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ON THE DECOMPOSITION AND DIRECT SUMS OF MODULES

MAHMOUD AHMED KAMAL

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1. Introduction. This paper studies direct sums of CS-modules. We give a number of necessary and sufficient conditions for such sums to be CS-, or quasi-continuous, modules. This question was settled in [7], in a very satisfactory way, in case the ring is commutative Noetherian. The case dealt with here is more general.

Direct sums of indecomposable modules have been investigated in great detail, in long series of papers, by M. Harada and K. Oshiro, and by B.J. Muller and S.T. Rizvi.

The well known Matlis-Papp's Theorem, for injective modules, was generalized to continuous modules in [10], and to extending modules, so to quasi-continuous modules, in [11]. The present paper generalizes such a Theorem to 1-quasi-continuous modules. As a result, we obtain that, over a right Noetherian ring, 1-quasi-continuity is equivalent to the extending property for independent family of modules.

All modules here are right-modules over a ring R. m^o denotes the annihilator in R of the element $m \in M$. $X \subseteq^e M$ and $Y \subseteq^{\oplus} M$ signify that X is an essential submodule, and Y is a direct summand, of M. A submodule A is closed in M if it has no proper essential extensions in M.

A module M is called a *CS-module* (*n-CS-module*), if every closed submodule A of M (A of M with U-dim$(A) \leq n$) is a direct summand. M is quasi-continuous (n-quasi-continuous) if it is CS- (n-CS) module, and satisfies the following; (C_3) ((n-C_3)): For all $X, Y \subseteq^{\oplus} M$ (for all $X, Y \subseteq^{\oplus} M$, with U-dim$(X)$, U-dim$(Y) \leq n$), where $X \cap Y = 0$, one has $X \oplus Y \subseteq^{\oplus} M$. A direct sum $\oplus_{i \in I} N_i$ of submodules of M is called a local direct summand if $\oplus_{i \in F} N_i \subseteq {}^{\oplus} M$, for all finite subsets F of I.

For a decomposition $M = \oplus_{i \in I} M_i$, we recall the following conditions:

(A_2): For any choice of $x \in M_i$ ($i \in I$), and $m_j \in M_{i_j}$ for distinct $i_j \in I$, $j \in N$, such that $m_j^0 \supseteq x^o$, the ascending sequence $\cap_{j \geq n} m_j^o$ ($n \in N$) becomes stationary.

(A_3): For any choice of distinct $i_j \in I$ ($j \in N$) and $m_j \in M_{i_j}$, if the the sequence m_j^o is ascending, then it becomes stationary.

(lsTn) (locally semi-T-nilpotent): For every sequence f_n: $M_{i_n} \to M_{i_{n+1}}$, $n \in N$, of non-isomorphisms, with all i_n distinct, and every $x \in M_{i_0}$, there exists $k \in N$ such that $f_k f_{k-1} \cdots f_0(x) = 0$.

2. The decomposition theorem. In this section, we show that every 1-quasi-continuous module, over a right Noetherian ring, is a direct sum of uniform submodules.

Lemma 1. *Let M be a* 1-CS-*module. Then every closed submodule of M of the form $\oplus_{i=1}^{n} A_i$,* with all *A_i uniform, is a direct summand.*

Proof. By induction. Assume that the claim holds true for n, and let $A = \oplus_{i=0}^{n} A_i$ be a closed submodule of M. By induction, $A^* =: {}_{i=1}^{n} A_i$ is a direct summand. Write $M = A^* \oplus M^*$, it follows that $A = A^* \oplus A \cap M^*$. It is clear that $A \cap M^*$ is a closed and uniform submodule of M. Since direct summands of 1-CS-modules are 1-CS-modules, we have $A \cap M^* \subseteq^{\oplus} M$. Therefore $A \subseteq^{\oplus} M$.

Lemma 2. *Let M be a* 1-CS-*module. Then every non-zero closed submodule of M, of finite uniform dimension, contains a uniform summand.*

Proof. Let $A \neq 0$ be a closed submodule of M, with U-dim$(A) < \infty$. Let A_1 be a uniform submodule of A, and let U be a maximal essential extension of A_1 in A. Since U is closed in A, and A is closed in M; we have that U is closed in M. Since M is a 1-CS-module, we obtain $U \subseteq^{\oplus} M$; and therefore $U \subseteq^{\oplus} A$.

