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SPEED LIMIT OPERATORS FOR OSCILLATING SPEED FUNCTIONS

BERNDT FERNOW and ULRICH KRENGEL*

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1. Introduction

As shown by Krengel and Lin [5], order preserving integral preserving operators in L_1^+ generalize the traditional model of Markov operators in such a way that interaction of the movement of mass particles is permitted. In the same paper, the study of the asymptotic properties of such operators was initiated, and speed limit operators for monotonely decreasing speed functions φ on the line were introduced, serving as examples and counterexamples. Roughly speaking, an integrable function $f \geq 0$ on the line describes a mass distribution. The mass particles move to the right. A given speed function $\varphi(x)$ assigns the highest permitted speed at the location x. However, particles can move more slowly if they are slowed down by slower particles in front of them.

In the present paper, we begin the study of speed limit operators for speed limits φ which need not be monotone. We assume that φ is piecewise constant and takes finitely many (at least two) different values. In a preliminary section we describe the evolution $T_t f$ of f when φ is monotonely increasing. In this case, there is no interaction, and T_t is linear. When φ oscillates, the mass is "spread out" in points of increase of φ, and points of decrease of φ cause interaction over possibly long distances. It is no longer possible to consider the movement of the various "levels" of the mass distribution separately.

The general non-monotonic case seems difficult. When φ is piecewise constant and the length of the intervals on which φ has a fixed value is assumed bounded below, an explicit recursive definition of $T_t f$ is possible for small $t > 0$ when f belongs to the class $\mathscr{F}$ of left-continuous piecewise constant nonnegative step functions. $T_t f$ again belongs to $\mathscr{F}$. The recursive definition works until there is a discontinuity in the "law of motion" of $T_t f$. A crucial step is to show that there are only finitely many

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such discontinuities in any finite time interval. This allows to define $T_t f$ for all $t>0$. T_t extends to L_1^+ by continuity. Once the construction of $T_t f$ on the line is clear, it can be transferred to motions on the circle, and, in fact, to general proper flows on abstract measure spaces. $\{T_t, t\geq 0\}$ is a semigroup of order preserving and integral preserving operators in L_1^+, the T_t are positively homogeneous and they act nonexpansively in all L_p^+-spaces ($1\leq p\leq\infty$). (Definitions are given below.)

We explore the asymptotic behaviour of $T_t f$ for $f\in L_1^+\cap L_\infty$. On the circle, $T_t f$ converges uniformly and exponentially fast to the constant function having the same integral as f. On $\boldsymbol{R}$, $T_t f$ converges uniformly to 0 provided the length of the intervals of constancy of φ is also bounded above.

The general idea of this paper is to provide a natural model of a certain kind of motion of mass particles subject to interaction. The classical linear Markov operators are by now rather well explored. However, it seems that realistic models should frequently take interaction into consideration. It therefore seems desirable to develop rigorous mathematical models of such phenomena.

We remark that a different construction of the speed limit operators T_t for piecewise constant φ on $\boldsymbol{R}$ was discussed in the masters thesis [2] of the first named author, written under the direction of the second named author.

2. Monotonely increasing and decreasing speed limits

Nonnegative measureable functions f on the line (with Lebesgue measure λ) can be interpreted as mass distributions. The integral $\int_B f\,d\lambda$ measures the mass contained in $B\subset\boldsymbol{R}$. It will be useful to think of $f\in L_1^+$ as a set of particles. At the location x, there is a particle at each level y with $0\leq y\leq f(x)$.

Our aim is to study a class of operators T_t in $L_1^+(\lambda)$ which describe the movement of mass distributions f subject to a speed limit φ on $\boldsymbol{R}$. $T_t f$ represents the distribution at time t when f is the distribution at time 0. Heuristically, the mass located at x tries to move to the right with speed $\varphi(x)>0$. However, it can be slowed down if it touches, at the same level, particles at its right side which move more slowly. We do not allow that a particle rises to a higher level in order to "pass" a slow particle. In other words, its potential energy can never increase. (A decrease of the potential energy shall be possible, though.) We assume that the total mass is preserved.

If φ is a constant $v>0$, T_t is a translation

$$(T_t f)(x) = f(x - tv).$$

In this section, we look at speed functions $\varphi > 0$ which are monotonely increasing or decreasing, not necessarily in the strict sense. In the oscillating case studied later, the movement shall be locally determined as in the increasing or decreasing case.

The increasing case is very simple. The particles to the right of a particle at location x move at least as fast as this particle and cannot slow it down. Thus, the speed limit is equal to the actual speed. A particle located at x_0 at time 0 needs the time

$$t(x_0, x_1) = \int_{x_0}^{x_1} \frac{dx}{\varphi(x)}$$

to reach the location x_1. If $u(x, t)$ denotes the location at time 0 of particles which arrive at location x at time t, then

$$t = \int_{u(x,t)}^{x} \frac{dy}{\varphi(y)}.$$

This condition determines $u = u(x,t)$ uniquely. (The existence of u follows since φ is increasing.) For fixed $t, u(x,t)$ must be differentiable with the exception of at most countably many points x, and

$$\frac{du(x,t)}{dx} = \frac{\varphi(u(x,t))}{\varphi(x)}.$$

We put

$$(T_t f)(x) = f(u(x,t)) \cdot \frac{\varphi(u(x,t))}{\varphi(x)}.$$

Then the total mass of $T_t f$ in $[x, x+dx]$ equals

$$(T_t f)(x)dx = f(u(x,t))du.$$

T_t carries the mass of f contained in the interval $(u, u+du)$ into the interval $(x, x+dx)$. It follows that T_t is *integral preserving*:

$$\int T_t f \, d\lambda = \int f \, d\lambda.$$

Clearly, T_t is linear. It is easily checked that $T_{t+s}=T_tT_s$ $(t,s\geq 0)$. The semigroup $\{T_t, t\geq 0\}$ is continuous in L_p for $1\leq p<\infty$, but not in L_∞. The norm of the operators T_t in L_p is ≤ 1 for $1\leq p\leq\infty$.

We just mention some results about $\{T_t\}$ which will not be needed below. The proofs are not difficult.

If φ is continuously differentiable, the domain of the generator A of the semigroup $\{T_t,\ t\geq 0\}$ contains all functions f on $\boldsymbol{R}$ with bounded support which are continuously differentiable. For such an f, Af is given by

$$Af(x)=-(f'(x)\varphi(x)+f(x)\varphi'(x)).$$

If the increasing function $\varphi>0$ is unbounded, and $f\in L_\infty^+$ has bounded support, then $\|T_tf\|_\infty\to 0$ as $t\to\infty$. Similarly, for $f\in L_p^+$ with $1<p<\infty$, $\|T_tf\|_p\to 0$ as $t\to\infty$. If f does not vanish, the unboundedness of φ is also necessary for such a limit behaviour.

We remark that the above definitions can also be used to define a semigroup $\{T_t,\ t\geq 0\}$ of linear operators in L_1 for more general φ. It is enough to assume that φ is strictly positive and of bounded variation, and that in addition $\int_I\varphi(y)dy$ is finite for bounded intervals I and infinite for $I=(-\infty,0]$. However, the resulting operators shall be different from the speed limit operators studied here when the condition that φ is increasing is violated. If the above definitions are applied for non-increasing φ, T_t is an operator which describes the motion of mass with speed φ rather than with speed limit φ. Particles can then increase their potential energy. $\|T_tf\|_\infty$ can then exceed $\|f\|_\infty$. In fact, $T_t$need no longer map L_p^+ into L_p^+ for $1<p\leq\infty$. As we are chiefly interested in the nonlinear model involving interaction we shall have to proceed differently.

The simplest case with interaction is the case where φ is *decreasing*. It was first studied in [5], see also [3]. It seems important to understand this case before proceeding to the case of oscillating functions φ. We recall the basic definitions.

