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THE PERIPHERAL SUBGROUP AND THE SECOND HOMOLOGY OF THE GROUP OF A KNOTTED TORUS IN S^4

To the memory of Professor Masahisa Adachi

TAIZO KANENOBU* and KEN-ICHIRO KAZAMA

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1. Introduction

Let F be a torus embedded in S^4 and X its exterior. The peripheral subgroup of F is the image of $\pi_1(\partial X)$ by the map i_* induced by the inclusion i: $\partial X \subsetneq X$. So it is isomorphic to the direct sum of the infinite cyclic group $\boldsymbol{Z}$, which is generated by a meridian, and some quotient of $\pi_1(F)=\boldsymbol{Z}\oplus\boldsymbol{Z}$. We denote by τF the second summand and call it the *type* of F; see [10, §3]. If F is unknotted, that is, F bounds a solid torus in S^4 (see [7]), then $\tau F=0$. If F is a torus constructed by spinning a non-trivial classical knot, then $\tau F=\boldsymbol{Z}$, cf. [3, 10, 11]. Asano [1] and Litherland [10] constructed examples with $\tau F=\boldsymbol{Z}\oplus\boldsymbol{Z}$. And Boyle [3] showed that there are tori of type $\boldsymbol{Z}_n$ for $n=2,5$ and 10 by attaching a 2-dimensional 1-handle to the 5-twist-spun trefoil.

We abbreviate the group $\pi_1(S^4-F)$ of F as πF. It is known that the second homology of the group $H_2(\pi F)$ is a quotient of $H_2(S^4-F)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$, and several authors gave examples having non-trivial second homology [2, 4], see also [12]. Litherland [10] showed that $H_2(\pi F)$ is a quotient of τF and that any quotient of $\boldsymbol{Z}\oplus\boldsymbol{Z}$ is realizable as $H_2(\pi F)$ for some torus F of type $\boldsymbol{Z}\oplus\boldsymbol{Z}$.

For abelian groups A and B, we write $A\leq B$ if A is a quotient of B. It is natural to ask

QUESTION. *For any abelian groups A and B such that $A\leq B\leq\boldsymbol{Z}\oplus\boldsymbol{Z}$, does there exist a torus F in S^4 with $H_2(\pi F)\cong A$ and $\tau F\cong B$?*

As a partial answer to this question, we show

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Theorem. *Suppose that A and B are abelian groups satisfying $A \leq B \leq \boldsymbol{Z} \oplus \boldsymbol{Z}$ and one of the following conditions*:
(1) *B is either free or finite cyclic*;
(2) *The rank of B is one, and A is finite.*
Then there exists a torus F in S^4 with $H_2(\pi F) \cong A$ and $\tau F \cong B$.

Thus the remaining cases are
(3) The ranks of A and B are one except for $A=B=\boldsymbol{Z}$, and
(4) B is finite and not cyclic.

The proof of Theorem is divided into four lemmas except for the above mentioned Litherland's case ($B=\boldsymbol{Z} \oplus \boldsymbol{Z}$). In Sect. 2, we give a torus F with $H_2(\pi F)=0$ and $\tau F \cong \boldsymbol{Z}_p, p \geq 0$, which is constructed by attaching a 1-handle to the 6-twist-spun trefoil (Lemma 1). In Sect. 3, we define Litherland's satellite torus, and show that the satellite tori Σ (Lemma 2) and Σ' (Lemma 3) with suitable patterns and companion F satisfy $H_2(\pi\Sigma) \cong \boldsymbol{Z}_q$ and $\tau\Sigma \cong \boldsymbol{Z}_p$, and $H_2(\pi\Sigma') \cong \boldsymbol{Z}_q$ and $\tau\Sigma' \cong \boldsymbol{Z}_p \oplus \boldsymbol{Z}$ where either $q(>0)$ divides $p(>0)$ or $p=q=0$. Furthermore, we show that the satellite torus Σ^* with companion Σ' satisfies $H_2(\pi\Sigma^*) \cong \boldsymbol{Z}_q \oplus \boldsymbol{Z}_r$ and $\tau\Sigma^* \cong \boldsymbol{Z}_s \oplus \boldsymbol{Z}$ with $q(>0)$ dividing $s(>0)$ and $r>0$ (Lemma 4).

We work in piecewise-linear or smooth category. Both B^n and D^n denote the unit balls and S^n denotes the unit sphere. For $0,1 \in B^n$ or D^n, 0 is the center and 1 is a boundary point.

All the homology groups are taken with integer coefficient. If G is a group, $H_2(G) \cong H_2(X)$ where X is a $K(G,1)$. If X is any CW-complex with $\pi_1(X) \cong G$, $H_2(G)$ is isomorphic to the cokernel of the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$.

We use $\langle \cdots | \cdots \rangle$ for the group presentation, and $\langle G | R \rangle$ means the group obtained by adding new relations R to G. For $a,b \in G$, $[a,b]$ denotes the commutator of a and b; $[a,b]=aba^{-1}b^{-1}$. If $A,B \subseteq G$, then $[A,B]$ denotes the normal closure of $\{[a,b] \mid a \in A, b \in B\}$. We use $\langle\!\langle \cdots | \cdots \rangle\!\rangle$ for the presentation of an additively written abelian group.