Corollary 3. *Let M be a module over a right Noetherian ring. If M is a* 1-*CS*-module, then *every non-zero closed submodule contains a uniform direct summand.*

Proposition 4. *If M is a* 1-CS-*module, then M is an n-*CS-*module.*

Proof. Lemma 1, and Lemma 2.

Lemma 5 ([7], *Lemma* 17). *Let $M = X \oplus Y$ be a module, where Y is X-injective. Let N be a submodule of M, with $N \cap Y = 0$. Then there is a homomorphism $f: X \to Y$ such that $N \subseteq X^* =: \{x + f(x) : x \in X\} \cong X$, and that $M = X^* \oplus Y$.*

Lemma 6 ([9], *Theorem* 7). *For a module with a decomposition $M = \oplus_{i \in I} M_i$, and with all M_i indecomposable, The following are equivalent:*

1) *M is quasi-continuous;*
2) *the M_i are quasi-continuous and M_j-injective ($j \neq i \in I$), and* (A_2) *holds.*

Lemma 7. *Let M be a* 1-*quasi-continuous module. Then for every family*

$\{A_i\}_{i=1}^n$ of uniform direct summands of M, with $\Sigma_{i=1}^n A_i$ direct, one has $\oplus_{i=1}^n A_i \subseteq^{\oplus} M$.

Proof. By induction. Let the claim hold true for n, and consider $\oplus_{i=0}^n A_i$, with all A_i uniform summands of M. By induction, $\oplus_{i=1}^n A_i \subseteq^{\oplus} M$. By Proposition 4, M is an n-CS-module; and hence $\oplus_{i=0}^n A_i \subseteq^e K \subseteq^{\oplus} M$. Write $K = \oplus_{i=1}^n A_i \oplus K_0$. Since direct summands of 1-quasi-continuous modules are 1-quasi-continuous modules; then, by Lemma 6, $\oplus_{i=1}^n A_1$ is K_0-injective. Hence, by Lemma 5, $K = \oplus_{i=1}^n A_i \oplus K_0^*$; where $A_0 \subseteq K_0^* \cong K_0$. Since A_0 is a uniform direct summand of M, it follows that $A_0 = K_0^*$. Therefore $\oplus_{i=0}^n A_i = K \subseteq^{\oplus} M$.

Lemma 8. *Let M be a module over a right Noetherian ring R. Then every local direct summand of M is a closed submodule of M.*

Proof. Let $L = \oplus_{i\in I} L_i$ be a local direct summand of M, and let $L \subseteq^e K \subseteq M$. Consider an arbitrary $x \in K$, and let $J =: \{r \in R\colon xr \in L\}$. Since J is a finitely generated right ideal of R, it follows that (for some finite subset F of I) $xJ \subseteq \oplus_{i\in F} L_i$. Since $\oplus_{i\in I} L_i$ is a local direct summand of M, we have that $K = \oplus_{i\in F} L_i \oplus K^*$. Hence $x = a + b$, where $a \in \oplus_{i\in F} L_i$, and $b \in K^*$. It is clear that $J = \{r \in R\colon br \in L\}$. If $b \neq 0$, then there is $r \in J$ such that $0 \neq br = ar - xr \in \oplus_{i\in F} L_i \cap K^* = 0$, which is a contradiction. Thus $x = a \in L$. Therefore $L = K$.