Let $\mathscr{F}_0$ denote the class of nonnegative integrable functions f on $\boldsymbol{R}$ assuming at most two values, piecewise constant, and continuous from the left. In other words, f belongs to $\mathscr{F}_0$ if there are numbers

$$a_1<b_1\leq a_2<b_2\leq\cdots\leq a_k<b_k$$

and $\alpha>0$ such that

$$f=\alpha\sum_{i=1}^{k} I(]a_i,b_i])$$

where $I(A)$ denotes the indicator function of a set A. $I(A)(x)=1$ for $x\in A$ and $=0$ for $x\in A^c$. If a particle starts at location b_i and moves with speed $\varphi(x)$ when passing through x, let $c_i(t)$ be the distance it covers until time t. $c_i(t)$ is the unique number with

$$b_i=u(b_i+c_i(t),t).$$

When φ is decreasing, interaction comes in as follows. We require that particles shall not move faster than neighboring particles to their right. In particular, the lefthand endpoint of an interval travels at the same speed as its righthand endpoint. If a righthand endpoint catches up with the lefthand endpoint of the interval to its right, the two intervals combine into one and move with the speed determined by the righthand endpoint of the combined interval. If $d_i(t)$ denotes the distance travelled by b_i until time t subject to these restrictions, we obtain the recursive relations

$$d_k(t)=c_k(t)$$
$$d_i(t)=\min(c_i(t),\ d_{i+1}(t)+(a_{i+1}-b_i)),\qquad i=k-1,k-2,\cdots,1.$$

Put $a_i(t)=a_i+d_i(t)$, $b_i(t)=b_i+d_i(t)$ and $T_tf:=\alpha\sum_{i=1}^{k} I(]a_i(t),\ b_i(t)])$.
Let $\mathscr{F}$ denote the set of functions f which are finite sums $\sum_{j=1}^{m} f_{(j)}$ of elements $f_{(j)}\in\mathscr{F}_0$. Without restriction of generality we can write the sum in such a way that

$$\{f_{(1)}>0\}\supset\{f_{(2)}>0\}\supset\cdots\supset\{f_{(m)}>0\}.$$

If the $f_{(j)}$ are chosen subject to this condition, put $T_tf=\sum_{j=1}^{m} T_tf_{(j)}$ $\qquad(t\geq 0)$.
As T_t is *nonexpansive* with respect to the L_p-norm, that is

$$\|T_tf-T_tf'\|_p\leq\|f-f'\|_p\qquad(1\leq p\leq\infty)$$

and $\mathscr{F}$ is dense in L_p^+ for $1\leq p<\infty$, T_t can be expanded to L_p^+ by continuity. The operators $\{T_t,\ t\geq 0\}$ form a semigroup. Each T_t is *order preserving* ($f\leq g\Rightarrow T_tf\leq T_tg$) and *positively homogeneous* ($T_t(\beta f)=\beta(T_tf)$ for $\beta\geq 0$). The operators T_t are not linear except when φ is constant. The proofs of these assertions are given in [5].

Above, we have defined T_tf by cutting f into horizontal slices $f_{(j)}$. If φ is not decreasing, some of the mass of f can be spread out and change its level when passing through points in which φ increases. Thus, we shall have to handle interaction of slices which do not keep their level. So far, we have been able to handle this problem only when φ is piecewise constant.

3. Construction of T_t for moderately oscillating speed limits on $\boldsymbol{R}$

We call a function φ on $\boldsymbol{R}$ *moderately oscillating* if there exists a $\delta>0$ and a sequence $(\zeta_i,\ i\in\boldsymbol{Z})$ with $\zeta_{i+1}-\zeta_i\geq\delta>0$ for all i such that φ is constant on the intervals $[\zeta_i,\ \zeta_{i+1}[$, bounded on $\boldsymbol{R}$, and bounded below by a positive constant $\sigma>0$ on $\boldsymbol{R}$. In this section, we define speed limit operators in $\mathscr{F}$ determined by moderately oscillating speed limits φ. Later, we extend them to $L_1^+(\boldsymbol{R},\lambda)$. Subsequently, we shall refer to this construction in order to define speed limit operators on other measure spaces.

Clearly, $\zeta_i\to-\infty$ for $i\to-\infty$ and $\zeta_i\to+\infty$ for $i\to+\infty$. If β_i denotes the value of φ on $[\zeta_i,\zeta_{i+1}[$,

$$\varphi=\sum_{i\in\boldsymbol{Z}}\beta_i I([\zeta_i,\zeta_{i+1}[)$$

The $\zeta_i(i\in\boldsymbol{Z})$ shall be called *ζ-points*.

We are going to describe the definition of T_tf for $f\in\mathscr{F}$ and $0\leq t\leq t_1$ where $t_1>0$ depends on f. t_1 will be called the first instant of a change of the law of motion of f. $f^{(1)}=T_{t_1}f$ belongs to $\mathscr{F}$. When t_i, the i-th instant of a change of the law of motion of f, and $f^{(i)}=T_{t_i}f$ have been defined, let $t_1^{(i)}$ be the first instant of a change of the law of motion of $f^{(i)}$ and $t_{i+1}=t_i+t_1^{(i)}$. For $t_i<t\leq t_{i+1}$, T_tf is defined by

$$T_tf=T_{t-t_i}f^{(i)}.$$

We shall show that, for any $f\in\mathscr{F}$, $t_i\to\infty$. Therefore, T_tf will be well-defined for all $t\geq0$.

$f\in\mathscr{F}$ can be written as a sum $f=\sum_{j=1}^{k}f_j$ of "boxes" $f_j=\alpha_jI(]a_j,a_{j+1}])$ with $\alpha_j\geq0$ and $a_1<a_2<\cdots<a_{k+1}$. We can assume that the ζ-points which are contained in the interval $[a_1,a_{k+1}]$ coincide with an a_j. If this is not the case to begin with, we can split one of the boxes into two boxes of equal height. Put

$$F_i=\sum_{j=i}^{k}f_j.$$

We inductively define T_tF_i starting with $i=k$ and proceeding backwards. Note that $F_i=f_i+F_{i+1}$. When T_tF_{i+1} has been defined for small $t>0$, we know the movement of the mass to the right of f_i. The movement of f_i is governed by the constructions in the cases with increasing or decreasing φ except if this movement is slowed down by the mass to the right. (Locally, φ is decreasing or increasing.)

DEFINITION OF T_tF_k: If a_{k+1} is not one of the ζ-points there exists an index l with

$$\zeta_l \leq a_k < a_{k+1} < \zeta_{l+1}.$$

In the interval $[\zeta_l,\zeta_{l+1}[$ the speed limit φ takes the value β_l. Hence we put

$$T_tF_k = \alpha_k I(]a_k+\beta_l t, a_{k+1}+\beta_l t])$$

for $0 \leq t \leq t_{1,k} := \beta_l^{-1}(\zeta_{l+1}-a_{k+1})$. $t_{1,k}$ is the time when the right hand side of the box F_k reaches the nearest ζ-point.

If a_{k+1} is a ζ-point, say $a_{k+1}=\zeta_{l+1}$, and $\beta_{l+1} \leq \beta_l$ the box F_k can only move with the speed β_{l+1} because its right hand side instantly enters the interval with the lower value of φ. (This is the case with decreasing φ.) Hence,

$$T_tF_k = \alpha_k I(]a_k+\beta_{l+1} t, a_{k+1}+\beta_{l+1} t])$$

for $t \leq t_{1,k} := \beta_{l+1}^{-1}(\zeta_{l+2}-\zeta_{l+1})$.

If $a_{k+1}=\zeta_{l+1}$ and $\beta_{l+1} > \beta_l$ we are in the case of increasing φ for the beginning of the motion of F. The construction in section 2 yields

$$T_tF_k = \alpha_k I(]a_k+\beta_l t, \zeta_{l+1}]) + \alpha_k \beta_l \beta_{l+1}^{-1} I(]\zeta_{l+1}, \alpha_{k+1}+\beta_{l+1} t])$$

for $t \leq t_{1,k} := \min(\beta_l^{-1}(a_{k+1}-a_k), \beta_{l+1}^{-1}(\zeta_{l+2}-\zeta_{l+1}))$. (The mass which has not yet passed ζ_{l+1} is translated with speed β_l, the mass which has passed ζ_{l+1} is translated with speed β_{l+1}. The factor $\beta_l\beta_{l+1}^{-1}$ comes in because the total mass must remain constant. At time $t_{1,k}$, either the left hand end of the support of T_tF_k reaches ζ_{l+1}, or the right hand end reaches ζ_{l+2}.)

To motivate the formulation of the inductive step, we first look at an example: Assume $\zeta_l < a_{k-1} < a_k < a_{k+1} = \zeta_{l+1}$, $\beta_{l+1} < \beta_l$, and $\alpha_k < \alpha_{k-1}$. Then the mass of $F_k = \alpha_k I(]a_k, a_{k+1}])$ moves with the slower speed β_{l+1}. The box $f_{k-1} = \alpha_{k-1} I(]a_{k-1}, a_k])$ is higher than F_k. Its lower part $\alpha_k I(]a_{k-1}, a_k])$ is slowed down by the slowly moving mass of F_k just as in the construction with the decreasing speed function. It can therefore move with speed β_{l+1} only. However, the remaining part of f_{k-1}, above level α_k, can move with speed β_l for a little while. We see that two "slices" of f_{k-1} move with different speed.

In general, we will have to start the inductive step with a situation in which the box f_{i+1}, for which the movement was defined in the previous step splits into different "slices" which move with different speeds.

