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Osaka J. Math.
32 (1995), 97–119

MARKOV PROCESSES ASSOCIATED WITH SEMI-DIRICHLET FORMS

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(Received May 10, 1993)

1. Introduction

Recently, in [1], [11] an analytic characterization of all (non-symmetric) Dirichlet forms (on general state spaces) which are associated with pairs of special standard porocesses has been proved extending fundamental results in [8], [9], [18], [5], [10] (cf. also the literature in [11]). These Dirichlet forms are called *quasi-regular* (cf. Section 3 below). The processes forming the pairs are in duality w.r.t. the reference (*speed*) measure of the Dirichlet form. From a probabilistic point of view, however, this duality is quite restrictive. It arises from the fact that a Dirichlet form by the definition in [1], [11] exhibits a contraction property in *both* of its arguments. More precisely, we recall that a coercive closed form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(E;m)$ (cf. Section 2 below) is called a *Dirichlet form* if for all $u\in D(\mathscr{E})$ we have $u^+\wedge 1\in D(\mathscr{E})$ and

$$\mathscr{E}(u+u^+\wedge 1, u-u^+\wedge 1)\geq 0 \tag{1.1}$$

$$\mathscr{E}(u-u^+\wedge 1, u+u^+\wedge 1)\geq 0. \tag{1.2}$$

The purpose of this paper is to show that quasi-regularity is also sufficient and necessary for the existence of an associated special standard process if the given coercive closed form is merely a *semi-Dirichlet form*, i.e., only (1.1) (or (1.2)) holds. The existence of a (Hunt) process associated with a semi-Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ was first proved in [5] in the case where E is a locally compact separable metric space under much more stronger assumptions on $(\mathscr{E},D(\mathscr{E}))$.

Let us now briefly describe the contents of the single sections of this paper in more detail. In section 2 we first prove a few new results for the (one sided) analytic potential theory of semi-Dirichlet forms which are needed later. Here we only require that E is a measurable space in contrast to earlier work on this subject (cf. [5], [2], [3], where e.g. the measure representation of potentials was crucial which could only be obtained because E was assumed to be locally compact). In particular, we give a new proof for the characterization of α-excessive functions in terms of the semi-Dirichlet form in this purely measure theoretic context (Theorem 2.4 below). This proof does not use the "dual structure" of the semi-Dirichlet (i.e., the dual semigroup $(\hat{T}_t)_{t>0}$, generator $\hat{L}$ etc.) at all giving rise to possible extensions to more general situations. Furthermore, we show that the infimum of an α excessive function in $L^2(E;m)$ and a function in $D(\mathscr{E})$ belongs to $D(\mathscr{E})$ (which appears to be new even if $(\mathscr{E},D(\mathscr{E}))$ is a Dirichlet form; cf. Theorem 2.6). Another important result is the characterization of $\mathscr{E}$-nests in terms of a suitably defined capacity without duality (cf. Theorem 2.14) if E is a topological space. Section 2 furthermore contains a description of the general setting and a review of the underlying terminology. Based on these results the construction of the process and the proof of necessity is then analaogous to the case considered in [1], [11]. The corresponding theorems are formulated in Section 3 where we also summarize the necessary facts on quasi-regularity. Finally we want to emphasize that due to the results of this paper, all results in [11] carry over to semi-Dirichlet forms.

This paper was motivated both by the results in [5], [20] for finite dimensional state spaces and by applications to cases with infinite dimensional state spaces, more precisely to measure valued diffusions, in particular the construction of Fleming-Viot processes with selection. The situation in Section II.3 of [20] cannot be handled within the theory of Dirichlet forms, but only with the help of semi-Dirichlet forms. We describe our results, which extend the examples in [5] and a part of the results in [20], in Subsection 3.4 below, where we also sketch the applications to the Fleming-Viot processes. The details of the latter are contained in a forthcoming joint paper of the two last-named authors and B. Schmuland.

2. Analytic potential theory of semi-Dirichlet forms

In this section we state the definiton of semi-Dirichlet forms and develop the necessary potential theoretic tools on which the construction of the process will be based. As far as the proofs in [11] apply (i.e., only use (1.1), not (1.2)) we just quote them and concentrate on the new parts.

2.1. Semi-Dirichlet forms and excessive functions

Let $(E,\mathscr{B},m)$ be a measure space. Let $\mathscr{E}$ be a bilinear form with domain $D(\mathscr{E})$ on the (real) Hilbert space $L^2(E;m)$ with inner product $(\,,\,)$. We set $\mathscr{E}_\alpha := \mathscr{E} + \alpha(\,,\,)$, $\alpha > 0$.

Definition 2.1. $(\mathscr{E},D(\mathscr{E}))$ with $D(\mathscr{E})$ dense in $L^2(E;m)$ is called a *coercive closed form* if:

(i) $(\tilde{\mathscr{E}},D(\mathscr{E}))$ is positive definite and closed on $L^2(E;m)$, where

$\tilde{\mathscr{E}}(u,v):=1/2(\mathscr{E}(u,v)+\mathscr{E}(v,u))$ is the *symmetric part* of $\mathscr{E}$.

(ii) (*Sector condition*). There exists a constant $K>0$ such that $|\mathscr{E}_1(u,v)|\leq K\mathscr{E}_1(u,u)^{1/2}\mathscr{E}_1(v,v)^{1/2}$ $\forall u,v\in D(\mathscr{E})$.

$(\mathscr{E},D(\mathscr{E}))$ is called a semi-Dirichlet form on $L^2(E;m)$ if in addition:

(iii) (*Semi-Dirlchlet property*) For every $u\in D(\mathscr{E})$, $u^+\wedge 1\in D(\mathscr{E})$ and $\mathscr{E}(u+u^+\wedge 1, u-u^+\wedge 1)\geq 0$.

From now on we fix a semi-Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(E;m)$. Below $D(\mathscr{E})$ is always equipped with the norm $\tilde{\mathscr{E}}_1^{1/2}$.

REMARK 2.2. (i) Let $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$ denote the *semigroup* and *resolvent* (*of operators*) associated with $(\mathscr{E},D(\mathscr{E}))$ as in [11, Diagram 2, page 27]. By [11, I.4.4] the semi-Dirlchlet property is equivalent to the *sub-Markov property* of T_t and αG_α for all $t,\alpha>0$, i.e., $0\leq f\leq 1$ implies $0\leq T_t f,\alpha G_\alpha f\leq 1$.

(ii) If also the dual form $\hat{\mathscr{E}}(u,v):=\mathscr{E}(v,u)$ satisfies 2.1 (iii) then $(\mathscr{E},D(\mathscr{E}))$ is a *Dirlchlet form* (cf. [11, Chapter I.4]). This, however, is not always the case as the following example shows (cf. [12, 1.4.3a]).
Let dx denote Legesgue measure. Consider on $L^2(]0,1[,dx)$ the coercive closed form $\mathscr{E}(u,v)=\int_0^1 u'v'dx+\int_0^1 bu'vdx, D(\mathscr{E})=H_0^{1,2}(]0,1[)$, with $b(x):=\sqrt{x}$. Let $(T_t)_{t>0}$, $(\hat{T}_t)_{t>0}$ be the strongly continous contraction semigroups associated with $(\mathscr{E},D(\mathscr{E}))$, $(\hat{\mathscr{E}},D(\mathscr{E}))$ respectively (cf. (i)). Let $(L,D(L))$ be the L^2-generator of $(T_t)_{t>0}$ (cf. [11, Chapter I]). Suppose $(\hat{T}_t)_{t>0}$ is sub-Markovian. Then for all $t>0$

$$\int|T_t u|dx\leq\int T_t|u|dx\leq\int|u|\hat{T}_t 1dx\leq\int|u|dx,$$

i.e., the operators T_t are $L^1(]0,1[,dx)$-contractive. Hence its L^1-generator is accretive (cf. [14, Theorem X 48]) and, since it coincides on $C_0^\infty(]0,1[)$ with L, we obtain that

$$-\int_0^1 u''dx+\int_0^1 bu'dx>0 \quad \text{for all } u\in C_0^\infty(]0,1[),\ u\geq 0$$

Since $\int_0^1 u''dx=0$ integration by parts implies that $b'=1/2x^{-1/2}$ is negative on $]0,1[$. Therefore, $(\hat{T}_t)_{t>0}$ cannot be sub-Markovian.