$\boldsymbol{Z}$ denotes the infinite cyclic group or the integers. $\boldsymbol{Z}_n$ denotes the cyclic group of order n; especially, $\boldsymbol{Z}_1$ means the trivial group 0 and $\boldsymbol{Z}_0=\boldsymbol{Z}$. $\boldsymbol{Z}\langle x \rangle$ and $\boldsymbol{Z}_n\langle x \rangle$ denote the cyclic groups with generator x. For integers $m(>0)$ and $n(\geq 0)$, $m|n$ means that m is a divisor of n.

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2. Tori obtained by attaching 1-handles to 2-knots

Lemma 1. *For any non-negative integer p, there is a torus F such that $H_2(\pi F)=0$ and $\tau F \cong \boldsymbol{Z}_p$.*

For $p=0$, such an example is given by a spun torus; see [3, 10]. In this section, we construct a torus by attaching a 1-handle to the 6-twist-spun trefoil. We give a brief summary of what we need. The readers are referred to Boyle's paper [3] for the details.

Let K be a 2-knot in S^4, and h a 1-handle on K. We denote by $K+h$ the resulting torus. We can assume that the attaching disks of h are very near on K, and that the core of h represents an element g of $\pi=\pi_1(S^4-K)$. There exists an element g' of π', the commutator subgroup of π, such that $g=t^n g'$, where t is a meridian of π. We set $T(h)=\{t^n g' t^{-n} \mid n\in \boldsymbol{Z}\}$, and call it the orbit of h. Conversely, for any element $g'\in\pi'$, there exists a 1-handle h such that $g'\in T(h)$, and two 1-handles h_1 and h_2 are equivalent iff $T(h_1)=T(h_2)$.

From [3, Lemmas 9 and 10], we have

Proposition 1. *The fundamental group of the complement of $K+h$, $\pi(K+h)$, is isomorphic to $\pi/[t,T(h)]$. The peripheral subgroup of $K+h$ is generated by t and $T(h)$ in $\pi(K+h)$, and thus the type of $K+h$ is $\boldsymbol{Z}_p$, where p is the order of $g'\in T(h)$ in $\pi/[t,T(h)]$.*

For the remainder of this section, K denotes the 6-twist-spun trefoil. The group π is presented by

$$\langle x, y \mid xyx=yxy, [x^6,y]=1 \rangle,$$

where x and y are meridians; see [14]. Letting $a=yx^{-1}$, this becomes

$$\langle x, a \mid xax^{-1}=a(x^2ax^{-2}), x^6ax^{-6}=a \rangle.$$

Putting $a_i=x^iax^{-i}$, the commutator subgroup π' is presented by

$$\langle a_0, a_1, \cdots, a_5 \mid a_0=a_5a_1, a_1=a_0a_2, \cdots, a_5=a_4a_0 \rangle.$$

Letting $b=a_1$, we have

$$a_2=a^{-1}b, a_3=b^{-1}a^{-1}b, a_4=ab^{-1}a^{-1}, a_5=ab^{-1},$$

from which, we see that π' becomes

$$\langle a, b, c \mid [a,c]=[b,c]=1, c=[a,b] \rangle, \tag{1}$$

which is given in [13, p. 307] as a presentation of the fundamental group of the 6-fold covering space of S^3 branched over the trefoil knot. It is easy to see the second commutator subgroup π'' of π is just the center of π' and is the infinite cyclic group generated by c. The abelianized group

π'/π'' is the free abelian group of rank 2. For the group (1), see also [8, p. 60]. The following are easy.

Claim 1. *Any element of π' is uniquely expressible in the form $a^k b^l c^m, k, l, m \in \boldsymbol{Z}$.*

Claim 2. *In π', for any $k, m, n \in \boldsymbol{Z}$, the following holds:*

$$(b^m a^n)^k = a^{nk} b^{mk} c^{-mnk(k+1)/2}.$$

We denote by h_w the 1-handle on K corresponding to $w \in \pi'$.

Claim 3. *The orbit of $h_{c^m}, m \in \boldsymbol{Z}$, has length one:*

$$T(h_{c^m}) = \{c^m\},$$

and each of the other one has length six:

$$T(h_w) = \{w, w_1, \cdots, w_5\},$$

where $w = a^k b^l c^m, (k,l) \neq (0,0)$, and $w_i = x^i w x^{-i}$, which are written as follows:

$$\begin{aligned} w_1 &= a^{-1} b^{k+l} c^{kl + l(l-1)/2 + m}; \\ w_2 &= a^{-k-l} b^k c^{(k-1)l + k(k-1)/2 + m}; \\ w_3 &= a^{-k} b^{-l} c^{-k-l+m}; \\ w_4 &= a^l b^{-k-l} c^{kl - k + l(l-1)/2 + m}; \\ w_5 &= a^{k+l} b^{-k} c^{kl + k(k-1)/2 + m}. \end{aligned} \tag{2}$$

Denote $G(k,l,m)$ by the group of $K + h_w$, where $w = a^k b^l c^m$. Then $G(0,0,m) = \pi$, and $G(k,l,m), (k,l) \neq (0,0)$, is presented by

$$\langle x, a, b, c \mid xax^{-1} = b, xbx^{-1} = a^{-1}b, [a,c] = [b,c] = 1, c = [a,b], w = w_1 = \cdots = w_5 \rangle.$$

