Miyazawa, Y.
Osaka J. Math.
32 (1995), 193–206

ARF INVARIANTS OF STRONGLY INVERTIBLE KNOTS OBTAINED FROM UNKNOTTING NUMBER ONE KNOTS

YASUYUKI MIYAZAWA

(Received May 27, 1993)

Introduction

Two techniques, branched covering space and Dehn surgery, are known as popular methods by which we obtain new 3-manifolds from a link in S^3. In 1977, Montesinos [8] showed the following relationship between the 2-fold cyclic branched covering of S^3 branched over a link and the closed, orientable 3-manifold which is obtained by doing surgery on a link in S^3. A link in S^3 is called strongly invertible if there is an orientation preserving involution of S^3 which induces in each component of L an involution with two fixed points.

Theorem (MONTESINOS). *Let M be a closed, orientable 3-manifold that is obtained by doing surgery on a strongly invertible link L of n components. Then M is a 2-fold cyclic covering of S^3 branched over a link of at most $n+1$ components. Conversely, every 2-fold cyclic branched covering of S^3 can be obtained in this fashion.*

A nontrivial knot is called of unknotting number one if there exists a crossing which is exchanged to deform the knot into a trivial knot. From the proof of this theorem, as a special case, we have the following: the 2-fold cyclic branched covering of S^3 branched over an unknotting number one knot can be obtained by doing surgery on a strongly invertible knot. In this paper, we give a relationship between these two knots.

We define the Conway polynomial $\nabla(z)(\in \mathbf{Z}[z])$ [1] by the following recursive formulas:

(1) For three links L_+, L_- and L_0 which differ only in one place as shown in Fig. 1,

$$\nabla_{L_+}(z)-\nabla_{L_-}(z)=z\nabla_{L_0}(z).$$

(2) For a trivial knot U, $\nabla_U(z)=1$.

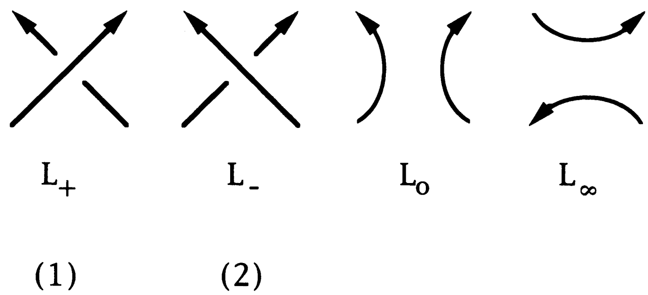


Fig. 1

Then, for a link L, the Conway polynomial $\nabla_L(z)$ of L can be written as

$$\sum_{n=r-1}^{\infty} a_n(L)z^n,$$

where r is the number of the components of L and $a_n(L)=0$ except a finite number of n. If L is a knot, the Conway polynomial $\nabla_L(z)$ can be written as $\Sigma_{n=0}^{\infty} a_{2n}(L)z^{2n}$.

For a knot K, we define the arf invariant $Arf(K)$ [11] [4] of K by

$$Arf(K) \equiv a_2(K) \pmod 2.$$

For a knot K in S^3, we denote by $\Sigma_2(K)$ the 2-fold branched covering space of S^3 branched over K. Let K be an unknotting number one knot. Then there is a 3-ball B in S^3 for which we can change K into a trivial knot U by applying the modification as shown in Fig. 2. For the 2-fold cyclic branched cover $f: \Sigma_2(U)(=S^3) \to S^3$ branched over U, let C be the core of the solid torus $V=f^{-1}(B)$ in $\Sigma_2(U)$. It is easy to see that C is a strongly invertible knot. By Montesinos' theorem, $\Sigma_2(K)$ can be obtained by doing surgery on C. We call the strongly invertible knot C a surgical knot for $\Sigma_2(K)$. Then we have:

Main Theorem. *Let K be an unknotting number one knot and C be a surgical knot for $\Sigma_2(K)$. Then*

$$Arf(C) \equiv a_4(K) \pmod 2.$$

In §1, we construct a surgical knot C for $\Sigma_2(K)$. In §2, we calculate the arf invariant of C.