INDUCTIVE STEP: We now assume that T_tF_{i+1} has been defined for small enough t, say for $0 \leq t \leq t_{1,i+1}$. In the previous step, the movement of f_{i+1} subject to its interaction with T_tF_{i+2} was defined. For this purpose, f_{i+1} has been split into, say, $p \geq 1$ horizontal slices $f_{i+1,1}, f_{i+1,2}, \cdots, f_{i+1,p}$ each of them a multiple of f_{i+1}. (In the case $\alpha_{i+1}=0$, there is no interaction and we can define T_tf_i in the same way as T_tF_k above.)

First assume that a_{i+2} is not a ζ-point, or a_{i+2} is a ζ-point in which φ decreases. Then the contribution of $f_{i+1,\nu}$ to T_tF_{i+1} is the translate of $f_{i+1,\nu}$ by $t \cdot \gamma_\nu$ where γ_ν is the speed of this slice. The lower slices move more slowly than the upper slices, that is, we have $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_p$. If π_ν denotes the height of the ν-th slice,

$$f_{i+1,\nu} = \pi_\nu I(]a_{i+1}, a_{i+2}]).$$

If a_{i+2} is a ζ-point in which φ increases, this description is valid to the left of a_{i+2} only.

Now we describe how to move f_i subject to its interaction with T_tF_{i+1}. We may assume $\alpha_i > 0$. (If $\alpha_i = 0$, then $F_i = F_{i+1}$.)

CASE 1: a_{i+1} is not a ζ-point. Then the speed limit in $[a_i, a_{i+2}[$ is equal to some fixed $\beta > 0$. We have $\gamma_p \leq \beta$ because no portion of f_{i+1} was allowed to move faster than with speed β left of a_{i+2}. If $\alpha_i \leq \pi_1$, there shall be only one slice $f_{i,1} = f_i$ of f_i and it moves with speed γ_1. Let $f_{i,1}^t$ denote its position at time t. Then $f_{i,1}^t(x) = f_{i,1}(x - t\gamma_1)$. If $\pi_1 < \alpha_i \leq \alpha_{i+1}$ find ν with

$$\pi_1 + \cdots + \pi_\nu < \alpha_i \leq \pi_1 + \cdots + \pi_{\nu+1}.$$

Cut f_i into $\nu+1$ slices $f_{i,1}, \cdots, f_{i,\nu+1}$ with height $\pi'_1 = \pi_1, \cdots, \pi'_\nu = \pi_\nu$ and $\pi'_{\nu+1} = \alpha_i - (\pi_i + \cdots + \pi_\nu)$. Move $f_{i,j}$ with speed γ_j as long as possible. We may take $t_{1,i} = t_{1,i+1}$ in this case. (After this time T_tF_{i+1} is no longer defined.)

If $\alpha_i > \alpha_{i+1}$ and $\gamma_p = \beta$, cut f_i into p slices with heights $\pi_1, \cdots, \pi_{p-1}$ and $\alpha_i - (\pi_1 + \cdots + \pi_{p-1})$. They move with speed $\gamma_1, \cdots, \gamma_p$. If $\alpha_i > \alpha_{i+1}$ and $\gamma_p < \beta$, cut f_i into $p+1$ slices with heights $\pi_1, \cdots, \pi_p$ and $\alpha_i - (\pi_1 + \cdots + \pi_p)$, moving with speed $\gamma_1, \cdots, \gamma_p, \beta$.

CASE 2: a_{i+1} is a ζ-point at which φ decreases: Proceed as in case 1 when β denotes the speed $\varphi(a_{i+1}+0)$ to the right of a_{i+1}.

CASE 3: a_{i+1} is a ζ-point at which φ increases: Put $\beta'=\varphi(a_{i+1}-0)$ and $\beta=\varphi(a_{i+1})$, then $\beta'<\beta$. Roughly speaking, those of the slices $f_{i+1,v}$ which have speeds $\leq\beta'$ shall be continued in the interval $(a_i,a_{i+1}]$ as long as α_i is large enough. For the excess, we must apply the construction with increasing speed limits.

Let $\gamma_1,\gamma_2,\cdots,\gamma_j$ be $\leq\beta'$ and $\gamma_{j+1},\cdots,\gamma_p>\beta'$. For $v=1,\cdots,j$ we define the height of slices of f_i as follows: Put

$$\pi'_v=\begin{cases}\pi_v & \text{if } \pi_1+\cdots+\pi_v\leq\alpha_i\\ (\alpha_i-\pi_1-\cdots-\pi_v)^+ & \text{if } \pi_1+\cdots+\pi_v\geq\alpha_i.\end{cases}$$

In other words, the numbers π'_v are such that

$$\pi'_1+\cdots+\pi'_v=\alpha_i\wedge(\pi_1+\cdots+\pi_v).$$

Put

$$f_{i,v}=\pi'_v I(]a_i,a_{i+1}]) \tag{3.1}$$

and move $f_{i,v}$ with speed γ_v. If some of the π'_v are $=0$ because α_i is small, the corresponding slices vanish and can be deleted.

If α_i is $\leq\pi_1+\cdots+\pi_j$, the sum of the slices $f_{i,v}$ with $v\leq j$ is f_i, and we need no additional ones. If $\alpha_i>\pi_1+\cdots+\pi_j$ holds, we have $\pi'_v=\pi_v$ for $v\leq j$, and we need additional slices, which will be accelerated when passing the ζ-point a_{i+1}.

If they have speed β' to the left of a_{i+1} and speed $\gamma_v>\beta'$ to the right of a_{i+1}, and if their thickness after the passage through a_{i+1} shall be π_v, then their thickness prior to the passage should be

$$\pi''_v=\frac{\pi_v\gamma_v}{\beta'} \quad \text{for} \quad v=j+1,\cdots,p.$$

We cut as many slices with heights π''_v from the remainder of f_i as possible: Let π'_v be the numbers with

$$\pi'_1+\cdots+\pi'_v=\alpha_i\wedge(\pi_1+\cdots+\pi_j+\pi''_{j+1}+\cdots+\pi''_v)$$

for $v=j+1,\cdots,p$. Again, define $f_{j,v}$ by (3.1), and delete the slices which vanish. If α_i is $>\pi_1+\cdots+\pi_j+\pi''_{j+1}+\cdots+\pi''_p$, we need one additional slice $f_{j,p+1}$ with height

$$\pi'_{p+1}=\alpha_i-(\pi_1+\cdots+\pi_j+\pi''_{j+1}+\cdots\pi''_p)$$

We now describe the movement of $f_{i,v}$ for $v\geq j+1$:

If $\pi'_\nu = \pi''_\nu$, the contribution of $f_{i,\nu}$ to T_tF_i is given by

$$(3.2)\qquad f^t_{i,\nu} = \pi'_\nu\left(I(]a_i + t\beta', a_{i+1}]) + \frac{\beta'}{\gamma_\nu} I(]a_{i+1}, a_{i+1} + t\gamma_\nu])\right).$$

This corresponds to the construction with increasing speed limits when the speed right of a_{i+1} is γ_ν and the speed left of a_{i+1} is β'. Note that the portion of $f^t_{i,\nu}$ to the right of a_{i+1} just fills the space emptied by the movement of $f_{i+1,\nu}$ because $\pi_\nu = \pi'_\nu\beta'/\gamma_\nu$.

There is at most one ν with $j+1 \le \nu \le p$ and $0 < \pi'_\nu < \pi''_\nu$. For this ν, we again use the definition (3.2).

If $\pi'_{p+1} > 0$, then $\pi'_\nu = \pi''_\nu$ for $1 \le \nu \le p$ and

$$\sum_{\nu=1}^{p} \frac{\pi'_\nu \beta'}{\gamma_\nu} = \sum_{\nu=1}^{p} \pi_\nu = \alpha_{i+1}.$$

In this case, put

$$(3.3)\qquad f^t_{i,p+1} = \pi'_{p+1}\left(I(]a_i + t\beta', a_{i+1}]) + \frac{\beta'}{\beta} I(]a_{i+1}, a_{i+1} + t\beta])\right).$$

(There are no slices of f_{i+1} which slow the movement of $f_{i,p+1}$ down.) Finally, put

$$T_tF_i = T_tF_{i+1} + \sum_\nu f^t_{i,\nu}.$$

This definition can be used as long as T_tF_{i+1} is well-defined and all the slices move with unchanged speed. The definition of T_tF_{i+1} is guaranteed up to time $t_{1,i+1}$, and up to this time the slices $f_{i+1,\nu}$ keep their speed (by induction). We can therefore define $t_{1,i}$ as the infimum of $t_{1,i+1}$ and the first time when the left or right side of an $f^t_{i,\nu}$ reaches a ζ-point. For example, if $\pi'_{p+1} > 0$, then

$$t_{1,i} = \min\left(t_{1,i+1}, \frac{a_{i+1} - a_i}{\beta'}, \frac{\zeta_{l+1} - \zeta_l}{\beta}\right)$$

where $\zeta_l = a_{i+1}$.