Thus the commutator subgroup $G'(k,l,m)$ of $G(k,l,m)$ is presented by

$$\langle a, b, c \mid [a,c] = [b,c] = 1, c = [a,b], w = w_1 = \cdots = w_5 \rangle.$$

Claim 4. *In $G'(k,l,m), c^k = c^l = 1$.*

Proof. From $w = w_1$, we get $b^k = a^{k+l} c^{-kl - l(l-1)/2}$, which commutes with a. On the other hand, by Claim 2, $ab^k a^{-1} b^{-k} = aa^{-1} b^k c^k b^{-k} = c^k$, which is trivial. In the same way, $w_1 = w_2$ implies $c^l = 1$, completing the proof. □

Thus (2) are rewritten as follows:

$$\begin{aligned} w_1 &= a^{-l} b^{k+l} c^{l(l-1)/2+m}; \\ w_2 &= a^{-k-l} b^{k} c^{k(k-1)/2+m}; \\ w_3 &= a^{-k} b^{-l} c^{m}; \\ w_4 &= a^{l} b^{-k-l} c^{l(l-1)/2+m}; \\ w_5 &= a^{k+l} b^{-k} c^{k(k-1)/2+m}. \end{aligned} \tag{3}$$

Let C be the subgroup of $G'(k,l,m)$ generated by c. Then C is just the commutator subgroup, and $G'(k,l,m)/C$ is an abelian group of order k^2+kl+l^2. Thus we have

Claim 5. *If $(k,l)\neq(0,0)$, then $G'(k,l,m)$ is a finite group.*

It is easy to see that $G'(k,0,m)$ is isomorphic to $G'(0,k,m)$. In $G'(k,0,m)$, w and (3) are as follows:

$$\begin{aligned} w &= a^{k} c^{m}; \\ w_1 &= b^{k} c^{m}; \\ w_2 &= a^{-k} b^{k} c^{k(k-1)/2+m}; \\ w_3 &= a^{-k} c^{m}; \\ w_4 &= b^{-k} c^{m}; \\ w_5 &= a^{k} b^{-k} c^{k(k-1)/2+m}, \end{aligned}$$

and so $G'(k,0,m)$ is presented by

$$\langle a,b,c \mid [a,c]=[b,c]=1,\, c=[a,b],\, a^k=b^k=c^{k(k-1)/2},\, c^k=1 \rangle,$$

whose abelianized group is

$$\langle a,b \mid a^k=b^k=[a,b]=1 \rangle \cong \boldsymbol{Z}_{|k|}\oplus\boldsymbol{Z}_{|k|}.$$

Since it is easy to see that $G'(k,0,m)$ and $G'(-k,0,m)$ are isomorphic, we shall consider $G'(k,0,m)$ only for $k>0$ in Claims 6–8 below.

Claim 6. *If k is odd, then $G'(k,0,m)$ is presented by*

$$\langle a,b,c \mid [a,c]=[b,c]=1,\, c=[a,b],\, a^k=b^k=c^k=1 \rangle,$$

whose commutator subgroup C is

$$\langle c \mid c^k = 1 \rangle;$$

and thus the order of $G'(k,0,m)$ is k^3 and that of $w = c^m$ is $|k/\gcd(k,m)|$.

Proof. There is a representation φ of $G(k,0,m)$ onto the group of the matrices

$$\left\{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \middle| r,s,t \in \boldsymbol{Z}_k \right\}$$

defined by

$$\varphi(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \varphi(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \varphi(c) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

cf. [8, Chapter 5, Exercise 4]. Thus the order of c is k. □

Claim 7. *If k is even, say $2r$, then $G'(k,0,m)$ is presented by*

$$\langle\, a,b,c \mid [a,c] = [b,c] = 1, c = [a,b], a^{2r} = b^{2r} = c^r, c^{2r} = 1 \,\rangle,$$

which is an extension of

$$N = \langle\, b,c \mid [b,c] = 1, b^{2r} = c^r, c^{2r} = 1 \,\rangle$$

by the cyclic group of order $2r$, and thus the order of $G'(k,0,m)$ is k^3 and that of $w = c^{m+r}$ is $2r/|\gcd(2r,m+r)|$.

Proof. First, notice that

$$\begin{aligned} N &= \langle b,d \mid [b,d] = b^{4r} = d^r = 1 \rangle \\ &\cong \boldsymbol{Z}_{4r} \oplus \boldsymbol{Z}_r, \end{aligned}$$

where $c = b^2 d^{-1}$. We define an automorphism σ of N by

$$\sigma(b) = bc \text{ and } \sigma(c) = c,$$

and thus

$$\sigma(b)=b^3d^{-1} \text{ and } \sigma(d)=b^4d^{-1}.$$

Since σ^{2r} is the identity mapping and $\sigma(c^r)=c^r$, by [5, Theorem 15.3.1], there exists an extension G of N by the cyclic group $\langle a|a^{2r}=1\rangle$, and G has the same presentation as that of $G'(k,0,m)$; cf. [8, Chapter 10, Proposition 1]. This completes the proof. □

REMARK. $G'(2,0,m)$ is the quarternion group of order 8.