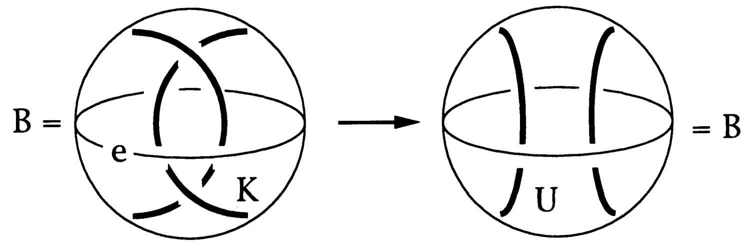


Fig. 2

1. Construction of a surgical knot

Let K be an unknotting number one knot. Then there exists a crossing which is exchanged to deform K into a trivial knot. We can choose the crossing as shown in Fig. 3. Near the crossing, we modify K as shown in Fig. 3, where R is a 2-string tangle. That is, R is a pair (B_1, k_1), where B_1 is a 3-ball and k_1 is a pair of disjoint arcs in B_1 with $\partial B_1 \cap k_1 = \partial k_1$. Let (B_0, k_0) be the complementary tangle. Thus $(S^3, K) = (B_0, k_0) \cup (B_1, k_1)$. If we replace R with a tangle $S = (B_2, k_2)$ with a band b as shown in Fig. 4, we obtain a trivial knot U with b; $(S^3, U) = (B_0, k_0) \cup (B_2, k_2)$. Let $h: \partial B_2 \to \partial B_0$ be a homeomorphism such that $(S^3, K) = (B_0, k_0) \underset{h}{\cup} (B_2, k_2)$. (we may take h as $h_1 \circ h_1$, where h_1 is given in [12, pp. 300–302].) Let C^* be the core of b whose endpoints meet the trivial knot U. If we transform U into the standard form, C^* becomes an arc coiled around U. By an isotopy, we can deform C^* as shown in Fig. 5(1), where P is a tangle. Considering the 2-fold branched cover $f: \Sigma_2(U) \to S^3$ branched over U, the preimage $f^{-1}(C^*)$ of C^* is a strongly invertible knot as shown in Fig. 5(2), where the tangle ꟼ is obtained by flipping P. This is a surgical knot for $\Sigma_2(K)$, which we denote by C.

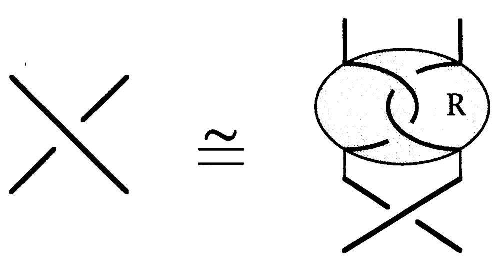


Fig. 3

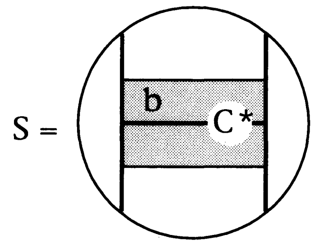


Fig. 4

We orient U and C^* as shown in Fig. 5(1). For each crossing, we give the signature as follows: If a crossing is positive as shown in Fig. 1(1), then the signature is $+1$. If a crossing is negative as shown in Fig. 1(2), then the signature is -1.

Let σ be the sum of the signatures of the crossings of U and C^*. (Thus we do not count for the self-crossings of U or of C^*.) If σ is not zero, we can coil C^* around U near an endpoint of C^* so that σ is equal to zero. Thus we assume that σ is zero.