Figure 1 illustrates the construction in the case $p=4$, $j=2$, $\gamma_2 \le \beta' < \gamma_3 < \gamma_4$, $\pi'_3 = \pi''_3, \pi'_4 < \pi''_4$. The dotted boxes indicate the positions of $f^t_{i,1}, \cdots, f^t_{i,4}$, the contributions of $f_{i,1}, \cdots, f_{i,4}$ to T_tF_i.

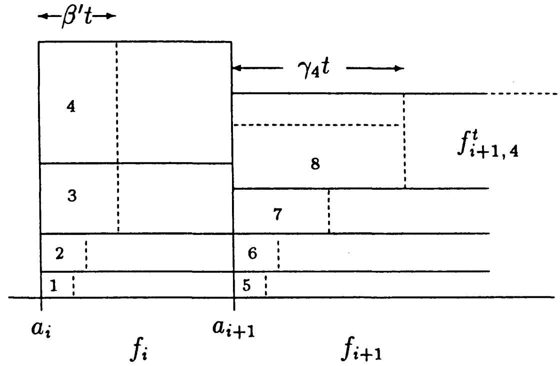

Fig. 1

The area of the rectangle 3 is equal to that of rectangle 7, that of rectangle 4 is equal to that of rectangle 8, etc.

Assume that, prior to the above construction, $f_{i+1,\nu}$ is cut into two slices of equal speed horizontally. Then, in the construction also the corresponding $f_{i,\nu}$ is split into two pieces, but the definition of T_tF_i is not affected.

Applying the above construction successively for $i=k-1,\cdots,1$ we arrive at the desired definition of $T_tf=T_tF_1$ for $t\le t_1:=t_{1,1}$. t_1 is the first instant of a change of the law of motion of f mentioned in the beginning of this section. As explained above, we must now show that there are only finitely many changes of the law of motion in any finite time interval. This is done in the next section.

It is intuitively clear that the above construction has the property

$$T_{s+t}f=T_sT_tf \text{ for } s,t>0 \text{ with } s+t\le t_1. \tag{3.4}$$

The formal argument is a bit tedious because the boxes of T_tf need not correspond to boxes of f. For example, looking at figure 1, we see that, at time t, the image of the box f_i consists of the pieces $f^t_{i,\nu}$. Their sum is a complicated function and not a box. Therefore, the inductive argument cannot simply be a passage from F_{i+1} to F_i. The boxes must be split into subboxes horizontally or, sometimes, vertically.

Such splittings occur already in the most elementary situations. For example, consider $f=I(]a,b])$ with $\zeta_1<a<b=\zeta_2<\zeta_3$ and assume $\beta_1>\beta_2$. The function just moves with speed β_2 up to time t_1. However, T_tf consists of *two* boxes $I(]a+t,\zeta_2])$ and $I(]\zeta_2,\zeta_2+t])$. Therefore the definition of $T_s(T_tf)$ involves two steps. On the other hand, in the definition of the contribution of $I(]a+t,\zeta_2])$ to $T_s(T_tf)$ one looks to the right and proceeds as if $T_t f$ consisted of only one box.

Inductively, the semigroup property (3.4) is not just verified for the F_i, but for the slices moving with the different speeds. It seems unnecessary to give the details.

As soon as we know that there are only finitely many changes of the law of motion, the semigroup property (3.4) follows for all $s,t \geq 0$.

4. Finiteness of the construction

We call the instants t_i of a change of the law of motion *change times*. We shall now show that, if we work with proper representations of the functions $f^{(i)}$, there are only finitely many change times in any finite time interval.

Note that t_1 depends on the representation of f as a sum of boxes f_j. If we split a box $f_j = \alpha_j I(]a_j, a_{j+1}])$ into two boxes of height α_j by splitting the interval $]a_j, a_{j+1}]$ into two nonempty subintervals, the t_1 derived from the new representation may be smaller than the original one. However, we shall not work with unnecessarily many points a_j. In other words, we assume that the points a_j are points of discontinuity of f or ζ-points.

(A little reflection shows that the definition of $T_t f$ for small t is not affected by the splitting of f_j, but we do not want to introduce artificial change points.)

As φ is bounded, we can assume $\varphi \leq 1$ by a change of the time scale. Recall that $\delta > 0$ was a lower bound for the length of the distances $\zeta_{i+1} - \zeta_i$. It will suffice to show that there are only finitely many change points in $[0,\delta]$.

We say that y is a *d-point* at time t, or that (y,t) is a d-point, if $f^t = T_t f$ is well defined, and y is a point of discontinuity of f^t in which f^t decreases, that is if $f^t(y) > f^t(y+0)$. (y,t) is called an *i-point* if $f^t(y) < f^t(y+0)$.

There are two kinds of d-points, those which move with piecewise constant positive speed, and those which correspond to the middle discontinuities a_{i+1} in (3.2) and (3.3). The latter shall be called *stationary* discontinuities, they can occur only in ζ-points, in which the speed φ is increasing. The former shall be called *moving* d-points. Let $\boldsymbol{D}_s, \boldsymbol{D}_m$ and $\boldsymbol{I}$ denote the sets of stationary d-points, moving d-points, and i-points in $\mathscr{E} = \boldsymbol{R} \times [0,\delta]$. The sets $\boldsymbol{D}_m$ and $\boldsymbol{I}$ are formed by strictly increasing (pieces of) straight lines, and $\boldsymbol{D}$ is formed by vertical (pieces of) straight lines. These lines shall be called *d_s-lines*, *d_m-lines* and *i-lines* respectively.

Call ζ_i a *ζ^+-point* if the speed function increases in ζ_i, that is $\varphi(\zeta_i) > \varphi(\zeta_i - 0)$, and a *$\zeta^-$-point* if the speed function decreases in ζ_i. The

ζ^+-lines (ζ^--lines) are the sets $\{(\zeta_i,t)\colon\ 0\le t\le\delta\}$ where ζ_i is a ζ^+-point (resp. a ζ^--point). The d_s-lines are subsets of ζ^+-lines. The discontinuity diagram of f presents the graph of all these lines. $t\le 1$ can be a change time if there is a y such that two of the lines meet in (y,t).

For fixed $f\in\mathscr{F}$, the support of all functions f^t $(0\le t\le\delta)$ stays in a finite subinterval R of $\boldsymbol{R}$. Within R, φ takes only finitely many values. If a discontinuity moves with some speed β, the slope dt/dy of the corresponding line in $\mathscr{E}$ equals β^{-1}. Thus, only finitely many values of the slopes of the lines occur. A d_m-line through (y,t) must have slope $\varphi(y)^{-1}$ in y because the mass to the left of this discontinuity is not slowed down by mass to the right of y. For the same reason, for fixed y the slope of a d_m-line through (y,t_1) must be smaller or equal to the slope of any i-line through (y,t_2). Therefore, if a d_m-line is strictly to the right of an i-line at some time t, these lines cannot meet at a later time.

At time $t=0$, only finitely many lines can start, but in the moment when two lines meet new lines can be generated even in distant places. We must therefore list carefully exactly what can happen. We begin with the crucial case:

(1) *A d_m-line meets an i-line*: Figure 2 presents an f and the corresponding discontinuity diagram for small t in a case which shows the main possibilities. f has support in $[\zeta_1,\zeta_4]$ and we assume $\beta_3<\beta_1<\beta_2$.