(iii) By [11, I.4.4] we know that 2.1 (iii) is equivalent with the following:

For all $u\in D(\mathscr{E})$ and $\alpha\geq 0$, $u\wedge\alpha\in D(\mathscr{E})$ and $\mathscr{E}(u\wedge\alpha,\ u-u\wedge\alpha)\geq 0$.

In particular, $u^+,u^-,|u|\in D(\mathscr{E})$, and thus $u\wedge v, u\vee v\in D(\mathscr{E})$ for all $v\in D(\mathscr{E})$. Since for all $u\in D(\mathscr{E}),\alpha>0$,

$$\mathscr{E}_1(u\wedge\alpha, u\wedge\alpha)\le\mathscr{E}_1(u\wedge\alpha, u)\le K\mathscr{E}_1(u\wedge\alpha, u\wedge\alpha)^{1/2}\mathscr{E}_1(u,u)^{1/2},$$

it follows that

$$\mathscr{E}_1(u\wedge\alpha, u\wedge\alpha)^{1/2}\le K\mathscr{E}_1(u,u)^{1/2}.$$

Hence, since $|u|=u^+ + u^- = -((-u)\wedge 0)+(u\wedge 0)$, by the triangle inequality we obtain that

$$\mathscr{E}_1(|u|,|u|)\le 4K^2\mathscr{E}_1(u,u) \quad \text{for all } u\in D(\mathscr{E}). \tag{2.1}$$

(iv) A detailed study of the analytic theory of semi-Dirichlet forms in case E is a locally compact separable metric space and $\mathscr{B}$ its Borel σ-algebra $\mathscr{B}(E)$, can be found in [2], [3].

Below we write $f\le g$ or $f<g$ for $f,g\in L^2(E;m)$ if the inequality holds m-a.e. for corresponding m-versions. We say that f is positive if $f\ge 0$. Let us now consider excessive functions.

DEFINITION 2.3. Let $\alpha\in]0,\infty[$. $u\in L^2(E;m)$ is called α-excessive if $e^{-\alpha t}T_t u\le u$ for all $t>0$.

It is easy to check that an α-excessive function u is positive. Furthermore, we have:

REMARK. Let $u\in L^2(E;m)$. Then u is α-excessive if and only if $\beta G_{\beta+\alpha}u\le u$ for all $\beta>0$. The "only if" part is clear since (cf. [11. Chapter I.1])

$$G_\alpha=\int_0^\infty e^{-\alpha t}T_t dt, \ \alpha>0.$$

The "if" part is shown as follows. By the resolvent equation we have that

$$\begin{aligned} e^{-\alpha t}T_t G_{\beta+\alpha}u &= e^{-\alpha t}T_t(G_\alpha(u-\beta G_{\beta+\alpha}u)) \\ &\le G_\alpha(u-\beta G_{\beta+\alpha}u) \\ &= G_{\beta+\alpha}u \end{aligned}$$

where the inequality follows by assumption, since $G_\alpha f$ is α-excessive for every $f\in L^2(E;m)$, $f\ge 0$. Hence by the strong continuity

$$e^{-\alpha t}T_t u=\lim_{\beta\to\infty} e^{-\alpha t}T_t(\beta G_{\beta+\alpha}u)\le \lim_{\beta\to\infty}\beta G_{\beta+\alpha}u=u.$$

Also in the case of semi-Dirichlet forms it is possible to characterize α-excessive functions purely in terms of the form in this purely measure theoretic context. This will be crucial below in order to show that reduced functions are 1-excessive.

REMARK. Also "(ii)$\Rightarrow$(i)" of the following theorem is proved in [11, III.1.2]. One only has to realize that the dual semigroup $(\hat{T}_t)_{t>0}$ and resolvent $(\hat{G}_\alpha)_{\alpha>0}$ are still positivity preserving in our more general situation. For possible later generalizations, however, we present a different new proof here which does not use the dual structure of the semi-Dirichlet form at all.

Theorem 2.4. *Let $u \in D(\mathscr{E})$ and $\alpha > 0$. The following assertions are equivalent.*

(i) *u is α-excessive.*

(ii) $\mathscr{E}_\alpha(u,v) \geq 0$ *for all* $v \in D(\mathscr{E})$, $v \geq 0$.

Proof. (i)$\Rightarrow$(ii) see [11, III.1.2.].
(ii)$\Rightarrow$(i):

Claim 1: $u \geq 0$:

We have that

$$\begin{aligned}&\mathscr{E}_\alpha(u-(u\vee 0), u-(u\vee 0))\\&=\mathscr{E}_\alpha((-u)-((-u)\wedge 0), (-u)-((-u)\wedge 0))\\&=\mathscr{E}_\alpha((-u), (-u)-((-u)\wedge 0))-\mathscr{E}_\alpha((-u)\wedge 0, (-u)-((-u)\wedge 0)).\end{aligned}$$

The second term is positive by the semi-Dirichlet property (see 2.2 (iii)) and because $\alpha\int[((-u)\wedge 0)\cdot(-u)-(u^+)^2]dm=0$. The first term is negaitve by assumption (ii). Therefore, the strict positivity of $\mathscr{E}_\alpha$ implies $u=u\vee 0$, i.e., $u \geq 0$.

Claim 2:

Let $\mathscr{H}$ be the topological dual of $(D(\mathscr{E}), \tilde{\mathscr{E}}_1^{1/2})$. For $\alpha > 0$ let $U_\alpha: \mathscr{H} \to D(\mathscr{E})$ be the linear map defined as follows: If $J \in \mathscr{H}$ then

$$\mathscr{E}_\alpha(U_\alpha(J), w) = J(w) \quad \text{for all } w \in D(\mathscr{E}).$$

The existence of U_α follows by the Theorem of Lax-Milgram and it is obvious that U_α is a bijection. Furthermore for $\alpha, \beta > 0$ we have a "resolvent equation"

$$U_\beta(J) - U_\alpha(J) = (\alpha-\beta)G_\alpha U_\beta(J). \tag{2.2}$$

Indeed, for all $w \in D(\mathscr{E})$ we have that

$$\begin{aligned}&\mathscr{E}_\alpha(U_\beta(J)-(\alpha-\beta)G_\alpha U_\beta(J),w)\\&=\mathscr{E}_\alpha(U_\beta(J),w)+(\alpha-\beta)(U_\beta(J),w)-(\alpha-\beta)(U_\beta(J),w)\\&=J(w)=\mathscr{E}_\alpha(U_\alpha(J),w).\end{aligned}$$

In particular, (2.2) implies that $G_\alpha U_\beta(J)=G_\beta U_\alpha(J)$.

Claim 3: If $J(v)\geq 0$ for all $v\in D(\mathscr{E})$ with $v\geq 0$, then $U_\alpha(J)\geq 0$ m-a.e. for all $\alpha>0$.

The positivity of J implies $\mathscr{E}_\alpha(U_\alpha(J),v)\geq 0$ for all $v\in D(\mathscr{E})$ with $v\geq 0$. Hence by Claim 1 $U_\alpha(J)\geq 0$ m-a.e. for all $\alpha>0$.

Claim 4: u is α-excessive.

Let $v\in D(\mathscr{E})$, $v\geq 0,\beta>0$.

$$\begin{aligned}(u-\beta G_{\beta+\alpha}u,v)&=\mathscr{E}_\alpha(G_\alpha u-\beta G_\alpha G_{\beta+\alpha}u,v)\\&=\mathscr{E}_\alpha(G_{\beta+\alpha}u,v) &&\text{(resolvent equation)}\\&=\mathscr{E}_\alpha(G_{\beta+\alpha}U_\alpha(U_\alpha^{-1}u),v) &&(U_\alpha \text{ is a bijection})\\&=\mathscr{E}_\alpha(G_\alpha U_{\beta+\alpha}(U_\alpha^{-1}u),v) &&\text{(Claim 2)}\\&\geq 0,\end{aligned}$$

because $U_{\beta+\alpha}(U_\alpha^{-1}u)$ is positive by Claim 3 and assumption (ii) and because $G_\alpha f$ is α-excessive, if f is positive.
For $v=\lambda G_\lambda f$ with $0\leq f\in L^2(E;m)$ we thus have that $0\leq(u-\beta G_{\beta+\alpha}u,\lambda G_\lambda f)$, which yields as $\lambda\to\infty$ that $0\leq(u-\beta G_{\beta+\alpha}u,f)$ for all $f\in L^2(E;m)$, $f\geq 0$, and (i) follows by he remark preceding this theorem. □

REMARK 2.5. (i) If $u\in D(L)$, the crucial point in Claim 4 can be prove without using U_α. $(u-\beta G_{\beta+\alpha}u,v)=\mathscr{E}_\alpha(G_{\beta+\alpha}u,v)=\mathscr{E}_\alpha(G_{\beta+\alpha}G_\alpha(\alpha-L)u,v)=\mathscr{E}_\alpha(G_\alpha G_{\beta+\alpha}(\alpha-L)u,v)$ ≥ 0, since because of

$$((\alpha-L)u,v)=\mathscr{E}_\alpha(u,v)\geq 0\quad\text{for all } v\in D(\mathscr{E}),v\geq 0,$$

the function $(\alpha-L)u$ is positive. By the Markov property $G_{\alpha+\beta}(\alpha-L)u$ is still positive which implies that $G_\alpha G_{\beta+\alpha}(\alpha-L)u$ is α-excessive.