Theorem 9. *Let M be a 1-quasi-continuous module over a right Noetherian ring R. Then every closed submodule of M is a direct sum of uniform submdules. In particular M is a direct sum of uniform modules.*

Proof. Let N be a closed submodule of M. Let $\mathscr{S} =: \{N_\alpha\colon \alpha \in \Lambda\}$ be the family of all uniform direct summands of N. By Corollary 3, $\mathscr{S}$ is not empty. We call a subset J of Λ direct, if the sum $\Sigma_{\alpha\in J} N$ is direct. Consider the collection of all direct subsets of Λ, ordered by inclusion. An application of Zorn's Lemma, yields a maximal direct subset I of Λ. Again by Corollary 3, it follows that $\oplus_{i\in I} N_i \subseteq^e N$. Since $N_i \subseteq^{\oplus} N$ and since N is closed in M, we have that N_i is closed in M. Since M is a 1-CS-module, we get that $N_i \subseteq^{\oplus} M$ for all $i \in I$. Thus, by Lemma 7 and since M has (1-C_3), we obtain that $\oplus_{i\in I} N_i$ is a local direct summand of M. Therefore, by Lemma 8, $\oplus_{i\in I} N_i = N$.

The following is a generalization of Matlis-Papp's Theorem.

Corollary 10. *A ring R is right Noetherian if and only if every 1-quasi-continuous R-module is a direct sum of uniform submodules.*

We recall; The property (E) (extending property for independent families of submodules) requires that $\oplus_{i\in I} A_i \subseteq M$, yields the existance of $A_i \subseteq^e A_i^*$ such that

$\oplus_{i\in I} A_i^* \subseteq^{\oplus} M$.

Proposition 11. *Let M be a module over a right Noetherian ring. Then the following are equivalent*:
1) *M is 1-quasi-continuous*;
2) *M has the extending property* (E);
3) *M is quasi-continuous.*

Proof. 1) $\Rightarrow$ 2). Let $\oplus_{i\in I} A_i \subseteq M$. Let A_i^* be a maximal essential extension of the A_i in M. Denote $A^* = \oplus_{i\in I} A_i^*$, and consider a complement B of A^* in M. It follows that $A^*\oplus B \subseteq^e M$. By Lemma 7, and Theorem 9, $A^*\oplus B$ is a direct sum of uniform submodules, which is a local direct summand of M. Therefore, by Lemma 8, $M = A^*\oplus B$.

2) $\Rightarrow$ 3), and 3) $\Rightarrow$ 1) are obvious.

Proposition 12. *Let M be a module over a right Noetherian Ring. Then the following are equivalent*:
1) *every closed submodule of the form $\oplus_{i\in I} U$, with all U_i uniform, is a direct summand*;
2) *M is a* 1-CS-*module, and every direct sum of uniform submodules of M, which is a local direct summand, is a direct summand.*
3) *M is a* 1-CS-*module, and every local direct summand of M is a direct summand.*
4) *M is a* CS-*module.*

Proof. Corollary 3, and Lemma 8.

A module M is called a *D-R-I-module*, if X is Y-injective whenever $M = X\oplus Y$. It is clear that every quasi-continuous module is a D-R-I-module. There are D-R-I-modules, which are not quasi-continuous.

Proposition 13. *Let M be a* D-R-I-*module. Then M is a* 1-CS-*module if and only if M is 1-quasi-continuous.*

Proof. Let U_1, $U_2 \subseteq^{\oplus} M$, where the U_i $(i=1,2)$ are uniform with $U_1 \cap U_2 = 0$. Write $M = U_1 \oplus M_1$. By Lemma 5, and since U_1 is M_1-injective, we have $M = U_1 \oplus M_1^*$, where $U_2 \subseteq^{\oplus} M_1^*$. Therefore $U_1 \oplus U_2 \subseteq^{\oplus} M$.

Corollary 14. *Let M be a* D-R-I-*module over a right Noetherian ring. Then the following are equivalent*:
1) *M is a* 1-CS-*module*;
2) *M has the property* (E);
3) *M is a direct sum of uniform submodules.*

Proof. Propositions 11, 13.

3. Direct sums of modules. In this section we study direct sums $M=\oplus_{i\in I} M_i$ of CS-modules.

For a given decomposition $M=\oplus_{i\in I} M_i$, and a subset J of the index set I, $\mathrm{M}(J)$ stands for $\oplus_{i\in J} M_i$.