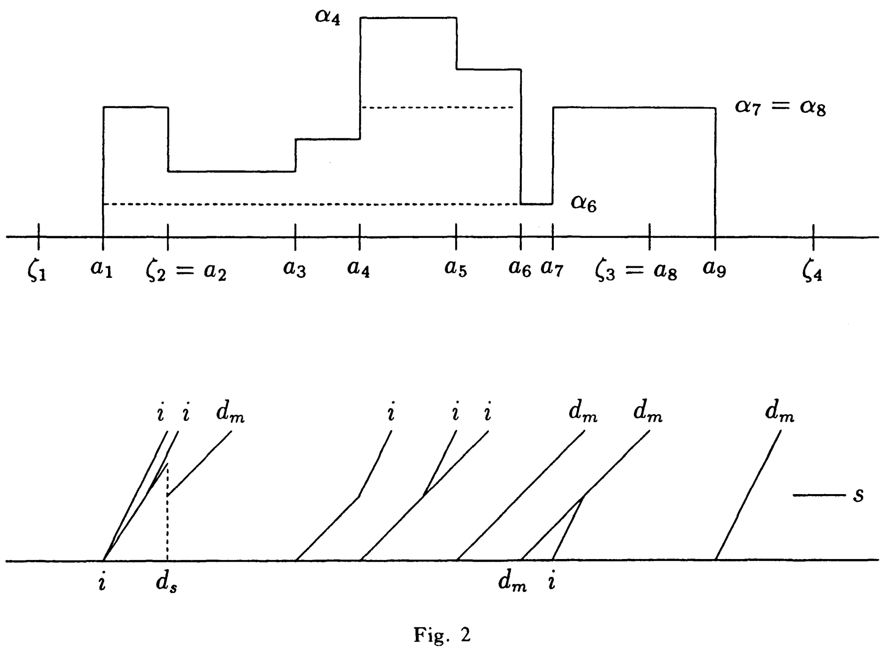

Fig. 2

The discontinuity at a_9 produces the rightmost d_m-line. It does not interfere with anything on the left. At time $(\zeta_4 - a_9)/\beta_3$ it shall produce a d_s-line at ζ_4 if $\beta_4 > \beta_3$ holds, but we consider a shorter time interval. No line is drawn which starts at $\zeta_3 = a_8$. It has simply been a convenience for the definition of T_t to assume that any f_j has support between two neighboring ζ_i's but the boxes $f_7 + f_8$ move like just one here. At time $s = (a_7 - a_6)/(\beta_2 - \beta_3)$, the i-line starting at a_7 and the d_m-line starting at a_6 meet. Now also the upper part of f_5 is slowed down. As $\alpha_5 > \alpha_7$ holds, the i-line ends. (In the case $\alpha_5 < \alpha_7$ the d_m-line would end instead.) The i-line starting at a_4 is split into two i-lines, because the portion of $f_4 + f_5$ above level α_7 can still move with speed β_2, while the lower portion has speed β_3 now. The i-line starting at a_3 just changes the direction getting steeper. As all the mass below level α_2 now moves with speed β_3 only, a new d_m-line starts at time s in ζ_2. One of the two i-lines starting in a_1 at time 0 is split into two i-lines at time s. When the faster one arrives at ζ_2 it meets the d_s-line and they both end. In this case, $t_1 = s$.

The important facts to remember are: If a d_m-line meets an i-line, new d_m-lines can only be generated at ζ^+-points to the left (assuming there is a d_s-line). i-lines to the left can be split or change their direction. Nothing new appears to the right of the d_m-line meeting the i-line. If an i-line splits, all the branches are at least as steep as the part of the i-line before the splitting, because the splitting is due to the fact that some of the mass on the right moves more slowly. The only time when an i-line can get more flat (start to move faster) is when it meets a ζ^+-line.

(2) *A d_m-line meets a ζ^--line*: This case is almost identical to the previous one. The d_m-line gets more steep from that moment on, but—to the left—only the same things can happen.

(3) *A d_m-line meets a ζ^+-line*: The d_m-line changes the direction, getting a smaller slope. A d_s-line starts in the corresponding ζ-point. Nothing happens to te left since the mass left of the ζ^+-point still must move as slow as before.

(4) *An i-line meets a ζ^--line*: Nothing happens since the line must have been slow before.

(5) *An i-line meets a ζ^+-line*: If there is no d_s-line on that portion of the ζ^+-line, nothing new happens since the mass to the right of the ζ^+-line must have been slowed down to move no faster than on the left

side of it. Thus the i-line continues with the same direction. If there is a d_s-line, it may but need not end. The i-line may split into several i-lines having different slopes or it may just change its slope.

It is not hard to see that an i-line never meets another i-line except where they originate. Therefore, we have listed all possibilities, and can start to complete the proof of the finiteness of the construction.

Begin with the rightmost ζ-interval $[\zeta_m, \zeta_{m+1}[$ intersecting with the support of f. At time 0, only finitely many d_m-lines start there, and each of them can meet only finitely many i-lines to its right up to time δ, and a ζ-line at ζ_{m+1}. This may produce finitely many new lines on the left, but no new d_m-lines starting in the interval $[\zeta_m, \zeta_{m+1}[$ exept those starting at the ζ-line l_m through ζ_m. Consider such a new d_m-line g_1. It may hit an i-line and generate yet another d_m-line g_2 in a point of l_m, and this one may generate, in a similar way, g_3, and so on. We have to exclude the possibility that, in this way, an infinite sequence of d_m-lines g_k is generated up to time δ.

Assume h_1 is the i-line hit by g_1. h_1 may also be hit by g_2, but by this time h_1 is further away from l_m. If $s_{1,i}$ denotes the time when h_1 is hit by g_i, then $s_{1,i+2} - s_{1,i+1} > s_{1,i+1} - s_{1,i}$. Thus, h_1 can be hit only by finitely many g_k.

There are only finitely many i-lines starting at time 0 in $[\zeta_m, \zeta_{m+1}[$ and they have only finitely many branches. As we can argue with any of them like with h_1, they can be responsible only for finitely many change times.

There may be some additional i-lines coming from the interval $[\zeta_{m-1}, \zeta_m[$. Can they cause trouble? Look at the one starting closest to ζ_m at time 0. Before it hits the line l_m it might split. The flattest (fastest) branch can split only at the change times listed so far. Thus, only finitely many branches can grow out of it, and they all must be steeper. When the flattest branch passes the line l_m it may start to produce finitely many l_m-lines (like g_1 above). Now the next lowest branch is treated in the same way, and then the next one. (They are ordered lexicographically.) In this way, we see that the i-line starting closest to ζ_m is, again, only responsible for finitely many change times. This way we continue from right to left. (We need only consider $[\zeta_{m-1}, \zeta_m[$ because the i-lines even further left cannot reach l_m before time δ.) It follows that there are only finitely many change times for which the lines meet at a location in $[\zeta_m, \zeta_{m+1}[$.

The argument is now repeated with $[\zeta_{m-1}, \zeta_m[$. Note, that there may be more lines than at time 0 due to the change times above. However, the finiteness was preserved.

5. Properties of the speed limit operators

It is clear from the definition, that the operators T_t are *positively homogeneous*.

Let us show that T_t is *order preserving* in $\mathscr{F}$: Assume that $f, \tilde{f} \in \mathscr{F}$ satisfy $f \leq \tilde{f}$. We may assume that $\tilde{f}$ is a sum of boxes $\tilde{f}_j = \tilde{\alpha}_j I(]a_j, a_{j+1}])$ with the same a_j's as in the representation of f, and with $\tilde{\alpha}_j \geq \alpha_j$. As T_t is positively homogeneous, $T_t\tilde{F}_k \geq T_tF_k$ for $t \leq \min(t_1, \tilde{t}_1)$. Assume that $T_t\tilde{F}_{i+1} \geq T_tF_{i+1}$ has been proved. Cutting some of the slices $f_{i+1,\nu}$ or $\tilde{f}_{i+1,\nu}$ horizontally, we can assume $\pi_\nu = \tilde{\pi}_\nu$ for $\nu = 1, \cdots, p$. (There may be additional slices of $\tilde{f}_{i+1}$.) It follows from $T_t\tilde{F}_{i+1} \geq T_tF_{i+1}$ that $\tilde{\gamma}_\nu \leq \gamma_\nu$. Therefore, the slices $\tilde{f}_{i,\nu}$ are thinner than the slices $f_{i,\nu}$, and move at most equally fast. Hence

$$\tilde{f}_i - \sum_{\nu=1}^{p} \tilde{f}_{i,\nu} \geq f_i - \sum_{\nu=1}^{p} f_{i,\nu}.$$

It is then not hard to see from the construction of T_t, that $T_t\tilde{F}_i \geq T_tF_i$. By induction, we arrive at $T_t\tilde{f} \geq T_tf$.

In all steps of the construction, we have $\int f^t_{i,\nu}(x)dx = \int f_{i,\nu}(x)dx$. The operators T_t are therefore *integral preserving* in $\mathscr{F}$. By a simple lemma in [5], order preserving integral preserving operators are nonexpansive with respect to the L_1-norm. As $\mathscr{F}$ is dense in $L_1^+ = L_1^+(\boldsymbol{R}, \lambda)$ we can extend the range of definition of the operators T_t to all of L_1^+ by continuity. The properties proved so far (semigroup property, preservation of order and integrals, positive homogeneity) remain true for T_t in L_1^+.