(ii) Theorem 2.4 is well-known if E is a locally compact separable metric space and $\mathscr{B}$ its Borel σ-algebra $\mathscr{B}(E)$. The classical proof is based on measure representation of potentials (see [2]).

The domain of a Diriclet form is inf-stable. The next theorem extends this result to semi-Dirichlet forms under the restriction that one of the functions is 1-excessive but not necessarily in the Dirichlet space.

Theorem 2.6. *Let $u \in (\mathscr{E}, D(\mathscr{E}))$ and $h \in L^2(E;m)$, 1-excessive. Then $u \wedge h \in D(\mathscr{E})$ and $\mathscr{E}_1(u \wedge h, u) \geq \mathscr{E}_1(u \wedge h, u \wedge h)$. In particular, any 1-excessive function bounded by a function $u \in d(\mathscr{E})$ is itself in $D(\mathscr{E})$.*

Proof. Definite for $\alpha, \beta > 0, u, v \in L^2(E;m)$

$$(\mathscr{E}_\alpha)^{(\beta)}(u,v) := \beta((1-\beta G_{\beta+\alpha})u, v).$$

Fix $\beta > 0$. The identity $u = (u-h)^+ + u \wedge h$ implies that

$$(\mathscr{E}_1)^{(\beta)}(u \wedge h, u - u \wedge h) = (\mathscr{E}_1)^{(\beta)}(u \wedge h, (u-h)^+). \tag{2.3}$$

Since $(G_\alpha)_{\alpha>0}$ is sub-Markovian and h is 1-excessive it follows that

$$(u-h)^+(1-\beta G_{\beta+1})(u \wedge h) \geq (u-h)^+(h - \beta G_{\beta+1}h) \geq 0,$$

and hence

$$(\mathscr{E}_1)^{(\beta)}(u \wedge h, (u-h)^+) = \beta((1-\beta G_{\beta+1})(u \wedge h), (u-h)^+) \geq 0. \tag{2.4}$$

Furthermore,

$$\begin{aligned}(\mathscr{E}_1)_1^{(\beta)}((u-h)^+, (u-h)^+) &= (\mathscr{E}_1)_1^{(\beta)}(u, (u-h)^+) - (\mathscr{E}_1)_1^{(\beta)}(u \wedge h, (u-h)^+) \\ &\leq (\mathscr{E}_1)_1^{(\beta)}(u, (u-h)^+) \qquad \text{(by (2.4))} \\ &\leq (K+1)\mathscr{E}_2(u,u)^{1/2}(\mathscr{E}_1)_1^{(\beta)}((u-h)^+, (u-h)^+)^{1/2}\end{aligned}$$

where the last inequality follows by [11, I.2.11 (iii)] applied to the form $(\mathscr{E}_1, D(\mathscr{E}))$ and its resolvent $(G_{1+\alpha})_{\alpha>0}$. Consequently,

$$\sup_{\beta>0} (\mathscr{E}_1)^{(\beta)}((u-h)^+, (u-h)^+) < \infty,$$

which implies that $(u-h)^+ \in D(\mathscr{E})$ (by [11, I.2.13 (i)]), and hence $u \wedge h = u - (u-h)^+ \in D(\mathscr{E})$. By [11, I.2.13 (iii)]

$$\mathscr{E}_1(u \wedge h, u - u \wedge h) = \lim_{\beta \to 0} (\mathscr{E}_1)^{(\beta)}(u \wedge h, u - u \wedge h),$$

and the desired inequality follows by (2.3) and (2.4). □

REMARK 2.7. Theorem 2.6 generalizes the result of inf-stability of the Dirichlet space [11, I.4.11] as well as the result that an α-excessive function is in the Dirichlet space $D(\mathscr{E})$ if it is dominated by an α-coexcessive function in $D(\mathscr{E})$ (cf. [11, III.1.3 (ii)]).

2.2. Reduced functions

In this section we recall the definition of reduced functions and collect properties which remain true if $(\mathscr{E}, D(\mathscr{E}))$ is merely a semi-Dirichlet form. We assume from now on that E is a Hausdorff topological space and take $\mathscr{B}$ to be its Borel σ-algebra $\mathscr{B}(E)$. We also assume that m is a σ-finite positive measure on $\mathscr{B}(E)$.

Proposition 2.8. *Let h be a function on E. Define for $U \subset E$, U open,*

$$\mathscr{L}_{h,U} := \{w \in D(\mathscr{E}) | w \geq h \text{ } m\text{-a.e. on } U\}.$$

Suppose that $\mathscr{L}_{h,U} \neq \emptyset$. Then:

(i) *There exists a unique $h_U \in \mathscr{L}_{h,U}$ such that for all $w \in \mathscr{L}_{h,U}$*

$$\mathscr{E}_1(h_U, w) \geq \mathscr{E}_1(h_U, h_U).$$

(ii) *$\mathscr{E}_1(h_U, w) \geq 0$ for all $w \in D(\mathscr{E})$ with $w \geq 0$ m-a.e. on U. In particular, h_U is 1-excessive and $\mathscr{E}_1(h_U, w) = 0$ for all $w \in D(\mathscr{E})_{U^c}$, where*

$$D(\mathscr{E})_{U^c} := \{u \in D(\mathscr{E}) | u = 0 \text{ } m\text{-a.e. on } U\},$$

and $U^c := E \setminus U$.

(iii) *h_U is the smallest function u on E such that $u \wedge h_U$ is a 1-excessive function in $D(\mathscr{E})$ and $u \geq h$ m-a.e. on U. In particular, $(0 \leq) h_U \leq h$ (m-a.e. on E) if and only if $h \wedge h_U$ is a 1-excessive function in $D(\mathscr{E})$. In this case $h_U = h$ m-a.e. on U.*

Suppose that $V \subset U \subset E$, V open. Then:

(iv) *$\mathscr{L}_{h,V} \supset \mathscr{L}_{h,U} (\neq \emptyset)$, $h_V \leq h_U$, and*

$$\mathscr{E}_1(h_V, h_V) \leq K^2 \mathscr{E}_1(h_U, h_U).$$

(v) *If $h \wedge h_U$ is a 1-excessive function in $D(\mathscr{E})$, then $(h_U)_V = h_V$.*

(vi) *If $g: E \to \boldsymbol{R}$, with $\mathscr{L}_{g,U} \neq \emptyset$ and $g \geq h$ m-a.e. on U, then $g_U \geq h_U$ (m-a.e. on E).*

Proof. Because of Theorem 2.4, 2.6, the proofs in [11, III.1.5 (i)-(v) and III.1.6 (iii)] carry over to the case of semi-Dirchlet forms. □

2.3. Capacities

We first recall the notion of "$\mathscr{E}$-nest" and "$\mathscr{E}$-quasi-continuity".

DEFINITION 2.9.

(i) An increasing squence $(F_k)_{k\in N}$ of closed subsets of E is called an *$\mathscr{E}$-nest* if $\bigcup_{k\geq 1} D(\mathscr{E})_{F_k}$ is dense in $D(\mathscr{E})$.

(ii) An $\mathscr{E}$-nest $(F_k)_{k\in N}$ is *regular* if for all $k\in N$, $U\subset E$, U open, $m(U\cap F_k)=0$ implies that $U\subset F_k^c$.

(iii) A subset $N\subset E$ is called *$\mathscr{E}$-exceptional* if $N\subset \bigcap_{k\geq 1} F_k^c$ for some $\mathscr{E}$-nest $(F_k)_{k\in N}$. We say that a property of points in E holds *$\mathscr{E}$-quasi-everywhere* (abbreviated *$\mathscr{E}$-q.e.*), if the property holds outside some $\mathscr{E}$-exceptional set.