Lemma 15 ([9], *Corollary* 2). *Let* $M=\oplus_{i\in I} M_i$. *Then* $\mathrm{M}(J)$ *is* $\mathrm{M}(I-J)$-*injective, for all* $J\subseteq I$, *if and only if the* M_i *are* M_j-*injective for all* $i\neq j\in I$, *and* (A_2) *holds.*

Proposition 16. *Let* $M=\oplus_{i\in I} M_i$ *be an R-module, where all the* M_i *are* 1-CS-*modules and* M_j-*injective for all* $i\neq j\in I$. *Let* (A_2) *be hold. Then* M *is an* n-CS-*module.*

Proof. By Proposition 4, it is enough to show that M is a 1-CS-module. Let A be a closed and uniform submodule of M. Let $0\neq a\in A$, it follows that $aR\subseteq \mathrm{M}(F)$; F is a finite subset of I. Since A is uniform, we have that $A\cap M(I-F)=0$. By Lemma 15, $\mathrm{M}(I-F)$ is $\mathrm{M}(F)$-injective; and thus, by Lemma 5, $M=\mathrm{M}(I-F)\oplus \mathrm{M}^*(F)$, where $A\subseteq M^*(F)\cong \mathrm{M}(F)$. Write $\mathrm{M}^*(F)=\oplus_{i\in F} M_i^*$, where $M_i^*\cong M_i$ $(i\in F)$. It follows, again by Lemma 5, that $A\subseteq N\cong M_j^*$ (for some $j\in F$), and that $N\subseteq^{\oplus} \mathrm{M}^*(F)$. Therefore $A\subseteq^{\oplus} M$.

Corollary 17. *Let* $M=\oplus_{i\in I} M_i$ *be a module over right-Noetherian ring, where all the* M_i *are* 1-CS-*modules and* M_j-*injective,* $i\neq j$. *Then* M *is a* CS-*module if and only if every local summand of* M *is a summand.*

Proof. Propositions 12, 16.

Lemma 18 ([6],*Proposition* 24). *Let* $M=\oplus_{i\in I} M_i$, *where all the* M_i *are* CS-*modules and* M_j-*injective,* $i\neq j\in I$. *Then* $\mathrm{M}(F)$ *is a* CS-*module for all finite subsets* F *of* I.

Proposition 19. *Let* $M=\oplus_{i\in I} M_i$ *be an R-module, where all the* M_i *are* CS-*modules and* M_j-*injective,* $i\neq j\in I$. *Let* (A_2) *hold. If every local direct summand of* M *is a summand, then* M *is a* CS-*module.*

Proof. Let A be a closed submodule of M. Zorn's Lemma yields a maximal member $\oplus_{k\in K}A_k$ of the family of all submodules of A of the form $\oplus_{\alpha\in L} N_\alpha$, which is a local direct summand of M. By assumption $\oplus_{k\in K}A_k$ is a direct summand of M, hence a direct summand of A. Write $A=\oplus_{k\in K} A_k\oplus A^*$. If $A^*\neq 0$, then, for some $0\neq x\in A^*$, $xR\subseteq \mathrm{M}(F)$ (where F is a finite subset of I). Consider a maximal essential extension $(xR)^*$ of xR in A^*. It follows that $(xR)^*$ is closed in M, with $(xR)^* \cap M(I-F)=0$. By Lemma 15, $M(I-F)$ is $M(F)$-injective. By Lemma 5, $M=M(I-F)\oplus M^*(F)$; where $(xR)^* \subseteq M^*(F) \cong M(F)$. Since, by Lemma 18, $M(F)$ is a CS-module, it follows that $(xR)^* \subseteq^{\oplus} M^*(F)$. Hence $A=\oplus_{k\in K} A_k \oplus (xR)^* \oplus B$, where $\oplus_{k\in K} A_k \oplus (xR)^*$ is a local direct summand of M; which contradicts the maximality of $\oplus_{k\in K} A_k$. Therefore $A=\oplus_{k\in K} A_k \subseteq^{\oplus} M$.

Corollary 20. *Let R be a right Noetherian ring. Let $M=\oplus_{i\in I} M_i$ be an R-module, where the M_i are 1-CS-modules and M_j-injective for all $i\neq j\in I$. Then M is a CS-module if and only if every local direct summand of M is a direct summand.*

In the following, we obtain the same equivalent conditions, 3) $\Rightarrow$ 4), as in proposition 12, for a weaker A.C.C. .

