Nakamura, I.
Osaka J. Math.
31 (1994), 787–829

MOISHEZON FOURFOLDS HOMEOMORPHIC TO Q_C^4

IKU NAKAMURA

(Received March 30, 1993)

Introduction

In general, there are many different complex manifolds having the same underlying topological or differentiable structure. However there are a few exceptional cases where we can expect that homeomorphy to a given compact complex manifold implies analytic isomorphism to it, for instance, an irreducible compact Hermitian symmetric space. Among irreducible Hermitian symmetric spaces, the complex projective space $\boldsymbol{P}_{\boldsymbol{C}}^n$ and a smooth hyperquadric $\boldsymbol{Q}_{\boldsymbol{C}}^n$ in $\boldsymbol{P}_{\boldsymbol{C}}^{n+1}$ seem to be nice exceptions which we can handle with algebraic methods. In [15] we studied the complex projective space $\boldsymbol{P}_{\boldsymbol{C}}^n$, while in the present article we study a smooth hyperquadric $\boldsymbol{Q}_{\boldsymbol{C}}^n$ in $\boldsymbol{P}_{\boldsymbol{C}}^{n+1}$ in the same way as in [15]. A goal we have in mind is the following

Conjecture MQ_n. *Any Moishezon complex manifold homeomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^n$ is isomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^n$.*

The conjecture has been solved partially by Brieskorn [3] under the assumption that the manifold in question is *Kählerian* and odd-dimensional. In the even-dimensional *Kählerian* case, there still remains a possibility of manifolds of general type. Recently Kollár [7] and the author [13] solved Conjecture MQ_3 in the affirmative, each supplementing the other. Peternell [16][17] also asserts the same consequence. See [7,5.3.6].

Theorem 1. *Any Moishezon threefold homeomorphic to* $\boldsymbol{Q}_{\boldsymbol{C}}^3$ is isomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^3$.

The main purpose of the present article is to give a partial solution to the above conjecture MQ_4 in dimension 4. We prove,

Theorem 2. *Let X be a Moishezon fourfold homeomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^4$,* *and L a line bundle on X with $L^4=2$.* Assume $h^0(X, O_X(L)) \geq 5$. *Then X is isomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^4$.*

Corollary 3. *Any global deformation of $\boldsymbol{Q}_{\boldsymbol{C}}^4$ is isomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^4$.*

It is easy to see that any complex analytic (global) deformation of $\boldsymbol{Q}_{\boldsymbol{C}}^n$ is Moishezon. However it is possible that there appears a *non-projective* or a *non-Kählerian* Moishezon manifold of dimension $n \geq 3$ as a global deformation of a *projective* or a *Kählerian* manifold (Hironaka [6]). This is one of the reasons why we consider a possibly *non-projective* or a possibly *non-Kählerian* Moishezon manifold as in Theorem 1 and Theorem 2. We easily derive Corollary 3 from Theorem 2. In fact, any global deformation of $\boldsymbol{Q}_{\boldsymbol{C}}^4$ not only in any complex analytic family but also in any differentiable family is isomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^4$.

Now we give an outline of our proof of Theorem 2. Let X be a Moishezon fourfold homeomorphic to $\boldsymbol{Q}_{\boldsymbol{C}}^4$. Then we have a unique line bundle L on X such that $\operatorname{Pic} X \simeq \boldsymbol{Z}L$, $c_1(X)=4c_1(L)$, and $L^4=2$. Let $m:=h^0(X, O_X(L))-1 \geq 4$. We consider the rational map h: $X \rightarrow \boldsymbol{P}_{\boldsymbol{C}}^m$ associated to the linear system $|L|$. Let W be the closure of the image $h(X \backslash \operatorname{Bs}|L|)$. Let $d=\deg W$. Then $d \geq m+1-\dim W$. Since $\operatorname{Pic} X \simeq \boldsymbol{Z}L$, we have $\dim W \geq 2$. Let τ be a complete intersection $D \cap D'$ for general D and $D' \in |L|$. Then τ is connected, pure two-dimensional and Gorenstein.

Assume first $\dim W=2$. Then we have reduced irreducible components Z_i $(1 \leq i \leq d)$ of τ outside $B:=\operatorname{Bs}|L|$. We note that $d \geq m-1 \geq 3$. Each Z_i is nonsingular outside B by Bertini's theorem. Let ν_i: $Y_i \rightarrow Z_i$ be the normalization of Z_i, f_i: $S_i \rightarrow Y_i$ the minimal resolution of Y_i and $g_i:=\nu_i \cdot f_i$. We see that $K_{S_i}=-2g_i^*(L)-A_i$ for some effective divisor A_i with $supp(A_i) \subset g_i^{-1}(B)$. Since $g_i^*(L)$ is effective by $m \geq 2$, $S_i \simeq \boldsymbol{P}_{\boldsymbol{C}}^2$ or S_i is ruled.

If $S_i \simeq \boldsymbol{P}_{\boldsymbol{C}}^2$, then $S_i \simeq Y_i \simeq Z_i$, $g_i^*(L)=A_i \in |O_{\boldsymbol{P}^2}(1)|$. If moreover $Z_i \cap Z_j \neq \emptyset$ for $i \neq j$, then $d=2$, which contradicts $d \geq 3$. If $Z_i \cap Z_j=\emptyset$ for $i \neq j$, then W turns out to be a cone over a smooth variety of minimal degree by the Del Pezzo-Bertini classification [5]. Any such W has a reducible or noreduced hyperplane section for $d \geq 3$, which contradicts $\operatorname{Pic} X \simeq \boldsymbol{Z}L$. If S_i is ruled, then we can derive a contradiction similarly.

Similarly we can disprove $\dim W=3$. Consequently $\dim W=4$. Bertini's theorem shows that a scheme-theoretic complete (not necessarily proper) intersection l of general 3 members of $|L|$ is pure one-dimensional and irreducible nonsingular outside B. We infer from $c_1(X)=4c_1(L)$ that l has a rational curve C with $LC=2$ as an irreducible component outside B. Then applying (in sections 3 an 4) the same argument as, in fact simpler than, in [15], we can study the morphism h in detail. Subsequently we see that $h^0(X,L)=6$ and that h is an isomorphism of X onto a smooth hyperquadric Q_C^4 in P_C^5.

The article is organized as follows. In sections one and two, we study a scheme-theoretic complete intersection l_V of general $(n-1)$-members in $|L|$ along reduced curve-components.

In sections 3 and 4, we study Moishezon manifolds of dimension n with the second Betti number $b_2(X)$ equal to one, and with $c_1(X)=nc_1(L)$ for some line bundle L on X. In section 5, we prove Theorem 2 by applying the results in the previous sections.

NOTATION. The notation is indexed at the end of the article.

ACKNOWLEDGEMENT. The author would like to express his hearty gratitude to A. Fujiki, F. Hidaka and I. Shimada for their advices during the preparation of the article. Fujiki kindly showed simpler proofs for some of our proofs in the first version of the article.

1. A complete intersection l_V (1) —local structure—

(1.1) BASIC ASSUMPTIONS. Let X be a complete nonsingular algebraic variety of dimension n, L a line bundle on X. We assume

(1.1.1) $\qquad c_1(X)=dc_1(L) \qquad$ for some integer d,

(1.1.2) $\qquad h^0(X,L)\geq n.$

Let V be an $(n-1)$-dimensional subspace of $H^0(X,L)$, $l:=l_V$ a scheme-theoretic complete intersection associated with V. This means that the ideal I_l of O_X defining l is defined by $I_l=\Sigma_{s\in V}sO_X$.

We say that C is *a reduced curve-component of* l if C is an irreducible one-dimensional component of l along which l is reduced generically. We assume that

(1.1.3) $\qquad l$ has *a reduced curve-component* outside B $(:=\mathrm{Bs}\,|L|)$.

(1.2) TORSION SHEAVES Q_C, Q'_C AND Q''_C. Let C a reduced curve-component of l_V, and I_C the ideal sheaf of O_X defining C with $\sqrt{I_C}=I_C$. Let ν: $\tilde{C}\to C$ be the normalization of C. Then we have a natural exact sequence

(1.2.1) $\qquad 0\to(I_l/I_l^2)\otimes O_{\tilde{C}}(\simeq O_{\tilde{C}}(-\nu^*L)^{\oplus(n-1)})\xrightarrow{\phi_C}[(I_C/I_C^2)\otimes O_{\tilde{C}}]\to Q_C\to 0$

where $Q_C:=\operatorname{Coker}\phi_C$ and $[J]:=J/O_{\tilde{C}}$-torsions for an $O_{\tilde{C}}$-module J. We also have a natural exact sequence

$$0\to Q''_C\to \Omega^1_C\otimes O_{\tilde{C}}\xrightarrow{\eta}\Omega^1_{\tilde{C}}\to Q'_C\to 0 \tag{1.2.2}$$

where $Q'_C=\operatorname{Coker}\eta$, $Q''_C=\operatorname{Ker}\eta$.

We define $l(F):=\dim_{\mathbf{C}}F$ for a torsion $O_{\tilde{C}}$-module F and $l(F,p):=l(F_p)$ for a stalk F_p of F at p. See (1.10) and Theorem 1.11. Then by [15, §1 and 2], we have,

Lemma 1.3. *Under the notation and assumptions in* (1.1), *let C be a reduced curve-component of l_V. Let $Q^{(0)}_C:=Q_C$, $Q^{(1)}_C:=Q'_C$ and $Q^{(2)}_C:=Q''_C$. Then,*

(1.3.1) $(d-n+1)LC+c_1(\Omega^1_{\tilde{C}})+l(Q_C)+l(Q''_C)-l(Q'_C)=0.$

(1.3.2) $l(Q^{(\nu)}_C)=\Sigma_{p\in C}l(Q^{(\nu)}_C,p)\qquad(\nu=0,1,2).$

(1.3.3) $l(Q^{(\nu)}_C,p)\geq 0$, $l(Q''_C,p)\geq l(Q'_C,p)$ *for any* $p\in C$.

(1.3.4) $l(Q_C,p)=0$ *if and only if* $(C,p)\simeq(l,p)$.

(1.3.5) $l(Q''_C,p)\geq l(Q'_C,p)$ *for any* $p\in C$, *where equality holds if and only if* (C,p) *is irreducible nonsingular. If* (C,p) *is nonsingular, then* $l(Q''_C,p)=l(Q'_C,p)=0$.

(1.3.6) *If* (C,p) *is irreducible and singular, then* $l(Q''_C,p)\geq l(Q'_C,p)+2$.

(1.3.7) *Assume that* (C,p) *is reducible. Let* (C_λ,p) $(\lambda\in\Lambda)$ *be all the irreducible components of* (C,p), *and* Λ_{ns} (resp. Λ_s) *the subset of* Λ *consisting of all* λ *with* (C_λ,p) *nonsingular* (*resp. singular*). *Then* $l(Q''_C,p)\geq l(Q'_C,p)+2\sharp(\Lambda_s)+\sharp(\Lambda_{ns})$.

The proof of (1.3.5)–(1.3.7) is partially based on the following Lemma 1.4. See [15, §2] for the details.

