Yamada, Y.
Osaka J. Math.
32 (1995), 185–192

AN EXTENSION OF WHITNEY'S CONGRUENCE

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(Received July 1, 1993)

1. Introduction and Main results

Throughout this paper, we will work in the *PL* category, and all embeddings will be locally flat.

Let M be a connected and oriented 4-manifold, F a closed and connected surface of Euler characteristic $\chi(F)$. For a given embedding of F into $M (F \subset M)$, let $e(M,F)$ be the normal Euler number of it, and let $[F]$ be the element in $H_2(M;Z_2)$ represented by F in M. We are interested in the ralation between $e(M,F)$ and $[F]$. In the case of $M=S^4$, the following theorem is well-known.

Theorem 1.1 (H. Whitney [8]: Whitney's congruence). If $M=S^4$,

$$e(M,F)+2\chi(F)\equiv 0 \quad mod\ 4.$$

For some time, we assume that M is closed and $H_1(M;Z)=\{0\}$. We will define a Z_4-quadratic map q from $H_2(M;Z_2)$ to Z_4 as follows. By the assumption $H_1(M;Z)=\{0\}$, the mod 2-reduction map p_2 from $H_2(M;Z)$ to $H_2(M;Z_2)$ is surjective. For a given element α in $H_2(M;Z_2)$, we define $q(\alpha)$ by

$$q(\alpha)\equiv\tilde{\alpha}\circ\tilde{\alpha} \quad \text{mod } 4,$$

where $\tilde{\alpha}$ is an element of ${p_2}^{-1}(\alpha)$ and $\circ$ is the intersection form on $H_2(M;Z)$.

The well-definedness of q is easy to see, and q is Z_4-quadratic, i.e.,

$$q(\alpha+\beta)\equiv q(\alpha)+q(\beta)+2(\alpha\bullet\beta) \quad \text{mod } 4,$$

where $\bullet$ is (Z_2-valued) intersection form on $H_2(M;Z_2)$, and 2: $Z_2\to Z_4$ is the natural embedding.

Using the quadratic function q, we extend Theorem 1.1 as follows:

Theorem 1.2.

$$e(M,F)+2\chi(F)\equiv q([F]) \quad mod\ 4.$$

It is well-known that if $F \subset M$ is characteristic (i.e., $[F]$ is dual to the 2nd Stiefel-Whitney class $w_2(M)$), then $\sigma(M) \equiv [F] \circ [F]$ mod 8, where $\sigma(M)$ is the signature of M. Thus we have

Corollary 1.3 (V.A. Rochlin [5], see also [4]: Generalized Whitney's congruence). *If $F \subset M$ is characteristic,*

$$e(M, F) + 2\chi(F) \equiv \sigma(M) \quad mod\ 4.$$

Theorem 1.2 can be extended to the general case in which the only assumption on M is its orientability; we need not assume that $H_1(M; Z) = \{0\}$ nor that M is closed. M can even be non-compact. In fact, we can prove the following.

Theorem 1.4. *Let M be an oriented 4-manifold. A map which assigns $e(M, F) + 2\chi(F)$ mod 4 to an embedding $F \subset M$ induces a Z_4-quadratic map from $H_2(M; Z_2)$ to Z_4. We will also call it q.*

For immersions from F into M, we have the following.

Corollary 1.5. *Let M be an oriented 4-manifold, F an closed surface immersed in M with only normal crossings. Then*

$$e(M, F) + 2\chi(F) + 2\# self(F) \equiv q([F]) \quad mod\ 4,$$

where $\# self(F)$ is the number of self-intersection points of F.

After writing the first version of this paper, we were informed by Prof. B.-H. Li that he found a general formula which includes our theorem 1.4([3]). In fact, he works in $(2n, n)$-dimensional case. His proof is homotopy-theoretic, on the other hand, ours is geometric.

2. A Connected sum formula

For a given embedding $F \subset M$, assume that there is a connected sum decomposition of M:

$$M = M_1 \# M_2 = \text{punc}\, M_1 \bigcup\nolimits_{\partial} \text{punc}\, M_2,$$

such that each embedding $F_i \subset \text{punc}\, M_i$ is proper (i.e., $F_i \bigcap \partial(\text{punc}\, M_i) = \partial F_i$), where $\text{punc}\, M_i$ is M_i with an open 4-ball deleted, and $F_i = F \bigcap \text{punc}\, M_i$, for $i = 1, 2$. Here we assume that F intersects $\partial(\text{punc}\, M_i)$ transversely. The symbol $\bigcup_{\partial}$ on the right-hand side means disjoint union with boundary identified by an orientation reversing homeomorphism.

Then

$$(\partial \text{punc}\, M_1, \partial F_1) = (\partial \text{punc}\, M_2, \partial F_2) \cong (S^3, L)$$

for a certain link L in S^3.

