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ON SMOOTH K-GONAL CURVES WITH ANOTHER FIXED PENCIL

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Let M_g be the moduli scheme of complex smooth complete curves of genus g and $M_{g,k}$ its integral subvariety parametrizing the k-gonal curves. For any integer a with $k \leq a \leq g/2$, let $U(k,a;g)$ be the constructible subset of $M_{g,k}$ parametrizing the k-gonal curves with base point free, simple and complete g_a^1; in particular, note that the g_a^1 will not be composed with the g_k^1; hence we have $g \leq (a-1)(k-1)$. In section 2 we will consider $U(k,a;g)$ very briefly (using [1]).

Our main result is the following theorem.

Theorem 0.1. *Fix non negative integers g, k, a, u, r, with $4 \leq k \leq a \leq g/2$. Set $n := (a-1)(k-1) - g$ and assume $g \leq (a-1)(k-1)$ i.e. $n \geq 0$. Assume the following condition:*

$$3g \geq 2ak - 4a - 4k + 3 \tag{1}$$

Then there is an integral family T of genus g curves in $U(k,a;g)$ such that for a general $C \in T$ (with g_k^1 and g_a^1 as linear systems) we have

$$\begin{aligned} \dim |rg_k^1 + ug_a^1| = (r+1)(u+1) + n - (k-u-1)(a-r-1) + \\ + \max(0, \max(k-u-1, 0) \cdot \max(a-r-1, 0) - n) - 1 \end{aligned} \tag{2}$$

In particular $|rg_k^1 + ug_a^1|$ is not special if $n \geq (k-1-u)(a-1-r)$.

As will be clear from the proof of 0.1 (or essentially just by Riemann-Roch and Serre duality) the value given by (2) corresponds to the minimum possible value (which by semicontinuity will then be the value for $|rg_k^1 + ug_a^1|$ in an open subset of $U(k,a;g)$); indeed we will just prove the existence of one such curve, C, and then use semicontinuity to claim the same result for a general element in any component, T, of $U(k,a;g)$ containing C.

In section 3 we will prove Theorem 0.1. For a related statement (which gives the existence of large families in $U(k,a;g)$ for which the value for $|rg_k^1 + ug_a^1|$ is given and different from the one in eq. (2)), see Theorem 3.1. The proof of 0.1 is just a very classical game with singular curves contained as divisors of type

(k,a) on a smooth quadric $Q=\boldsymbol{P}^1\times\boldsymbol{P}^1$. This proof is an extension of the proof of [3], Prop. 1, which in turn is an extension of the very first step in the proof of the main theorem of [15]. The statemant of [3], Prop. 1, was generalized from another point of view in [8], Prop. 1.1, (again only for general k-gonal curves). The first section of this paper shows (see Proposition 1.1 and Remark 1.2) how easily the proof of [8], Prop. 1.1, is extended to our setting. Indeed we consider a problem to find nice subvarieties, T, of M_g such that every curve $C\in T$ has some natural property and, viceversa, to show that a property known to hold in general (say for $M_{g,k}$) holds outside subvarieties of a given codimension.

In section 4 we will consider very informally a few loose ends related to Theorems 0.1 and 3.1: curves with pairs of suitable Weierstrass points or with suitable Weierstrass pairs (in the sence of [2], p. 365). Then we will list a few open questions (on which we hope to work).

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1. As an interesting (in our opinion) example of this program of bounding the dimension of subvarieties of $M_{g,k}$ formed by curves with some bad property, we give (see 1.1) the extension of [8], Prop. 1.1, in our setting; the proof is just to check that these are exactly the conditions used in the proof of [8], Prop. 1.1. In order to keep our discussion as elementary and self-contained as possible we provide the proof which is basically the same as in [8], Prop. 1.1.

Proposition 1.1. *Fix non negative integers g, k, t, m, n with $g\geq 4$, $g\geq 2m+n(k-1)$, $t\geq 2m+(n+2)k+g-n-5$, $2k\leq g+1$. Let T be an integral subvariety of $M_{g,k}$ with $\dim(T)=t$ and let $D\in C^{(m)}$ (the m-fold symmetric product of C). Assume that for a general $C\in T$ the (or the chosen one) g_k^1 is simple and $\dim|ng_k^1|=n$. Assume that there is no $E\in g_k^1$ with $E\leq D$. (For example this holds if $m<k$.) Assume either $n>0$ or that C has a unigue g_k^1 and no other base point free pencil with negative Brill-Noether number. Then $\dim|ng_k^1+D|=n$.*

Proof. For $n=0$, this follows from the last assumption that C has a unique g_k^1 and no other base point free pencil with negative Brill-Noether number. So we assume $n\geq 1$. We break up the proof into two parts.