Lemma 1.4. *Assume that* (C,p) *is irreducible and singular. Let* $x_1,\cdots,x_n$ *be a local coordinate system of* (X,p). *We* (*may*) *assume that the normalization* ν: $\tilde{C}\to C$ $(\subset X)$ *is locally given by*

$$\begin{aligned}&x_1=t^m\\&x_j=f_j(t)=t^{m_j}g_j(t),\qquad g_j(0)\neq 0\qquad(2\leq j\leq s)\\&x_j=0\qquad(s+1\leq j\leq n)\end{aligned}$$

for some $f_j, g_j \in O_{\bar{C}}$, where $2 \le m < m < m_2 < m_3 < \cdots < m_s$, none of m_j and none of $m_j - m_k$ is an integral multiple of m, while s is the embedding dimension of (C,p). Let q be the unique positive integer such that $m \le qm < m_2 < (q+1)m$. Then

$$l(Q'_C, p) = m - 1. \tag{1.4.1}$$

$$l(Q''_C, p) \ge min(2qm, m_3) + m - m_2 \ge m + 1. \tag{1.4.2}$$

Proof. See [15, (2.3)] for the details. We recall the proof only for the later use. By the prrof of [15, (2.3)], $l(Q'_C, p) = l(\Omega^1_{\bar{C},\bar{p}} / \Omega^1_{C,p} \otimes O_{\bar{C},\bar{p}}) = m - 1$. Let

$$e_j = dx_j \otimes 1 \in \Omega^1_X \otimes O_{\bar{C}}, \qquad \bar{e}_j = dx_j \otimes 1 \in \Omega^1_C \otimes O_{\bar{C}}.$$

Then the element $\sigma_j = (f'_j(t)/mt^{m-1})\bar{e}_1 - \bar{e}_j$ is contained in Q''_C.

Since Q''_C is a torsion sheaf, we (can) choose the minimal integer $N \ge 0$ such that $t^N \sigma_2 = 0$. By definition $l(Q''_C, p) \ge N$. The condition $t^N \sigma_2 = 0$ means that there exist some $F_i \in \mathbf{C}[[t]]$ and $\varphi_i \in I_C$ $(1 \le i \le l)$ such that

$$t^N((f'_2(t)/mt^{m-1})e_1 - e_2) = \sum_{j=1}^{s} (\sum_{i=1}^{l} F_i(t) v^*(\partial\varphi_i/\partial x_j)) e_j. \tag{1.4.3}$$

The coefficient of e_1 in the right hand side of (1.4.3) is equal to $\Sigma_{i=1}^{l} F_i(t) v^*(\partial\varphi_i/\partial x_1)$. Take any element $\varphi \in I_C \cap \mathbf{C}[[x_1, \cdots, x_s]]$ $(\subset m_p^2)$ with its expansion given by

$$\varphi = \sum_{i_1 + \cdots + i_s \ge 2} a_{i_1 \cdots i_s} x_1^{i_1} \cdots x_s^{i_s}.$$

Then $\varphi \in I_{C,p}$ implies that $a_{j0\cdots0} = 0$ $(1 \le j \le 2q)$, $a_{j10\cdots0} = 0$ $(1 \le j \le q)$. Hence

$$\partial\varphi/\partial x_1 = (2q+1)ax_1^{2q} + (q+1)bx_1^q x_2 + cx_2^2 + dx_3 + \cdots.$$

$$\deg v^*(\partial\varphi/\partial x_1) \ge \min(2qm, qm + m_2, 2m_2, m_3) = \min(2qm, m_3)$$

for some constants a, b, c and d. Hence $\deg t^{N-m+1} f'_2(t) \ge \min(2qm, m_3) \ge m_2 + 1$, which completes the proof of (1.4.2). q.e.d.

(1.5) THREE CASES. Let C be a reduced curve-component of l, p a point of C. Assume $d = n$, $LC \ge 0$ and that (C,p) is singular. Then by (1.3.1) we have,

$$l(Q_C)+l(Q''_C)-l(Q'_C)\leq -c_1(\Omega^1_C)\leq 2.$$

If (C,p) is irreducible, then $l(Q_C)=0$, and $l(Q''_C)-l(Q'_C)=2$. If (C,p) is reducible, (C,p) has by (1.3.7) exactly two irreducble components, and any irreducible component of it is nonsingular. Thus we have only to consider the following three cases:

CASE A. $\quad l(Q_C,p)=1, \quad l(Q''_C,p)=l(Q'_C,p)=0.$

CASE B. $\quad l(Q_C,p)=0, \quad l(Q''_C,p)-l(Q'_C,p)=2.$

CASE C. $\quad l(Q_C,p)=2, \quad l(Q''_C,p)=l(Q'_C,p)=0.$

Lemma 1.6 (CASE A). *Assume that $l(Q_C,p)=1$, $l(Q''_C,p)=l(Q'_C,p)=0$. Then (l,p) has two irreducible components (C,p) and (C',p), and there exists a local parameter system $x_1\cdots,x_n$ such that*

$$I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C,p}=(x_1,\cdots,x_{n-1}),\ I_{C',p}=(x_1,\cdots,x_{n-2},x_n).$$

Proof. By (1.3.5), the germ (C,p) is nonsingular, so that we can choose local parameters $x_1,\cdots,x_{n-1}$ such that $I_{C,p}=(x_1,\cdots,x_{n-1})$. The condition $l(Q_C,p)=1$ implies that we may assume $x_i\in I_{l,p}$ $(1\leq i\leq n-2)$. Moreover we can choose an $(n-1)$-th generator f_{n-1} of $I_{l,p}$ such that $f_{n-1}\bmod I_C^2$ has a single zero at p as a local section of I_C/I_C^2. Therefore by choosing an n-th local coordinate x_n at p suitably, we may assume $I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n)$. It follows that l has another irreducible component (C',p) as above. q.e.d.

Lemma 1.7 (CASE B). *Assume that $l(Q_C,p)=0$, $l(Q''_C,p)=l(Q'_C,p)+2$. Then there exists a local parameter system $x_1,\cdots,x_n$ such that one of the following is true.*

(1.7.1) $\quad I_{l,p}=I_{C,p}=(x_1,\cdots,x_{n-2},x^3_{n-1}-x^2_n),$

(1.7.2) $\quad I_{l,p}=I_{C,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n).$

Proof. By $l(Q_C,p)=0$, we have $I_{l,p}+I^2_{C,p}=I_{C,p}$. By Nakayama's lemma we see $I_{l,p}=I_{C,p}$, whence $(l,p)\simeq(C,p)$. There are two subcases CASE B-1 and CASE B-2 according as (C,p) is irreducible or not.

CASE B-1. Assume that (C,p) is irreducible. We use the same notation as in Lemma 1.4. Then by the proof of Lemma 1.4

$$l(Q'_C,p)=m-1, \qquad l(Q''_C,p)\geq N\geq m+1.$$

Consequently we have

(1.7.3) $$N=l(Q''_C,p)=m+1\geq 3.$$

Moreover by the proof of Lemma 1.4, we see that there exists $\varphi\in I_{C,p}$ such that

(1.7.4) $$N-m+m_2=\deg \nu^*(\partial\varphi/\partial x_1)=\min(2qm,m_3).$$

CASE B-1-1. First we consider the case where $2qm\leq m_3$. Then $N-m+m_2=2qm$. Since $N=m+1$ and $qm<m_2<(q+1)m$, we have $q=1$, $m_2=2m-1$, and $m_3\geq 2m+1$. In view of the proof of Lemma 1.4 the expansions of φ and $\partial\varphi/\partial x_1$ are given by

$$\varphi=ax_1^3+bx_1^2x_2+cx_1x_2^2+dx_1x_3+ex_2^2+\cdots,$$
$$\partial\varphi/\partial x_1=3ax_1^2+2bx_1x_2+cx_2^2+dx_3+\cdots.$$

Since $m_3\geq 2m+1$, $\nu^*x_1^2$ is the unique monomial term of degree $2m$ in the right hand side of $\partial\varphi/\partial x_1$. Since $\deg \nu^*\partial\varphi/\partial x_1=2qm=2m$ by (1.7.4), we have $a\neq 0$. On the other hand since $\nu^*\varphi=0$, there is another nontrivial term of degree $3m$ besides x_1^3 in the right hand side of φ, which is just ex_2^2 by the choice of m_j in Lemma 1.4. Therefore $e\neq 0$. Hence we have $\varphi=x_1^3-x_2^2+\cdots$ by modifying x_2 and x_3 by constant multiples. Therefore $3m=2m_2=4m-2$, $m=2$ and $m_2=3$. It follows that the normalization ν: $\tilde{C}\to C$ is given by

$$x_1=t^2,\qquad x_2=t^3g_2(t)$$

for some holomorphic $g_2(t)$ with $g_2(0)=1$. Then there exist $g_{21}(x_1)$ and $g_{22}(x_1)\in m_p$ such that

$$\nu^*x_2=t^3+g_{21}(t^2)+t^3g_{22}(t^2)=t^3(1+\nu^*g_{22}(x_1))+\nu^*g_{21}(x_1).$$

By taking $x'_2=(x_2-g_{21}(x_1))(1+g_{22}(x_1))^{-1}$ instead of x_2, the normalization ν: $\tilde{C}\to C$ is given by

$$x_1=t^2,\qquad x_2=t^3.$$

Since any monomial t^n $(n\geq 4)$ is a product of t^2 and t^3, we may assume $x_j=0$ $(j\geq 3)$, so that the embedding dimension of (C,p) is equal to 2. Thus we see that

$$I_{l,p}=I_{C,p}=(x_1^3-x_2^2,x_3,x_4,\cdots,x_n),$$

$$l(Q''_C,p)=3, \qquad l(Q'_C,p)=1.$$

CASE B-1-2. Next we consider the case where $2qm>m_3$. By (1.7.3) and (1.7.4), we see $N=m+1$, $m_3=m_2+1$. Moreover there exists $\varphi\in I_{C,p}\cap m_p^2$ such that $\deg v^*(\partial\varphi/\partial x_1)=m_3$. Since $\varphi\in I_{C,p}$ has at least two monomial terms of minimum degree in t by the condition $v^*\varphi=0$, we see

$$\varphi=ax_1x_3+bx_2^2+\cdots,$$

where $a\neq 0$, $b\neq 0$, and $m+m_3=2m_2$ by the choice of m_j in Lemma 1.4. It follows that $m_2=m+1$, $m_3=m+2$. Hence φ has exactly two monomial terms x_1x_3 and x_2^2 of minimum degree. We may assume $\varphi=x_1x_3-x_2^2$ +(higher terms) by choosing x_2, x_3 suitably. Since $\varphi_j\in I_{C,p}$ $(1\leq j\leq l)$ in the right hand side of (1.4.3), by the above argument we can write $\varphi_j=c_j\varphi+\varphi_j^*$ where c_j is a constant and φ_j^* has no monomial terms x_1x_3 and x_2^2. Let $c=\Sigma_{i=1}^l c_iF_i(0)$. Then we have

$$\deg v^*(\partial\varphi_j^*/\partial x_1)\geq m_3+1\geq m+3,\ \deg v^*(\partial\varphi_j^*/\partial x_3)\geq\min(2m,m_2)\geq m+1.$$

It follows from (1.7.3) that the coefficient of e_1 (resp. e_3) in the right hand side of (1.4.3) starts with ct^{m+2} (resp. $-ct^m$), where $N=m+1$ implies that $c\neq 0$. However the coefficient of e_3 in the left hand side of (1.4.3) is equal to 0, which implies $c=0$, a contradiction.