Proposition 21. *Let $M=\oplus_{i\in I} M_i$ be an R-module, where the M_i are uniform. Let (A_3) hold. If $\oplus_{i\in I} M_i$ complements direct summands, then the following are equivalent:*

1) *M is a CS-module,*
2) *M is a 1-CS-module, and every local summand of M is a summand.*

Proof. 1)) $\Rightarrow$ 2). Let $L=\oplus_{j\in J} L_j$ be a local direct summand of M. Since M is a CS-module, we have $L \subseteq^e L^* \subseteq^{\oplus} M$. Since $\oplus_{i\in I} M_i$ complements direct summands, there exists a subset K of I such that $L^*=\oplus_{k\in K} N_k$, and with $N_k \cong M_k$ for all $k\in K$. Hence, without loss of generality, we may consider $L \subseteq^e M$. We show that $M= L$. If $M\neq L$, then we shall derive a contradiction to (A_3) by inductively constructing a sequence $\{m_n\}$ such that $m_n \in M_{i_n}\backslash L$ for distinct i_n, and that $m_1^o \subset m_2^o \subset \cdots \subset m_n^o \cdots$. To this end assume that $m_1, m_2, \cdots, m_n$ have been constructed. Since $L \subseteq^e M$, there exist $s_1, s_2, \cdots, s_n \in R$ such that $0\neq m_i s_i \in L$ $(i=1,2,\cdots,n)$. Since $m_n s_n \in L(F) \subseteq^{\oplus} M$, for some finite subset F of J, we have $M=L(F)\oplus M(K)$, $K \subseteq I$. Thus $m_n = l+\Sigma_{i\in K} y_i$, where $l\in L(F)$ and $\Sigma_{i\in K} y_i \in M(K)$. It is clear that $m_n^o \subseteq y_i^0$ for each $i\in K$. But since $m_n s_n \in L(F)$, we deduce that $m_n^o \subset y_i^o$; $i\in K$. Observe that not all y_i are in L (due to $m_n \notin L$). Then for some $i_0 \in K$, $y_{i_o}=m_{n+1}$ will satisfy the desired condition.

2) $\Rightarrow$ 1). Let A be a closed submodule of M. By Zorn's Lemma, let $\oplus_{j\in J} A_j$ be a maximal local direct summand of M, which is a submodule of A. By assumption, $M=\oplus_{j\in J} A_j \oplus M(K)$; where $K \subseteq I$. Thus $A=\oplus_{j\in J} A_j \oplus A\cap M(K)$. Now if $A\cap M(K)\neq 0$, then $A\cap M(F)\neq 0$ for a finite subset F of K. It follows that $A\cap M(F)$, hence A, contains a uniform submodule U. Let U^* be a maximal essential extension of U in $A\cap M(K)$. It is clear that U^* is closed in M. Since M is a 1-CS-module, we get that $U^* \subseteq^{\oplus} M$; and thus $A \supseteq \oplus_{j\in J} A_j \oplus U^*$, which is a local summand of M. This contradicts the maximality of $\oplus_{j\in J} A_j$, therefore $A=\oplus_{j\in J} A_j \subseteq^{\oplus} M$.

Corollary 22. *For a module $M=\oplus_{i\in I}M_i$, where the M_i are uniform, for all $i\in I$, the following are equivalent*:
1) *M is quasi continous*;
2) *M is 1-quasi-continuous, and every local summand of M is a summand.*