Next, we show that the operators T_t are nonexpansive in $\mathscr{F}$ with respect to the L_∞-norm. As the T_t are order preserving, we need only prove

$$\| T_tf - T_t\tilde{f} \|_\infty \leq \| f - \tilde{f} \|_\infty \tag{5.1}$$

for $f, \tilde{f} \in \mathscr{F}$ with $f \leq \tilde{f}$ as above. Put $\alpha^* = \| f - \tilde{f} \|_\infty$. Fix t, and let ζ_m be a ζ-point with $\zeta_m > a_k + t$. As the speed is bounded by 1 in $\boldsymbol{R}$, T_tf and $T_t\tilde{f}$ are not changed if we modify the value of the speed function φ in $[\zeta_m, \zeta_m + t[$. We can therefore assume that $\varphi(s) = \delta$ on $[\zeta_m, \zeta_m + t[$. The function

$$f^* = \alpha^* I(]a_1, \zeta_m]) + f$$

has the properties $f \leq \tilde{f} \leq f^*$ and $\| f - f^* \|_\infty = \alpha^*$. In the construction of T_tf^* the lowest slice $\alpha^* I(]a_1, \zeta_m])$ moves with constant speed δ, and on top of this the remainder of f^* is just f and moves exactly like f. Hence,

$$T_t f^* = \alpha^* I(]a_1 + \delta t, \zeta_m + \delta t]) + T_t f.$$

(5.1) now follows from $T_t f \leq T_t \tilde{f} \leq T_t f^*$ and $\| T_t f - T_t f^* \|_\infty = \alpha^*$. By a result of Browder (see [5]) the operators T_t are now nonexpansive with respect to the L_p-norm for $1 \leq p < \infty$. (They extend to L_p^+.) We have proved:

Theorem 5.1. *The operators T_t in $\mathscr{F}$ constructed in section 3 extend continuously to $L_p^+(\boldsymbol{R}, \lambda) = L_p^+$ for $1 \leq p < \infty$. They form a semigroup of integral preserving, order preserving and positively homogeneous operators. The operators T_t are nonexpansive with respect to the L_q-norm on $L_p^+ \cap L_q$ for all p, q with $1 \leq p < \infty$, $1 \leq q \leq \infty$.*

We remark that the operators T_t are linear if and only if φ is increasing.

6. Convergence of $T_t f$

We now study the asymptotic properties of $T_t f$ for $t \to \infty$. If φ is asymptotically constant, we cannot expect convergence of $T_t f$ since T_t may behave asymptotically like a translation. However, we shall obtain uniform convergence to 0 for bounded integrable f if φ oscillates sufficiently much.

The key to convergence is the fact that the "peaks" of f are spread out when passing through ζ^+-points. In order to give a precise quantitative meaning to this statement, we look at the restriction of $f \in \mathscr{F}$ to an interval $[\zeta_i, \zeta_{i+2}]$, in which ζ_{i+1} is a ζ^+-point; see figure 3.

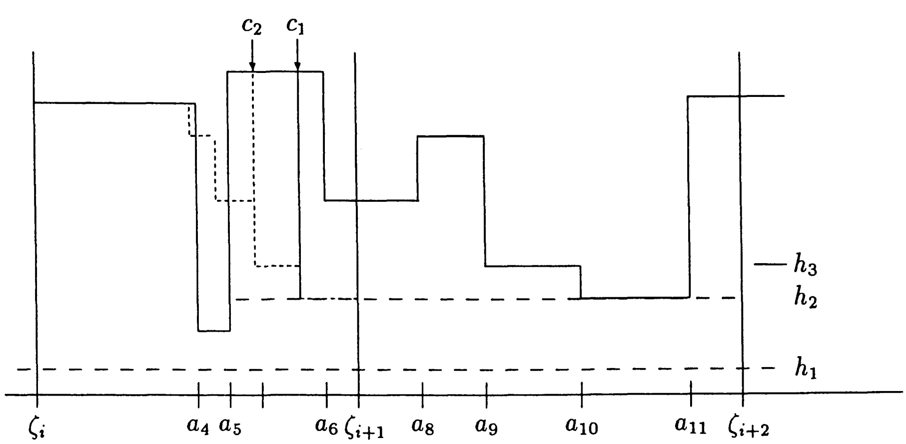

Fig. 3

We have $\beta_i < \beta_{i+1}$ because ζ_{i+1} is a ζ^+-point. The mass below the broken lines at level h_1 and h_2 may move slowly due to influence from right of ζ_{i+2}. But certainly the mass above level h_2 between $a_7 = \zeta_{i+1}$ and a_{11} moves with full speed β_{i+1} at least until time $(a_{11} - a_{10})/\beta_{i+1} = r_1$.

Up to this time the mass above level h_2 passing ζ_{i+1} is not slowed down below speed β_{i+1}. Let $c_1 = \zeta_{i+1} - r_1\beta_i$. By time r_1 the mass above level h_2 between c_1 and ζ_{i+1} has passed ζ_{i+1}. The height of this piece of mass was reduced by a factor β_i/β_{i+1}. After time r_1 the mass between level h_2 and h_3 in the interval $[\zeta_{i+1}, \zeta_{i+2}]$ may also move slowly, but the mass above level h_3 which sits left of a_9 at time 0 still moves fast at least up to time $r_2 = (a_{11} - a_9)/\beta_{i+1}$. Therefore, an additional piece of the mass left of ζ_{i+1} can pass ζ_{i+1} taking up the full speed β_{i+1} in the moment of the passage.

In these considerations, we have argued as if the slice between level h_1 and h_2 moved with a speed $\leq \beta_i$. If this speed is between β_i and β_{i+1} description of the passage through ζ_{i+1} is more complicated; see figure 1. However, the total reduction of the height is at least as large as in our argument.

For the validity of the argument above, it is not important that the subset of $[\zeta_{i+1}, \zeta_{i+2}[$ on which f takes the value h_2 corresponds to a single block. Its total length enters the formula for c_1. We summarize the conclusion as follows:

For any $h > 0$, put

$$v_i(h) = \lambda(\{x \in [\zeta_{i+1}, \zeta_{i+2}]: f(x) < h\}).$$

Then the mass at level h to the left of ζ_{i+1} can take up the speed β_{i+1} at the time of the passage through ζ_{i+1} until time $r(h) = v_i(h)/\beta_{i+1}$. This affects the mass in the interval $]\zeta_{i+1} - \beta_i v_i(h)/\beta_{i+1}, \zeta_{i+1}[$. Put

$$w(x) := \inf\{h > 0: \ x + \frac{\beta_i v_i(h)}{\beta_{i+1}} > \zeta_{i+1}\}.$$

Then, for $x \in]\zeta_i, \zeta_{i+1}]$, the mass between $w(x)$ and $f(x)$ is spread out by a factor β_{i+1}/β_i when passing through ζ_{i+1}. In figure 3, the function f dips below the level w in the interval $]a_4, a_5]$. This does not affect our argument. In fact, due to this dip even some of the mass to the left of a_4 and *below* w will be spread out. Also additional mass may be spread out because the pile of mass to the left of a_{11} actually move, while our estimates would work even if it didn't move. We only determined a sufficient condition for the spreading.

In this section, we need only a corollary of the above estimates: Assume $h>0$ is such that

$$\lambda(\{x\in[\zeta_{i+1},\zeta_{i+2}]\colon f^t(x)>h\})\leq\frac{\delta}{2}.$$

Then all the mass of f^t in $]\zeta_{i+1}-\delta\beta_i/(2\beta_{i+1}),\zeta_{i+1}]$ above the level h will pass ζ_{i+1} in the time interval $[t,t+\delta/(2\beta_{i+1})[$ and will be spread out in this moment.

Theorem 6.1: *Assume that the speed function φ assumes only finitely many distinct positive values, and that there exist $0<\delta,\delta^*<\infty$ with $\delta<\zeta_{i+1}-\zeta_i<\delta^*$ for all $i\in\boldsymbol{Z}$. Then f^t tends to 0 uniformly mod nullsets in* $\boldsymbol{R}$ for bounded integrable $f\geq 0$.

Proof. We can assume $0<\sigma\leq\varphi\leq 1$. As the operators T_t are order preserving and nonexpansive with respect to the L_1- and L_∞-norm, theorem 3.3 in [5] (or, rather, a simple extension of this theorem to continuous time) implies that f^t converges in distribution for $t\to\infty$. In other words: There exists a decreasing function $F\geq 0$ on $]0,\infty[$ with

$$\lambda(\{x\in\boldsymbol{R}\colon f^t(x)\geq h\})\to F(h)$$

for all $h>0$ in a set D which is dense in $]0,\infty[$ and contains all points of continuity of F. Put

$$h^*=\inf\{h>0\colon F(h)=0\}.$$

We shall first show $h^*=0$. Assume $h^*>0$.

CASE 1: F is continuous at h^*.