CASE B-2. Assume that (C,p) is reducible. Let (C_λ,p) $(\lambda\in\Lambda)$ be all the irreducible components of (C,p). In view of (1.3.7), $\Lambda=\Lambda_{\mathrm{ns}}$, $\sharp(\Lambda)=2$. Let $\Lambda=\{0,1\}$. Then for $\lambda=0,1$, we have

$$l(\mathrm{Ker}(\Omega_C^1\otimes O_{C_\lambda}\to\Omega_{C_\lambda}^1))=1 \tag{1.7.5}$$

by the proof of (1.3.7) [15, §2]. We choose a local coordinate system $x_1,\cdots,x_n$ at p such that $I_{C_0,p}=(x_1,\cdots,x_{n-1})$. The normalization $v\colon\tilde{C}_0\to C_0$ is clearly given by

$$x_i=0 \qquad (1\leq i\leq n-1), \qquad x_n=t.$$

Let $S:=O_{C_0}\{d\varphi;\varphi\in I_C,\ v^*\dfrac{\partial\varphi}{\partial x_n}=0\}$. Then we have

$$\mathrm{Ker}(\Omega_C^1\otimes O_{C_0}\to\Omega_{C_0}^1)\simeq O_{C_0}dx_1+\cdots+O_{C_0}dx_{n-1}/S. \tag{1.7.6}$$

By (1.7.5) we may assume that $x_i\in I_{C,p}$ $(1\leq i\leq n-2)$ and that the right hand side of (1.7.6) is generated by dx_{n-1}. Moreover there exists $\varphi\in I_{C,p}$ such that

$$v^*(\partial\varphi/\partial x_{n-1})=t, \qquad v^*(\partial\varphi/\partial x_i)=0 \ (1\le i\le n-2 \text{ or } i=n).$$

It follows that $\varphi=x_{n-1}x_n \bmod (x_1,\cdots,x_{n-1})^2$. Therefore by choosing $\varphi\in I_{C,p} \bmod(x_1,\cdots,x_{n-2})$ and $x_n \bmod(x_1,\cdots,x_{n-1})$ suitably we may assume $\varphi=x_{n-1}x_n\in I_{C,p}$. Then we have

$$I_{C,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n).$$

In fact, let $J=(x_1,\cdots,x_{n-2},x_{n-1}x_n)$ and $O_{C'}:=O_{X,p}/J$. Then (C',p) is a reduced subvariety of (X,p) with two irreducible components C_0 and C_1 at p, so that $supp(C')=supp(C)$. Hence $l(I_{C,p}/J)$ is finite. Therefore $I_{C,p}\supseteq J=\sqrt{J}=\sqrt{I_{C,p}}\supseteq I_{C,p}$, whence $J=I_{C,p}$ and $(C,p)\simeq(C',p)$. Thus (C,p) has two irreducible components C_0 and C_1 defined by

$$I_{C_0,p}=(x_1,\cdots,x_{n-2},x_{n-1}), \qquad I_{C_1,p}=(x_1,\cdots,x_{n-2},x_n).$$

q.e.d.

(1.8) EXAMPLE. Let s be an integer ≥ 1. Consider a germ (C,p) defined by

$$I_{C,p}=(x_1x_3-x_2^2,x_1^{s+2}-x_2x_3^s,x_1^{2s+3}-x_3^{2s+1}).$$

The normalization v: $(\tilde{C},p)\to(C,p)$ is given by

$$x_1=t^{2s+1}, \qquad x_2=t^{2s+2}, \qquad x_3=t^{2s+3}.$$

Let $e_i^*:=dx_i\otimes 1\in Q''_{C,p}$. The torsion sheaf $Q''_{C,p}$ is generated by two elements

$$\sigma_1:=t^2e_1^*-2te_2^*+e_3^*, \qquad \sigma_2:=(2s+2)te_1^*-(2s+1)e_2^*$$

where $t^{2s}\sigma_1=0$, $t^{2s^2+3s}\sigma_2=0$. Thus we have $l(Q''_C,p)=2s^2+5s$ and $l(Q'_C,p)=2s$. Compare the proof of Lemma 1.7 CASE B-1-2.

Lemma 1.9 (GENERIC CASE IN CASE C). *Assume that (l,p) is sufficiently general and that (l,p) is reduced, nonsingular and pure one-dimensional outside $B:=\mathrm{Bs}\,|L|$. Let C be a movable reduced curve-component of l. If $l(Q_C,p)=2$, $l(Q''_C,p)=l(Q'_C,p)=0$, then there exists a local coordinate system $x_1,\cdots,x_n$ at p such that*

$$I_{C,p}=(x_1,\cdots,x_{n-1}),$$

and that one of the following is true.

(1.9.1) $I_{l,p}=(x_1,\cdots,x_{n-3},x_{n-2},x_{n-1}x_n^2)$, $I_{B,p}=(x_1,\cdots,x_{n-3},x_{n-2},x_n^2)$.

(1.9.2) $I_{l,p}=(x_1,\cdots,x_{n-3},x_{n-2},(x_n^2-x_{n-1}^s)x_{n-1})$,
$I_{B,p}=(x_1,\cdots,x_{n-3},x_{n-2},x_n^2-x_{n-1}^s)$ $(s\geq 2)$.

(1.9.3) $I_{l,p}=(x_1,\cdots,x_{n-3},x_{n-2}x_n,x_{n-1}x_n)$, $I_{B,p}=(x_1,\cdots,x_{n-3},x_n)$.

(1.9.4) $I_{l,p}=(x_1,\cdots,x_{n-3},x_{n-2}x_n,(x_n+a(x))x_{n-1})$

for some $a(x)$ $(\neq 0)\in \boldsymbol{C}[[x_{n-2},x_{n-1}]]\cap m_p$. *The germ* (l,p) *has at least* 3 *irreducible components. Among them, there are at most* 4 *movable components of* (l,p).

(1.9.4.1) *If* (l,p) *has exactly* 3 *irreducible components, then* $a(x_{n-2},x_{n-1})=x_{n-2}^{e_{n-2}}x_{n-1}^{e_{n-1}}$ *for some* $e_{n-2}\geq 0$, $e_{n-1}\geq 0$ *with* $e_{n-2}+e_{n-1}\geq 1$.

(1.9.4.2) *If* (l,p) *has exactly* 4 *movable components* (C_j,p) $(0\leq j\leq 3)$ *with* $C_0=C$, *then* $a(x_{n-2},x_{n-1})=x_{n-2}+x_{n-1}$ *so that* (l,p) *has no fixed components, and* $I_{B,p}=(x_1,\cdots,x_{n-3},x_{n-2},x_{n-1},x_n)$.

Proof. Since (C,p) is nonsingular by (1.3.5), we can choose a local coordinate system $x_1,\cdots,x_n$ at p such that

$$I_{C,p}=(x_1,\cdots,x_{n-1}),\qquad I_{l,p}=(x_1,\cdots,x_{n-3},\varphi_{n-2},\varphi_{n-1})$$

for some $\varphi_j\in I_{l,p}$. Since $l(Q_C,p)=2$, we may assume by choosing φ_{n-2} and φ_{n-1} suitably that one of the following is true;

CASE C-1. $\varphi_{n-2}=x_{n-2}$, $\varphi_{n-1}=x_n^2x_{n-1}$

CASE C-2. $\varphi_{n-2}=x_{n-2}$, $\varphi_{n-1}=(x_n^2-x_{n-1}^s)x_{n-1}$ $(s\geq 2)$

CASE C-3. $\varphi_{n-2}=x_nx_{n-2}$, $\varphi_{n-1}=x_nx_{n-1}$

Case C-4. $\varphi_{n-2}=x_nx_{n-2}$, $\varphi_{n-1}=(x_n+\psi(x_{n-2},x_{n-1},x_n))x_{n-1}$

where ψ $(\neq 0)\in \boldsymbol{C}[[x_{n-2},x_{n-1},x_n]]\cap m_p$.

CASE C-1. Since any nonreduced component of (l,p) is contained in B_{red}, B_{red} passes through p and (1.9.1) follows.

CASE C-2. If s (≥ 3) is odd, then $x_n^2-x_{n-1}^s$ is irreducible. Hence (l,p) has an irreducible component (C',p) besides (C,p) defined by

$$I_{C,p}=(x_1,\cdots,x_{n-2},x_n^2-x_{n-1}^s).$$

Since (C',p) is singular, (C',p) does not belong to the same algebraic family as a movable (C,p), whence (C',p) is contained in B_{red}. Thus (1.9.2) for s odd follows.

If $s=2q$ (≥ 4) is even, then we have, in addition to (C,p), two irreducible components (C',p) and (C'',p) of (l,p). Although (C,p) and (C',p) (or (C'',p)) intersect transversally, (C',p) and (C'',p) have a contact. That is, $I_{C',p}+I_{C'',p}\neq m_p$, where m_p is the maximal ideal of $O_{X,p}$. Hence none of (C',p) and (C'',p) belongs to the same algebraic family as (C,p), whence both (C',p) and (C'',p) are contained in B_{red}. Therefore (1.9.2) follows.

CASE C-3. In this case, B_{red} passes through p and (B,p) is a surface defined by $I_{B,p}=(x_1,\cdots,x_{n-3},x_n)$. Therefore (1.9.3) follows.

CASE C-4. By modifying x_n by a suitable unit, and by deleting some multiples of $x_1,\cdots,x_{n-3}$ from φ_j, we may assume

$$\varphi_{n-2}=x_nx_{n-2}, \qquad \varphi_{n-1}=(x_n+a(x_{n-2},x_{n-1}))x_{n-1}$$

for some $a(x_{n-2},x_{n-1})\in \boldsymbol{C}[[x_{n-2},x_{n-1}]]\cap m_p$. We have 3 components C_j $(0\leq j\leq 2)$, $C_0=C$ and the rest C' defined by

$$\begin{aligned}
&I_{C_0,p}=(x_1,\cdots,x_{n-3},x_{n-2},x_{n-1})\\
&I_{C_1,p}=(x_1,\cdots,x_{n-3},x_{n-2},x_n+a(0,x_{n-1}))\\
&I_{C_2,p}=(x_1,\cdots,x_{n-3},x_n,x_{n-1})\\
&I_{C',p}=(x_1,\cdots,x_{n-3},x_n,a(x_{n-2},x_{n-1})).
\end{aligned}$$

where C' can be reducible or nonreduced, and it may contain C_1 and C_2.

Let $C'_{\text{red}}=C_3+\cdots+C_m$ and let $C''=C_3+\cdots+C_d$ $(d\leq m)$ be the union of movable components of C'. Any movable component of (l,p) is algebraically equivalent to (C,p), any (C_i,p) $(3\leq i\leq d)$ is nonsingular, so that there exists $(a_i,b_i)\in \boldsymbol{C}^2\backslash(0,0)$ such that

$$I_{C_i,p}=(x_1,\cdots,x_{n-3},x_n,a_ix_{n-2}+b_ix_{n-1}+(\text{higher terms})).$$

We see $C_j\neq C_1,C_2$ for $3\leq j\leq d$ and that $\text{emb.dim}(C_2+C_j+C_k)=2$ for $3\leq j<k\leq m$.

Claim 1.9.5. *Let C_j be an irreducible component of C'. Then*

(1.9.5.1) $\text{emb.dim}(C_0+C_1+C_j)=2$ *if and only if* $C_j=C_1$.

(1.9.5.2) $\text{emb.dim}(C_0+C_2+C_j)=2$ *if and only if* $C_j=C_2$.

Proof. We may assume $n=3$ without loss of generality. We let $I_j:=I_{C_j}$ and $x:=x_1$, $y=x_2$, $z=x_3$. We let

$$I_0=(x,y), \qquad I_1=(x,z+a(x,y)), \qquad I_2=(y,z), \qquad I_j=(z,h(x,y))$$

where $a(x,y)$ is divisible by $h(x,y)$ in $O_{X,p}$.

First we prove (1.9.5.1). let $u:=z+a(x,y)$. Then we have $I_1=(x,u)$, $I_j=(u,h)$ and $I_0\cap I_1=(x,yu)$. Therefore emb.dim$(C_0+C_1+C_j)=2$ implies the existence of an element $b\in O_{X,p}$ such that $x+byu\in I_j$. Hence x is divisible by h in $O_{X,p}$, so that $h=x$ up to a unit multiple.Therefore $I_1=I_j$. The converse is obviously true.