Lemma 2.10.

(i) *Let $U\subset E$, U open, and let $(F_k)_{k\in N}$ be an $\mathscr{E}$-nest. Let h be a* 1*-excessive function in $D(\mathscr{E})$. Then $h_{U\cup F_k^c}\to h_U$ in $D(\mathscr{E})$ as $k\to\infty$.*

(ii) *Let $h\in D(\mathscr{E})$ and $U_n\subset E$, U open, $U_n\uparrow U$. Then $h_{U_n}\to h_U$ in $D(\mathscr{E})$ as $n\to\infty$.*

Proof. (i): Since by Proposition 2.8 (iv) $(h_{U\cup F_k^c})_{k\in N}$ is a decreasing sequence of functions, $\lim_{k\to\infty} h_{U\cup F_k^c}=:h_\infty$ exists m-a.e. and in $L^2(E;m)$. Since by the inequality in Proposition 2.8 (iv) $(h_{U\cup F_k^c})_{k\in N}$ is bounded in $\tilde{\mathscr{E}}_1^{1/2}$-norm, it follows that it weakly converges to h_∞ in $(D(\mathscr{E}),(\tilde{\mathscr{E}}_1)$ (cf. [11, I.2.12]).

Step 1. Assume $U=\emptyset$.

For every $w\in \bigcup_{k\in N} D(\mathscr{E})_{F_k}$

$$\mathscr{E}_1(h_\infty,w)=\lim_{k\to\infty}\mathscr{E}_1(h_{F_k^c},w)=0.$$

Becasue $(F_k)_{k\in N}$ is an $\mathscr{E}$-nest, this implies $h_\infty=0$. By Proposition 2.8. (ii) we see that

$$\limsup_{k\to\infty}\mathscr{E}_1(h_{F_k^c},h_{F_k^c})=\lim_{k\to\infty}\mathscr{E}_1(h_{F_k^c},h)=0,$$

hence

$$h_{F_k^c}\to 0 \qquad \text{in } D(\mathscr{E}) \text{ as } k\to\infty.$$

Step 2. Assume $U\subset E$, open.

We have by Proposition 2.8 (iii)

$$h_{U\cup F_k^c}\leq h_U+h_{F_k^c},$$

hence $h_\infty\leq h_U$. But it is obvious that $h_\infty\geq h$ m-a.e. on U and that h_∞ is 1-excessive. Therefore, $h_\infty\geq h_U$ and consequently, $h_\infty=h_U$.
Because h is 1-excessive, we have that $h_{U\cup F_k^c}=h=h_U$ m-a.e. on U by Proposition 2.8 (iii), and therefore, by Proposition 2.8 (ii)

$$\begin{aligned}
&\mathscr{E}_1(h_U-h_{U\cup F_k^c},h_U-h_{U\cup F_k^c})\\
&=\mathscr{E}_1(h_{U\cup F_k^c},h_{U\cup F_k^c})-\mathscr{E}(h_{U\cup F_k^c},h_U)\\
&=\mathscr{E}_1(h_{U\cup F_k^c},h)-\mathscr{E}_1(h_{U\cup F_k^c},h_U)\\
&\underset{k\to\infty}{\to}\mathscr{E}_1(h_U,h-h_U)=0.
\end{aligned}$$

(ii): $(h_{U_n})_{n\in N}$ is increasing and bounded in $(D(\mathscr{E}),\tilde{\mathscr{E}}_1)$ by Proposition 2.8 (iv). Let h_∞ be the pointwise and weak limit. Then the weak convergence of $(h_{U_n})_{n\in N}$ in $(D(\mathscr{E}),\tilde{\mathscr{E}}_1)$ yields

$$\begin{aligned}
&\limsup_{n\to\infty}\mathscr{E}_1(h_\infty-h_{U_n},h_\infty-h_{U_n})\\
&=\limsup_{n\to\infty}\mathscr{E}_1(h_{U_n},h_{U_n})-\mathscr{E}_1(h_\infty,h_\infty)\\
&\leq\limsup_{n\to\infty}\mathscr{E}_1(h_{U_n},h_\infty)-\mathscr{E}_1(h_\infty,h_\infty)=0,
\end{aligned}$$

where the inquality follows from Proposition 2.8 (i) since clearly $h_\infty\geq h$ m-a.e. on each U_n. □

The description of "small sets" by Definition 2.9 (iii) is essentially sufficient to formulate quasi-regularity and to construct the process. But for the proofs we need to "quantify" $\mathscr{E}$-nests. Therefore, we introduce a capacity whose zero sets are exactly the $\mathscr{E}$-exceptional sets.

DEFINITION 2.11. Let $\phi\in L^2(E;m)$ such that $0<\phi\leq 1$ m-a.e. and set $h:=G_1\phi(>0)$. Then h is a 1-excessive function in $D(\mathscr{E})$ and strictly positive m-a.e. Define for $U\subset E$, *U open*,

$$\text{cap}_\phi(U):=(h_U,\phi)$$

and for any $A\subset E$

$$\mathrm{cap}_\phi(A) := \inf\{\mathrm{cap}_\phi(U) | A \subset U,\ U \textit{ open}\}.$$

REMARK 2.12. (i) A cap_ϕ-zero set is also an m-zero set, since

$$\mathrm{cap}_\phi(A) \geq \inf_{A\subset U} \int_U h\phi dm \geq \int_A h\phi dm.$$

(ii) Note that if $(\mathscr{E}, D(\mathscr{E}))$ is a Dirichlet form and if we set $g := \hat{G}_1\phi$, then by (2.5), 2.8 (ii) and [11, III.2.4] $\mathrm{cap}_\phi(A) = \mathrm{Cap}_{h,g}(A)$, where $\mathrm{Cap}_{h,g}$ is the capacity defined in [11, III.2.4].

(iii) We have for $U \subset E$, U open,

$$\mathrm{cap}_\phi(U) = \mathscr{E}_1(h_U, \hat{G}_1\phi) \leq K\mathscr{E}_1(\hat{G}_1\phi, \hat{G}_1\phi)^{\frac{1}{2}}\mathscr{E}_1(h_U, h_U)^{\frac{1}{2}}, \tag{2.5}$$

where $(\hat{G}_\alpha)_{\alpha>0}$ is the resolvent of $\hat{\mathscr{E}}$ (cf. 2.2 (ii)).

Proposition 2.13. (i) *If $U \subset W$, U and W open, with $m(W\backslash U) = 0$, then* $\mathrm{cap}_\phi(U) = \mathrm{cap}_\phi(W)$.

(ii) $$A \subset B \Rightarrow \mathrm{cap}_\phi(A) \leq \mathrm{cap}_\phi(B).$$

(iii) $$U_n \uparrow U, U_n \textit{ open} \Rightarrow \lim_{n\to\infty} \mathrm{cap}_\phi(U_n) = \mathrm{cap}_\phi(U).$$

(iv) $$A_n \subset E \Rightarrow \mathrm{cap}_\phi\Big(\bigcup_{n\in N} A_n\Big) \leq \sum_n \mathrm{cap}_\phi(A_n).$$

Proof. (i): Clearly, $\mathscr{L}_{h,U} = \mathscr{L}_{h,W}$, hence $h_W = h_U$ m-a.e. and the assertion follows.

(ii): It is sufficient to consider open sets $A \subset B$. The assertion follows then from $h_B \geq h_A$, (cf. Proposition 2.8 (iv)).

(iii): Trivial by Lemma 2.10 (ii).