In this case $F(h^*)=0$. As there are only finitely many possible values of φ there exist numbers $\theta,\theta^*>1$ with $\theta<\beta_{i+1}/\beta_i<\theta^*$ whenever ζ_{i+1} is a ζ^+-point. If $h'<h^*$ is close enough to h^*, $F(h')<\delta/2$. We may assume that h' is a point of continuity of F. Hence, for large enough t, we have $\lambda(\{x\colon f^t(x)\geq h'\})<\delta/2$. This means that, from some time t on, all the mass above level h' is spread out as soon as it passes a ζ^+-point. This reduces a value of $f^t(x)$ to at most $h'+\theta^{-1}(f^t(x)-h')$. By the assumption on φ, there exists, for any $K>0$ some $t(K)$, such that within a time interval of length $t(K)$ any mass particle passes at least K ζ^+-points. For K large enough,

$$h' + \theta^{-K}(\sup_x f(x) - h') \leq \frac{h' + h^*}{2} < h^*.$$

It follows, that for $t \geq t' + t(K)$, $\sup_x f^t(x) \leq \frac{h'+h^*}{2}$. This implies that $F(\frac{h'+h^*}{2}) = 0$, contradicting the definition of h^*.

CASE 2: h^* is a point of discontinuity of F.

There exists $p > 0$ with $F(h^* - 0) = p$, and $F(h) = 0$ for $h > h^*$. There exists $0 < h' < h^*$ with $p - F(h' - 0) < \delta/8$. Put $h'' = (h^* + h')/2$, $h''' = h'' + (h^* - h'')/\theta$, $\Delta = (h^* - h''')/8$ and $h^\circ = h''' + \Delta$.

Then we have the following property: If, at some ζ^+-point ζ_{i+1}, the mass above level h'', which sits to the right of ζ_{i+1} and close to ζ_{i+1}, can move with full speed β_{i+1} for some time, then the mass left of ζ_{i+1} between level h'' and $h^* + \Delta$ will be spread out during the passage through ζ_{i+1}, and its new level will be $\leq h^\circ$. We have $h^* - h^\circ = 7\Delta$.

As the distribution of f^t tends to F, there exists, by the choice of h', a number $v > 0$ such that $t \geq v$ implies

$$\lambda(f^t \in [h' + \Delta,\ h^* - \Delta]) \leq \frac{\delta}{4}. \tag{6.1}$$

Choosing $v > 0$ sufficiently large we can also assume

$$\int (f^v - (h^* + \Delta))^+ d\lambda \leq \mathrm{Min}\left(\frac{\Delta\delta}{8}, \frac{\Delta p}{8}\right). \tag{6.2}$$

There are arbitrarily large ζ^+-points. As f^v is integrable, we can find a ζ^+-point ζ_{i+1} with

$$\int f^v I(]\zeta_{i+1}, \infty[) d\lambda \leq \mathrm{Min}\left(\frac{\Delta\delta}{8}, \frac{\Delta p}{8}\right). \tag{6.3}$$

We can split $\zeta_{i+2} - \zeta_{i+1} \geq \delta$ into a finite number of pieces of equal length u with $\delta/8 \leq u \leq \delta/4$. Put $w = u/\beta_{i+1}$, and consider $T_s f^v = f^{s+v}$ for $0 \leq s \leq w$. In view of (6.3) the portion of f^v above the level h'' can pass ζ_{i+1} without being slowed down by mass to the right. It follows that

$$T_w((f^v \wedge (h^* + \Delta)) I(]-\infty, \zeta_{i+1}]))(x) \leq h^\circ \qquad \text{for } x > \zeta_{i+1}.$$

Using the fact that T_w is order preserving and integral preserving, we find using (6.2) and (6.3) that

$$\lambda(\{x \geq \zeta_{i+1}:\ T_w f^v(x) \geq h^* - \Delta\}) \leq \frac{\delta}{4}.$$

Together with (6.1) this implies that

$$\lambda(\{x \geq \zeta_{i+1}:\ f^{v+w}(x) \geq h''\}) \leq \frac{\delta}{2}.$$

Therefore the mass above level h'' passing ζ_{i+1} can take up full speed for at least a second time interval of length w.

We can now proceed by induction. Assume that all the mass of f^v above level h'' which is passing ζ_{i+1} before time mu can take up speed β_{i+1} to the right of ζ_{i+1}. This is even more true for the smaller function $\hat{f} = (f^v \wedge (h^* + \Delta))I(]-\infty, \zeta_{i+1}])$. The passage of $\hat{f}$ through ζ_{i+1} yields functions $T_t\hat{f}$ $(t \leq m\,u)$ with value $\leq h^\circ$ in $]\zeta_{i+1}, \infty[$. We obtain

$$\lambda(\{x \geq \zeta_{i+1}:\ T_{mu} f^v(x) \geq h^* - \Delta\}) \leq \frac{\delta}{4}.$$

Together with (6.1) this yields the assertion until time $(m+1)u$.

We can conclude that the limit distribution of $T_t\hat{f}$ for $t \to \infty$ is contained in the interval $[0, h^\circ]$. Using the bound $\Delta p/8$ in (6.2) and (6.3) we arrive at a contradiction to the assumption $F(h^* - 0) = p$.

Hence, $h^* = 0$ has been proved in both cases.

The remaining part of the proof is simple. Let $f \geq 0$ be bounded by $C > 0$ and integrable. It follows from $h^* = 0$ that there exists, for any $\varepsilon > 0$, a $t(\varepsilon)$ with

$$\lambda(f^t \geq \frac{\varepsilon}{2}) < \frac{\delta}{2} \qquad \text{for } t \geq t(\varepsilon).$$

Thus, after time $t(\varepsilon)$, none of the mass above level $\varepsilon/2$ is even slowed down below the local permitted speed by mass to its right side. Let N denote the number of distinct possible values of the β_i's. As all β_i are $\geq \sigma > 0$, and as δ^* is an upper bound for the differences $\zeta_{i+1} - \zeta_i$, all the mass passes at least one ζ^+-point in any time interval of length $L = (N+1)\delta^*/\sigma$. Therefore $\varepsilon/2 + (C - \varepsilon/2)\cdot\theta^{-k}$ is a bound for f^t as soon as $t \geq t(\varepsilon) + kL$. As $\varepsilon > 0$ was arbitrary the uniform convergence of f^t to 0 follows.

Corollary 6.2: *If φ satisfies the conditions in theorem 6.1, $T_t f = f^t$ converges to 0 in L_p-norm for $f \in L_p^+$, $p > 1$, $t \to \infty$.*

Proof. For bounded integrable f this follows from theorem 6.1 because the uniform convergence of f^t to 0 together with $\int f^t d\lambda = \int f d\lambda$ $(t \geq 0)$ implies the convergence in L_p-norm. For general $f \in L_p^+$ approximate with bounded integrable functions and use the nonexpansiveness with respect to the L_p-norm.

7. Speed limits on the circle

We now study speed limit operators for speed functions φ on the circle $[0,1[$ with addition mod 1. We assume that $[0,1[$ splits into finitely many subintervals $[\zeta_i, \zeta_{i+1}[$ $(i=0,\cdots,k-1)$, and φ assumes a constant positive value $\beta_i > 0$ on $[\zeta_i, \zeta_{i+1}[$. Without restriction of generality, we assume $\beta_0 \leq \beta_j$ $(j=1,\cdots,k-1)$.

We can start the definition of T_t (for $t \leq (\zeta_1 - \zeta_0)/2\beta_0$) by putting, for $f \in F$ with support in $[\zeta_0, (\zeta_1 + \zeta_0)/2[= I_0$, $(T_t f)(x) = f(x - \beta_0 t)$. As β_0 is the smallest speed, T_t acts simply as a translation as long as one stays in $[\zeta_0, \zeta_1[$. This defines the movement of the boxes in I_0, and it is now possible to continue as in section 3 with the boxes to the left of I_0 (mod 1). After finitely many steps, also the movement of the boxes in $[(\zeta_0 + \zeta_1)/2, \zeta_1[$ is defined. As usual, we can define T_t for larger t by the semigroup property. The assertions in theorem 5.1 remain true in the present case.

On the circle, we do not only obtain uniform convergence. It turns out that the uniform convergence holds with exponential speed, and most remarkably, the speed is independent of f as long as we consider functions bounded by a fixed constant.

Theorem 7.1: *Assume that φ is as above and assumes at least two distinct values. For $f \in L_\infty^+([0,1[, \lambda) = L_\infty^+$, let $\bar{f}$ denote the constant function on $[0,1[$ with the value $\int f\, d\lambda$. There exist $0 < \rho < 1$ and $C > 0$ (independent of f) such that*

$$\|f^t - \bar{f}\|_\infty \leq C \rho^t \|f\|_\infty$$

holds for all $t > 0$ and all $f \in L_\infty^+$.

Proof. For $f \in L_\infty^+$ put $\operatorname{ess\,inf}(f) = \sup\{h \geq 0\colon \lambda(f \geq h) = 1\}$, and $\Delta(f) = \|f\|_\infty - \operatorname{ess\,inf}(f)$. For any constant $\alpha \geq 0$

$$T_t(f + \alpha) = \alpha + T_t f \qquad (t \geq 0) \tag{7.1}$$

because the mass of the function $f+\alpha$ below level α rotates with constant speed β_0, and the portion of $f+\alpha$ above this invariant slice is transformed exactly like f.