Step (1): Let C be any smooth curve possessing a complete and base point free g_k^1 and $D\in C^{(m)}$ such that $|D-g_k^1|=\emptyset$, $\dim|ng_k^1+D|>n$ and such that $|ng_k^1+D|$

is base point free and simple. Then we can construct a plane model of C as follows.

Take $A_1, A_2; F_1, \cdots, F_{n-1} \in g_k^1$ general and $G \in |ng_k^1 + D|$ general. Let $D_i := A_i + F_1 + F_2 + \cdots + F_{n-1} + D$ for $i = 1,2$. Consider the net $g_{nk+m}^2 := <D_1, D_2, G> \subseteqq |ng_k^1 + D|$ spanned by D_1, D_2 and G. Since $|ng_k^1 + D|$ is base point free and simple so is g_{nk+m}^2. Let C' be the plane model of C determined by g_{nk+m}^2. Since $g_k^1 + F_1 + F_2 + \cdots + F_{n-1} + D = <D_1, D_2> \subseteqq g_{nk+m}^2$ the plane curve C' has a singular point s of multiplicity $(n-1)k + m$ such that the pencil of lines through s induces the g_k^1 on C. In particular, there is a line L_i in $\boldsymbol{P}^2$ through s corresponding to $F_i \in g_k^1$, for $i = 1, \cdots, n-1$. Let $\nu' : X' \to \boldsymbol{P}^2$ be the blowing up of $\boldsymbol{P}^2$ at s and let E' be the exceptional divisor on X'. Then Γ' denotes the proper transform of $C' \subseteqq \boldsymbol{P}^2$ on X' and L_0' (resp. $L_1', \cdots, L_{n-1}'$) the proper transform of a general line of $\boldsymbol{P}^2$ (resp. of $L_1, \cdots, L_{n-1}$) on X'. The curve Γ' is contatined in the linear series $\boldsymbol{P}' := |(nk+m)L_0' - ((n-1)k+m)E'|$.

The intersection of Γ' with E' (resp. with L_i', $0 < i < n$) gives rise to the divisor $F_1 + \cdots + F_{n-1} + D$ (resp. F_i) on C. But L_i' $(i > 0)$ and E' intersect transversally at exactly one point q_i, and so q_i is a singular point of Γ' of multiplicity k.

Let $\nu : X \to X'$ be the blowing up of the surface X' at these points $q_1, \cdots, q_{n-1}$ and let $E_1, \cdots, E_{n-1}$ be the associated exceptional divisors on X. By L_0 (resp. Γ, E) we denote the proper transform of L_0' (resp. Γ', E') on X. We then have

$$\Gamma \in \boldsymbol{P} := |(nk+m)L_0 - ((n-1)k+m)E - \sum_{i=1}^{n-1} kE_i|.$$

Step (2). We now assume that the statement of the theorem is not true. Then there are $0 \leq m \in Z$, $n \in N$ with $g \geq 2m + n(k-1)$ such that for a curve C corresponding to the general point of T, there exists $D \in C^{(m)}$ with $|D - g_k^1| = \varnothing$ and $\dim |ng_k^1 + D| > n$. Since we may assume that $|ng_k^1 + D|$ is base point free and simple, we can construct plane models of C using two-dimensional linear subseries of $|ng_k^1 + D|$.

The Grassmannian $\boldsymbol{Gr}(2; |ng_k^1 + D|)$ parametrizes two dimensional linear subspaces of $|ng_k^1 + D|$. Consider the rational map $\Theta : (g_k^1)^{n-1} \times |ng_k^1 + D| \to \boldsymbol{Gr}(2; |ng_k^1 + D|)$, defined by $\Theta((F_1, \cdots, F_{n-1}), G) = <G \cup (g_k^1 + F_1 + \cdots + F_{n-1} + D)>$. The two-dimensional linear subseries we considered in Step (1) give rise to a dense quasi-projective subset of the image Z of Θ. Moreover, for a general element γ in Z, $(F_1, \cdots, F_{n-1}) \in (g_k^1)^{n-1}$ is completely determined by γ while G varies in an open subset of the two-dimensional linear subspace γ of $|ng_k^1 + D|$, and so $\dim \Theta^{-1}(\gamma) = 2$. Since $\dim |ng_k^1 + D| \geq n+1$ this implies $\dim Z \geq (n-1) + (n+1) - 2 = 2n - 2$.