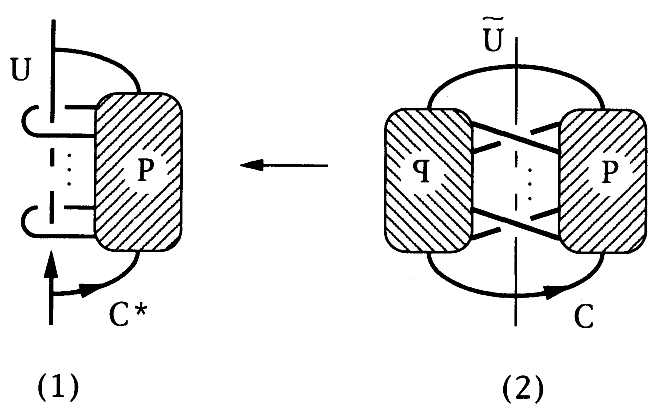


Fig. 5

Let $\tilde{U}=f^{-1}(U)$. Let C_+, C_-, C_0 and C_∞ be the oriented links as shown in Fig. 6, which are identical outside a 3-ball Q inside it are as shown in Fig. 7 and P is the same tangle as in Fig. 5. Note that, for each link, the orientations of strings in Q extend to the link compatibly. The three links C_+, C_- and C_0 have period 2 with periodic map the covering translation of $\Sigma_2(U)$. Let C_+^*, C_-^* and C_0^* be the factor knots of C_+, C_- and C_0, respectively. Since the absolute value of the linking number $|lk(C_\pm^*, U)|$ of $C_\pm^*$ and U is 1, C_+ and C_- are knots. And so C_0 is a 2-component link. Now C_∞ is a strongly invertible knot and its knot type is the same one as the surgical knot C as shown in Fig. 5(2). So we can regard C as a knot closely related to the three links C_+, C_- and C_0 with period 2. In §2, we will calculate the arf invariant of a surgical knot from this point of view.

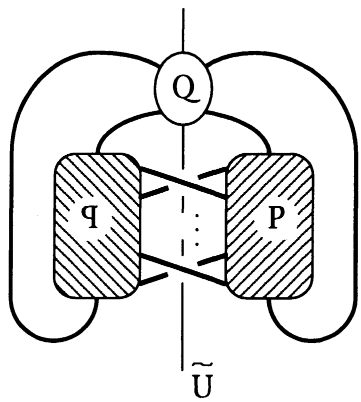


Fig. 6

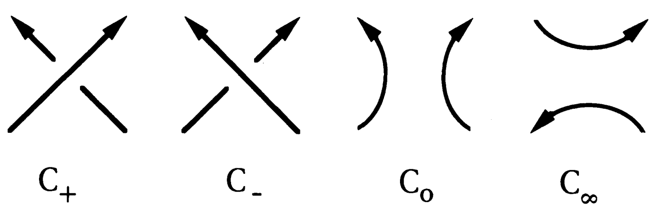


Fig. 7

2. Calculation of the arf invariant

We give some lemmas before proving Main Theorem. We define the Jones polynomial $V(t)(\in Z[t^{\pm 1/2}])$ [2] by the following recursive formulas:

(1) For three links L_+, L_- and L_0 which differ only in one place as shown in Fig. 1,

$$t^{-1}V_{L_+}(t)-tV_{L_-}(t)=(t^{1/2}-t^{-1/2})V_{L_0}(t),$$

(2) For a trivial knot U, $V_U(t)=1$.

Let L be an oriented link and D be its diagram. We denote the writhe of D by $w(D)$, which is the sum of the signatures of all the crossings of D. The Kauffman polynomial $F_L(a,z)(\in Z[a^{\pm 1}, z^{\pm 1}])$ [5] of L is defined by $a^{-w(D)}\Lambda_D(a,z)$, where $\Lambda_D(a,z)$ is a regular isotopy invariant of D determined by the following properties:

(1) $\Lambda_{\bigcirc}(a,z)=1$,

(2) $\Lambda_{\wp}(a,z)=a\Lambda_{)}(a,z)$, $\quad \Lambda_{\wp}(a,z)=a^{-1}\Lambda_{)}(a,z)$,

(3) $\Lambda_{\times}(a,z)+\Lambda_{\times}(a,z)=z(\Lambda_{)(}(a,z)+\Lambda_{\asymp}(a,z))$.