Claim 8. *The second homology of the group $G(k,0,m)$ is trivial.*

Proof. We only prove for $k=2r$. Let H be the subgroup of π generated by $a^{2r}c^{-r}$, $b^{2r}c^{-r}$ and c^{2r}, which we put α,β, and γ, respectively. Then we have:

$$\begin{array}{llll} x\alpha x^{-1}=\beta, & a\alpha a^{-1}=\alpha, & b\alpha b^{-1}=\alpha\gamma^{-1}, & c\alpha c^{-1}=\alpha, \\ x\beta x^{-1}=\alpha^{-1}\beta\gamma^{2r(r-1)}, & a\beta a^{-1}=\beta\gamma, & b\beta b^{-1}=\beta, & c\beta c^{-1}=\beta, \\ x\gamma x^{-1}=\gamma, & a\gamma a^{-1}=\gamma, & b\gamma b^{-1}=\gamma, & c\gamma c^{-1}=\gamma, \end{array}$$

from which we see that H is a normal closure of $\{\alpha,\beta,\gamma\}$ in π, and $\pi/H=G(k,0,m)$. Since π is a 2-knot group, $H_2(\pi)=0$ [9], and clearly $H\subset\pi'$. Thus by [2, Lemma 1.4.1], $H_2(\pi/H)\cong H/[H,\pi]$. In this group from the above formulas, we have $\alpha=\beta=\gamma=1$. This completes the proof. □

Combining Claims 6–8, we obtain Lemma 1 and also we have:

Corollary 1. *For any positive integer n, there exist infinitely many 1-handles $h_i,i=1,2,\cdots$, on the 6-twise-spun trefoil K such that the group G_i of the knotted torus $K+h_i$ satisfy the following*:

(i) *the peripheral subgroup of $K+h_i$ is $\boldsymbol{Z}\oplus\boldsymbol{Z}_n$*;

(ii) *G_i and G_j are not isomorphic if $i\neq j$; and*

(iii) *the second homology of G_i is trivial.*

For the 1-handle h_{c^m} whose orbit has length one, we have the following; cf. [11; 3, Theorem 13].

Corollary 2. *The knotted tori $K+h_{c^m}$, $m=1,2,\cdots$, sharing the same group π are mutually inequivalent.*

Proof. We have remarked before Claim 1 that the center of π' is the infinite cyclic group generated by c. Thus any automorphism of π sends c to $c^{\pm 1}$. In other words, there is no automorphism of π taking c^m to $c^{\pm n}$ if $m \neq n$. This completes the proof. $\square$

3. Satellite tori

Following [10], we define a satellite torus. Let us regard S^3 as the union of two standard solid tori: $S^3 = D^2 \times \partial B^2 \cup \partial D^2 \times B^2$. Let κ be a knot contained in a solid torus $D^2 \times \partial B^2$. We set $T = \kappa \times S^1 \subset D^2 \times \partial B^2 \times S^1$. Let F be a torus in S^4, and let ϕ: $\partial B^2 \times S^1 \to F$ be a homeomorphism, and ψ: $D^2 \times \partial B^2 \times S^1 \to S^4$ its canonical extension, that is, ψ is an extension of ϕ by identifying $\partial B^2 \times S^1$ with $0 \times \partial B^2 \times S^1$ such that the restriction $\psi|$: $1 \times \partial B^2 \times S^1 \to S^4 - F$ induces the zero map on the first homology. Then $\psi(T)$ is called a satellite of F patterned on T and denoted by $\Sigma(\phi, \kappa, F)$.

We compute the group $\pi\Sigma(\phi, \kappa, F)$ using the van Kampen theorem to the exterior of F and $D^2 \times \partial B^2 \times S^1 - T$. We set

$$G = \pi_1(D^2 \times \partial B^2 \times S^1 - T) = \pi_1(D^2 \times \partial B^2 - \kappa) \times \boldsymbol{Z}\langle t \rangle,$$

where t is represented by $1 \times 1 \times S^1$, and

$$A = \pi_1(\partial D^2 \times \partial B^2 \times S^1) = \boldsymbol{Z}\langle x \rangle \oplus \boldsymbol{Z}\langle y \rangle \oplus \boldsymbol{Z}\langle z \rangle,$$

where x, y and z are represented by $\partial D^2 \times 1 \times 1, 1 \times \partial B^2 \times 1$ and $1 \times 1 \times S^1$, respectively. The fundamental groups take $1 \times 1 \times 1$ as the base points. Let $\phi_+ : A \to \pi F$ be the map induced by the restriction $\psi|(\partial D^2 \times \partial B^2 \times S^1)$ of ψ, and i_*: $A \to G$ the map induced by the inclusion i: $\partial D^2 \times \partial B^2 \times S^1 \subsetneq D^2 \times \partial B^2 \times S^1 - T$. Then we have

$$\pi\Sigma(\phi, \kappa, F) \cong \langle \pi F * G \mid \phi_+(a) = i_*(a)\ (a \in A) \rangle.$$

Note that if the type of F is not $\boldsymbol{Z} \oplus \boldsymbol{Z}$, that is, ϕ_+ is not injective, then this does not display an amalgamated free product of πF and G.