Let S be a (connected) Seifert Surface for L in S^3, and regard it as being in $S^3 = \partial B^4 : S \subset S^3 = \partial B^4 \subset B^4$. Let $(M_i, \hat{F}_i)$ denote $(\text{punc}\, M_i, F_i) \bigcup_\partial (B^4, S)$, for $i = 1, 2$. Now, we have

Lemma 2.1 Connected Sum Formula.
Let M_i, F, $\hat{F}_i$ be as above. Then

$$e(M_1 \# M_2, F) = e(M_1, \hat{F}_1) + e(M_2, \hat{F}_2).$$

In particular,

$$e(M_1 \# M_2, F_1 \# F_2) = e(M_1, F_1) + e(M_2, F_2).$$

Proof. Let ν be a non-zero, normal vector field over S in S^3. We can take a transverse push-off F' of F in M such that $F' \cap (\partial \text{punc}\, M_1) = \nu(L)$. Then

$$e(M, F) = \sum_{p \in F \cap F'_1} sign(p) + \sum_{p \in F \cap F'_2} sign(p)$$

$$= \sum_{p \in F_1 \cap F'_1} sign(p) + \sum_{p \in F_2 \cap F'_2} sign(p),$$

where F'_i is $F' \bigcap \text{punc}\, M_i$, for $i = 1, 2$. On the other hand, if we regard $F'_i \bigcup_\partial \nu(S)$ as a push-off of $\hat{F}_i$ in M_i, then

$$e(M_i, \hat{F}_i) = \sum_{p \in F_i \cap F'_i} sign(p).$$

Thus we have the lemma. ∎

3. Proof of Theorem 1.2

The proof is devided into 3 steps. We are given an embedding $F \subset M$.

(Step 1) We will show the theorem for $M = mCP^2 \# n\overline{CP^2} (m + n > 0)$. We have a standard handlebody decomoposition of M:

$$M = H^0 \bigcup \left(\bigcup_{i=1}^{m+n} H_i^2\right) \bigcup H^4,$$

where H^r is an r-handle, and fix an identification

$$h_i^2 : D^2 \times D^2 \xrightarrow{\cong} H_i^2.$$

Without loss of generality (by general position augument), we can assume the following.

(1) $F\cap H^4=\phi$.

(2) $F\cap H_i^2=h_i^2(D^2\times\{\text{finite points in int}\,D^2\})$,

i.e., each component of $F\cap H_i^2$ is parallel to the core of H_i^2.

We regard M as $S^4\# M$ by $H^0\cup(\bigcup_{i=1}^{m+n}H_i^2\cup H^4)=B^4\cup_\partial \text{punc}\,M$, and use the notation "$M_i, F_i, \hat{F}_i$" as in the last section ($M_i=S^4$, $M_2=M$). Note that F_2 consists of some proper disks, and F_1 is F with some open 2-disks deleted.

We orient all the components of F_2, and take a Seifert surface S so that the orientation of S is compatible with that of F_2. Note that $\hat{F}_2(=S\cup F_2)$ is an orientable closed surface.

In the situation above, we have the following equalities.

(1) $e(M,F)=e(S^4,\hat{F}_1)+e(M,\hat{F}_2)$

(2) $e(S^4,\hat{F}_1)+2\chi(\hat{F}_1)\equiv 0 \mod 4$

(3) $e(M,\hat{F}_2)+2\chi(\hat{F}_2)\equiv e(M,\hat{F}_2)\equiv q([\hat{F}_2]) \mod 4$

(4) $[\hat{F}_2]=[F]$ in $H_2(M;Z_2)$

(5) $\chi(\hat{F}_1)+\chi(\hat{F}_2)\equiv\chi(F) \mod 2$

The first holds by the connected sum formula, the second by Theorem 1.1, the third follows from the orientability of $\hat{F}_2$, and the others are easy to verify. Now the theorem in this case follows from these equalities.

(Step 2) We will prove the theorem for a simply-connected manifold. We use the following fact [7], [4: Fact (2)].

Fact. *Let M be a simply-connected, closed and oriented 4-manifold. Then there exist integers $l,m,n\geq 0$ such that*

$$M\#(l+1)\overline{CP^2}=mCP^2\#n\overline{CP^2}.$$

For a given embedding $F\subset M$, we take a connected sum $M\#(l+1)CP^2\#l\overline{CP^2}$ disjointly from the neighborhood of F. It is easy to see that $e(M,F)$, $\chi(F)$ and $q([F])$ are unchanged by the connected sum. Thus the proof is reduced to the first step.