Lemma 1 ([3, Lemma 1]). *Let L_+, L_-, L_0 and L_∞ be four links which differ only in one place as shown in Fig. 1. If L_+ is a knot and L_0 is a 2-component link $J_1 \cup J_2$, then*

$$a_2(L_\infty)=-\frac{1}{2}(a_2(L_+)+a_2(L_-))+2(a_2(J_1)+a_2(J_2))+\frac{1}{2}a_1(L_0)^2.$$

Proof. Let D_+, D_-, D_0 and D_∞ be diagrams of L_+, L_-, L_0 and L_∞, respectively. We may assume that the four diagrams differ only in one place as shown in Fig. 1. Let d be the writhe of D_0, and $\lambda=lk(J_1,J_2)=a_1(L_0)$. Then the writhes of D_+, D_- and D_∞ are $d+1, d-1$ and $d-4\lambda$, respectively. From the definition, we have:

$$\Lambda_{D_+}(a,z)+\Lambda_{D_-}(a,z)=z(\Lambda_{D_0}(a,z)+\Lambda_{D_\infty}(a,z)).$$

Hence

$$\begin{aligned} a\{a^{-(d+1)}\Lambda_{D_+}(a,z)\}&+a^{-1}\{a^{-(d-1)}\Lambda_{D_-}(a,z)\} \\ &=z(a^{-d}\Lambda_{D_0}(a,z)+a^{-4\lambda}\{a^{-(d-4\lambda)}\Lambda_{D_\infty}(a,z)\}). \end{aligned}$$

And thus

$$aF_{L_+}(a,z)+a^{-1}F_{L_-}(a,z)=z(F_{L_0}(a,z)+a^{-4\lambda}F_{L_\infty}(a,z)).$$

Since $V_L(t)=F_L(-t^{-3/4}, t^{1/4}+t^{-1/4})$ [6], we have

$$-t^{-3/4}V_{L_+}(t)-t^{3/4}V_{L_-}(t)=(t^{1/4}+t^{-1/4})(V_{L_0}(t)+t^{3\lambda}V_{L_\infty}(t)).$$

Taking the second derivative of both sides at $t=1$, we obtain

$$\begin{aligned} &-(\frac{21}{16}V_{L_+}(1)-\frac{3}{2}V_{L_+}^{(1)}(1)+V_{L_+}^{(2)}(1)) \\ &-(-\frac{3}{16}V_{L_-}(1)+\frac{3}{2}V_{L_-}^{(1)}(1)+V_{L_-}^{(2)}(1)) \\ &=\frac{1}{8}(V_{L_0}(1)+V_{L_\infty}(1)) \\ &\quad+2\{V_{L_0}^{(2)}(1)+3\lambda(3\lambda-1)V_{L_\infty}(1)+6\lambda V_{L_\infty}^{(1)}(1)+V_{L_\infty}^{(2)}(1)\}, \end{aligned}$$

where $V^{(1)}(1)$ and $V^{(2)}(1)$ are the first and second derivatives of $V(t)$ at $t=1$, respectively. It is shown in [9] that, for an oriented r-component link $L=K_1\cup\cdots\cup K_r$,

$$V_L^{(1)}(1) = -3(-2)^{r-2}\sum_{i<j}\lambda_{ij}(L)$$

and

$$\begin{aligned}V_L^{(2)}(1) = {} & 3(-2)^r\sum_{i=1}^{r}a_2(K_i)\\ & +3(-2)^{r-1}\sum_{i<j}\lambda_{ij}(L)^2\\ & +9(-2)^{r-3}\sum_{\substack{i<j,s<t\\(i,j)\neq(s,t)}}\lambda_{ij}(L)\lambda_{st}(L)\\ & +3(-2)^{r-2}\sum_{i<j}\lambda_{ij}(L)\\ & +(r-1)(-2)^{r-3},\end{aligned}$$

where $\lambda_{ij}(L)$ is the linking number of K_i and $K_j, i\neq j$. Since L_0 is a 2-component link and L_+, L_- and L_∞ are knots, we obtain the following:

$$\begin{aligned}&V_{L_0}^{(2)}(1) = 12(a_2(J_1)+a_2(J_2)) - 6a_1(L_0)^2 + 3a_1(L_0) - \frac{1}{2};\\ &V_{L_+}^{(1)}(1) = V_{L_-}^{(1)}(1) = V_{L_\infty}^{(1)}(1) = 0;\\ &V_{L_+}^{(2)}(1) = -6a_2(L_+);\\ &V_{L_-}^{(2)}(1) = -6a_2(L_-);\\ &V_{L_\infty}^{(2)}(1) = -6a_2(L_\infty).\end{aligned}$$

Using these foumulas, we have the result.

We consider the links C_+, C_-, C_0 and $C_\infty(=C)$ given in §1.