Let p and q be positive integers with $q|p$ or $p = q = 0$. There exists a torus F with $H_2(\pi F) = 0$ and $\tau F \cong \boldsymbol{Z}_p$ by Lemma 1. Let ϕ be a homeomorphism such that

$$\ker \phi_+(A \to \pi F) = \begin{cases} \boldsymbol{Z}\langle y \rangle \oplus \boldsymbol{Z}\langle z^p \rangle & \text{if } p > 0; \\ \boldsymbol{Z}\langle y \rangle & \text{if } p = 0. \end{cases}$$

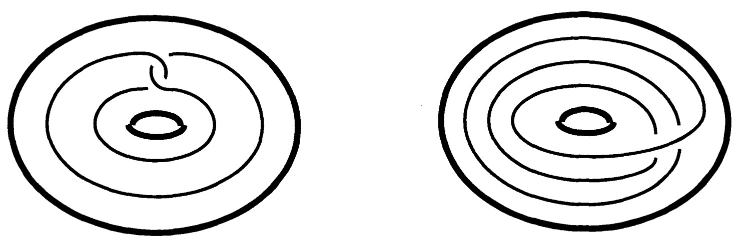
Let κ be a knot contained in a solid torus $D^2 \times \partial B^2$ such that the following conditions are satisfied:

(i) κ represents q times a generator of $H_1(D^2 \times \partial B^2)$.

(ii) κ is trivial in S^3.

(iii) $\partial D^2 \times 1$ is homotopically essential in $S^3 - \kappa$, so the map i_* is injective.

If $q > 0$, then (iii) follows from (i). An example of such κ is given in Figure 1, which indicates a $(q,1)$-torus knot for $q \geq 1$.



$q=0$ $q=3$

Figure 1

Lemma 2. *Let* $\Sigma = \Sigma(\phi, \kappa, F)$. *Then* $H_2(\pi\Sigma) \cong \boldsymbol{Z}_q$, *and* $\tau\Sigma \cong \boldsymbol{Z}_p$.

Proof. Let $\bar{G}$ and $\bar{A}$ be the groups defined by:

$$\bar{G} = \langle G \,|\, i_*(y) = i_*(z^p) = 1 \rangle,$$

$$\bar{A} = A/\ker \phi_+ \cong \boldsymbol{Z}\langle x \rangle \oplus \boldsymbol{Z}_p\langle z \rangle,$$

and let $\bar{\phi}_+$ and $\bar{i}_*$ be the maps such that the following diagram is commutative:

$$\begin{tikzcd} \pi F & A \arrow[l, "\phi_+"'] \arrow[r, "i_*"] \arrow[d] & G \arrow[d] \\ \pi F \arrow[u, equal] & \bar{A} \arrow[l, "\bar{\phi}_+"'] \arrow[r, "\bar{i}_*"] & \bar{G} \end{tikzcd}$$

where the vertical arrows are canonical projections. We use the same letters x, y, z and t for their image by the canonical projections. Since $\langle \pi_1(D^2 \times \partial B^2 - \kappa) \,|\, i_*(y) = 1 \rangle$ is the fundamental group of the space obtained from $D^2 \times \partial B^2 - \kappa$ by attaching a 2-cell along the loop $1 \times \partial B^2$, we see that $\bar{G} = \pi_1(S^3 - \kappa) \times \langle t \,|\, t^p = 1 \rangle = \boldsymbol{Z}\langle \mu \rangle \oplus \boldsymbol{Z}_p\langle t \rangle$, where μ is a meridian of

$\pi_1(S^3-\kappa)$. On the other hand $\bar{\iota}_*$ maps x to μ^q, z to t. Then $\bar{\Phi}_+$ and $\bar{\iota}_*$ are injective, and so $\pi\Sigma$ is an amalgamated free product of πF and $\bar{G}$ along $\bar{A}$.

We use the Mayer-Vietoris exact sequence for the amalgamated free product of groups; see [10, Lemma 7]. It is easy to see that $(\bar{\iota}_*, -\bar{\Phi}_+)\colon H_1(\bar{A}) \to H_1(\bar{G}) \oplus H_1(\pi F)$ is injective, and thus

$$H_2(\bar{A}) \to H_2(\bar{G}) \oplus H_2(\pi F) \to H_2(\pi\Sigma) \to 0$$

is exact. For any group H, $K(H\times \boldsymbol{Z}, 1)$ is homotopy equivalent to $K(H,1)\times S^1$. So the Künneth formula implies $H_2(H\times\boldsymbol{Z}) \cong H_2(H)\otimes H_0(\boldsymbol{Z}) \oplus H_1(H)\otimes H_1(\boldsymbol{Z})$. Thus both $H_2(\bar{A})$ and $H_2(\bar{G})$ split into $H_1(\boldsymbol{Z})\otimes H_1(\boldsymbol{Z}_p) \cong \boldsymbol{Z}\oplus\boldsymbol{Z}_p$, and $\bar{\iota}_*$ also splits into the product of maps on each factor, multiplication by q on $\boldsymbol{Z}$ and the identity on $\boldsymbol{Z}_p$. For the homology of the cyclic group, see for example [6, p. 76]. Since $H_2(\pi F)=0, H_2(\pi\Sigma)$ is isomorphic to the cokernel of $\bar{\iota}_*$, and thus $H_2(\pi\Sigma) \cong \boldsymbol{Z}_{\gcd(p,q)} = \boldsymbol{Z}_q$.