(1.9.5.2) is proved similarly. In fact, $I_0\cap I_2=(y,xz)$. Hence emb.dim$(C_0+C_2+C_j)=2$ implies that y is divisible by h in $O_{X,p}$, so that $I_2=I_j$. q.e.d.

Now we go back to the proof of Lemma 1.9 CASE C-4.

CASE C-4-1. We consider the case where C_2 is a movable component of l. Assume $d\geq 3$ and $m\geq 4$. Any movable component of l belongs to one and the same algebraic family, whence emb.dim$(C_2+C_3+C_4)=$ emb.dim$(C_0+C_2+C_4)=3$ by (1.9.5.2), a contradiction. Hence $d\leq 3$. Moreover if $d=3$, then $m=3$ and $C'=C''=C_3$.

Assume $d=3$. We let $a(x_{n-2},x_{n-1})=a_3x_{n-2}+b_3x_{n-1}+$(higher terms). If $a_3=0$, then C_2 and C_3 have a contact, while if $b_3=0$, then C_1 and C_3 have a contact, which contradicts transversal intersection of C_0, C_1 and C_2 in either case. Hence $a_3\neq 0$, $b_3\neq 0$. We may choose $a_3=b_3=1$ and $a(x)=x_{n-2}+x_{n-1}$ by multiplying x_{n-2} and x_{n-1} by some units. This proves (1.9.4.2) in this case.

CASE C-4-2. We consider the case where C_2 is fixed. If $d\geq 4$, then emb.dim$(C_2+C_3+C_4)=$emb.dim$(C_0+C_2+C_3)=3$ by Claim 1.9.5, a contradiction. Hence we have $d\leq 3$. Therefore l has at most 3 movable irreducible components.

The remaining assertions of (1.9.4) are easy to prove. q.e.d.

Appendix. Local invariants $e(Q_V^{(\nu)},B_U)$

(1.10) NOTATION. Let C be an irreducible curve, ν: $\tilde{C}\to C$ the normalization, F a torsion $O_{\tilde{C}}$-module, p (resp. q) a point of C (resp. $\tilde{C}$). Then we define $e(F,q)$, $l(F,p)$ and $l(F)$ as follows,

$$e(F,q)=l(F_q)=\dim_{\mathbf{C}} F_q, \qquad l(F,p)=\sum_{q\,\text{above}\,p} l(F_q).$$

Then we recall

Theorem 1.11 [15, (2.6)].

(1.11.1) $l(Q'_C,p)=\sum_{q\,\text{above}\,p} e(Q'_C,q)$, $l(Q''_C,p)=\sum_{q\,\text{above}\,p} e(Q''_C,q)$ *for any* $p\in C$.

(1.11.2) *If* (C,p) *is irreducible, then* $e(Q''_C,q)\geq e(Q'_C,q)$ *for a unique point* q *above* p. *Equality holds if and only if* (C,p) *is nonsingular. If* (C,p) *is singular, then* $e(Q''_C,q)\geq e(Q'_C,q)+2$.

(1.11.3) *Under the same notation and assumptions in* (1.3.7), *let* q *be a unique point of the normalization* $\tilde{C}_\lambda$ *of* C_λ *above* p. *Then*

(1.11.3.1) $\quad e(Q''_C,q)\geq 1,\quad e(Q'_C,q)=0$ for $\lambda\in\Lambda_{ns}$,

(1.11.3.2) $\quad e(Q''_C,q)\geq e(Q''_\lambda,q)\geq e(Q'_C,q)+2$ for $\lambda\in\Lambda_s$,

(1.12) TORSION SHEAVES Q'_V AND Q''_V. Let Z be an irreducible reduced algebraic variety, $\nu\colon Y\to Z$ the normalization. Let $U=Y\backslash\mathrm{Sing}\,Y$, $V=\nu(U)$. Then we have an exact sequence

$$0\to Q''_V\to \nu^*\Omega^1_Z\otimes O_U\xrightarrow{\phi}\Omega^1_U\to Q'_V\to 0 \tag{1.12.1}$$

where $Q''_V:=\mathrm{Ker}\,\phi$ and $Q'_V:=\mathrm{Coker}\,\phi$. Now take an arbitrary prime Weil divisor B of Y (resp. $\bar{B}$ of Z) with $\nu(B)=\bar{B}$. We define $e(F,B)$ (resp. $e(F,\bar{B})$) to be the length of a torsion sheaf F at a generic point of B (resp. $\bar{B}$) as a $k(B)$-module (resp. as a $k(\bar{B})$-module).

Let $B_U:=B\cap U$ and $\bar{B}_V:=\bar{B}\cap V$. Then we have

$$e(Q'_V,B_U)=\inf_{C,q} e(Q'_C,q),\qquad e(Q''_V,B_U)=\inf_{C,q} e(Q''_C,q) \tag{1.12.2}$$

where p ranges over $\bar{B}_V$, C is a local curve of V intersecting $\bar{B}_V$ transversally at p, and q is a point of B_U above p.

By Theorem 1.11 we have

$$e(Q''_V,B_U)\geq e(Q'_V,B_U). \tag{1.12.3}$$

(1.13) A TORSION SHEAF Q_V. Let X be a smooth algebraic variety of dimension n, D_i a reduced irreducible divisor of X ($1\leq i\leq m$). Assume that the scheme-theoretic complete intersection $\tau=D_1\cap\cdots\cap D_m$ has an irreducible component $Z=Z_{\mathrm{red}}$ of dimension $n-m$ along which τ is reduced generically. Let $\nu\colon Y\to Z$ be the normalization of Z, $U=Y\backslash\mathrm{Sing}$

Y, and $V := \nu(U)$. Let I_{D_i} (resp. I) be the ideal sheaf of O_X defining D_i (resp. Z) and let $I_\tau = I_{D_1} + \cdots + I_{D_m}$. So we note $\sqrt{I_{D_i}} = I_{D_i}$ and $\sqrt{I} = I$.

Then we have an exact sequence

$$0 \to \bigoplus_{i=1}^{m} O_U(-\nu^* D_i) \to [\nu^*(I/I^2) \otimes O_U] \to Q_V \to 0 \tag{1.13.1}$$

where $[F \otimes O_U] := F/O_U$-torsions in F. If Z intersects one of the irreducible components of τ other than Z along a prime Weil divisor $\bar{B}$ of V, then

$$e(Q_V, B_U) \geq 1 \tag{1.13.2}$$

for any prime Weil divisor B above $\bar{B}$.

Moreover by (1.12.1) and (1.13.1), we have

Theorem 1.14 [15, (2.A)]. *Under the notation in* (1.12) *and* (1.13),*let* $i\colon U \to Y$ *be the inclusion map, and let* $\Delta := \sum_B (e(Q_V, B_U) + e(Q''_V, B_U) - e(Q'_V, B_U))B$. *Then* Δ *is an effective divisor of* Y *and we have*

$$K_Y := i_*(K_U) \simeq \nu^* K_X + \sum_{i=1}^{m} \nu^* C_i - \Delta.$$

(1.15) REMARK. If Z is singular along a prime Weil divisor $\bar{B}$, then by Theorem 1.11 $e(Q''_V, B_U) \geq e(Q'_V, B_U) + 1$ for any prime Weil divisor B of Y lying over $\bar{B}$. If Z intersects one of the irreducible components of τ other than Z along a prome Weil divisor $\bar{B}$, then by the definition $e(Q_V, B_U) \geq 1$ for any prime Weil divisor B lying over $\bar{B}$. Thus we see that supp $(\nu_* \Delta)$ is the union of all the Weil divisors of Z whose supports are contained in either Sing Z or one of the irreducible components of τ other than Z. See [15, (2.A)] for the detail.

2. A complete intersection l_V (2) —global structure—

Lemma 2.1. *Assume* $d = n$, *and* $h^0(X, L) \geq n$. *Let* l *be a scheme-theoretic complete intersection of* $(n-1)$*-members of* $|L|$ *and* $B := \mathrm{Bs}\ |L|$. *Assume that* l *has a reduced curve-component* C *outside* B *with* $LC \geq 1$. *Then one of the following cases occurs.*

(2.1.1) $LC = 2$, $C \simeq \boldsymbol{P}^1$, $N_{C/X} \simeq O_C(2)^{\oplus(n-1)}$, *$C$ is a connected component of* l, (2.1.2) *$LC=1$, $C\simeq \boldsymbol{P}^1$, $N_{C/X}\simeq O_C\oplus O_C(1)^{\oplus(n-2)}$, and C intersects B at a point p transversally, where*

$$I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C,p}=(x_1,\cdots,x_{n-2},x_{n-1}),$$
$$I_{B,p}=(x_1,\cdots,x_{n-2},x_n)$$

by choosing a suitable local coordinate $x_1,\cdots,x_n$ at p.

(2.1.3) *There is another component C_1 of such that $C_i\simeq \boldsymbol{P}^1$, $C=C_0$, $LC_i=1$, $N_{C_i/X}\simeq O_{C_i}\oplus O_{C_i}(1)^{\oplus(n-2)}$ $(i=0,1)$. The components C_0 and C_1 intersect transversally at a point p where*

$$I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C_0,p}=(x_1,\cdots,x_{n-2},x_{n-1}),$$
$$I_{C_1,p}=(x_1,\cdots,x_{n-2},x_n),$$
$$I_{B,p}=(x_1,\cdots,x_{n-2},x_{n-1},x_n)$$

in terms of suitable coordinates at p.

(2.1.4) *There is a chain of $m+1$ (≥ 2) smooth rational curves C_i $(0\leq i\leq m)$ such that*

$$C=C_0,\quad LC_0=LC_m=1,\quad LC_i=0\ (1\leq i\leq m-1)$$

$$N_{C_i/X}\simeq\begin{cases} O_{C_i}\oplus O_{C_i}(1)^{\oplus(n-2)} & (i=0,m)\\ O_{C_i}(-2)\oplus O_{C_i}^{\oplus(n-2)} \ \text{ or } \ O_{C_i}(-1)^{\oplus 2}\oplus O_{C_i}^{\oplus(n-3)} & (1\leq i\leq m-1).\end{cases}$$

The curves C_j and C_i $(j<i)$ intersect nowhere unless $j=i-1$, while C_{i-1} and C_i intersect transversally at a point p_i where

$$I_{l,p_i}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),$$
$$I_{C_{i-1},p_i}=(x_1,\cdots,x_{n-2},x_{n-1}),$$
$$I_{C_i,p_i}=(x_1,\cdots,x_{n-2},x_n)$$

in terms of suitable local coordinates at p_i. Moreover $C_0+\cdots+C_m$ is a connected component of l with $C_i\cap B_{\mathrm{red}}=\phi$ $(1\leq i\leq m-1)$.

NOTE. (2.1.1)–(2.1.3) are known to exist for (not necessarily complete) general linear systems. However there are no examples of (2.1.4) except for $m=1$ or 2, $n\geq 3$. We also note that (2.1.3) and (2.1.4) with $m=1$ are distinguished by the condition that $C_0\cap C_1$ is a base point or not.

Proof. By (1.3.6), we have $l(Q''_C)=l(Q'_C)$ or $l(Q''_C)\geq l(Q'_C)+2$. Hence there are two cases by (1.3.1).

CASE 1. $LC=2$, $C\simeq \boldsymbol{P}^1$, $l(Q_C)=l(Q''_C)=l(Q'_C)=0$.

CASE 2. $LC=1$, $\tilde{C}\simeq \boldsymbol{P}^1$, $l(Q_C)=1$, $l(Q''_C)=l(Q'_C)=0$.

CASE 1. In this case C is nonsingular by (1.3.5). ϕ_C in (1.2.1) is an isomorphism by $Q_C=0$, so that $I_C=I_l$ along C by Nakayama's lemma. This implies that C is a connected component of l. It is clear that $N_{C/X}=(I_C/I_C^2)^\vee \simeq O_C(2)^{\oplus(n-1)}$.