Lemma 2.

$$a_2(C_\infty) \equiv -\frac{1}{2}a_1(C_0) \pmod 2.$$

Proof. By Lemma 1, we obtain

$$a_2(C_\infty) = -\frac{1}{2}(a_2(C_+)+a_2(C_-)) + 2(a_2(J_1)+a_2(J_2)) + \frac{1}{2}a_1(C_0)^2,$$

where J_1 and J_2 are the components of C_0. Since $a_2(C_+)-a_2(C_-)=a_1(C_0)$, the right-hand side is equal to

$$-a_2(C_-)+2(a_2(J_1)+a_2(J_2))-\frac{1}{2}a_1(C_0)+\frac{1}{2}a_1(C_0)^2.$$

Two knots C_+ and C_- have period 2 for the covering translation of $\Sigma_2(U)$. C_+^* and C_-^* are the factor knots of C_+ and C_-, respectively, and $U=f(\tilde{U})$. Since $|lk(C_+^*, U)|=1$, we have the following relationship between the Conway polynomials of C_+ and C_+^* [10]:

$$\nabla_{C_+}(z)\equiv\nabla_{C_+^*}(z)^2 \pmod 2.$$

Since $\nabla_{C_+^*}(z)^2=1+2a_2(C_+^*)z^2+O(z^4)$, we obtain $a_2(C_+)\equiv 2a_2(C_+^*)\equiv 0 \pmod 2$. We have the same result for C_-. Hence

$$a_1(C_0)=a_2(C_+)-a_2(C_-)\equiv 0 \pmod 2.$$

This completes the proof.

In order to prove Main Theorem, by Lemma 2, we have only to calculate $a_1(C_0)$, the linking number of C_0. Before doing this, we consider the writhe of C and the second coefficient of the Conway polynomial of K.

We consider the oriented knot U and the arc C^* as shown in Fig. 5(1). Suppose the tangle P in Fig. 5(1) consists of n strings and has q crossings. If we trace the arc C^* from the bottom endpoint according to its orientation, we can number the n strings in the tangle P in the order of passage. We denote the string with number i by γ_i, $i=1,2,\cdots,n$. We devide the q crossings in the tangle P into two types. A crossing where γ_i and γ_j, $i\equiv j \pmod 2$, intersect, is called of Type I. A crossing which is not of Type I is called of Type II. Let α_+(resp. α_-) be the number of Type I crossings with positive (resp. negative) signatures. Let β_+(resp. β_-) be the number of Type II crossings with positive (resp. negative) signatures. Then it is clear that $q=\alpha_+ +\alpha_- +\beta_+ +\beta_-$. From Fig. 5(1), U and C^* intersect at $2p(=2(n-1))$ crossings.

First we calculate the writhe of C.

Lemma 3. *Let w be the writhe of C. Then*

$$w=2(\alpha_+ -\alpha_- -\beta_+ +\beta_-)+p.$$

Proof. We notice the knot C is oriented as shown in Fig. 5(2). Among the strings in the tangle P in Fig. 5(2), γ_i, $i\equiv 0 \pmod 2$, has the orientation opposite to the original one in Fig. 5(1). So, the signature of a Type II crossing in the tangle P in Fig. 5(1) changes in Fig. 5(2). However, the signature of a Type I crossing does not change in Fig. 5(2). Hence the sum of the signatures of all the crossings in the tangle P is $\alpha_+ -\alpha_- -\beta_+ +\beta_-$. As for the other tangle ꟼ , we have the same result. If we consider the p crossings which are not included in these two tangles, the sum of the signatures is p since all the p crossings are positive. This completes the proof.

Next, we calculate the second coefficient of the Conway polynomial of K. To do this, we transform K. Replacing the tangle R in Fig. 3 with S in Fig. 4, K is deformed to a trivial knot U with C^* as shown in Fig. 5(1). We deform U along the band b so that the tangle S becomes small as shown in Fig. 4. If we replace S with the tangle R as shown in Fig. 3, we obtain K again. Since we can gather crossings derived from twists of the deformed band b, we may assume that K has a diagram as shown in Fig. 8, where $\boldsymbol{P}$ is the 2n-string tangle obtained from P by replacing each arc γ_i with parallel two arcs in the projection plane and W is a tangle whose strings are twisted. We orient K as shown in Fig.8.

