Pyrih, P.
Osaka J. Math.
32 (1995), 121–124

TWO COUNTEREXAMPLES TO CORNEA'S CONJECTURE ON THIN SETS

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(Received July 5, 1993)

1. Introduction

In the paper of Cornea ([1], p. 836) is the following conjecture: A set $A \subset \boldsymbol{R}^d$ is thin at 0 if there exist $v_1, v_2, v_3 \in \boldsymbol{R}^d$ linearly independent (pairwise, if $d=2$) with $\|v_j\|=1$ and such that $T_{v_j}(A)$ is thin at $0, j=1,2,3$, where $T_v(x):=x-\langle x,v\rangle v$. We show that this conjecture fails.

We recall that the *fine topology* on $\boldsymbol{R}^d$ is the smallest topology on $\boldsymbol{R}^d$ for which all superharmonic functions are continuous in the extended sense. A set $E \subset \boldsymbol{R}^d$ is *thin* at x if x is not a fine limit point of E. The *Wiener test* relates thinness of E to the *capacity* of certain subsets of E. We note that thinness of a set at a point is related to irregularity of boundary points relative to the Dirichlet problem. For general information see [2], [3].

2. An example in $\boldsymbol{R}^2$

We denote P_x, P_y, P_z and P_w the orthogonal projections which map $\boldsymbol{R}^2$ onto a line through the origin in such a way that the points (0,1), (1,0), (1,−1) and (1,1), respectively are mapped to the origin. We set $I_2:=\{(x,y)\in \boldsymbol{R}^2,\ 0\le x\le 1,\ 0\le y\le 1\}$, cap denotes the logarithmic capacity.

Lemma 2.1. *Given* $\varepsilon>0$ *there exists a set* $E\subset I_2$ *such that* $\operatorname{cap}(P_xE)<\varepsilon$, $\operatorname{cap}(P_yE)<\varepsilon$, $\operatorname{cap}(P_zE)=0$ *and* $\operatorname{cap}(E)\ge\operatorname{cap}(P_wE)\ge\sqrt{2}/8$.

Proof. We set $A:=\{(x,0)\in I_2,\ x\in \boldsymbol{Q}\}$, A is countable, hence $\operatorname{cap}(A)=0$. There exists an open set $U\supset A$ in $\boldsymbol{R}^2$ such that $\operatorname{cap}(U)<\varepsilon$. Denote $V:=\{(x,0))\in I_2\}\cap U$. We set $E:=\{(x,y)\in I_2,\ (x,0)\in V,\ 0\le y\le\varepsilon,\ x+y\in \boldsymbol{Q}\}$. Then

(i) $P_xE=V\subset U$, hence $\operatorname{cap}(P_xE)<\varepsilon$;

(ii) $P_yE=\{(0,y)\in I_2,\ 0\le y\le\varepsilon\}$, hence $\operatorname{cap}(P_yE)<\varepsilon$;

Research Supported by the Grant No. 201/93/2174 of Czech Grant Agency and by the Grant No. 354 of Charles University.

(iii) P_zE is countable, hence $\operatorname{cap}(P_zE)=0$.

Denote l the segment joining points (0,0) and $(1/2, -1/2)$. Then $l \subset P_wE$, hence $\operatorname{cap}(E) \geq \operatorname{cap}(P_wE) \geq \operatorname{cap}(l) = \sqrt{2}/8$. □

REMARK 2.2. We show that the set E can be constructed to be compact: We find real numbers $0=\alpha_0<\beta_0<\cdots<\alpha_n<\beta_n=1$ such that $\alpha_j-\beta_{j-1}<\varepsilon$, for $j=1,\cdots,n$, and

$$\operatorname{cap}(\{(x,0)\in I_2,\ \alpha_j \leq x \leq \beta_j \text{ for some } j=0,\cdots,n\})<\varepsilon.$$

(Here we use the Wiener capacity, which is countably subadditive.)

For each $j=0,1,\cdots,n$ we construct lines $l_1^j,\cdots,l_{k_j}^j$ with slopes -1 such that the point $(\beta_j,0)\in l_1^j$, $(\alpha_j,\varepsilon)\in l_{k_j}^j$ and the distance between l_p^j and l_{p+1}^j is less than $\sqrt{2}/(\beta_j-\alpha_j)$. We set

$$\tilde{E}:=\bigcup_{j=0}^{n}\bigcup_{p=1}^{k_j}(l_p^j\cap\{(x,y)\in I_2,\ \alpha_j\leq x\leq\beta_j,\ 0\leq y\leq\varepsilon\}).$$

The set $\tilde{E}$ is compact (consists only of finitely many segments) and the estimates of $\operatorname{cap}(P_x\tilde{E})$, $\operatorname{cap}(P_y\tilde{E})$, $\operatorname{cap}(P_z\tilde{E})$ and $\operatorname{cap}(P_w\tilde{E})$ can be obtained similarly as in the proof of Lemma 2.1.