(Step 3) The general case ($H_1(M;Z)=\{0\}$). For a given embedding $F\subset M$, and an element γ of $\pi_1(M)$, we take an embedded circle c in M such that

(1) c represents the element γ,

(2) $c\cap F=\phi$, and

(3) c bounds an immersed oriented surface G in M which satisfies the following condition: for each generator x of $H_2(M;Z)$, there is a representing surface T_x such that $G\circ T_x=0$ and $G\circ F=0$.

This is possible because of the assumption $H_1(M;Z)=\{0\}$ and $\partial G\neq\phi$. We do surgery on M along c, and repeat it till $\pi_1(M)$ becomes trivial. At each surgery, $e(M,F)$, $\chi(F)$ and $q([F])$ remain unchanged. We see it as follows ([6]). Suppose that we get

$$M'=D^2\times S^2\bigcup_{\varphi|\partial}\{M\setminus \mathrm{int}\varphi(S^1\times D^3)\}$$

from M by surgery along c, where φ is a trivialization of a tubular neighborhood of c. Then the homology of M changes into

$$H_2(M';Z)\cong H_2(M;Z)\oplus Z\langle x\rangle\oplus Z\langle y\rangle,$$

where $x=[(D^2\times *)\bigcup G]$ and $y=[*\times S^2]$.
Thus the intersection form changes as

$$H_2(M';Z)\cong H_2(M,Z)\oplus\begin{pmatrix} * & 1\\ 1 & 0\end{pmatrix}.$$

Under the isomorphism, the corespondence of $[F]_{old}$ and $[F]_{new}$ is:

$$[F]_{new}\leftrightarrow[F]_{old}+0+0.$$

Thus $q([F])$ is unchanged, and the proof is reduced to the second step. ■

4. Proof of Theorem 1.4

This proof is devided into 3 steps. We are given an embedding $F\subset M$. If M is closed and $H_1(M;Z)=\{0\}$, then Theorem 1.2 applies. We will consider other cases step by step.

(Step 1) Suppose that M is closed but $H_1(M;Z)\neq\{0\}$. We perform surgery along embedded circles c which are disjoint from F and represent non-zero elements of $H_1(M;Z)$.

Suppose that two embedded surfaces $F_1,F_2\subset M$ satisfy $[F_1]=[F_2]$ in $H_2(M;Z_2)$ and they are in general position. In Z_2-coefficient chain complex, we can take a 3-chain Δ^3 whose boundary is F_1+F_2. We will show that we can do surgery (along c to get M' from M) so that F_1 and F_2 also satisfy $[F_1]=[F_2]$ in $H_2(M';Z_2)$.

Since Δ has boundary F_1+F_2 with Z_2-coefficient, the small normal circle of F_i intersects Δ at an odd number of points. If necessary, by connecting c with the small normal circle along Δ, we can choose c such that the geometric intersection number $\sharp(c\bigcap\Delta)$ is even and $N(c)\bigcap\Delta$ consists of an even number of 3-balls B, where $N(c)$ is a thin tubular neighborhood of c. Then in $D^2 \times S^2$ which is to be attached to $M \backslash \operatorname{int} N(c)$, we can take $\{$half as many proper arcs$\} \times S^2$ whose boundary is the same as $(\partial N(c), \partial B)$. Then it is clear that $F_1 + F_2$ bounds a new 3-chain Δ' in M' with Z_2-coefficient.

(Step 2) Suppose that M is compact but $\partial M \neq \phi$.
Let DM be the double of $M (DM = M \bigcup_\partial -M)$. The mapping q is already well-defind over DM by Step 1. Over M, the mapping is the composition $q \circ i_*$, where i_* is the homology homomorphism: $H_2(M; Z_2) \to H_2(DM; Z_2)$, induced by canonical inclusion i.

(Step 3) Suppose that M is non-compact. There is a seqence of countably many compact oriented 4-manifolds and inclusions:

$$M_1 \subset M_2 \subset M_3 \subset \cdots \text{ such that } \bigcup_{i=1}^{\infty} M_i = M.$$

Since F is compact, there is a sufficiently large n such that $F \subset M_n$. We can apply the method in Step 2. ■

5. Proof of Corollary 1.5

We are given an immersed surface F in M with only normal crossings. For a crossing point p, we take a 4-ball neighborhood B around p. To remove the crossing at p, we cut out $\operatorname{int} B \bigcap F$ from F, where $\partial B \bigcap F \subset \partial B$ is a Hopf link, and glue in an annulus $A \subset \partial B$. We call this new surface $\tilde{F}$. By the construction, $[\tilde{F}] = [F]$ in $H_2(M; Z_2)$, $\chi(\tilde{F}) \equiv \chi(F)$ mod 2 and $\# self(\tilde{F}) = \# self(F) - 1$.