Lemma 4.

$$a_2(K)=\begin{cases}-(r+\alpha_+-\alpha_-+\beta_+-\beta_-) & \textit{if } t(W)=2r,\\ r+1+\alpha_+-\alpha_-+\beta_+-\beta_- & \textit{if } t(W)=2r+1,\end{cases}$$

where $t(W)$ is the number of the half twists in W and $r\in \boldsymbol{Z}$

Proof. First, we prove for the case $t(W)=2r$. The signatures of the two crossings in the tangle R are positive. Since the link L_-, obtained by changing a crossing in R, is a trivial knot and the link L_0, obtained by smoothing the crossing, is a 2-component link, the second coefficient $a_2(K)$ of the Conway polynomial of K is equal to the linking number of L_0 from the recursive formula of the Conway polynomial. Let $L_0=K_1\cup K_2$, where K_1 is the component which passes the point at infinity. The sums of the signatures of the crossings where K_1 and K_2 intersect in W and $\boldsymbol{P}$ are $-2r$ and $-2(\alpha_+-\alpha_-+\beta_+-\beta_-)$, respectively. Since the sum of the signatures of the other crossings where K_1 and K_2 intersect is zero, the linking number of L_0 is equal to $-(r+\alpha_+-\alpha_-+\beta_+-\beta_-)$. Thus $a_2(K)=-(r+\alpha_+-\alpha_-+\beta_+-\beta_-)$.

Next, we prove for the case $t(W)=2r+1$. The signatures of the two crossings in R are negative. Considering in the same way, $-a_2(K)$ is equal to the linking number of the 2-component link L_0 obtained by smoothing a crossing in R. Let $L_0=K_1\cup K_2$, where K_1 is the component which passes the point at infinity. The sums of the signatures of the crossings where K_1 and K_2 intersect in W and $\boldsymbol{P}$ are $-(2r+1)$ and $-2(\alpha_+-\alpha_-+\beta_+-\beta_-)$, respectively. Since the sum of the signatures of the other crossings where K_1 and K_2 intersect is -1, the linking number of L_0 is equal to $-(r+1+\alpha_+-\alpha_-+\beta_+-\beta_-)$. Hence $a_2(K)=r+1+\alpha_+-\alpha_-+\beta_+-\beta_-$. This completes the proof.

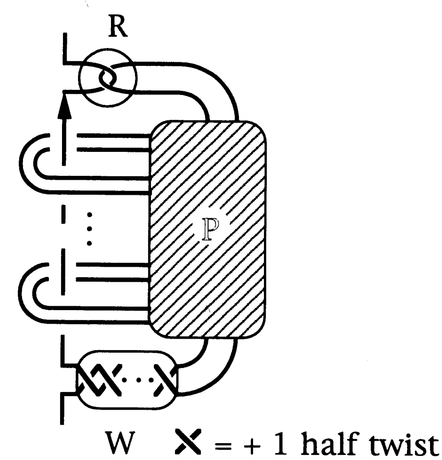


Fig. 8

In order to obtain $\Sigma_2(K)$ by doing surgery on C, we need a surgery coefficient for C. It is shown in [7] that a surgery coefficient for C is $\frac{\pm|H_1(\Sigma_2(K))|}{2}$, where $|H_1(\Sigma_2(K))|$ is the order of the first homology group of $\Sigma_2(K)$. Note that $|H_1(\Sigma_2(K))|=|\Delta_K(-1)|$, where $\Delta_K(-1)$ is the value of the normalized Alexander polynomial $\Delta_K(t)$ at $t=-1$. Note that $\Delta_K(t)=\nabla_K(t^{1/2}-t^{-1/2})$. Let $N(C)$ be a tubular neighbourhood of C. Let l and m be a preferred longitude (see p. 31 of [12]) and a meridian of $N(C)$, respectively. We assume that l and C have parallel orientation and $lk(m, C)=1$. The tangle $R=(B_1, k_1)$ in Fig. 8 is deformed as in Fig. 9, where e is the equator given in Fig. 2. Let V be the solid torus which is the 2-fold branched covering of B_2 branched over k_2. Let $g:\Sigma_2(K)\to S^3$ be the 2-fold branched cover branched over K induced by f in §1. Then the homeomorphism $h:\partial B_2\to\partial B_0$ given in §1 is covered by a longitudinal twist $\tilde{h}$ of a solid torus. Thus $\Sigma_2(K)=(S^3-intN(C))\cup_{\tilde{h}} V$, and $\tilde{h}$ sends a meridian of ∂V to the curve m' which is homologous to $m+2e'$ in $\partial N(C)$, where e' is one of the components of $g^{-1}(e)$. From Fig. 9, e' is homologous to $l+(t(W)+p+w)m$, and so m' is homologous to $2l+(2t(W)+2p+2w+1)m$. Therefore, $\Sigma_2(K)$ is obtained by $k/2$ surgery on C, where $k=2(t(W)+p+w)+1$.