A general element γ of Z gives rise to a plane model C' for C with a singular point s of multiplicity $(n-1)k + m$, as in Step (1). Hence, fixing s, from γ we obtain a 6-dimensional family of models for C on X' belonging to $\boldsymbol{P}'$. Each such model defines a closed point of $\underline{\mathrm{Hom}}_{X'}^{\boldsymbol{P}'}(\pi)$ (cf. [8], (0.3.3) for the precise definition and related basic facts). Varying γ in Z and C in some irreducible component of the inverse image of T in our fine moduli space S of curves of genus g, we obtain a quasi-projective subset $\Omega' \subseteqq \underline{\mathrm{Hom}}_{X'}^{P'}(\pi)$ of dimension at least

$$\dim T + \dim Z + 6 \geq 2m + (n+2)k + g - n - 5 + (2n-2) + 6 = 2m + (n+2)k + g + n - 1.$$

Each point on Ω' induces $n-1$ points of $q_1, \cdots, q_{n-1}$ on E' as explained in Step (1), and so we obtain a morphism $v: \Omega' \to (E')^{(n-1)}$. Let Ω be a non-empty fibre of v, and let $v(\Omega) = q_1 + \cdots + q_{n-1}$. We then have

$$\dim \Omega \geq \dim \Omega' - (n-1) \geq 2m + (n+2)k + g.$$

Blowing up X' at $q_1, \cdots, q_{n-1}$ as we did in Step (1) we get a quasi-projective irreducible subset M of $\underline{\mathrm{Hom}}_X^P(\pi)$ of dimension $\dim M \geq 2m + (n+2)k + g$.
In particular, $\dim M > g$. On the other hand [8], (0.3.3) implies that, for $\Gamma \in \boldsymbol{P}$, we have

$$\dim M \leq (\Gamma \cdot -K_X) + g - 1 =$$

$$(((nk+m)L_0 - ((n-1)k+m)E - \sum_{i=1}^{n-1} kE_i) \cdot (3L_0 - E - \sum_{i=1}^{n-1} E_i)) + g - 1$$

$$= 3(nk+m) - ((n-1)k+m) - (n-1)k + g - 1 = 2m + (n+2)k + g - 1.$$

This contradiction finishes the proof of the theorem. ◆

There are two main differences between this statement and [8], Prop. 1.1: here we are forced to assume $|ng_k^1| = n$ (which for general k-gonal curves was proved in [3], Prop. 1) and to make the last assumption (used in the first part of the proof of [8], Prop. 1.1 and as far as we know it is not automatically satisfied under our assumptions).

Remark 1.2. The assumption that the g_k^1 on C is simple comes for free in several interesting cases (often for trivial reasons). Here we will list a few cases. First of course k cannot be a prime. Since the g_k^1 is complete to find a contradiction we may assume $k = sb$, and that C is a degree s covering of a curve C' of genus $g' \geq 1$ and that the g_k^1 is induced from a g_b^1 on C'. By [10], Prop. 3, this implies $t \leq 2g - 2 - b(g'-1)$ and in particular $t \leq 2g-2$. If either k is odd or C has only finitely many g_k^1, then we must have $g' > 1$, hence $t \leq 2g - 2 - b$. And so on.

2. Here we will show (using [1]) that for a general $C \in U(k, a; g)$, the image of the map induced by the 2 pencils is a nodal curve. We work in characteriistic 0.

Consider a curve $C \in U(k, a; g)$; the g_k^1 is simple, the pair (g_k^1, g_a^1) induces a birational morphism $f: C \to Q := \boldsymbol{P}^1 \times \boldsymbol{P}^1$; set $Y := f(C)$.

Proposition 2.1. *For a dense set of curves $C \in U(k, a; g)$ the image Y of the birational morphism $f: C \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ induced by the g_k^1 and the g_a^1 has only nodes as singularities.*

Proof. This follows easily from standard deformation theory: see for instance line 3 from the bottom of page 347 of [1], i.e. the vanishing of suitable obstruction spaces to defrom any triple (curve + pair of pencils). ◆

REMARK 2.2. The same proof easily gives the dimension of the set of triples (curves + pair of pencils); as usual, for a constructible set Z, by $\dim(Z)$ we will denote the maximal dimension of any integral subvariety of a finite partition of Z into locally closed subset of an ambient scheme and irreducibility for a constructible subset means irreducibility of its closure.