Proof. One can easily show that (C_3) implies $\oplus_{i\in I}M_i$ complements direct summands. Hence by Lemma 6 and Proposition 21, 1)$\Rightarrow$2) follows.
2)$\Rightarrow$1). Let A be a closed submodule of M. By the same argument as in Theorem 9, there exists $\oplus_{j\in J}A\subseteq^e A$; where all A_j are uniform summands of M. By Lemma 7, $\oplus_{j\in J}A_j$ is a local direct summand of M. Hence, by assumption, $\oplus_{j\in J}A_j\subseteq^{\oplus}M$, and thus $A=\oplus_{j\in J}A_j\subseteq^{\oplus}M$. Therefore M is a CS-module. To show (C_3) holds, let X, $Y\subseteq^{\oplus}M$ with $X\cap Y=0$. From the above argument, we may consider X and Y are direct sums of uniform submodules. Hence, by Zorn's Lemma, $X\oplus Y\oplus M(K)\subseteq^e M$ for some $K\subseteq I$. Thus by Lemma 7, $X\oplus Y\oplus\mathrm{M}(K)$ as an indecomposable decomposition is a local direct summand of M, hence a direct summand by assumption. Therefore $X\oplus Y\oplus\mathrm{M}(K)=M$.

Corollary 23. *Let $M=\oplus_{i\in I}M_i$ be a module, where all the M_i are uniform. Let* (A_3) *be hold. Then the following are equivalent*:
1) *M has* (n-C_3), *and the decomposition complements direct summands,*
2) *M has* (C_3).

Proof. 1)$\Rightarrow$2). Let $X, Y\subseteq^{\oplus}M$, with $X\cap Y=0$. Since the $\oplus_{i\in I}M_i$ complements direct summands, we may consider $X=\oplus_{s\in S}X$ and $Y=\oplus_{k\in K}Y_k$; where X_s and Y_k are uniform for all s, k. By Zorn's Lemma, there exists $J\subseteq I$ such that $X\oplus Y\oplus\mathrm{M}(J)\subseteq^e M$. By $(n\text{-}C_3)$, $\oplus_{s\in S}X_s\oplus\oplus_{k\in K}Y_k\oplus\mathrm{M}(J)$ is a local direct summand of M. If $X\oplus Y\oplus\mathrm{M}(J)\neq M$, then, by the same argument as in proposition 21, we can derive a contradiction to (A_3). Therefore $X\oplus Y\oplus\mathrm{M}(J)=M$.
2)$\Rightarrow$1) is obvious.

Proposition 24. *Let $M=\oplus_{i\in I}M_i$ be a module, where all the M_i are uniform with* end (M_i) *local. Let M_i be M_j-injective for all $j\neq i\in I$. Then the following are equivalent*:
1) *M is a* 1-CS-*module,*
2) (A_2) *holds.*

Proof. 1)$\Rightarrow$2). By Lemma 15, (A_2) is equivalent to $\mathrm{M}(I-j)$ is M_j-injective for all $j\in I$. To show this, we have to extend an arbitrary homomorphism $f\colon N\to\mathrm{M}(I-j)$ from a non-zero submodule N of M_j, to all M_j. If f is monomorphism, then $N\cong f(N)$ is a uniform submodule of $M(I-j)$. Since M a 1-CS-module, we have $f(N)\subseteq^e K\subseteq^{\oplus}\mathrm{M}(I-j)$. Since K is uniform; thus $K\cong M_k$ for some $k\in I-j$, by the Krull-Schmidt-Azumaya Theorem. It follows that K is M_j-injective, and thus

there exists $\bar{f}$: $M_j \to K \subseteq M(I-j)$ extends f. On the other hand if f is not monomorphism, then $N^* \cap M_j \neq 0$, where $N^* =: \{x + f(X) : x \in N\} \cong N$. Since M is a 1-CS-module and N^* is uniform, we get that $N^* \subseteq^e L \subseteq^{\oplus} M$. Consequently End (L) is local, which yields L to have the exchande property. Hence $M = L \oplus M(I-j)$ (due to $L \cap M_j \neq 0$). Let π: $L \oplus M(I-j) \to M(I-j)$ be the projection, it follows that $\pi(x + f(x)) = 0$; i.e. $-\pi(x) = f(x)$ for all $x \in N$. Therefore $-\pi_{|M_j}$ extends f.
2) $\Rightarrow$ 1) follows from Proposition 16.