CASE 2. In this case C is nonsingular by (1.3.5). Consider the homomorphism ϕ_C

$$\phi_C: O_C(-L)^{\oplus(n-1)}(\simeq (I_l/I_l^2)\otimes O_C) \to I_C/I_C^2.$$

In view of $l(Q_C)=1$, there is a unique point p of C such that $l(\mathrm{Coker}\ \phi_{C,p})=1$. By Lemma 1.6, we can choose a local coordinate system $x_1,\cdots,x_n$ of X at p such that

$$I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n),\quad I_{C,p}=(x_1,\cdots,x_{n-1}).$$

It is easy to see $N_{C/X}\simeq O_C\oplus O_C(1)^{\oplus(n-2)}$. Therefore we have another irreducible component C_1 of l whose defining ideal $I_{C_1,p}$ is given by

$$I_{C_1,p}=(x_1,\cdots,x_{n-2},x_n).$$

In particular, l is generically reduced along C_1. Then there are two subcases $C_1\subset B_{\mathrm{red}}$ or $C_1\not\subset B_{\mathrm{red}}$.

CASE 2–1. If $C_1\subset B_{\mathrm{red}}$, then (2.1.2) is true.

CASE 2–2. If $C_1\not\subset B_{\mathrm{red}}$, then $LC_1\geq 0$. If $LC_1\geq 2$, then by CASE 1 above, we see that $LC_1=2$ and C_1 is a connected component of l, which is absurd. If $LC_1=1$, then $C_1\simeq \boldsymbol{P}^1$ and by the same argument as above, p is a unique point of C_1 such that $\mathrm{Coker}\,\phi_{C_1,p}\neq 0$. The union of C and C_1 is a connected component of l.

If $LC_1=0$, then $C_1\simeq \boldsymbol{P}^1$, $C_1\cap B_{\mathrm{red}}=\emptyset$ and $l(Q_{C_1})=2$, $l(Q''_C)=l(Q'_C)=0$ by (1.3.6). Hence there is another point p_2 of C_1 $(p_2\neq p)$ such that $\mathrm{Coker}\,\phi_{C_1,p_2}\neq 0$. In fact, by (1.3.1) we have $l(Q_{C_1,p})=l(Q_{C_1,p_2})=1$. Therefore by Lemma 1.6, we can choose a local coordinate system $y_1,\cdots,y_n$

at p_2 such that

$$I_{l,p_2}=(y_1,\cdots,y_{n-2},y_{n-1}y_n), \qquad I_{C_1,p_2}=(y_1,\cdots,y_{n-2},y_{n-1})$$

Hence we have the third reduced curve-component C_2 of l with $I_{C_2,p_2}=(y_1,\cdots,y_{n-2},y_n)$. Since $C_1\not\subset B_{\text{red}}$, we see $C_2\not\subset B_{\text{red}}$. As before $C_2\simeq \boldsymbol{P}^1$ and $LC_2=0$ or 1. If $LC_2=1$, then C_0 $(:=C)+C_1+C_2$ is a connected component of l.

If $LC_2=0$, then by repeating the same argument, we eventually obtain a chain of rational curves $C_0,C_1,\cdots,C_m$ with $LC_0=LC_m=1$, $LC_i=0$ $(1\le i\le m-1)$ such that C_{i-1} and C_i intersect transversally at a point p_i $(1\le i\le m)$, $p_1:=p$ and $C_0+C_1+\cdots+C_m$ is a connected component of l with $C_i\cap B_{\text{red}}=\emptyset$ $(1\le i\le m-1)$. We also see that

$$N_{C_i/X}\simeq O_{C_i}\oplus O_{C_i}(1)^{\oplus(n-2)} \qquad (i=0,m)$$

$$N_{C_i/X}\simeq O_{C_i}(-2)\oplus O_{C_i}^{\oplus(n-2)} \text{ or } O_{C_i}(-1)^{\oplus 2}\oplus O_{C_i}^{\oplus(n-3)} \qquad (1\le i\le m-1).$$

q.e.d.

Proposition 2.2. *Assume $h^0(X,L)\ge n$ and let l be a scheme-theoretic complete intersection of general $(n-1)$-members of $|L|$. Assume moreover that l has a reduced curve-component C not contained in B_{red} with $LC=0$. Then C is a nonsingular elliptic curve with $C\cap B_{\text{red}}=\emptyset$.*

Proof. If $C\cap B_{\text{red}}\ne\emptyset$, then C is contained in B_{red} by $LC=0$, which is absurd. If l is general enough, then by Bertini's theorem, Sing l is contained in B_{red}. Hence l is nonsingular along C, whence C is nonsingular and it is a connected component of l. q.e.d.

3. Moishezon manifolds with $c_1(X)=nc_1(L)$ and $b_2=1$ (1)

The purpose of this and the next sections is to prove:

Proposition 3.1. *Let X be a Moishezon manifold of dimension n (≥ 3) with $b_2=1$, and L a line bundle on X. Assume that $c_1(X)=nc_1(L)$ and $h^0(X,L)\ge n+1$. If a scheme-theoretic complete intersection l of general $(n-1)$-members of $|L|$ has an irreducible curve-component C with $LC\ge 2$ outside* Bs$|L|$, *then $X\simeq \boldsymbol{Q}^n$.*

In this section we prove Proposition 3.1 assuming $h^0(X,L)\ge n+2$. Our proof of Proposition 3.1 in this section is completed in (3.8). In the next section we disprove $h^0(X,L)=n+1$.

Lemma 3.2. *Let $m=h^0(X,L)-1$ $(\geq n+1)$, and let $B:=\mathrm{Bs}\,|L|$ be the scheme-theoretic base locus of $|L|$, and h: $X\to \boldsymbol{P}^m$ the rational map associated with $|L|$. Then $m=n+1$ and h is a birational map of X onto a hyperquadric W in $\boldsymbol{P}^{n+1}$.*

Proof. STEP 1. Let W be the closure of $h(X\backslash B)$, and $d=\deg W$. Then $d\geq m+1-\dim W$. By the assumption $\dim W\geq n-1$. Hence by choosing general $(n-2)$-members $D_i\in|L|$ $(1\leq i\leq n-2)$, we have reduced irreducible components Z_i $(1\leq i\leq e)$ of $\tau:=D_1\cap\cdots\cap D_{n-2}$ outside B. Each Z_i is nonsingular outside B by Bertini's theorem. Let ν_i: $Y_i\to Z_i$ be the normalization of Z_i, f_i: $S_i\to Y_i$ the minimal resolution of Y_i, $g_i=\nu_i\cdot f_i$. Let $Z=Z_1$, $Y=Y_1$, $S=S_1$, $f=f_1$, $\nu=\nu_1$, and $g=g_1$. Then there exist by Theorem 1.14 an effective Weil divisor $\triangle$ on Y, effective Cartier divisors E and G on S with no components in common such that the canonical sheaves K_Y and K_S are given by

$$K_Y=O_Y(\nu^*(K_X+(n-2)L)-\triangle),\qquad K_S=O_S(g^*(K_X+(n-2)L)-E-G)$$

where $f_*(E)=\triangle$, $f_*(G)=0$ and E is finite over $\triangle$. Let $\Sigma:=f^{-1}(\triangle)\cup g^{-1}(\mathrm{Sing}\,Z)$. Then Σ contains $supp(E+G)$ and $g_{|S\backslash\Sigma}$ is an isomorphism. We also note that the base locus $\mathrm{Bs}\,g^*|L|$ contains $supp(E+G)$ if D_i's are sufficiently general. Since $h^0(X,L)\geq n$ and $Z\not\subset B$, g^*L is effective. Since $c_1(S)=2c_1(g^*L)+c_1(E+G)$ and S is projective, we have $P_m(S)=0$. Therefore $S\simeq\boldsymbol{P}^2$ or S is ruled, that is, S has a morphism onto an algebraic curve with general fiber $\simeq\boldsymbol{P}^1$. Since any Z_i is algebraically equivalent to each other, $S_i\simeq\boldsymbol{P}^2$ for any i or S_i is reled for any i.

STEP 2. Assume $S\simeq\boldsymbol{P}^2$. Then we have $G=0$ and $S\simeq Y$. Let $H\in g^*|L|$. Then $K_S=-2H-E$. Since $K_{\boldsymbol{P}^2}$ is indivisible by 2, we have $E\neq 0$ and $H=E\in|O_{\boldsymbol{P}^2}(1)|$ in view of $E_{\mathrm{red}}\subset H_{\mathrm{red}}$. This shows that $(D_1\cap\cdots\cap D_{n-1})_{\mathrm{red}}\subset B$ for any D_{n-1}, which contradicts the assumption that general l contains a curve-component outside B.

STEP 3. By STEP 2, S has a morphism π: $S\to T$ onto an algebraic curve T with F $(\simeq\boldsymbol{P}^1)$ a general fiber of π. Let $H\in g^*|L|$, and let M (resp. N) be the movable part (resp. the fixed part) of H in $g^*|L|$. Since $F\simeq\boldsymbol{P}^1$, we have

$$-2=K_SF+F^2=K_SF=-2HF-(E+G)F.$$

Since $E_{\mathrm{red}}+G_{\mathrm{red}}\subset H_{\mathrm{red}}$, we have $HF=1$, $EF=GF=0$. Therefore there exists a unique irreducible component Γ of H with $\Gamma F=1$ and $\Gamma\not\subset E+G$. If $\Gamma\subset N$, then $MF=0$, whence M is a sum of general fibers. Choose $D_{n-1} \in |L|$ such that $H = g^* D_{n-1}$. Then $g(F)$ is a general movable component of $l := D_1 \cap \cdots \cap D_{n-1}$ with $(Lg_*(F))_X = (g^* LF)_S = 1$. This contradicts the assumption that there exists a component C of l with $LC \geq 2$. Hence $\Gamma \subset M$. Consequently $\Gamma^2 \geq 0$, and $M = \Gamma$, $NF = HF - MF = 0$.

STEP 4. Assume $\dim W = n-1$. Then τ is smooth and irreducible outside B for general D_i by Bertini's theorem. Since M is irreducible, we have $\deg W = d = 1 \geq m-n+2 \geq 3$, a contradiction. Hence $\dim W = n$. Moreover $\Gamma^2 > 0$. In fact, if $\Gamma^2 = 0$, then we have $\Gamma \cap \Gamma' = \emptyset$ for any general $\Gamma' \in |\Gamma|$, whence $Z \backslash B$ is mapped onto a curve by h. Hence $\dim W = n-1$, a contradiction.

We also have,

$$K_S \Gamma + \Gamma^2 = -\Gamma^2 - (2N + E + G)\Gamma \leq -1.$$

Therefore the inclusion $E_{\text{red}} + G_{\text{red}} \subset N_{\text{red}}$ shows that $\Gamma \simeq \boldsymbol{P}^1$, $\Gamma^2 = 2$, $K_S \Gamma = -4$, $N\Gamma = E\Gamma = G\Gamma = 0$ and $g^*(L)\Gamma = H\Gamma = (\Gamma + N)\Gamma = 2$.

Since $\dim W = n$, $C := g(\Gamma)$ is an irreducible component of a general complete intersection $l := D_1 \cap \cdots \cap D_{n-1}$ outside B. Clearly $(LC)_X = (g^*(L)\Gamma)_S = 2$, while we have an obvious relation

$$(g^*(L)\Gamma)_S = \deg(h \cdot g)_{|\Gamma} \deg W + \deg \operatorname{Bs} g^*|L|_\Gamma.$$

Since $g^*(L)\Gamma \geq d \geq m-n+1 \geq 2$ by the assumption $m \geq n+1$, we have $d = \deg W = 2$, $\deg(h \cdot g)_{|\Gamma} = 1$, $m = n+1$ and $\operatorname{Bs} g^*|L|_\Gamma = \emptyset$.