(iv): Let $U_1, \cdots, U_k$ be open subsets of E. Then

$$\sum_{n=1}^k h_{U_n} \geq h_{\overset{k}{\underset{n=1}{\cup}} U_n}$$

and hence

$$\sum_{n=1}^k (h_{U_n}, \phi)) \geq (h_{\overset{k}{\underset{n=1}{\cup}} U_n}, \phi).$$

Letting $k \to \infty$ we obtain the assertion by (ii) if $A_n = U_n$ is open. Then the assertion trivially follows for all sets in E. □

The crucial result connecting cap_ϕ with $\mathscr{E}$-nests is the following

Theorem 2.14. *An increasing squence $(F_k)_{k\in N}$ of closed subsets of E is an $\mathscr{E}$-nest if and only if $\lim_{k\to\infty}\text{cap}_\phi(F_k^c)=0$.*

Proof. The "only if" part follows by Lemma 2.10 (i) (with $U=\emptyset$). To prove the converse let $u\in D(\mathscr{E})$ such that $\mathscr{E}_1(w,u)=0$ for all $w\in\bigcup_k D(\mathscr{E})_{F_k}$. By the theorms of Hahn-Banach and Lax-Milgram it is enough to show $u=0$.
Let $g\le h, g\in D(\mathscr{E})$, the $g_U\le h_U$ by Proposition 2.8 (vi) for every open set $U\subset E$. Hence

$$0\le(g_{F_k^c},\phi)\le(h_{F_k^c},\phi).$$

Hence by assumption, since $(g_{F_k^c})_{k\in N}$ is decreasing by Proposition 2.8 (iv), $g_{F_k^c}\to 0$ in $L^2(E;m)$ as $k\to\infty$. But $\sup_k\mathscr{E}_1(g_{F_k^c},g_{F_k^c})<\infty$ (by Proposition 2.8 (iv)); hence by [11, I,2.12] $g_{F_k^c}\to 0$ weakly in $(D(\mathscr{E}),\tilde{\mathscr{E}}_1)$. Now we specify g as $G_1(1_A\phi)$ where $A\in\mathscr{B}(E)$. Then

$$0=\mathscr{E}_1(g-g_{F_k^c},u)\underset{k\to\infty}{\to}\mathscr{E}_1(g,u)=\int_A\phi u\,dm.$$

Because $A\in\mathscr{B}(E)$ is arbitrary and $\phi>0$ m-a.e., it follows that $u=0$. □

2.4. $\mathscr{E}$-quasi continuity

Given an $\mathscr{E}$-nest $(F_k)_{k\in N}$ we define

$$C(\{F_k\}):=\{f\colon A\to \mathbf{R}\mid \bigcup_{k\ge 1}F_k\subset A\subset E, f|_{F_k}\text{ is continuous for every }k\in \mathbf{N}\}.$$

DEFINITION 2.15. An $\mathscr{E}$-q.e. defined function f on E is called *$\mathscr{E}$-quasi-continuous* if there exists an $\mathscr{E}$-nest $(F_k)_{k\in N}$ such that $f\in C(\{F_k\})$.

Proposition 2.16. *Let S be a countable family of $\mathscr{E}$-quasi-continuous functions on E. Then there exists an $\mathscr{E}$-nest $(F_k)_{k\in N}$ such that $S\subset C(\{F_k\})$.*

Proof. (cf. [11, III.3.3]) Let $S=\{f_l\mid l\in \mathbf{N}\}$. Choose for every $l\in \mathbf{N}$ an $\mathscr{E}$-nest $(F_{lk})_{k\in N}$ such that

$$\operatorname{cap}_\phi(F_{lk}^c) \le \frac{1}{2^l k}.$$

The sets $F_k := \cap_l F_{lk}$ satisfy the assertion by the sub-additivity of cap_ϕ. □

The following analogue of the Chebychev inequality is crucial.

Proposition 2.17. *Let $u \in D(\mathcal{E})$ such that it has an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$. Then for all $\lambda > 0$*

$$\operatorname{cap}_\phi(\{|\tilde{u}| > \lambda\}) \le \frac{4K^4}{\lambda} \mathcal{E}_1(\hat{G}_1\phi, \hat{G}_1\phi)^{\frac{1}{2}} \mathcal{E}_1(u,u)^{1/2}.$$

Proof. Let $(F_k)_{k\in N}$ be an $\mathcal{E}$-nest such that $\tilde{u} \in C(\{F_k\})$. Consider the open set $U_k := \{|\tilde{u}| > \lambda\} \cup F_k^c$ and the function $u_k := \lambda^{-1}|\tilde{u}| + h_{F_k^c} \in D(\mathcal{E})$, which dominates h on U_k m-a.e. (since $h = G_1\phi \le 1$). Then

$$\begin{aligned}
\mathcal{E}_1(h_{U_k}, h_{U_k}) &\le \mathcal{E}_1(h_{U_k}, u_k) \\
&= \lambda^{-1}\mathcal{E}_1(h_{U_k}, |u|) + \mathcal{E}_1(h_{U_k}, h_{F_k^c}) \\
&\le K\mathcal{E}_1(h_{U_k}, h_{U_k})^{1/2}(\lambda^{-1}\mathcal{E}_1(|u|,|u|)^{1/2} + \mathcal{E}_1(h_{F_k^c}, h_{F_k^c})^{1/2}).
\end{aligned}$$

Therefore, using Proposition 2.13 (ii) and (2.5) we obtain that

$$\begin{aligned}
&\operatorname{cap}_\phi(\{|\tilde{u}| > \lambda\}) \le \operatorname{cap}_\phi(U_k) \\
&\qquad \le K^2 \mathcal{E}_1(\hat{G}_1\phi, \hat{G}_1\phi)^{\frac{1}{2}}(\lambda^{-1}\mathcal{E}_1(|u|,|u|)^{1/2} + \mathcal{E}_1(h_{F_k^c}, h_{F_k^c})^{1/2}).
\end{aligned}$$

The second summand tends to 0 as $k \to \infty$ by Theorem 2.14, hence (2.1) implies the assertion. □

We can now prove the remaining necessary results of the analytic potential theory of semi-Dirichlet forms in exactly the same way as in [11, III.3].

Proposition 2.18.

(i) *Let $u_n \in D(\mathcal{E})$, which have $\mathcal{E}$-quasi-continuous m-vesions $\tilde{u}_n$, $n \in N$, such that $u_n \to u \in D(\mathcal{E})$ as $n \to \infty$ w.r.t. $\tilde{\mathcal{E}}_1^{1/2}$. Then there exists a subsequence $(u_{n_k})_{k\in N}$ and an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ of u such that $(\tilde{u}_{n_k})_{k\in N}$ converges $\mathcal{E}$-quasi-uniformly to $\tilde{u}$, i.e., there is an $\mathcal{E}$-nest $(F_k)_{k\in N}$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$ uniformly on each F_k.*

(ii) *Suppose h in Definition 2.11 has an $\mathcal{E}$-quasi-continuous m-version $\tilde{h}$ and let*

$(F_k)_{k\in N}$ *be an* $\mathscr{E}$*-nest such that* $\tilde{h}\in C(\{F_k\})$. *Let* $(\delta)_{k\in N}$ *be a decreasing sequence of positive numbers and that* $\lim_{k\to\infty}\delta_k=0$. *Then there exists an* $\mathscr{E}$*-nest* $(\tilde{F}_k)_{k\in N}$ *such that* $\tilde{F}_k\subset F_k$ *and* $\tilde{h}\geq\delta_k$ *on* $\tilde{F}_k$ *for every* $k\in N$. *In particular,* $\{\tilde{h}=0\}$ *is* $\mathscr{E}$*-exceptional.*

(iii) *Let* $(F_k)_{k\in N}$ *be an* $\mathscr{E}$*-nest. Suppose that the relative topology on each* F_k *is strongly Lindelöf (i.e., every open cover of any given open set has a countable subcover). Set* $\tilde{F}_k:=\text{support}\,[1_{F_k}\cdot m]$, *then* $(\tilde{F}_k)_{k\in N}$ *is a regular* $\mathscr{E}$*-nest, such that* $\tilde{F}_k\subset F_k$ *for all* $k\in N$.

(iv) *Suppose* $(F_k)_{k\in N}$ *is a regular* $\mathscr{E}$*-nest and* $f\in C(\{F_k\})$. *If* $f\geq 0$ *m-a.e. on an open set* U *then* $f(z)\geq 0$ *for all* $z\in\bigcup_{k\geq 1}F_k\cap U$.