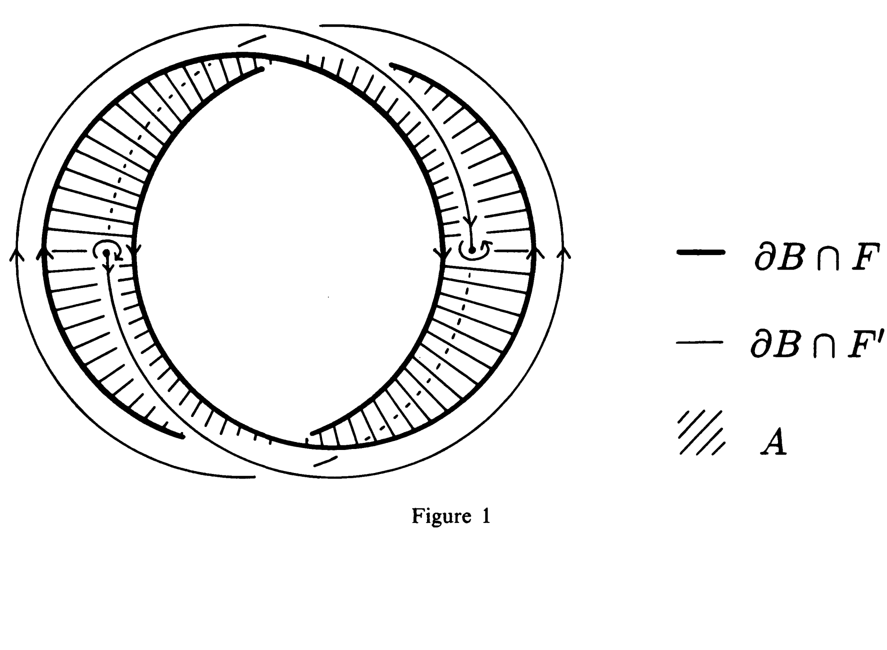

Figure 1

We show $e(M,\tilde{F})=e(M,F)\pm 2$. Let F' be a push-off of F. We can assume that F' is parallel to F near p and in particular $\partial B\bigcap F'$ gives a trivial framing for each component of $\partial B\bigcap F$ in $\partial B\cong S^3$. Then we can take an annulus A such that $\partial A=\partial B\bigcap F$ and $A\bigcap(\partial B\bigcap F')$ consists of two points whose signs are the same (Figure 1). Let A' be a push-off of A which is properly embedded in B^4 such that $\partial A'=\partial B\bigcap F'$. If we regard $(F'\setminus\mathrm{int}\,B)\bigcup_\partial A'$ as a push-off of $\tilde{F}$, we have the claim.

We can repeat the above process till $\#\mathit{self}(F)$ becomes zero without changing both sides of the congruence. Thus we can reduce the corollary to Theorem 1.2 or 1.4. ■

6. Examples

In this section, we will give two examples for Theorem 1.2.

Example 1. **([2])** Let $M=mCP^2\# n\overline{CP^2}(m+n>0)$, and identify its 2nd homology $H_2(M;Z_2)$ with $\oplus_{i=1}^m Z_2\langle\xi_i\rangle \oplus \oplus_{j=1}^n Z_2\langle\eta_j\rangle$. For an embedding $F\subset M$, such that $[F]\equiv\sum_{i=1}^k \xi_i+\sum_{j=1}^l \eta_j$,

$$e(M,F)+2\chi(F)\equiv k-l \mod 4.$$

Example 2. Let $M=S^2\times S^2$, and identify its 2nd homology $H_2(S^2\times S^2;Z)$ with $Z\langle x\rangle \oplus Z\langle y\rangle$, where $x\circ x=y\circ y=0$ and $x\circ y=y\circ x=1$. Let $S^2(m)\subset S^2\times S^2$ be an embedding of S^2 representing $1\cdot x+m\cdot y$, which is for instance the graph of a degree m map g_m: $S^2\to S^2$. Then

$$e(S^2\times S^2, S^2(m))=2m.$$

As an element of $H_2(S^2\times S^2;Z_2)\cong Z_2\langle\underline{x}\rangle\oplus Z_2\langle\underline{y}\rangle$,

$$[S^2(m)]\equiv\begin{cases}\underline{x} & \text{if } m \text{ is even}\\ \underline{x}+\underline{y} & \text{if } m \text{ is odd}\end{cases}.$$

Thus we have

$$q([S^2(m)])\equiv\begin{cases}0 & \text{if } m \text{ is even}\\ 2 & \text{if } m \text{ is odd}\end{cases} \mod 4.$$

Example 2 shows that our main theorem is optimal in a sense.

Acknowledgement. The author would like to thank to Professor Bang-He Li for sending his preprint [3]. Also the author would like to express sincere gratitude to Professors Yukio Matsumoto and Mikio Furuta for their valuable advice and encouragement.

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