We shall prove the existence of a number η with $0<\eta<1$, and of a number $r>0$ such that

$$\Delta(T_r g)\leq\eta \tag{7.2}$$

holds for all g with $0\leq g\leq 1$.

Let us check that this implies the assertion of the theorem. Put $f^{(0)}=f$. We can assume $0\leq f\leq 1$. When $f^{(n)}$ has been defined, put

$$\gamma_n=\|T_r f^{(n)}\|_\infty, \qquad \gamma'_n=\operatorname{ess\,inf} T_r f^{(n)}$$

and

$$f^{(n+1)}=\frac{T_r f^{(n)}-\operatorname{ess\,inf} T_r f^{(n)}}{\gamma_n-\gamma'_n}. \tag{7.3}$$

(If $\gamma_n=\gamma'_n$, let $f^{(n+1)}\equiv 0$.) It is easy to verify inductively that $T_{nr}f$ is of the form

$$T_{nr}f=\left(\Pi_{j=0}^{n-1}(\gamma_j-\gamma'_j)\right)f^{(n)}+c_n \tag{7.4}$$

where c_n is a constant function. The normalization in (7.3) implies $0\leq f^{(n+1)}\leq 1$. Hence, (7.2) yields $0\leq\gamma_j-\gamma'_j\leq\eta$ for all j. Using (7.4), we obtain $\Delta(T_{nr}f)\leq\eta^n$ for all n. Let $\rho=\eta^{1/r}$, and $C=\rho^{-r}$. Any $t>0$ can be written in the form $t=nr+s$ with $0\leq s<r$. We obtain

$$\Delta(T_t f)\leq\Delta(T_{nr}f)\leq\rho^t\rho^{-s}\leq C\rho^t. \tag{7.5}$$

Hence, $\Delta(T_t f)\to 0$. Note that, for any $t\geq 0$ and $t'\geq t$,

$$\operatorname{ess\,inf} T_{t'}f\geq\operatorname{ess\,inf} T_t f \qquad \textit{and} \quad \|T_{t'}f\|_\infty\leq\|T_t f\|_\infty.$$

Therefore $T_t f$ converges to some constant function $\bar{f}$. Clearly, for any $t\geq 0$,

$$\operatorname{ess\,inf} T_t f\leq\bar{f}\leq\|T_t f\|_\infty.$$

This, together with (7.5) implies the assertion.
It remains to verify (7.2).

STEP 1: Let $0<\mu<1$ be given. We first prove the existence of r

and η, depending on φ and μ, for which (7.2) holds for all g with $0 \leq g \leq 1$ and $\int g \, d\lambda \geq \mu$. For such a function g there exists $A \subset [0,1[$ *with* $\lambda(A) \geq \mu/2$ and $g \geq (\mu/2)I(A)$. It will be sufficient to show the existence of $r>0$ and $\eta<1$ with

$$\operatorname{ess\,inf} T_r\left(\frac{\mu}{2}I(A)\right) \geq 1-\eta>0.$$

As T_r is nonexpansive with respect to the L_1-norm, it is enough to consider sets A which are disjoint unions of half open intervals. Thus, what we must show is that there exists $\kappa>0$ and $r>0$ such that

$$\operatorname{ess\,inf} T_r I(A) \geq \kappa \tag{7.6}$$

holds if A is a disjoint union of half open intervals with $\lambda(A) \geq \mu/2$.

Let $A(t)$ denote the set $\{T_t I(A)>0\}$. We call an interval $[\zeta_i, \zeta_{i+1}[$ *slow* if $\beta_i = \beta_0$. The intervals between the slow intervals will be called *fast*. They may consist of several subintervals $[\zeta_i, \zeta_{j+1}[$ with $\beta_j > \beta_0$.

Let $H=[c_1, c_2[$ denote the longest fast interval. Let us say that H is *incompletely covered* until time s if $\lambda(H \backslash A(t))>0$ holds for $t \leq s$. As long as H is incompletely covered any mass which enters H in c_1 is spread out by a factor which is at least

$$\xi = \inf\{\beta_j/\beta_0 \colon \beta_j > \beta_0\}.$$

By time $1/\beta_0$ all the mass of $I(A)$ has passed c_1 at least once. Hence, there exists $t \leq 1/\beta_0$ with $\lambda((H \backslash A(t))=0$ or $\lambda(A(1/\beta_0)) \geq \lambda(A(0)) \cdot \xi$. This argument can be repeated with $A(1/\beta_0)$ instead of $A(0)$. There exists $m \geq 1$ with

$$\xi^m \cdot \frac{\mu}{2} > 1.$$

It follows that H cannot remain incompletely covered until time m/β_0. The height of the lowest slices in $T_t I(A)$ with $t \leq m/\beta_0$ is from a finite set of numbers depending on φ. Hence there exist $r_1 \leq m/\beta_0$ and $\kappa_1 > 0$ with $T_{r_1} I(A) \geq \kappa_1 I(H)$. We can take the same κ_1 for all A with $\lambda(A) \geq \mu/2$.

For the remainder of the proof we can assume $r_1 = 0$ and $A = H$. The set $H(t) = \{x \colon T_t I(H) > 0\}$ is an interval $[c_1(t), c_2(t)[$ on the *circle* for all t. (We might have $c_2(t) = 1/4$, $c_1(t) = 3/4$. In that case $H(t)$ is the set $[0, 1/4[\cup [3/4, 1[$. $c_2(t)$ is called the *right end point* of the interval.)

In the moment when the right end point of $H(t)$ enters a fast interval the length of $H(t)$ grows with constant speed, depending on the β_j's, and continues to do so as long as $c_2(t)$ stays in the fast interval. (This is true since H was the longest fast interval.) During any rotation this yields a minimal growth. Hence, there exists an $r_2>0$ with $H(r_2)=[0,1[$. Up to time r_2, the lowest slice has a height which is bounded below by some $\kappa_2>0$ depending only on φ and μ. We can now take $r=r_1+r_2$ and $\kappa=\kappa_1\kappa_2$.

STEP 2: The construction in step 1 yields the desired result if there exists a fast interval J with $\int gI(J)d\lambda \geq \lambda(J)/2$. We can therefore assume the existence of a fast interval J with $\int gI(J)d\lambda < \lambda(J)/2$ now.

We have $\lambda(J\cap\{g\leq 3/4\})\geq\lambda(J)/4$. Put $\beta^*=\max\beta_j$. Until time $r_1=\lambda(J)/4\beta^*$ the mass in the slow interval $[\zeta_i,\zeta_{i+1}[$ to the left of J which is above level 3/4 is not slowed down when passing ζ_{i+1}. It follows that

$$T_{r_1}g(x)\leq\frac{3}{4}+\frac{1}{4\xi}=:\ q_1<1 \qquad \text{for } x\in]\zeta_{i+1},\zeta_{i+1}+r_1\beta_0[.$$

This, in turn, implies that for additional $r_1\beta_0/\beta^*$ time units the mass passing ζ_{i+1} above level q_1 is not slowed down. Hence, if $r_2=r_1+r_1\beta_0/\beta^*$, then

$$T_{r_2}g(x)\leq q_1+\frac{1-q_1}{\xi}=q_2<1$$

for $x\in]\zeta_{i+1},\zeta_{i+1}+r_2\beta_0[$. For additional $r_2\beta_0/\beta^*$ time units the mass above level q_2 is not slowed down. In each step, the additional time intervals during which the mass above the corresponding level q_ν can spread out, get longer, and the *additional* length of the interval on which $T_{r_{\nu+1}}g(x)$ is $\leq q_{\nu+1}$ gets longer. Therefore, after finitely many steps we obtain $T_{r_m}g\leq q_m$ on $[0,1[$. Putting $r=r_m$ and $\eta=q_m$ we obtain (7.2) also in this case. This finishes the proof. □

Corollary 7.2: *Assume φ as in theorem 7.1. For $f\in L_p^+$ $(1\leq p<\infty)$, T_tf converges to $\bar{f}$ in L_p-norm as $t\to\infty$.*

Proof. Approximate f with bounded functions and use the nonexpansiveness of the operators T_t.

REMARK: Using the Ambrose-Kakutani representation of a proper measure preserving flow $\{\tau_t\colon t\in\boldsymbol{R}\}$ in a σ-finite measure space $(\Omega,\mathscr{A},\mu)$ (see [1] in the case of finite mesure spaces, and [4] in the general case), one can define speed limit operators for flows. The speed function should satisfy the present conditions on each orbit. In this case the uniform convergece (mod μ) for bounded integrable f can fail.

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Institut für Math. Stochastik,
Universität Göttingen
Lotzestr. 13,
D-37083 Göttingen,
Germany