STEP 5. STEP 4 shows that C $(= g(\Gamma))$ is the unique irreducible component of l outside B. $h(C)$ is an irreducible plane conic. Therefore $h(C) \simeq \boldsymbol{P}^1$, $C \simeq \boldsymbol{P}^1$, $\deg h(C) = 2$ and $|L|_C = |L_C|$. Moreover $\deg(h_{|C}) = 1$ and $\operatorname{Bs}|L|_C = \emptyset$ are clear from STEP 4. For D_i general, we have $\deg h = \deg(h_{|C}) = 1$. This completes the proof of Lemma 3.2. q.e.d.

Lemma 3.3. $\operatorname{Pic} X \simeq \boldsymbol{Z}L$.

Proof. First we prove $H^1(X, O_X) = 0$. Assume the contrary. Then since X is Moishezon, we have a nontrivial Albanese map alb: $X \to Alb(X)$ where $Alb(X)$ is projective. Since $b_2 = 1$, some multiple of L is a multiple of the pull back of an ample line bundle on $Alb(X)$. Therefore the morphism alb is generically finite, whence we have a nontrivial holomorphic two form on X. This contradicts $b_2 = 1$.

Now we consider an exact sequence

$$0 \to H^1(X,O_X)(=0) \to H^1(X,O_X^*) \xrightarrow{c_1} H^2(X,\boldsymbol{Z}) \to H^2(X,O_X).$$

Since $b_2=1$, $\mathrm{Coker}\, c_1$ is finite. As $H^2(X,O_X)$ is a $\boldsymbol{C}$-vector space, it has no torsions. Hence $\mathrm{Coker}\, c_1=0$. Therefore $\mathrm{Pic}\, X:=H^1(X,O_X^*)\simeq H^2(X,\boldsymbol{Z})$.

Next we prove $\mathrm{Tor}\, H^2(X,\boldsymbol{Z})=0$. Assume the contrary. Then we have an unramified covering ρ: $\tilde{X}\to X$ such that $\deg\rho=\sharp\mathrm{Tor}\, H^2(X,\boldsymbol{Z})\geq 2$ and $\rho^*\mathrm{Tor}\, H^2(X,\boldsymbol{Z})=0$. Letting $\tilde{L}:=\rho^*L$, we have $c_1(\tilde{X})=nc_1(\tilde{L})$, and $h^0(\tilde{X},\tilde{L})\geq n+2$. By Lemma 3.2, we have $\dim W=n$, $\deg W=2$, $\deg h=1$ and $h^0(X,L)=n+2$. Its proof makes no use of $b_2=1$, whence we have the same conclusion for $\tilde{X}$. Let $\tilde{h}$ be the rational map associated with $|\tilde{L}|$, $\tilde{B}:=\rho^{-1}(B)$, and $\tilde{W}$ the closure of $\tilde{h}(\tilde{X}\backslash\tilde{B})$. Then we have $\dim\tilde{W}=n$, $\deg\tilde{W}=2$, $\deg\tilde{h}=1$, and $h^0(\tilde{X},\tilde{L})=n+2$. It follows that $h^0(\tilde{X},\tilde{L})=h^0(X,L)$, $\tilde{h}=h\cdot\rho$, $\tilde{W}\simeq W$ and $\deg\rho=1$, This is a contradiction. Hence $\mathrm{Tor}\, H^2(X,\boldsymbol{Z})=0$, and $\mathrm{Pic}\, X\simeq H^2(X,\boldsymbol{Z})\simeq\boldsymbol{Z}$.

Finally we prove $\mathrm{Pic}\, X\simeq\boldsymbol{Z}L$. Choose a generator L_0 of $\mathrm{Pic}\, X$ so that $L=aL_0$ for some integer $a\geq 1$. Now we recall the proof of Lemma 3.2. Under the same notation as in Lemma 3.2, the surface S is ruled with F a fiber. Then $(g^*LF)_S=(HF)_S=1$. We have $(g^*LF)_S=a(g^*L_0F)_S$, whence $a=1$ and $\mathrm{Pic}\, X\simeq\boldsymbol{Z}L$. q.e.d.

Corollary 3.4. *Let W be the closure of $h(X\backslash B)$. Then W is a normal hyperquadric with Hessian-rank ≥ 5.*

Proof. If Hessian-rank $W\leq 4$, then W has a reducible or nonreduced hyperplane section, which contradicts Lemma 3.3. q.e.d.

(3.5) NOTATION. Let $\hat{X}$ be the normalization of the closure in $X\times W$ of the graph of h, $\hat{h}$: $\hat{X}\to W$ and $\varphi:\hat{X}\to X$ the natural morphisms. Let $\hat{B}=\varphi^{-1}(B)$ and $\hat{B}^*$ be the minimal subvariety of $\hat{X}$ containing $\hat{B}$ such that $\hat{h}$ is unramified on $\hat{X}\backslash\hat{B}^*$. Let $B^*=\varphi(\hat{B}^*)$ and $R^*=\hat{h}(\hat{B}^*)$. We note that $X\backslash B\simeq\hat{X}\backslash\hat{B}$, $X\backslash B^*\simeq\hat{X}\backslash\hat{B}^*\simeq W\backslash R^*$ and therefore $\hat{B}^*=\varphi^{-1}(B^*)=\hat{h}^{-1}(R^*)$.

Lemma 3.6. $\hat{B}^*=\hat{h}^{-1}(\mathrm{Sing}\, W)\cup\hat{B}$.

Proof. Let $\mathscr{H}=H^0(X,L)$. It is clear from $\hat{X}\backslash\hat{B}^*\simeq W\backslash R^*$ that $\mathrm{Sing}\, W\subset R^*$ and $\hat{h}^{-1}(\mathrm{Sing}\, W)\subset\hat{B}^*$. Assume that there exists a point $\hat{p}\in\hat{B}^*\backslash(\hat{h}^{-1}(\mathrm{Sing}\, W)\cup\hat{B})$. Then $q:=\hat{h}(\hat{p})$ is a smooth point of W and $\hat{h}^{-1}(q)$ is a connected subset of $\hat{X}$ with $\dim\hat{h}^{-1}(q)>0$. Let $p:=\varphi(\hat{p})\in B^*\backslash B$. Then it follows that $q=\hat{h}(\hat{p})=h(p)$. We infer from $\hat{X}\backslash\hat{B}\simeq X\backslash B$ that

$\dim h^{-1}(q)>0$ and that there exists a subset B' of B such that $\varphi(\hat{h}^{-1}(q))=h^{-1}(q)\cup B'$. Since $\hat{h}^{-1}(q)$ is connected, so is $\varphi(\hat{h}^{-1}(q))$.

Since q is a smooth point of W, we can choose a smooth conic l^* on W which is not contained in R^* and passes through q. Let $V=V(l^*)\in \mathrm{Grass}(n-1,\mathscr{H})$ be a linear subspace of $\mathscr{H}$ corresponding to l^* with $\dim V=n-1$. Let l_V be the subvariety of X defined by $I_{l_V}=\Sigma_{s\in V}sO_X$. Then l_V passes through p, and it is one-dimensional and nonsingular outside B^*. Let C be the unique irreducible component of l_V outside B such that $\overline{h(C\backslash B)}=l^*$. Since $LC=\deg(h_{|C})\deg W+\deg \mathrm{Bs}\,|L|_C$, we have $LC\geq 2$, whence by Lemma 2.1 $C\simeq \boldsymbol{P}^1$, $LC=2$ and that C is a connected component of l_V. Let $\hat{C}$ be the proper transform of C by φ^{-1}. Then since l^* passes through q and since $\hat{h}^{-1}(q)$ is connected, the union $\hat{C}\cup\hat{h}^{-1}(q)$ is a connected subset of $\hat{X}$, whence the union $\varphi(\hat{C})\cup\varphi(\hat{h}^{-1}(q))=C\cup h^{-1}(q)\cup B'$ is a connected subset of l_V. This contradicts that C is a connected component of l_V. q.e.d.

Lemma 3.7. $\hat{B}^*=\hat{h}^{-1}(\mathrm{Sing}\,W)$, $R^*=\mathrm{Sing}\,W$, $\hat{B}=\emptyset$, $B=\emptyset$ and $\hat{X}\simeq X$.

Proof. Assume $\hat{B}^*\neq\hat{h}^{-1}(\mathrm{Sing}\,W)$. Then $R^*\neq\mathrm{Sing}\,W$. Then we can choose a smooth conic l^* on W which is not contained in R^* and meets $R^*\backslash\mathrm{Sing}\,W$. Hence we can choose $V'\in\mathrm{Grass}(n-1,\mathscr{H})$ such that $l_{V'}$ is pure one dimensional and irreducible nonsingular outside B^* and $\overline{h(l_{V'}\backslash B^*)}=l^*$. Let q be a point of $(l^*\cap R^*)\backslash\mathrm{Sing}\,W$, C the unique irreducible component of $l_{V'}$ with $\overline{h(C\backslash B^*)}=l^*$. Note that $\hat{h}^{-1}(q)\subset\hat{B}$ by Lemma 3.6, whence $\varphi(\hat{h}^{-1}(q))\subset B$. Let $\hat{C}$ be the proper transform of C by φ^{-1}. Then $\hat{C}\cup\hat{h}^{-1}(q)$ is a connected subset of $\hat{X}$, whence $C\cup\varphi(\hat{h}^{-1}(q))$ is a connected subset of $l_{V'}$. Since $\varphi(\hat{h}^{-1}(q))\subset B$, this shows that $B\cap C\neq\emptyset$. Since $\overline{h(C\backslash B^*)}=l^*$, we have $LC\geq 2$. By Lemma 2.1, we have $LC=2$, and $B\cap C=\mathrm{Bs}\,|L|_C=\emptyset$, which is a contradiction. Hence $\hat{B}^*=\hat{h}^{-1}(\mathrm{Sing}\,W)\supset\hat{B}$ and $R^*=\mathrm{Sing}\,W$.

Next we prove $\hat{B}=\emptyset$. Let $\hat{p}\in\hat{B}$, and $q=\hat{h}(\hat{p})$. Since $\hat{B}\subset\hat{h}^{-1}(\mathrm{Sing}\,W)$, q is a singular point of W. A general (singular) conic l^* on W passing through q is a union of two lines. As before we choose $V\in\mathrm{Grass}(n-1,\mathscr{H})$ with $\overline{h(l_V\backslash B^*)}=l^*$. Let $\sigma:=C_0+C_m$ be a minimal subcurve of l_V with $\overline{h(\sigma\backslash B^*)}=l^*$. We notice that Lemma 2.1 is true if we only assume that C is a reduced curve-component of l with $LC\geq 1$. We have $LC_i=1$ and $C_i\cap B=\emptyset$ for $i=0,m$. Hence only (2.1.4) is possible. By (2.1.4), the connected component of l_V containing C_0 is a reducible curve $C_0+\cdots+C_m$ disjoint from B with $LC_i=0$ $(1\leq i\leq m-1)$. In the same manner as above we see that $\sigma\cup\varphi(\hat{h}^{-1}(q))$ is a connected subset of l_V intersecting B, which contradicts that $C_0+\cdots+C_m$ is a connected component of l_V disjoint from B. Hence $\hat{B}=\emptyset$, $B=\emptyset$ and $\hat{X}\simeq X$. q.e.d.