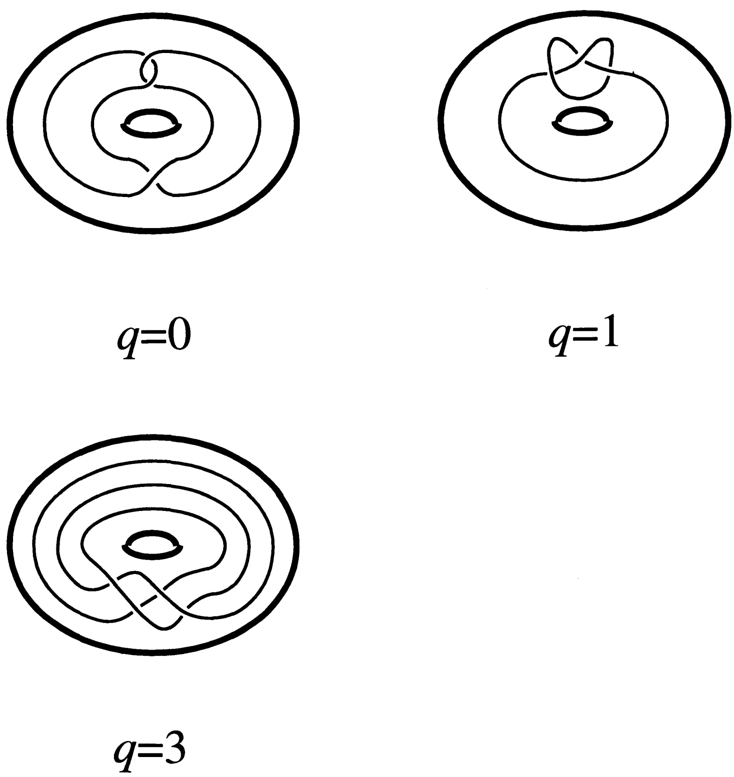


Figure 2

The peripheral subgroup of Σ is generated by the images of μ, t, and the longitude of κ. Since κ is trivial in S^3, the image of the longitude is trivial, and so the type of Σ is a cyclic subgroup generated by t, which has order p in $\bar{G}$, obtaining $\tau\Sigma \cong \boldsymbol{Z}_p$. This completes the proof. □

REMARK. A torus F with $H_2(\pi F) \cong \tau F \cong \boldsymbol{Z}$ is already constructed: There is a ribbon torus F with $H_2(\pi F) \cong \boldsymbol{Z}$ by [4, Theorem 1]. However such a torus must be of type $\boldsymbol{Z}$ by [10, Lemmas 2 and 3].

Let κ' be a knot contained in a solid torus $D^2 \times \partial B^2$ satisfying the above conditions (i), (iii) and
(ii') κ' is non-trivial in S^3.
An example of such κ' is given in Figure 2, which indicates a $(q, 2)$-torus knot for $q \geq 2$. See [10, pp. 429–430].

Lemma 3. *Let* $\Sigma' = \Sigma(\phi, \kappa', F)$. *Then* $H_2(\pi\Sigma') \cong \boldsymbol{Z}_q$, *and* $\tau\Sigma' \cong \boldsymbol{Z}_p \oplus \boldsymbol{Z}$.

Proof. The calculation for $H_2(\pi\Sigma')$ is the same as that in the proof of Lemma 2. However, the longitude of κ' injects to $\pi\Sigma'$, and so $\tau\Sigma'$ has the factor $\boldsymbol{Z}$. □

For the sake of Lemma 4 below, we observe the generators of $H_2(\pi\Sigma')$ and the peripheral subgroup of Σ' in more detail. Let U be a tubular neighbourhood of κ' in $D^2 \times \partial B^2$, and μ' and λ' be a meridian and a preferred longitude of κ' in ∂U. Let $G' = \pi_1(E \times S^1)$, where $E = D^2 \times \partial B^2 - \operatorname{int} U$, and so $G' = \pi_1(E) \times \boldsymbol{Z}\langle z \rangle$. Let i'_*: $A \to G'$ be the map induced by the inclusion i: $\partial D^2 \times \partial B^2 \times S^1 \subsetneq E \times S^1$. Letting $\bar{G}' = \langle G' \mid i'_*(y) = i'_*(z^p) = 1 \rangle$, we define $\bar{i}'_*$: $\bar{A} \to \bar{G}'$ as $\bar{i}_*$ in Lemma 2.

By the sphere theorem, E is aspherical. So $H_2(G') = H_2(E \times S^1)$, which has generators a', b', and c' represented by $\mu' \times \lambda'$, $\mu' \times S^1$ and $\lambda' \times S^1$. If we attach a 2-cell e^2 to E along a curve $1 \times \partial B^2$, then c' is trivialized. Identifying $E \cup e^2$ with $D^3 - \kappa'$, and then attaching a 3-cell e^3, we obtain $S^3 - \kappa'$. Then a' is trivialized in $H_2((S^3 - \kappa') \times S^1)$. The fundamental groups of $(D^3 - \kappa') \times S^1$ and $(S^3 - \kappa') \times S^1$ are both isomorpic to $\hat{G}' = \langle G' \mid i'_*(y) = 1 \rangle$. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
H_2(E \times S^1) & = & H_2(G') & & \\
\downarrow & & \downarrow & & \\
H_2((D^3 - \kappa') \times S^1) & \to & H_2(\hat{G}') & & \\
\downarrow & & \| & & \\
H_2((S^3 - \kappa') \times S^1) & \to & H_2(\hat{G}') & \to & H_2(\bar{G}')
\end{array}
$$

Each vertical arrow is induced by the inclusion of the space, and each left horizontal arrow is a natural map from the second homology group of the space to that of the fundamental group of the space. These are all surjective. The right horizontal arrow is induced by the natural projection of the group, which is surjective by [2, Lemma 1.4]; notice that $\hat{G}'=\pi_1(S^3-\kappa')\times \boldsymbol{Z}\langle z\rangle$ and $\bar{G}'=\pi_1(S^3-\kappa')\times \boldsymbol{Z}_p\langle z\rangle$. So only the generator of $H_2(G')$ corresponding to b' survives and generates $H_2(\bar{G}')$. Since $H_2(\pi F)=0$, the image of this generator of $H_2(G')$ generates $H_2(\pi\Sigma')$.