Proof. (i), (iii), and (iv) are proved as in [11, III.3.5, 3.8, 3.9]. To prove (ii) set

$$\tilde{F}_k:=\{\tilde{h}\geq\delta_k\}\cap F_k.$$

Then $(\tilde{F}_k)_{k\in N}$ is an increasing sequence of closed sets. Let

$$u_k:=(h\wedge\delta_k)+h_{F_k^c},\ k\in N.$$

Then for each $k\in N$, $u_k\in D(\mathscr{E})$, $u_k\geq h$ m-a.e. on $\tilde{F}_k^c$ and

$$\begin{aligned}(2.6)\qquad \mathscr{E}_1(h_{\tilde{F}_k^c},h_{\tilde{F}_k^c})&\leq\mathscr{E}_1(h_{\tilde{F}_k^c},u_k)\\ &=\mathscr{E}_1(h_{\tilde{F}_k^c},h\wedge\delta_k)+\mathscr{E}_1(h_{\tilde{F}_k^c},h_{F_k^c})\\ &\leq K\mathscr{E}_1(h_{\tilde{F}_k^c},h_{\tilde{F}_k^c})^{1/2}\\ &\qquad(\mathscr{E}_1(h\wedge\delta_k,h\wedge\delta_k)^{1/2}+\mathscr{E}_1(h_{F_k^c},h_{F_k^c})^{1/2}).\end{aligned}$$

Since by Remark 2.2 (iii)

$$\begin{aligned}(2.7)\qquad \mathscr{E}_1(h\wedge\delta_k,h\wedge\delta_k)&\leq\mathscr{E}_1(h\wedge\delta_k,h)\\ &\leq K\mathscr{E}_1(h\wedge\delta_k,h\wedge\delta_k)^{1/2}\mathscr{E}_1(h,h)^{1/2},\end{aligned}$$

it follows by [11, I.2.12] that $h\wedge\delta_k\underset{k\to\infty}{\to}0$ weakly in $(D(\mathscr{E}),\tilde{\mathscr{E}}_1)$, hence by (2.7) $h\wedge\delta_k\underset{k\to\infty}{\to}0$ w.r.t. $\tilde{\mathscr{E}}_1^{\frac{1}{2}}$. Now (2.6) and (2.5) imply the assertion. □

Proposition 2.19. *Suppose that the following condition holds.*

(2.8) *Every* $u\in D(\mathscr{E})$ *has an* $\mathscr{E}$*-quasi-continuous m-version denoted by* $\tilde{u}$.

Let f *be an* $\mathscr{E}$*-quasi-continuous function on* E.

(i) *If $f \geq 0$ m-a.e. on some open set $U \subset E$ then $f \geq 0$ $\mathscr{E}$-q.e. on U. In particular, $\mathscr{E}$-quasi-continuous m-versions of elements in $D(\mathscr{E})$ are unique $\mathscr{E}$-q.e.*

(ii) *Let $A \subset E$ and define*

$$\mathscr{L}_{f,A} := \{w \in D(\mathscr{E}) | \tilde{w} \geq f \ \mathscr{E}\text{-q.e. on } A\}.$$

Assume $\mathscr{L}_{f,A} \neq \emptyset$. Then there exists a unique $f_A \in \mathscr{L}_{f,A}$ such that for all $w \in \mathscr{L}_{f,A}$

$$\mathscr{E}_1(f_A, w) \geq \mathscr{E}_1(f_A, f_A).$$

This extends the notion of "reduced function on A" to arbitrary sets $A \subset E$. Furthermore, the correspondingly modified analogues of Proposition 2.8 *with A, f replacing U, h respectively and "$\mathscr{E}$-q.e." replacing "m-a.e." remain true.*

Proof. (i): The same arguments as in [4, Proposition I. 8.1.6] prove the assertion.

(ii): By Proposition 2.18 (i) the proof is analogous to that of Proposition 2.8. □

Theorem 2.20. *Let h be as in Definition* 2.11. *Suppose that condition* (2.8) *holds and let $A \subset E$. Then*

$$\operatorname{cap}_\phi(A) = (h_A, \phi) (= \mathscr{E}_1(h_A, \hat{G}_1 \phi)).$$

Proof. Let $U \subset E$, U open, with $A \subset U$. Then by Proposition 2.19 (ii) we have that $\tilde{h}_A \leq \tilde{h}_U$ $\mathscr{E}$-q.e. on E, hence

$$\operatorname{cap}_\phi(U) = (h_U, \phi) \geq (h_A, \phi).$$

Consequently, $\operatorname{cap}_\phi(A) \geq (h_A, \phi)$. To prove the dual inequality let $(U_n)_{n \in N}$ be a decreasing sequence of open sets in E such that $A \subset U_n$, $n \in N$, and

$$\operatorname{cap}_\phi(U_n) \downarrow \operatorname{cap}_\phi(A) \qquad \text{as } n \to \infty.$$

By Propositions 2.16 and 2.18 (ii) there exists an $\mathscr{E}$-nest $(F_k)_{k \in N}$ such that $\tilde{h}, \tilde{h}_A \in C(\{F_k\})$ and $\tilde{h} > 0$ on each F_k. Let $k \in N$ and define

$$V_k := \left\{ \tilde{h}_A > \left(1 - \frac{1}{k}\right) \tilde{h} \right\} \cup F_k^c.$$

Then V_k is open, $V_k \supset A \backslash N$ for some $\mathscr{E}$-exceptional set N and

$$h_A + \frac{1}{k} h + h_{F_k^c} \geq h \ m\text{-a.e.} \quad \text{on } V_k.$$

Since $h, h_A, h_{F_k^c}$ are 1-excessive, we conclude by Proposition 2.8 (iii) and (iv) that

$$h_A + \frac{1}{k} h + h_{F_k^c} \geq h_{V_k} \geq h_{U_n \cap V_k} \quad \text{for all } n \in N.$$

Hence

$$\begin{aligned}(h_A, \phi) &\geq \lim_{k\to\infty} \lim_{n\to\infty} (h_{U_n \cap V_k}, \phi) \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \operatorname{cap}_\phi(U_n \cap V_k) \\ &\geq \operatorname{cap}_\phi(A \setminus N) = \operatorname{cap}_\phi(A).\end{aligned}$$

□

Using Proposition 2.18 (i) we can prove the following lemma exactly as Lemma 2.10

Lemma 2.21. *Let $h \in D(\mathscr{E})$ and $A_n \subset E$, $A_n \uparrow A$.* Then $h_{A_n} \to h_A$ *in* $D(\mathscr{E})$ *as* $n \to \infty$.

Corollary 2.22. *Under the assumption of Theorem* 2.20 *we have that* cap_ϕ *is a Choquet capacity on E, i.e., it has the following two properties:*

(i) *If $(A_n)_{n\in N}$ is an increasing sequence of subsets of E then*

$$\operatorname{cap}_\phi\Big(\bigcup_{n\geq 1} A_n\Big) = \sup_{n\geq 1} \operatorname{cap}_\phi(A_n).$$

(ii) *If $(K_n)_{n\in N}$ is a decreasing sequence of compact subsets of E then*

$$\operatorname{cap}_\phi\Big(\bigcap_{n\geq 1} K_n\Big) = \inf_{n\geq 1} \operatorname{cap}_\phi(K_n).$$

Proof. (i): It is clear that the left-hand side dominates the right-hand side. The dual inequality follows immediately by Theorem 2.20 and Lemma 2.21.

(ii): Straightforward, cf. [11, II.2.8]. □

3. Quasi-regularity: a necessary and sufficient condition for the existence of an associated special standard process

In this section for simplicity we assume that $\mathscr{B}(E)$ is generated by the continuous functions on E. Let us first recall some notions from the general theory of Markov processes (cf. [11, IV.1], [16]).

3.1. Association of right processes and Dirichlet forms

DEFINITION 3.1.

(i) Let $M=(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},(X_t)_{t\geq 0},(P_z)_{z\in E_\Delta})$ be a normal strong Markov process with state space E, life time ζ, cemetery Δ, and shift operators $\theta_t, t\geq 0$. M is called a *right process* if for each $\omega\in\Omega$, $t\mapsto X_t(\omega)$ is right continuous on $[0,\infty[$.

(ii) Let μ be a positive measure on $(E_\Delta,\mathscr{B}(E_\Delta))$. A right process M is called *μ-tight* if there exists an increasing sequence $(K_n)_{n\in N}$ of compact metrizable sets in E such that

$$P_\mu\left[\lim_{n\to\infty}\sigma_{E\setminus K_n}<\zeta\right]=0,$$

where $\sigma_B:=\inf\{t>0|X_t\in B\}$ is the *first hitting time* of a subset B of E_Δ.

(iii) Let m be a σ-finite positive measure on $(E_\Delta,\mathscr{B}(E_\Delta))$. A right process M is called an *m-special standard process* if for one (and hence all) probability measures μ on $(E_\Delta,\mathscr{B}(E_\Delta))$, which are equivalent to m, it has the following additonal properties:

(a) (*left limits up to ζ*) $X_{t-}:=\lim_{\substack{s\uparrow t\\ s<t}} X_s$ exists in E for all $t\in]0,\zeta[$ $P_\mu-a.s.$.