(3.8) PROOF OF PROPOSITION 3.1 UNDER THE ASSUMPTION $h^0(X,L)\geq n+2$. The birational map h is defined everywhere by Lemma 3.7. It is easy to see by using $c_1(X)=nc_1(L)$ that $X\simeq W$ if W is smooth. So we consider the case where W is singular. We prove that *this case is impossible.* We recall Hessian-rank $W\geq 5$. We note that a complete intersection of general $(n-2)$ hyperplane sections of W passing through a singular point of W is a singular quadric surface Q $(\simeq\bar{\boldsymbol{F}}_2:=\boldsymbol{P}(O_{\boldsymbol{P}^1}\oplus O_{\boldsymbol{P}^1}(2))$ with (-2)-curve contracted).

Let $q\in W$ be a singular point of W. Let m_q be the maximal ideal of O_W defining q, $\Lambda:=h^*|m_qO_W(1)|$, and let $D_i\in\Lambda$ $(1\leq i\leq n-1)$ be general members. Let $\tau=D_1\cap\cdots\cap D_{n-2}$ be a scheme-theoretic intersection of D_i, Z the unique irreducible component of τ mapped onto a singular quadric surface Q passing through q, where q is the unique singular point of Q. Keeping the same notation S, Y, g, E and G as in Lemma 3.2, we let $H=g^*(D_{n-1})$, and M (resp. N) the movable part (resp. the fixed part) of H for $D_{n-1}\in\Lambda$. We see $S\not\simeq\boldsymbol{P}^2$ as in Lemma 3.2 CASE 1. We note that $\dim h^{-1}(q)\geq 1$.

Hence S is a ruled surface with F a general fiber of the ruling π. It follows that

$$2=-K_SF=(2H+E+G)F.$$

Since $E_{\mathrm{red}}+G_{\mathrm{red}}\subset H_{\mathrm{red}}$, we see that $HF=1$, $EF=GF=0$ and that there is a unique irreducible component Γ of H with $\Gamma F=1$. Assume that $\Gamma\subset M$. Then $M=\Gamma$ for general D_{n-1} because $MF=\Gamma F=1$. However $g(M)$ is by Lemma 3.7 a complete intersection of $(n-1)$-hyperplane sections of W passing through the point q. Hence it is a singular conic, that is, a union of two lines, which contradicts the irreducibility of M. Hence $\Gamma\subset N$. Since $B=\emptyset, l:=D_1\cap\cdots\cap D_{n-1}$ has by Lemma 2.1 a connected component $C_0+\cdots+C_m$ of type (2.1.4) for general D_i, where $g(C_i)$ $(i=0,m)$ is a line passing through q. Note that Lemma 2.1 is true only if C (C_0 or C_m in this case) is a reduced curve component of l but even if D_i is not general. Since $g(C_0)$ and $g(C_m)$ are algebraically equivalent as lines on W passing through q, C_0 and C_m intersect the same irreducible component, say C_1, of l for general D_{n-1}. Hence $m=2$. Since $B=\emptyset$ and h is birational, l is connected so that $l_{\mathrm{red}}=C_0+C_2+h^{-1}(q)_{\mathrm{red}}$. As $h(C_1)=q$ and $h^{-1}(q)$ is a connected subset of l, we have

$$l\simeq C_0+C_1+C_2,\quad C_0+C_2\simeq M,\quad h^{-1}(q)_{\mathrm{red}}\simeq C_1\simeq\Gamma\simeq\boldsymbol{P}^1.$$

In particular, $h^{-1}(q)\subset Z$. Since $\tau=h^{-1}(q)\cup Z$ by the choice of Q, this shows that τ is irreducible and pure two-dimensional, hence Gorenstein. Since it is generically reduced, it is reduced everywhere, whence $Z\simeq\tau$.

Moreover by (2.1.4) we have $I_{l,p}=(x_1,\cdots,x_{n-2},x_{n-1}x_n)$ at any point p of C_1, whence we may assume $I_{\tau,p}=(x_1,\cdots,x_{n-2})$. Hence τ is smooth everywhere along $h^{-1}(q)$. Since $\tau\backslash h^{-1}(q)\simeq Q\backslash\{q\}$, we have $\mathrm{Sing}\,\tau\subset h^{-1}(q)$. Thus τ is smooth everywhere, so that $S\simeq Y\simeq Z\simeq\tau$ and $E=G=0$. Hence we have $H\Gamma=g^*(L)\Gamma=0$, $K_S\Gamma=0$ and $\Gamma^2=-2$. We also have $HF=1$, $(H^2)_S=(L_Z^2)_Z=((h^*O_W(1))^n)_X=2$. Since $K_S=-2H$ and $H^2=2$, S is relatively minimal. Hence $\Gamma^2=-2$ implies that $S\simeq \boldsymbol{P}(\pi_*H)\simeq \boldsymbol{F}_2$. We also see that $h^{-1}(\mathrm{Sing}\,W)$ is a $\boldsymbol{P}^1$-bundle over $\mathrm{Sing}\,W$. We note that $\dim\mathrm{Sing}\,W=n+1-\text{Hessian-rank}\,W\leq n-4$.

Let $C=h^{-1}(q)$ for some $q\in\mathrm{Sing}\,W$. Then by (2.1.4) we have

$$\chi(C,N_{C/X})=n-1+\deg N_{C/X}=n-3>\dim\mathrm{Sing}\,W.$$

This shows that there exists an (at least) $(n-3)$-dimensional family of displacements $C(t)$ of C in X [9, Proposition 3]. Since $LC(t)=LC=0$, $h(C(t))$ is a point, so that any general $C(t)$ is not contained in $h^{-1}(\mathrm{Sing}\,W)$. However by Lemma 3.7 h is an isomorphism outside $h^{-1}(\mathrm{Sing}\,W)$, a contradiction. Thus it is impossible that W is singular. q.e.d.

4. Moishezon manifolds with $c_1(X)=nc_1(L)$ and $b_2=1$ (2)

The purpose of this section is to complete our proof of Proposition 3.1. In this section, we disprove the possibility of $h^0(X,L)=n+1$.

(4.1) NOTATION. In this section we always assume $h^0(X,L)=n+1$. We let $B:=\mathrm{Bs}\,|L|$ (resp. $\mathrm{Bs}\,|L|_C$) be the scheme-theoretic base locus of $|L|$ (resp. that of the restriction $|L|_C$ of $|L|$ to C), and $h\colon X\to\boldsymbol{P}^n$ the rational map associated with $|L|$, W the closure of $h(X\backslash B)$. We notice that the same argument as in Lemma 3.2 shows $\dim W=n$, that is, $W\simeq\boldsymbol{P}^n$. We define $\hat{X}$, $\hat{h}\colon\hat{X}\to W$, $\varphi\colon\hat{X}\to X$, $\hat{B}=\varphi^{-1}(B)$, $\hat{B}^*\subset\hat{X}$, $B^*=\varphi(\hat{B}^*)$ and $R^*=\hat{h}(\hat{B}^*)$ in the same manner as in Lemma 3.6. Then we have $\hat{B}^*=\varphi^{-1}(B^*)=\hat{h}^{-1}(R^*)$, $X\backslash B\simeq\hat{X}\backslash\hat{B}$ and $X\backslash B^*\simeq\hat{X}\backslash\hat{B}^*\simeq W\backslash R^*$.

Let l^* be a line on W not contained in R^*, $\hat{l}(l^*):=\hat{h}^{-1}(l^*)$, $\hat{C}(l^*)$ an irreducible component of $\hat{h}^{-1}(l^*)$ mapped onto l^*, $C(l^*):=\varphi(\hat{C}(l^*))$, $\sigma(l^*):=\varphi(\hat{h}^{-1}(l^*))$. Let $l(l^*)$ be a complete intersection of $(n-1)$-members of $|L|$ corresponding to l^*. We keep the same notation in Lemmas 4.2–4.3.

Lemma 4.2. *Under the notation in* (4.1), *we have* $W\simeq\boldsymbol{P}^n$ *and* $C(l^*)\simeq\boldsymbol{P}^1$.

(4.2.1) *$l(l^*)$ is irreducible outside B for general l^*.*

(4.2.2) *$C:=C(l^*)$ is the unique irreducible component of $l(l^*)$ outside B for general l^*. Then one of the following is true.*

(4.2.2.1) $\deg(h_{|C})=1,\quad LC=2,\quad \deg \mathrm{Bs}\,|L|_C=1.$

(4.2.2.2) $\deg(h_{|C})=2,\quad LC=2,\quad \deg \mathrm{Bs}\,|L|_C=0.$

(4.2.3) *Let C' be an irreducible component of $l(l^*)$ outside B. If $LC'=1$, then $\mathrm{Bs}\,|L|_{C'}=\emptyset$ (and l^* is not general).*

Proof. We note that $C(l^*)\simeq \boldsymbol{P}^1$ by Lemma 2.1. (4.2.1) is clear from Bertini's theorem. Next we prove (4.2.2). Let $C:=C(l^*)$. The first assertion is clear. We prove $LC=2$. If $LC=1$, then by the equality

$$LC=\deg(h_{|C})\deg W+\deg \mathrm{Bs}\,|L|_C,$$

we have $\deg(h_{|C})=1$ and $\mathrm{Bs}\,|L|_C=\emptyset$. The complete intersection $l(l^*)$ is therefore smooth along C, so that $l(l^*)\simeq C$ along C, which contradicts Lemma 2.1. Hence $LC=2$. It follows that $2=\deg(h_{|C})+\deg \mathrm{Bs}\,|L|_C$. The rest is clear. q.e.d.

In view of (4.2.2), we have $\deg h=\deg(h_{|C(l^*)})$ for general l^*. We disprove both the possibilities $\deg h=1$ and 2 respectively in Lemma 4.3 and Lemma 4.4.

Lemma 4.3. $\deg h=1$ *is impossible.*

We prove Lemma 4.3 in Claims 4.3.1–4.3.7 and (4.3.8).
Assume $\deg h=1$. Keeping the notation in (4.1), we first prove

Claim 4.3.1. *R^* is a hyperplane of W ($\simeq \boldsymbol{P}^n$).*

Proof. We keep the notation in (4.1) and Lemma 4.2. Assume first $\dim R^*\le n-2$. Then there is a general line l^* on W such that $l^*\cap R^*=\emptyset$ by Lemma 4.2. Let $C=C(l^*)$. Hence $\hat{h}^{-1}(l^*)\cap \hat{B}^*=\emptyset$, whence $B\cap C=\emptyset$. However we have $\deg \mathrm{Bs}\,|L|_C=LC-\deg h=1$, a contradiction. Hence there is an irreducible component of R^* of dimension $n-1$. Assume that R^* is not a hyperplane. Then R^* has another irreducible component or R^* is a hypersurface of degree greater than one. Therefore there is a line l^* of W such that $l^*\cap R^*$ contains two points q_i $(i=1,2)$. Then since $\hat{h}^{-1}(l^*)$ is a connected set containing $\hat{C}(l^*)$, $\varphi(\hat{h}^{-1}(l^*))$ is a connected subset of $l(l^*)$ containing C. Since C is a connected component of $l(l^*)$ by Lemma 2.1, we have $C=\varphi(\hat{h}^{-1}(l^*))$. Hence $p_i:=\varphi(\hat{h}^{-1}(q_i))$ is a point of C. If $p_i\notin B$, then $\hat{h}^{-1}(q_i)\cap \hat{B}=\emptyset$ so that $\hat{h}^{-1}(q_i)$ is a point $\hat{p}_i$ of $\hat{X}$ by the isomorphism $\hat{X}\backslash\hat{B}\simeq X\backslash B$. Since $\hat{h}$ is birational, $\hat{h}$ is unramified at $\hat{p}_i$, whence $\hat{p}_i\notin\hat{B}^*$ and $q_i\notin R^*$, a contradiction. Hence $p_i\in B$ $(i=1,2)$. However since $\deg(\hat{h}_{|\hat{C}})=\deg(h_{|C})=1$, we have $p_1\neq p_2$, which contradicts $\deg \mathrm{Bs}\,|L|_C=1$. Consequently R^* is a hyperplane. q.e.d.