Lemma 25. *Let $M = M_1 \oplus M_2$ be a module, where the M_i are uniform and with* End (M_i) *local, $i = 1, 2$. Then the following are equivalent*:
1) *M is a* CS-*module, and monomorphisms $M_i \to M_j$ are isomorphisms; $i \neq j$.*
2) *the M_i are M_j-injective, $i \neq j$.*

Proof. 1) $\Rightarrow$ 2). Let $f \in \mathrm{Hom}(E(M_i), E(M_j))$ be an arbitrary element, $i \neq j$. Let $X =: \{x \in M_i$: $f(x) \in M_j\}$. Then $A =: \{x + f(x)$: $x \in X\}$ is a closed and uniform submodule of M, by [6], Lemma 1. Since M is a CS-module, $M = A \oplus M_i$ or $M = A \oplus M_j$ (due to end (M_i) local). If $M = A \oplus M_i$, then $M_j = f(X)$; and hence f^{-1}: $M_j \to X \subseteq M_i$ is by assumption an isomorphism, i.e. $X = M_i$. On the other hand, if $M = A \oplus M_j$, then $X = M_i$.
2) $\Rightarrow$ 1) is obvious.

The following is an immediate consequence of Lemma 6, Proposition 24, and Corollary 25. It was observed in [9] where the proof was technical.

Corollary 26. *For a module $M = \oplus_{i \in I} M_i$, where the M_i are uniform and* End (M_i) *are local for all $i \in I$, the following are equivalent*:
1) *M is quasi-continuous,*
2) *M is 1-quasi-continuous,*
3) *M is* 1-CS-*module, and monomorphisms $M_i \to M_j$ are isomorphisms for $i \neq j$.*

Corollary 27. *Let $M = \oplus_{i \in I} M_i$, where the M_i are uniform and M_j-injective for all $i \neq j \in I$. Then the following are equivalent*:
1) *M is quasi-continuous,*
2) *M is* 1-CS-*module, and $\oplus_{i \in I} M_i$ complements uniform direct summands.*

Proof. A similar argument, to the one given in Proposition 24, shows that (A_2) holds. Hence 2) $\Rightarrow$ 1) follows. 1) $\Rightarrow$ 2) is trivial.

Corollary 28. *For a module $M = \oplus_{i \in I} M_i$, where the M_i are uniform for all $i \in I$, the following are equivalent*:
1) *M is quasi-continuous,*
2) *M is 1-quasi-continuous, and $\oplus_{i \in I} M_i$ complements uniform summands.*

Proposition 29. *Let $M=\oplus_{i\in I}M_i$ be an R-module, where all the M_i are uniform and M_j-injective; $i\neq j\in I$. If all M_i, but a finite number, are non-singular, then M is quasi-continuous.*

Proof. First we show that M is a 1-CS-module. Let A be a closed and uniform submodule of M. Let $0\neq a\in A$; then $aR\subseteq \mathrm{M}(F)$, F is a finite subset of I. By Lemma 15, $\mathrm{M}(F)$ is a quasi-continuous module; and thus $aR\subseteq^{e}(aR)^{*}\subseteq^{\oplus}\mathrm{M}(F)$. Hence $\mathrm{M}(F)=(aR)^{*}\oplus\mathrm{M}(F-k)$, for some $k\in K$. Then, without loss of generality, we may assume that $M=M_k\oplus\mathrm{M}(I-k)$, where $A\cap M_k\neq 0$. Let π_i be the projection of M onto M_i, $i\in I$. For each $a\in A$ $a=\pi_k(a)+\Sigma_{i\neq k}\pi_i(a)$, with $\pi_k(a)\neq 0$. Since $A\cap M_k\subseteq^{e}M_k$, it follows that $\pi_i(a)J=0$ for some essential right ideal J of R. Since all, but a finite number, of the M_i are non-singular, we obtain that the set $\{i\in I\colon\ \pi_i(a)\neq 0,\ a\in A\}$ is finite. Hence $A\subseteq \mathrm{M}(K)$, for some finite subset K of I. Therefore $A\subseteq^{\oplus}\mathrm{M}(K)\subseteq^{\oplus}M$.

The above argument also shows $\oplus_{i\in I}M_i$ complements uniform direct summands. Hence, by Corollary 27, M is quasi-continuous.

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Mathematics Department
Faculty of Education
Ain Shams University
Roxy, Cairo, Egypt