We consider the case $t(W)=2r$. Since, by Lemmas 3 and 4, $w=2(\alpha_+-\alpha_--\beta_++\beta_-)+p$ and $r=-a_2(K)-(\alpha_+-\alpha_-+\beta_+-\beta_-)$, we have

$$\begin{aligned}2r+p-(-w)&=-2a_2(K)-2(\alpha_+-\alpha_-+\beta_+-\beta_-)+p\\&\quad+2(\alpha_+-\alpha_--\beta_++\beta_-)+p\\&=-2a_2(K)-4(\beta_+-\beta_-)+2p.\end{aligned}$$

Hence $\frac{k-1}{4}=-a_2(K)-2(\beta_+-\beta_-)+p$. Note that $\Delta_K(-1)\equiv 1 \pmod 4$. So we obtain $k=\Delta_K(-1)$ and

$$-\frac{1}{2}\Big(\frac{\Delta_K(-1)-1}{4}+a_2(K)\Big)=(\beta_+-\beta_-)-\frac{p}{2}.$$

For the case $t(W)=2r+1$, we have

$$2r+p+1-(-w)=2a_2(K)-4(\beta_+-\beta_-)+2p-1.$$

Hence $\frac{k+1}{4}=a_2(K)-2(\beta_+-\beta_-)+p$. So we obtain $k=-\Delta_K(-1)$ and

$$\frac{1}{2}\Big(\frac{\Delta_K(-1)-1}{4}+a_2(K)\Big)=(\beta_+-\beta_-)-\frac{p}{2}.$$

Therefore we have:

Lemma 5.

$$\beta_+-\beta_--\frac{p}{2}=(-1)^{t(W)+1}\frac{1}{2}\Big(\frac{\Delta_K(-1)-1}{4}+a_2(K)\Big).$$

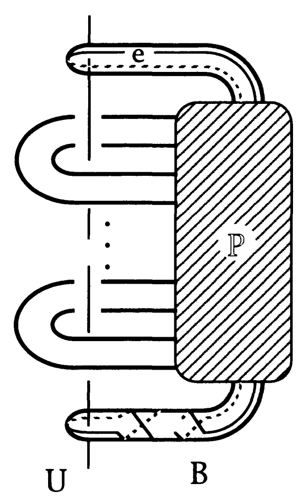


Fig. 9

We calculate the linking number of C_0.

Lemma 6.

$$a_1(C_0)=(-1)^{t(W)+1}\frac{1}{2}(\frac{\Delta_K(-1)-1}{4}+a_2(K)).$$

Proof. In the tangle P, the crossings where two different componets intersect are those of type II. Since the orientations of strings in P of Fig. 5(1) and Fig. 6 coincide, the sum of their signatures is $\beta_+-\beta_-$. For the other tangle, we have the same result. We consider the p crossings which are not included in these two tangles. Since they are the crossings at which two different components intersect and all the signatures of them are negative, the sum of thier signatures is $-p$. Hence the linking number of C_0 is equal to $\frac{1}{2}\{2(\beta_+-\beta_-)-p\}=\beta_+-\beta_--\frac{p}{2}$. By Lemma 5, we have the desired formula.