(b) (*quasi-left continuity up to ζ and special*) If τ,τ_n, $n\in N$, are $(\mathscr{F}_t^{P_\mu})$-stopping times such that $\tau_n\uparrow\tau$, then $X_{\tau_n}\to X_\tau$ as $n\to\infty$ P_μ-*a.s.* on $\{\tau<\zeta\}$ and X_τ is $\bigvee_{n\in N}\mathscr{F}_{\tau_n}^{P_\mu}$-measurable.

(iv) M is called a *special standard process* if it is a μ-special standard process for all probability measures μ on $(E_\Delta,\mathscr{B}(E_\Delta))$.

REMARK 3.2. $\mathscr{F}_t^{P_\mu}$ denotes the completion of $\mathscr{F}_t$ w.r.t. to $P_\mu:=\int P_z\mu(dz)$ and from now on we will assume without restriction that $(\mathscr{F}_t)_{t\geq 0}$ is the natural filtration of M, cf. [11, IV 1.10]. Let $E_z[\cdot]$ denote the expectation w.r.t. $P_z, z\in E_\Delta$.

DEFINITION 3.3. A right process M with state space E is said to be (*properly*) *associated* with a semi-Dirichlet form $(\mathscr{E},D(\mathscr{E}))$ on $L^2(E;m)$ iff $p_tf:=E[f(X_t)]$ is an ($\mathscr{E}$-quasi-continuous) m-version of T_tf for all $f\colon E\to R$, $\mathscr{B}(E)$-measurable, m-square integrable, and all $t>0$.

Proposition 3.4. *Let* $(p_t)_{t>0}$ *and* M *be as in* 3.3 *and let* $(\mathscr{E},D(\mathscr{E}))$ *be a semi-Dirichlet form with resolvent* $(G_\alpha)_{\alpha>0}$. *Let for* $f\colon E\to R$, $\mathscr{B}(E)$*-measurable, bounded,*

$$R_\alpha f(z) := \int_0^\infty e^{-\alpha t} p_t f(z) dt, \ z \in E, \ \alpha > 0,$$

(i.e., $(R_\alpha)_{\alpha>0}$ is the resolvent (of kernels) associated with $(p_t)_{t>0}$ resp. $\boldsymbol{M}$). Then the following are equivalent.

(i) *$\boldsymbol{M}$ is properly associated with $(\mathscr{E}, D(\mathscr{E}))$.*

(ii) *$R_\alpha f$ is an $\mathscr{E}$-quasi continuous m-version of $G_\alpha f$ for all $\alpha>0$ and all $\mathscr{B}(E)$-measurable, bounded, m-square integrable functions $f: E \to \boldsymbol{R}$.*

Proof. Because of Proposition 2.18 (i) the proof is the same as that of Proposition IV.2.8 in [11]. □

In order to state our main theorem we have to extend the notion of *quasi-regularity* to semi-Dirichlet forms.

DEFINITION 3.5. A semi-Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E; m)$ is called *quasi-regular* if:

(i) There exists an $\mathscr{E}$-nest $(E_k)_{k\in \boldsymbol{N}}$ consisting of compact sets.

(ii) There exists an $\tilde{\mathscr{E}}_1^{1/2}$-dense subset of $D(\mathscr{E})$ whose elements have $\mathscr{E}$-quasi-continuous m-versions.

(iii) There exist $u_n \in D(\mathscr{E})$, $n \in \boldsymbol{N}$, having $\mathscr{E}$-quasi-continuous m-versions $\tilde{u}_n$, $n \in \boldsymbol{N}$, and an $\mathscr{E}$-exceptional set $N \subset E$ such that $\{\tilde{u}_n | n \in \boldsymbol{N}\}$ separates the points of $E \setminus N$.

The next proposition collects the properties of quasi-regular semi-Dirichlet forms which are important for the construction of an assocaited process.

Proposition 3.6. *Let $(\mathscr{E}, D(\mathscr{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E; m)$. Then:*

(i) *There exists an $\mathscr{E}$-nest of metrizable compact sets.*

(ii) *$D(\mathscr{E})$ is separable w.r.t. $\tilde{\mathscr{E}}_1^{1/2}$.*

(iii) *Each element $u \in D(\mathscr{E})$ has an $\mathscr{E}$-quasi-continuous m-version denoted by $\tilde{u}$.*

(iv) *If f is $\mathscr{E}$-quasi-continuous and $f \geq 0$ m-a.e. on an open subset U of E, then $f \geq 0$ $\mathscr{E}$-q.e. on U. In particular, $\tilde{u}$ is $\mathscr{E}$-q.e. unique for all $u \in D(\mathscr{E})$.*

(v) *If $\mathscr{D}_1$ is a dense subset of $D(\mathscr{E})$, then there exists an $\mathscr{E}$-exceptional set $N \subset E$ and $\mathscr{E}$-quasi-continuous m-versions $\tilde{u}$, $u \in \mathscr{D}_1$, such that $\{\tilde{u} | u \in \mathscr{D}_1\}$ separates the points of $E \setminus N$.*

(vi) *There exists a countable subset* $\mathscr{D}_0^+$ *of* $D(\mathscr{E})$ *consisting of bounded* 1*-excessive functions such that* $\mathscr{D}_0^+ - \mathscr{D}_0^+$ *is dense in* $\mathscr{D}(\mathscr{E})$*, an* $\mathscr{E}$*-exceptional set* $N \subset E$ *and* $\mathscr{E}$*-quasi-continuous m-versions* $\tilde{u}$, $u \in \mathscr{D}_0^+$, *such that* $\{\tilde{u} | u \in \mathscr{D}_0^+\}$ *separates the points of* $E \backslash N$.

Proof. By virtue of our results in Section 2 the proofs of [11, IV 3.3, 3.4] carry over to the case of semi-Dirichlet forms. □

REMRK 3.7. (i) As shown in [11, IV 3.2 (iii)] the set $L^2(E;m)$ can canonically be identified with $L^2(Y;m)$, where $Y := \cup E_k$ and $(E_k)_{k \in N}$ is an $\mathscr{E}$-nest of metrizable compacts, because $m(E \backslash Y) = 0$ and $\mathscr{B}(Y) = \mathscr{B}(E) \cap Y$. The set Y is a topological Lusin space. Therefore, when dealing with quasi-regular semi-Dirichlet forms one could assume without loss of generality that E is a (topological) Lusin space.

(ii) It will turn out that for m-a.e. $z \in E$, P_z-a.e. $\boldsymbol{M}$ takes values in the Lusin space Y.

3.2. Sufficiency of quasi-regularity

Theorem 3.8. *Let* $(\mathscr{E}, D(\mathscr{E}))$ *be a quasi-regular semi-Dirichlet form on* $L^2(E;m)$. *Then there exists an m-tight special standard process* $\boldsymbol{M}$ *which is properly associated with* $(\mathscr{E}, D(\mathscr{E}))$.

After having developed all the necessary analytic potential theory of semi-Dirichlet forms in Section 2 and because of Propositions 3.4 and 3.6, the proof of Theorem 3.8 can be done by carrying over the proof of Theorem IV.3.5 in [11] (cf. also [1]) *word by word.*
The idea is to construct a set Y_2 with $E \backslash Y_2$ $\mathscr{E}$-exceptional and via a nice countable set $\mathscr{J}_0 \subset D(\mathscr{E})$ of 1-excessive $\mathscr{E}$-quasi-continuous functions a compactification $\bar{E}$ of $Y_2 \cup \{\Delta\}$ and from $(G_\alpha)_{\alpha > 0}$ a corresponding Ray-resolvent $(\bar{R}_\alpha)_{\alpha > 0}$ on $\bar{E}$. Then we show that the corresponding right process $\bar{\boldsymbol{M}}$ can be restricted to $Y_2 \cup \{\Delta\}$ and that this restriction is an m-tight special standard process properly associated with $(\mathscr{E}, D(\mathscr{E}))$.

3.3. Necessity of quasi-regularity

Theorem 3.9. *Suppose that there exists an m-special standard process* $\boldsymbol{M} = (\Omega, \mathscr{F}, (X_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ *with state space E and life time* ζ *which is m-tight and associated with* $(\mathscr{E}, D(\mathscr{E}))$. *Then* $(\mathscr{E}, D(\mathscr{E}))$ *is quasi-regular, i.e., satisfies* 3.5 (i)-(iii). *Moreover,* $\boldsymbol{M}$ *is properly associated with* $(\mathscr{E}, D(\mathscr{E}))$.