COUNTEREXAMPLE 2.3. *There is a set E in $\boldsymbol{R}^2$ such that E is not thin at the origin and the projections P_xE, P_yE and P_zE are thin at the origin.*

Proof. Let E_n be the set E in Lemma 2.1 constructed with $\varepsilon=1/2^{n^3}$. Set

$$E:=\bigcup_{n=3}^{\infty}\frac{1}{2^{n+1}}\cdot((1,1)+\frac{1}{2}\cdot E_n).$$

Denote $A_n:=\{a\in\boldsymbol{R}^2,\ 1/2^{n+1}\leq\|a\|\leq 1/2^n\}$. Then

$$\operatorname{cap}(P_xE\cap A_n)=\operatorname{cap}(P_x(E\cap A_n))<1/2^{n^3}.$$

$$\operatorname{cap}(P_yE\cap A_n)=\operatorname{cap}(P_y(E\cap A_n))<1/2^{n^3}.$$

$$\operatorname{cap}(P_zE\cap A_n)=\operatorname{cap}(P_z(E\cap A_n))=0, \text{ and}$$

$$\operatorname{cap}(E\cap A_n)\geq\operatorname{cap}(P_w(E\cap A_n))\geq\sqrt{2}/(16\cdot 2^{n+1}).$$

Hence P_xE, P_yE and P_zE are thin, and E is not thin at the origin due to the Wiener test. □

3. An example in $\boldsymbol{R}^3$

We denote P_{xy}, P_{xz}, P_{yz} and P_{wy} the orthogonal projections, which map $\boldsymbol{R}^3$ onto a plane through the origin in such a way, that (0,0,1), (0,1,0), (1,0,0) and (1,0,1), respectively are mapped to the origin. We set $I_3 := \{(x,y,z) \in \boldsymbol{R}^3,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$, c denotes the Newton capacity.

Lemma 3.1. *Given $\varepsilon > 0$ there exists a set $E \subset I_3$ such that $c(P_{xy}E) < \varepsilon$, $c(P_{xz}E) < \varepsilon$, $c(P_{yz}E) < \varepsilon$ and $c(E) \ge c(P_{wy}E) \ge c(P_{wy}\{(x,y,0) \in I_3\})$, $(=: b > 0)$.*

Proof. We set $A := \{(x,y,0) \in I_3, x \in \boldsymbol{Q}\}$, hence $c(A) = 0$. There exists an open set $U \supset A$ in $\boldsymbol{R}^3$ such that $c(U) < \varepsilon$. Denote $V := \{(x,y,0) \in I_3\} \cap U$.

We set $L := \{(0,y,0) \in I_3\}$, hence $c(L) = 0$. We find $\delta < \varepsilon$ such that $c(\{(0,y,z) \in I_3,\ 0 \le z \le \delta\}) < \varepsilon$.

We set $E := \{(x,y,z) \in I_3,\ (x,y,0) \in V,\ 0 \le z \le \delta\}$. Then

(i) $P_{xy}E \subset V$, hence $c(P_{xy}E) \le c(U) < \varepsilon$;

(ii) $P_{xz}E \subset \{(x,0,z) \in I_3, 0 \le z \le \delta\}$, hence $c(P_{xz}E) < \varepsilon$ due to construction of δ;

(iii) $P_{yz}E \subset \{(0,y,z) \in I_3, 0 \le z \le \delta\}$, hence $c(P_{yz}E) < \varepsilon$ due to construction of δ.

Nevertheless $P_{wy}E$ contains the set $P_{wy}(\{(x,y,0) \in I_3\})$, hence $c(E) \ge c(P_{wy}E) \ge b$. □

REMARK 3.2. We show that the set E can be constructed in such a way that it is a compact set consisting of finitely many rectangles (like in Remark 2.2).

COUNTEREXAMPLE 3.3. *There is a set E in $\boldsymbol{R}^3$ such that E is not thin at the origin and the projections $P_{xy}E$, $P_{xz}E$ and $P_{yz}E$ are thin at the origin.*

Proof. Let E_n be the set E in Lemma 3.1 constructed with $\varepsilon = 1/2^n$. Set

$$E := \bigcup_{n=10}^{\infty} \frac{1}{2^{n+1}} \cdot \left((1,1,1) + \frac{1}{2} \cdot E_n\right).$$

The rest of the proof runs like in the proof of Counterexample 2.3. □

REMARK 3.4. A counterexample to Cornea's conjecture in dimension $d > 3$ can be derived from the set E in Conterexample 3.3. It suffices to consider the set $E \times \boldsymbol{R}^{d-3} \subset \boldsymbol{R}^d$ because, for any set $F \subset \boldsymbol{R}^3$, $F \times \boldsymbol{R}^{d-3}$ is thin at 0 in $\boldsymbol{R}^d$ if and only if F itself is thin at 0 in $\boldsymbol{R}^3$ (in the sense of potential theory in $\boldsymbol{R}^3$).

REMARK 3.5. Cornea states in [1], Remark on the page 836, that the conjecture is true for a set A contained in a set of the form $\bigcup_{j=0}^{\infty} G_j$, where G_j is a Lipschitz manifold (graph of a Lipschitz function). It should be compared with

Counterexample 2.3, where the set obtained is contained in countably many lines.

References

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