Proof of Main Theorem. Since

$$-\frac{1}{2}a_1(C_0)=(-1)^{t(W)}\frac{1}{4}(\frac{\Delta_K(-1)-1}{4}+a_2(K))$$
$$\equiv\frac{1}{4}(\frac{\Delta_K(-1)-1}{4}+a_2(K)) \pmod 2,$$

we have only to check $\frac{1}{4}(\frac{\Delta_K(-1)-1}{4}+a_2(K))\equiv a_4(K) \pmod 2$.
Since $\Delta_K(-1)=\nabla_K(2\sqrt{-1})$,

$$\Delta_K(-1)=1-4a_2(K)+16a_4(K)-64a_6(K)+\cdots.$$

Hence

$$\frac{\Delta_K(-1)-1}{4}=-a_2(K)+4a_4(K)-16a_6(K)+\cdots.$$

And so,

$$\frac{1}{4}(\frac{\Delta_K(-1)-1}{4}+a_2(K))=a_4(K)-4a_6(K)+\cdots$$
$$\equiv a_4(K) \pmod 4.$$

This completes the proof of Main Theorem.

EXAMPLE. We consider the knot 8_{13} [12, Appendix C]. If we apply the modification as shown in Fig. 2 for the 3-ball B_1 or B_2 as shown in Fig. 10, we have a trivial knot. These two unknotting operations are not equivalent [13]. By an isotopy preserving B_1, we can deform the knot 8_{13} into the knot as shown in Fig. 8, where $p=2$, $t(W)=-11$, $\boldsymbol{P}$ is a tangle given in Fig. 11(1). And so, the surgical knot C_1 is the torus knot of type (2,7). The writhe of C_1 is -6, and so $\Sigma_2(8_{13})$ is obtained by $-29/2$ surgery on C_1. Similarly, using B_2, the knot 8_{13} is deformed into the knot as shown in Fig. 8, where $p=2$, $t(W)=2$, and $\boldsymbol{P}$ is a tangle given in Fig. 11(2). And so, the surgical knot C_2 is the knot 10_{124}, which is the torus knot of type (3,5). The writhe of C_2 is 10, and so $\Sigma_2(8_{13})$ is obtained by 29/2 surgery on C_2. Although the knot type of C_1 is different from that of C_2, from Main Theorem, we have $Arf(C_1) \equiv Arf(C_2) \equiv a_4(8_{13})$ (mod 2). In fact, $a_2(C_1)=6$, $a_2(C_2)=8$ and $a_4(8_{13})=2$.

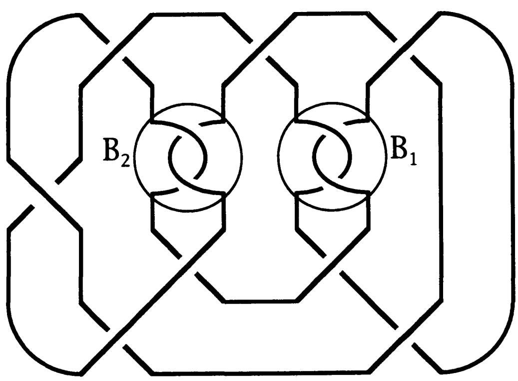
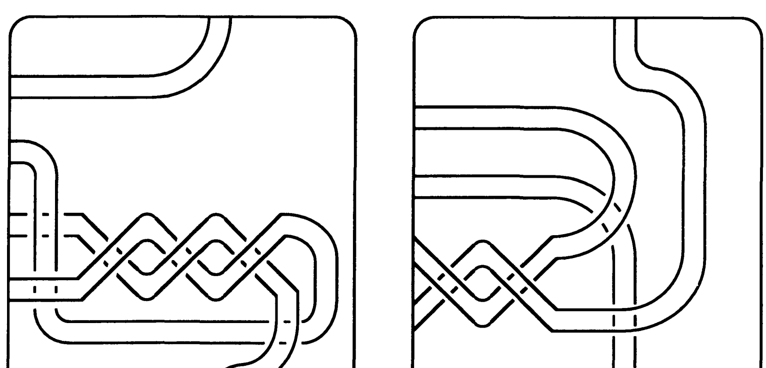


Fig. 10



(1) (2)

Fig. 11

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Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-Ku
Osaka 558, Japan

Current Address
Department of Mathematics
Yamaguchi University
Yamaguchi 753, Japan