3. We work over an algebraically closed field with characteristic 0 because we use as key results parts of [1] whose proof use the characteristic 0 assumption in an essential way. Let $Q = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be a smooth quadric surface of $\boldsymbol{P}^3$; $\boldsymbol{O}_Q(x, y)$ or $\boldsymbol{O}(x, y)$ or just (x, y) will denote the line bundle of bidegree x, y on Q; for any morphism $u: T \to Q$, $\boldsymbol{O}_T(x, y)$ will denote its pull-back to T. We fix the integers k, a, g, u, r and n as in the statement of Theorem 0.1.

Fix a curve $C \in U(k, a; g)$; the g_k^1 and the g_a^1 induce a birational morphism $f: C \to Q$; set $Y := f(C)$. Assume that Y has only nodes as singularities; by the adjunction formula it would have exactly n nodes. Viceversa, if we have such a nodal curve Y, its normalivation, C, will be a curve in $U(k, a; g)$ (at least if it has no g_b^1 with $b < k$; for this point, see the proof of 0.1). This is exactly the way in which we will prove the existence of C.

Proof of 0.1. First we want to check that for a general subset S of Q with $\mathrm{card}(S) = n$ there is an integral curve Y of type (k, a) on Q having exactly S as singular locus and only nodes on S. The assumption (1) in the statement of 0.1 means that $h^0(Q, (k, a)) \geq 3n + 1$; hence there is a, possibly non reduced or non integral curve of type (k, a) on Q with S contained in its singular locus. Thus the assumptions ii) and iii) of [1], Prop. 4.1, are satisfied. A curve of type (4,4) has arithmetic genus 9 and if it has at least 8 singular points it cannot have geometric genus $g \geq 4$. Since the only elliptic curves on Q are of type (2,2), the case $k = 4$, $a = 4$, $g = 1$ (i.e. $n = 8$) is excluded by the assumptions of 0.1. Hence, by [1], Prop. 4.1, we are sure of the existence of such an integral nodal curve Y of type (k, a); call C its normalization. To prove that $C \in U(k, a; g)$ we just have to check that the induced g_a^1 on C is complete; however, this will be a byproduct of the proof we will just give of the equality (2) for $r = 0$ and $u = 1$. The cohomology sequence of the exact sequence

$$0 \to \boldsymbol{O}_Q(-u-2,-r-2) \to \boldsymbol{O}_Q(k-u-2,a-r-2) \to \boldsymbol{O}_Y(k-u-2,a-r-2) \to 0 \quad (3)$$

easily computes $h^0(Q,\boldsymbol{O}_Q(k-u-2,a-r-2))=h^0(Y,\boldsymbol{O}_Y(k-u-2,a-r-2))$. Hence by Riemann-Roch, Serre duality on C and the adjunction formula, to prove Theorem 0.1 it is sufficitent to note that by the generality of S, the set S imposes the maximal possible number of conditions on $H^0(Q,\boldsymbol{O}_Q(k-u-2,\mathrm{a}-r-2))$. We have to check that the normalization, C, of a suitable nodal curve is indeed in $M_{g,k}$; usually by definition this implies that C has no g_b^1 with $b<k$. This is a statement of Brill-Noether type; to get around this difficulty, here we just assume $U(k,a;g)\neq\emptyset$ and just note that, as in the proof of Proposition 2.1, the image of a general enough curve will be nodal; then move the nodes (essentially as in [1]). ◆

It is obvious that under the assumptions of Theorem 0.1 for $(r,u)=(1,1)$ the g_k^1 and the g_a^1 are primitive in the sense of [8], i.e. base point free and adding to them just one point does not increase their dimension.

In the following result condition (6) is very restrictive.

Theorem 3.1. *Fix non negative integers g, k, a, u, r, ε with $4\leq k\leq a\leq g/2$. Set $n:=(a-1)(k-1)-g$ and assume $n\geq 0$. Set $\varepsilon':=\max(0,u-[k/2])$, $\varepsilon'':=\max(0,r-[a/2])$. Assume the following conditions*:

$$3g\geq 2ak-4a-4k+3 \tag{4}$$

$$\varepsilon a/2\leq n\leq (k-1-u)(a-1-r)+\varepsilon \tag{5}$$

$$0<\varepsilon\leq\varepsilon'+\varepsilon'' \tag{6}$$

Then there is an integral family T of curves in $U(k,a;g)$ with $\dim(T)\geq 2n-([k/2]+[a/2]+\varepsilon)+2$ and such that for a general $C\in T$ (with g_k^1 and g_a^1 as linear systems) we have

$$\begin{aligned}\dim|rg_k^1+ug_a^1| &= (r+1)(u+1)+n-(k-u-1)(a-r-1)+\\ &\quad+\max(0,\max(k-u-1,0)\cdot\max(a-r-1,0)-n+\varepsilon)-1 \end{aligned}\tag{7}$$