As was already seen, the peripheral subgroup of Σ' is $\boldsymbol{Z}\langle \mu'\rangle\oplus\boldsymbol{Z}\langle\lambda'\rangle\oplus \boldsymbol{Z}_p\langle t'\rangle$, where $\bar{i}'_*(z)=t'$.

Fix three positive integers q,r, and s with $q|s$ and let $p=rs$. Since $q|p$, by Lemma 3, the satellite torus $\Sigma'=\Sigma(\phi,\kappa',F)$ satisfies $H_2(\pi\Sigma')\cong\boldsymbol{Z}_q$ and $\tau\Sigma'\cong\boldsymbol{Z}_p\oplus\boldsymbol{Z}$. Let κ_r be the $(r,1)$-torus knot contained in a solid torus $D^2\times\partial B^2$. Let $i_*\colon A\to\pi_1((D^2\times\partial B^2-\kappa_r)\times S^1)$ be the map induced by the inclusion. Let $\phi^*\colon \partial B^2\times S^1\to\Sigma'$ be a homeomorphism such that $\phi^*_+\colon A\to\pi\Sigma'$ maps x,y, and z to μ',t', and λ', respectively, and thus $\ker\phi^*_+=\boldsymbol{Z}\langle y^p\rangle$.

Lemma 4. *Let* $\Sigma^*=\Sigma(\phi^*,\kappa_r,\Sigma')$. *Then* $H_2(\pi\Sigma^*)\cong\boldsymbol{Z}_q\oplus\boldsymbol{Z}_r$, *and* $\tau\Sigma^*\cong \boldsymbol{Z}_s\oplus\boldsymbol{Z}$.

Proof. We set

$$A^*=A/\ker\phi^*_+=\boldsymbol{Z}\langle x\rangle\oplus\boldsymbol{Z}_p\langle y\rangle\oplus\boldsymbol{Z}\langle z\rangle$$

and

$$G^*=\langle\, \pi_1((D^2\times\partial B^2-\kappa_r)\times S^1)\,|\, i_*(y^p)=1\,\rangle.$$

Putting $\Gamma=\langle\,\pi_1(D^2\times\partial B^2-\kappa_r)\,|\,i_*(y^p)=1\,\rangle$, we have $G^*=\Gamma\times\boldsymbol{Z}\langle t\rangle$. $\pi_1(D^2\times\partial B^2-\kappa_r)$ has a presentation $\langle u,\mu| u^r\mu u^{-r}=\mu\rangle$, where μ is a meridian of κ_r and u is represented by the core of the solid torus. Γ is the fundamental group of the space obtained from $D^2\times\partial B^2-\kappa_r$ by attaching a 2-cell e^2 along the loop $1\times\partial B^2$ by a map of degree p: $\Gamma=\pi_1((D^2\times\partial B^2-\kappa_r)\cup_p e^2)$. Since $u\mu$ is represented by this loop, putting $u\mu=y$, we have

$$\begin{aligned}\Gamma&=\langle\, u,\mu\,|\,u^r\mu u^{-r}=\mu,(u\mu)^p=1\,\rangle\\&=\langle\, u,y\,|\,u^r y u^{-r}=y, y^p=1\,\rangle.\end{aligned}$$

Thus $H_1(\Gamma)=\boldsymbol{Z}\langle\mu\rangle\oplus\boldsymbol{Z}_p\langle y\rangle$.

We define the injections $\bar{\phi}^*_+\colon A^*\to\pi\Sigma'$ and $\bar{i}_*\colon A^*\to G$ in the same way. Then $\pi\Sigma^*$ is an amalgamated free product of $\pi\Sigma'$ and G^* along A^*.

Since $(\bar{\imath}_*, -\bar{\Phi}^*_+)$: $H_1(A^*) \to H_1(G^*) \oplus H_1(\pi\Sigma')$ is injective, we have an exact sequence:

$$H_2(A^*) \to H_2(G^*) \oplus H_2(\pi\Sigma') \to H_2(\pi\Sigma^*) \to 0.$$

Now $H_2(A) = H_2(\partial D^2 \times \partial B^2 \times S^1) = \langle\!\langle a, b, c \mid \rangle\!\rangle$, where a, b, and c are represented by $\partial D^2 \times \partial B^2 \times 1, \partial D^2 \times 1 \times S^1$ and $1 \times \partial B^2 \times S^1$, respectively. Let a^*, b^*, and c^* be the images of a, b, and c by the canonical map $H_2(A) \to H_2(A^*)$. By the Künneth formula, we have

$$\begin{aligned} H_2(A^*) &= H_2(\boldsymbol{Z} \oplus \boldsymbol{Z}_p \oplus \boldsymbol{Z}) \\ &= H_2(\boldsymbol{Z} \oplus \boldsymbol{Z}_p) \otimes H_0(\boldsymbol{Z}) \oplus H_1(\boldsymbol{Z} \oplus \boldsymbol{Z}_p) \otimes H_1(\boldsymbol{Z}) \\ &= \langle\!\langle a^*, b^*, c^* \mid pa^* = pc^* = 0 \rangle\!\rangle, \end{aligned}$$

where b^* and c^* correspond to x and y, the generators of $H_1(\boldsymbol{Z} \oplus \boldsymbol{Z}_p)$.

