{(0)}\in B^s(\hbar)$ and $f(t)\in\mathscr{E}_t^0([0,T];B^s(\hbar))$. We can take sequences $\{u_j^{(0)}\}_{j=1}^{\infty}$ in $B^{s+2}(\hbar)$ and $\{f_j(t)\}_{j=1}^{\infty}$ in $\mathscr{E}_t^0([0,T];B^{s+2}(\hbar))$ such that $\lim_{j\to\infty}u_j^{(0)}=u^{(0)}$ in $B^s(\hbar)$ and $\lim_{j\to\infty}f_j(t)=f(t)$ in $\mathscr{E}_t^0([0,T];B^s(\hbar))$. Let $u_j(t)\in\mathscr{E}_t^0([0,T];B^s(\hbar))\cap\mathscr{E}_t^1([0,T];B^{s-2}(\hbar))$ be the solution of (3.14) to $u_j^{(0)}$ and $f_j(t)$. Then, noting the uniqueness of the solution, we have by (3.15)

$$\text{(3.16)}\qquad \|u_j(t)-u_k(t)\|_{B^s(\hbar)}$$
$$\leq C_s(T)(\|u_j^{(0)}-u_k^{(0)}\|_{B^s(\hbar)}+\int_0^T\|f_j(\theta)-f_k(\theta)\|_{B^s(\hbar)}d\theta).$$

Consequently there exists a $u(t)\in\mathscr{E}_t^0([0,T];B^s(\hbar))$ such that

$$\text{(3.17)}\qquad \lim_{j\to\infty}u_j(t)=u(t)\quad\text{in } \mathscr{E}_t^0([0,T];B^s(\hbar)).$$

Then we can prove as in the proof of the 1st step that this $u(t)$ belongs to $\mathscr{E}_t^1([0,T];B^{s-2}(\hbar))$ and satisfies (3.14) and (3.15). The uniqueness of the solution of (3.14) in $\mathscr{E}_t^0([0,T];B^s(\hbar))\cap\mathscr{E}_t^1([0,T];B^{s-2}(\hbar))$ has been proved in the 2nd step. Thus we could prove Theorem except for (1.2).

Let $u^{(0)}\in B^2(\hbar)$ and $u(t)\in\mathscr{E}_t^0([0,T];B^2(\hbar))\cap\mathscr{E}_t^1([0,T];L^2)$ be the solution of (0.1). Then we get

$$\|u(t)\|=\|u^{(0)}\|\quad(t\in[0,T])$$

as in the proof of (3.15). We can prove (1.2) from this by repeating the arguments above in the 3rd step. Thus the proof could be completed. Q.E.D.

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Section of Applied Mathematics
Department of Computer Science
Ehime University
Matsuyama, 790, Japan.