Proof. By the proof of 0.1 it is sufficient to find a suitable family, T', of nodal curves in $\boldsymbol{P}^1\times\boldsymbol{P}^1$ such that the set, S, of nodes of its general member is of the following type. We fix a line L of type (1,0) (i.e. inducing the g_a^1) and a line R of type (0,1). S is the union of a general subset S' of $\boldsymbol{P}^1\times\boldsymbol{P}^1$, a set A of at most $[a/2]$ points on L and a set A' of at most $[k/2]$ points on R in such a way that $A\cup A'$ loses exactly ε condition with respect to forms of type (u,r); this is possible by the definition of ε' and ε'' and by assumption (6). By the generality of S' and the non speciality assumption (5), ε will be the total loss of conditions imposed by S. To be sure to obtain integral nodal curves with this singular set, see the corresponding discussion (and quotation of [11], Th. 1.5) in the next section. ◆

We do not claim that the family T is maximal or that there is no other family with the same property for $|rg_k^1 + ug_a^1|$. Other constructions of suitable sets of nodes are possible, but they seem (to us) to be less efficient, except in particular cases.

4. In this section we will show (in a very informal way) the existence of subfamilies of large dimensions of $U(k,a;g)$ formed by curves, C, with exactly one of the following properties:

(A1) C has a Weierstrass point, P, inducing a g_a^1.

(A2) C has a Weierstrass pair $\{P', P''\}$ (in the sense of [2], p. 365) inducing a g_a^1; furthermore we may prescribe arbitrarily the multiplicities with whom P' and P'' occur in a divisor of the g_k^1.

(B1) C has a Weierstrass point, P, inducing a g_a^1.

(B2) C has a Weierstrass pair $\{P', P''\}$ (in the sense of [2], p. 365) inducing a g_k^1; furthermore we may prescribe arbitrarily the multiplicities with whom P' and P'' occur in a divisor of the g_k^1.

(U) C has both exactly one of the properties (A1) or (A2) and one of the properties (B1) or (B2).

To prove this assertion we have to make the construction used at the beginning of section 3 to prove Theorem 0.1 with the following restriction. We consider only case (U) (the more difficult case); just to fix the notations, we will prove the existence of C with both properties (A1) and (B2). We fix a line L of type (1,0) (i.e. inducting the g_a^1) and a point $P \in L$ and a line R of type (0,1) and two points P' and P'' on R. Then we consider only curves passing through P, P' and P'', smooth at these points and with the prescribed order of contact with L and R. Instead of using just the deformation theory used in [1] (or [16] or [9]), we use also a theorem of Kleppe (see [11], Th. 1.5) which describes exactly the differential of a map having the prescribed restriction to a subset, e.g. $L \cup R$; it is the classical one twisted by the ideal sheaf of the subscheme, here $L \cup R = \boldsymbol{O}_Q(1,1)$. The point is that $K_Q = \boldsymbol{O}_Q(-2,-2)$; hence the line bundle $K_Q^{-1} \otimes \boldsymbol{O}_Q(-1,-1)$ is still ample; thus we may apply again the theory in [16] and [1]: see in particular the statements and proofs of [16], Lemma 2.2, [1], Lemma 3.3, or [9], Prop. 2.1.

Several related open questions seems natural to us (and we hope to work on some of them in the future). First, there is the problem of the irreducibility of $U(k,a;g)$ (following the paths of [9], [12] and [13]). A very interesting problem is the extension to general enough curves in $U(k,a;g)$ of the Brill-Noether theory; this theory is still in its infancy (without even general conjectures) even for general k-gonal curves; the method of [5] (based on [14]) applies also to $U(k,a;g)$ in the range of genera in which the nodes may be taken in general position, but seems to give only very weak results; on $\boldsymbol{P}^2$ by far the more powerful methods and results are contained in [6] and [7]. Another question: as in the limit linear series of Eisenbud and Harris (or for pencils in the Harris-Mumford theory of generalized coverings) give good criteria for smoothing a reducible curve (not given as a curve in Q) with two "generalized pencils" preserving both pencils; in some very special case (e.g. when one of the two "generalized pencil" corresponds to a cyclic corering), this is rather easy, but we do not know it in general.

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