We first determine the map $\bar{\imath}_*$: $H_2(A^*) \to H_2(G^*)$. As we will see in Claim 9, $H_2(\Gamma) \cong \boldsymbol{Z}_p$, and so we have

$$\begin{aligned} H_2(G^*) &= H_2(\Gamma \times \boldsymbol{Z}\langle t \rangle) \\ &= H_2(\Gamma) \otimes H_0(\boldsymbol{Z}) \oplus H_1(\Gamma) \otimes H_1(\boldsymbol{Z}) \\ &= \langle\!\langle \alpha, \beta, \gamma \mid p\alpha = p\gamma = 0 \rangle\!\rangle, \end{aligned}$$

where α is the generator of $H_2(\Gamma)$ and β, γ correspond to μ, y, the generators of $H_1(\Gamma)$. Let i_1 and i_2 be the maps which are the restrictions of $\bar{\imath}_*$ to the first and the second factors of $H_2(A^*)$, respectively. Consider the following commutative diagram:

$$\begin{array}{ccc} H_2(\partial D^2 \times \partial B^2 \cup_p e^2) & \to & H_2((D^2 \times \partial B^2 - \kappa_r) \cup_p e^2) \\ \downarrow & & \downarrow \\ H_2(\boldsymbol{Z} \oplus \boldsymbol{Z}_p) & \to & H_2(\Gamma) \end{array}$$

Each vertiacal arrow is a natural map, which is surjective. The top horizontal arrow is induced by the inclusion, and is surjective. Therefore i_1 is surjective. The first factors are both isomorphic to $\boldsymbol{Z}_p$, and so i_1 is an isomorphism. Thus we may write $i_1(a^*) = \alpha$.

For the second factor, $\bar{\imath}_*$ maps x to μ^r and y to y. So we have $i_2(b^*) = r\beta$ and $i_2(c^*) = \gamma$.

Next we must determine the map $\bar{\Phi}^*_+$: $H_2(A^*) \to H_2(\pi\Sigma')$. This maps a^*, b^* and c^* to b', a', and c', respectively, the generators of $H_2(E \times S^1)$. As was already observed, only b' survives and generates $H_2(\pi\Sigma')$, and therefore we have $\bar{\Phi}^*_+(a^*) = b'$ and $\bar{\Phi}^*_+(b^*) = \bar{\Phi}^*_+(c^*) = 0$.

Hence we obtain

$$
\begin{aligned}
H_2(\pi\Sigma^*) &= \langle\!\langle \alpha,\beta,\gamma,b' \mid p\alpha=p\gamma=0, r\beta=0, \gamma=0, \alpha=b', qb'=0 \rangle\!\rangle \\
&= \langle\!\langle \alpha,\beta \mid q\alpha=r\beta=0 \rangle\!\rangle \\
&\cong \boldsymbol{Z}_q \oplus \boldsymbol{Z}_r.
\end{aligned}
$$

The peripheral subgroup of Σ^* is generated by μ and λ, a meridian and a longitude of κ_r, and t. By the choice of ϕ^*, t injects to $\pi\Sigma^*$. We can choose $\lambda = xy^r$ and $x=\mu^r$ in $\pi_1(D^2 \times \partial B^2 - \kappa_r)$, where x and y are represented by the loops $\partial D^2 \times 1$ and $1 \times \partial B^2$, respectively. Thus $\tau\Sigma^*$ is generated by y^r and t. The order of y is $p=rs$, and so that of y^r is s. This completes the proof. □

Claim 9. $H_2(\Gamma) \cong \boldsymbol{Z}_p$.

Proof. Let

$$
\begin{aligned}
G_1 &= \boldsymbol{Z}\langle v\rangle \oplus \boldsymbol{Z}_p\langle y\rangle, \\
G_2 &= \boldsymbol{Z}\langle u\rangle, \\
H &= \boldsymbol{Z}\langle v\rangle,
\end{aligned}
$$

and $\varphi_i\colon H \to G_i$, $i=1,2$, be the injections defined by $\varphi_1(v)=v$ and $\varphi_2(v)=u^r$. Then

$$
\begin{aligned}
\Gamma &= \langle v, y \mid vyv^{-1}=y, y^p=1, v=u^r \rangle \\
&= \langle G_1 * G_2 \mid v=u^r \rangle
\end{aligned}
$$

is an amalgamated free product of G_1 and G_2 along H. Since $(\varphi_{1*}, -\varphi_{2*})\colon H_1(H) \to H_1(G_1) \oplus H_1(G_2)$ is injective and $H_2(H)=H_2(G_2)=0$, using the Mayer-Vietoris exact sequence, we have $H_2(\Gamma) \cong H_2(G_1)$. The result follows from the Künneth formula. □

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Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-Ku
Osaka 558, Japan

Department of Mathematics
Kobe University
Nada-Ku
Kobe 657, Japan