After our preparation the proof of properties 3.5 (i) and (ii) can be done in exactly the same way as in [11, IV.5a and b]. Now let us turn to the proof of 3.5

(iii). We first note that by Proposition 2.18 (i) it follows from property 3.5 (ii), which we have just proved, that every $u \in D(\mathscr{E})$ has an $\mathscr{E}$-quasi continuous version $\tilde{u}$. Hence, we know by Corollary 2.22 that cap_ϕ is a Choquet capacity. Now the proof of property 3.5 (iii) parallels entirely the proof in [11, IV.5c]. The fact that cap_ϕ is a Choquet capacity was crucial in [11, IV.5c] for the proof of Theorem 5.28 (i), which implies that the resolvent of the process maps a bounded $\mathscr{B}(E)$-measurable funcion to an $\mathscr{E}$-continuous function.

REMARK 3.10. (i) For the same reasons as above all other results in [11, Chapters IV-VI] on Dirichlet form carry over to semi-Dirichlet forms (as far as they still make sense), like e.g. the *one-to-one correspondence* between semi-Dirichlet forms $(\mathscr{E},D(\mathscr{E}))$ and special standard processes $\boldsymbol{M}$, the equivalence of the *local property* of $(\mathscr{E},D(\mathscr{E}))$ with the continuity of the sample paths of $\boldsymbol{M}$, the *regularization/transfer method* developed in [11, VI] etc. We also mention that the crucial relation of the capacity with the hitting probabilities given by

$$\text{cap}_\phi(A) = \int_E E_x\Big[\int_{\tau_A}^{\zeta} e^{-s}\Phi(X_s)ds\Big]\phi(x)m(dx),$$

still holds. Here $A \in \mathscr{B}(E)$ and τ_A is the *first touching time of A* and Φ is an m-version of ϕ such that $\Phi(z)>0$ for all $z \in E$.

(ii) Using the regularization/transfer method mentioned above one can also derive a proof of the "sufficiency part" of our result, i.e., Theorem 3.8, from [5] (i.e., the "regular locally compact" case).

3.4. Examples

(i) Let U be an open (not necessarily bounded) subset in $\boldsymbol{R}^d$ and let dx denote d-dimensional Lebesgue measure. Let $a_{ij}, b_i, d_i, c \in L^1_{loc}(U;dx), 1 \le i,j \le d$, and define for $u,v \in C_0^\infty(U)$(:= the set of all infinitely differentiable functions with compact support in U)

$$\mathscr{E}(u,v) = \sum_{i,j=1}^{d}\int \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}a_{ij}dx + \sum_{i=1}^{d}\int \frac{\partial u}{\partial x_i}vb_i dx$$
$$+ \sum_{i=1}^{d}\int u\frac{\partial v}{\partial x_i}d_i dx + \int uvc\,dx.$$

Then $(\mathscr{E}, C_0^\infty(U))$ is a densely defined bilinear form on $L^2(U;dx)$. Set $\tilde{a}_{ij} := \frac{1}{2}(a_{ij}+a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij}-a_{ji})$, $\underline{b} := (b_1,\cdots,b_d)$, $\underline{d} := (d_1,\cdots,d_d)$ and let $\|\ \ \|$ denote the Euclidean distance. Suppose that

(3.1) (*strict ellipticity*) There exists $\nu \in]0,\infty[$ such that

$$\sum_{i,j=1}^{d} \tilde{a}_{ij}\xi_i\xi_j \geq \nu\|\underline{\xi}\|^2 \qquad dx\text{-a.e. for all } \underline{\xi}=(\xi_1,\cdots,\xi_d)\in \boldsymbol{R}^d.$$

(3.2) $\check{a}_{ij}\in L^\infty(U;dx), 1\leq i,j\leq d.$

(3.3) $\|\underline{b}\|, \|\underline{d}\| \in L^d_{loc}(U;dx), c\in L^{d/2}_{loc}(U;dx).$

(3.4) $\|\underline{b}-\underline{d}\| \in L^\infty(U;dx)\cup L^d(U;dx).$

(3.5) There exists $\alpha\in]0,\infty[$ such that

$$(c+\alpha)dx-\sum_{i=1}^{d}\frac{\partial d_i}{\partial x_i} \qquad \text{and} \qquad (c+\alpha)dx-\sum_{i=1}^{d}\frac{\partial \gamma_i}{\partial x_i}$$

are positive measures on $\mathscr{B}(U)$, where $\underline{\beta}:=(\beta_1,\cdots,\beta_d), \underline{\gamma}:=(\gamma_1,\cdots,\gamma_d), \beta_i,\gamma_i\in L^1_{loc}(U;dx)$, $1\leq i\leq d$, such that $\underline{b}=\underline{\beta}+\underline{\gamma}$ with $\|\underline{\beta}\|\in L^\infty(U;dx)\cup L^p(U;dx)$ for some $p\leq d$.

Then $(\mathscr{E}_\alpha, C_0^\infty(U))$ is closable on $L^2(U;dx)$ and its closure $(\mathscr{E}_\alpha, D(\mathscr{E}_\alpha))$ is a quasi-regular semi-Dirichlet form, (in fact it is regular in the sense of Fukushima [8], [9]). In particular, the corresponding semi-group $(T_t)_{t>0}$ is sub-Markovian and there exists a special standard process porperly associated with $(\mathscr{E}, D(\mathscr{E}))$ which is in fact a diffusion (cf.[11, V.1.5]). The poof of this statement is similar to that in [11, Chapter II.2d.], so we do not repeat it here. For a more general result including sub-elliptic possibly degenerate cases and its detailed proof we refer to [15]. Note that if $\underline{\beta}\not\equiv 0$, $(\mathscr{E}_\alpha, D(\mathscr{E}_\alpha))$ is in general not a Dirichlet form as shown in Remark 2.2 (ii). In [5], based on the classical results in [19], only the case, where in addition to our conditions $\tilde{a}_{ij}\in L^\infty(U;dx)$ and $\|\underline{b}\|, \|\underline{d}\| \in L^d(U;dx)$ (globally!), was treated. We were able to treat the more general case above because of the more refined closability results in [11] and [15]. In [20, Theorem II.3.8] the case $a_{ij}\in L^\infty(U;dx)$, $\check{a}_{ij}=0$ for $1\leq i,j\leq d, \underline{\gamma}, \underline{d}\equiv 0$, and $\underline{\beta}\in L^\infty(U;dx)$ was considered. We emphasize, however, that Stroock's result is stronger in this particular case since he even proves the corresponding semigroup to be strongly Feller and to have a density w.r.t. dx.

(ii) In a forthcoming paper [13] we shall use Theorem 3.8 to construct a Fleming-Viot process with *generalized selection* associated with a semi-Dirichlet form$(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;m)$, where E is the set of all probability measures on a polish space S and m the unique reversible measure of the Fleming-Viot process with neutral mutation, but without any selection, cf. [7], [17]. $(\mathscr{E}, D(\mathscr{E}))$ is defined by

$$\mathscr{E}(u,v)=\int dm(\mu)(\langle\nabla u(\mu),\nabla v(\mu)\rangle_\mu+u(\mu)\langle b(\mu),\nabla v(\mu)\rangle_\mu+\alpha u(\mu)v(\mu)),$$

with $D(\mathscr{E})$ the closure of the finitely based functions $\mathscr{F}\mathscr{C}_b^\infty$ w.r.t. the norm $\tilde{\mathscr{E}}_1^{\frac{1}{2}}$. The gradient $\nabla u(\mu)$ is the function in $L^2(S;\mu)$ defined by

$$\nabla_x u(\mu) = \frac{\partial u}{\partial \delta_x}(\mu),$$

where $\frac{\partial u}{\partial \delta_x}$ is the Gâteaux derivative in direction of the Dirac measure δ_x.
For two functions $f,g \in L^2(S;\mu)$, the scalar product $<f,g>_\mu$ is the covariance of f and g w.r.t. μ.
The only assumption on the "generalized selection" function b: {probability measures on $S\} \times S \to \boldsymbol{R}$ is that

$$\sup_\mu < b(\mu,\cdot), b(\mu,\cdot) >_\mu < \infty.$$

This is less restrictive than

$$\sup_\mu \sup_x b(\mu,x) < \infty,$$

which is assumed by D.A. Dawson [6, 7.2.2, 10.1.1] in order to construct Fleming-Viot processes with selection by a Girsanov type transformation. (D.A. Dawson, however, also shows uniqueness of the corresponding martingale problem.)

ACKNOWLEDGEMENT

We would like to thank Y. Le Yan for suggesting to consider this extension of [11]. We would like to thank D. Stroock for bringing the reference [20] to our attention which partly motivated this work.

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