Claim 4.3.2. *There exists a point $p_0\in X$ such that $B=\{p_0\}$, $\hat{X}\simeq Q_{p_0}(X)$ and $\hat{B}=\varphi^{-1}(p_0)\simeq R^*$.*

Proof. We first take and fix a point q_∞ of $W\backslash R^*$. Let $p_\infty:=h^{-1}(q_\infty)$ and $\hat{p}_\infty:=\hat{h}^{-1}(q_\infty)$. For any point $q\in R^*$ there exists a unique line l_q^* of W connecting q_∞ and q. Let $l_q:=l(l_q^*)$, $\hat{l}_q:=\hat{h}^{-1}(l_q^*)$, $C_q:=C(l_q^*)$ and $\sigma_q:=\varphi(\hat{l}_q)$. Since σ_q is a connected subset of l_q, we have by Lemma 2.1 $\sigma_q=C_q\simeq \boldsymbol{P}^1$ for general $q\in R^*$. Take a general point q_0 of R^*. Then by (4.2.1) and by the same argument as in Claim 4.3.1 $p_0:=\varphi(\hat{h}^{-1}(q_0))$ is a point of C_{q_0} with $\{p_0\}=\mathrm{Bs}\,|L|_{C_{q_0}}$ scheme-theoretically. Since C_{q_0} is a connected component of l_{q_0} by (2.1.1), p_0 is an isolated point of B so that $p_0\in\sigma_q$ for any q, and $\{p_0\}=\mathrm{Bs}\,|L|_{C_q}$ for general $q\in R^*$.

Next we prove that σ_q is smooth at p_0 for any $q\in R^*$. In fact, σ_q has a reduced curve component C_q. If $LC_q=2$, then $\sigma_q\simeq l_q\simeq C_q$ along C_q by Lemma 2.1, whence σ_q is smooth at p_0. If $LC_q=1$, then $B\cap C_q=\emptyset$ by (4.2.3), whence l_q along C_q is of type (2.1.4). Hence there exists a unique irreducible component C'_q of σ_q with $LC'_q=1$ containing p_0, where σ_q is smooth. Thus it turns out that σ_q is of type either (2.1.1) or (2.1.4) for any $q\in R^*$. We also see that σ_q is a connected component of l_q containing C_q. Moreover $\sigma_q\cap B'=\emptyset$ for any $q\in R^*$ where $B':=B\backslash\{p_0\}$.

Since $\mathrm{Bs}\,|L|_{C_q}=\{p_0\}$ for general q, there exist (general) n-members $D_1,\cdots,D_n$ of $|L|$ and a closed subset A of X such that $D_1\cap\cdots\cap D_n=p_0+A$ and $p_0\notin A$. Hence n equations defining D_i form a local coordinate at p_0 so that after blowing up X at p_0 we have a rational map of $Q_{p_0}(X)$ onto W induced from h, which is a morphism near the exceptional set $E:=Q_{p_0}(p_0)\simeq\boldsymbol{P}^{n-1}$. It follows that $\hat{X}\simeq Q_{p_0}(X)$ near E. In what follows we view E as a divisor of $\hat{X}$ by the above isomorphism. Then $E=\varphi^{-1}(p_0)$.

Next we prove that $\hat{h}_{|E}$ is an isomorphism of E onto R^*. In fact, since σ_q is smooth at p_0, $\hat{l}_q$ intersects E at a unique point $\hat{p}(q)$ with $\varphi(\hat{p}(q))=p_0$. Since R^* is normal, this defines a morphism $\hat{p}$: $R^*\to E$ such that $\hat{h}\cdot\hat{p}=\mathrm{id}_{R^*}$. This shows that $\hat{h}_{|E}$ is an isomorphism.

Finally we prove that $\hat{B}=E$. Assume the contrary. We define a closed subset $\hat{B}'$ of $\hat{X}$ by $\hat{B}':=\varphi^{-1}(B')$. As $\hat{h}(\hat{B}^*)=R^*$, $\hat{h}(\hat{B}')$ is a subset of R^*, from which we choose a point q. Since $\hat{h}$ is birational, $\hat{l}_q$ is a connected subset of $\hat{X}$ intersecting $\hat{B}'$. Therefore $\sigma_q\cap B'\neq\emptyset$, a contradiction. Hence $\hat{B}=E$, whence $B=\{p_0\}$. Consequently $\hat{X}\simeq Q_{p_0}(X)$. q.e.d.

Claim 4.3.3. *Let P be a general plane of W passing through the point q_∞ in Claim* 4.3.2, *and $Z(P):=\varphi(\hat{h}^{-1}(P))$. Then $Z(P)\simeq \boldsymbol{F}_0$ or $\boldsymbol{F}_2$.*

Proof. Let $Z=Z(P)$. First we note that $Z\setminus\{p_0\}$ is smooth by Bertini's theorem. As was shown in the proof of Claim 4.3.2, there exist general n-members $D_1,\cdots,D_n$ of $|L|$ such that $D_1\cap\cdots\cap D_n=p_0+A$, $p_0\notin A$ so that $D_1\cap\cdots\cap D_{n-2}$ is smooth at p_0. This proves that Z is smooth at p_0. Since $\sigma_q\simeq C_q\simeq \boldsymbol{P}^1$ is a member of $|L_Z|$ for general q, we have $(\sigma_q^2)=(L\sigma_q)=(LC_q)_X=2$. As we have $K_Z\simeq -2L_Z$, Z is a smooth relatively minimal rational ruled surface, isomorphic to either $\boldsymbol{F}_0$ or $\boldsymbol{F}_2$. q.e.d.

Claim 4.3.4. *Under the notation in Claim* 4.3.2,

(4.3.4.1) *σ_q is reduced for any $q\in R^*$.*

(4.3.4.2) Let $\triangle:=\{q\in R^*;\ \sigma_q$ *is reducible*$\}$. *If a general $Z\simeq \boldsymbol{F}_0$ (resp. $\boldsymbol{F}_2$), then $\triangle$ is a hypersurface of R^* with* $\deg\triangle=2$ (*resp.* 1).

Proof. First we prove (4.3.4.1). By the proof of Claim 4.3.2, σ_q is of type (2.1.1) or (2.1.4). In either case σ_q is reduced.

Next we prove (4.3.4.2). Assume that σ_q is reducible. Then by the proof of Claim 4.3.2, σ_q is of type (2.1.4), that is, $\sigma_q\simeq C_q+C_{q,1}+\cdots+C'_q$ with $LC_q=LC'_q=1$ and $LC_{q,i}=0$, where we may assume that $p_0\in C_q$, $h(C_q\setminus\{p_0\})=\{q\}$ and $h(C'_q)=l^*_q$. Since $\sigma_q\in|L_Z|$, we have

$$(C_q^2)_Z=(C'^2_q)_Z=(LC'_q)_X-1=0,\qquad (C_{q,i}^2)_Z=(LC_{q,i})_X-2=-2.$$

First we consider the case where $Z\simeq \boldsymbol{F}_0\simeq \boldsymbol{P}^1\times\boldsymbol{P}^1$. We identify $p_0=(0,0)$ and $L_Z=\pi_1^*O_{\boldsymbol{P}^1}(1)\otimes\pi_2^*O_{\boldsymbol{P}^1}(1)$ via the isomorphism where π_i $(i=1,2)$ is the i-th projection. Note that $|L|_Z=|m_{p_0}L_Z|$. The linear subsystem $\{\sigma_q\}_{q\in P\cap R^*}$ coincides with $|m_{p_\infty}m_{p_0}L_Z|$. Since σ_q is irreducible for q general, there are no fibers of π_i $(i=1,2)$ containing both p_0 and p_∞. Then by a direct computation we see that σ_q is reducible for exactly two (distinct) points of $P\cap R^*$. Thus $\triangle$ contains a hypersurface $\triangle_0$ of degree two in R^*. Similarly if $Z\simeq \boldsymbol{F}_2$, then σ_q is reducible for a unique point q of the line $P\cap R^*$, for which σ_q has exactly three components C_q, $C_{q,1}$, and C'_q by the above proofs. Hence $\triangle$ contains a hyperplane $\triangle_0$ of R^*.

Finally we prove that $\triangle=\triangle_0$. Assume the contrary. Then choose a point $q'\in\triangle\setminus\triangle_0$. Then $l_{q'}:=D_1\cap\cdots\cap D_{n-1}$ is of type (2.1.4) with $\sigma_{q'}\simeq C_{q'}+C_{q',1}+\cdots+C'_{q'}$. Let $V\subset H^0(X,L)$ be an $(n-1)$-dimensional subspace defining $l_{q'}$. Then since $\sigma_{q'}$ has at worst ordinary double singularities given in (2.1.4), we can choose a general $(n-2)$-dimensional subspace U of V such that the surface $(=:Z)$ defined as the common zeroes of U is smooth. (This is clear from the form of the ideal defining $\sigma_{q'}$ in (2.1.4)). In other words, there exists a plane P of W such that $Z:=Z(P)$ is a smooth surface containing $\sigma_{q'}$. Then by choosing U sufficiently general, we may assume that the line $P\cap R^*$ intersects $\triangle_0$ transversally. As we have seen above, $Z\simeq \boldsymbol{F}_0$ or $\boldsymbol{F}_2$. If $\deg\triangle_0=2$ (resp. $\deg\triangle_0=1$), then σ_q is reducible for at least three (resp. two) distinvt points of $P\cap\triangle$, a contradiction. Hence $\triangle=\triangle_0$. q.e.d.

Claim 4.3.5. *Under the notation in Claim* 4.3.4,

(4.3.5.1) *if* $\deg\triangle=2$, *then* σ_q *has two irreducible components for any* $q\in\triangle\backslash\operatorname{Sing}\triangle$, *while* σ_q *has three irreducible components for any* $q\in\operatorname{Sing}\triangle$.

(4.3.5.2) *If* $\deg\triangle=1$, *then* σ_q *has three irreducible components for any* $q\in\triangle$.

Proof. Let l^* be a line of R^*, $P:=P(l^*)$ a plane of W spanned by l^* and q_∞, and $Z:=Z(P)=\hat{h}^{-1}(P)$. As we saw in the proof of Claim 4.3.4, we can choose, for any $q\in\triangle$, a general line $l^*=l_q^*$ passing through q of R^* such that $Z=Z(P(l^*))$ is a smooth surface.

Assume $\deg\triangle=2$. For a smooth point q of $\triangle$, there exists a line l^* of R^* such that $l^*\cap\triangle=q+q'$ for some point q' $(\neq q)$. Obviously $q'\notin\operatorname{Sing}\triangle$. By the proof of Claim 4.3.4, $Z\simeq\boldsymbol{F}_0$, σ_q and $\sigma_{q'}$ are the only reducible curves in $|L|_Z$ passing through p_∞. Hence σ_q as well as $\sigma_{q'}$ has exactly two irreducible components. This proves the first part of (4.3.5.1).

If q is a singular point of $\triangle$, then there exists a general plane P with $P\cap\triangle=\{2q\}$ such that $Z=Z(P)$ is a smooth surface containing σ_q. (In fact, this is also clear from the form of the ideal defining σ_q in (2.1.4).) In the same way as in the proof of Claim 4.3.4, we see that $K_Z\simeq -2L_Z$, $(L_Z^2)_Z=(L\sigma_q)_X=2$ and $\sigma_q=C_q+C_{q,1}+\cdots+C'_q$ with $(C_q^2)_Z=(C'^2_q)_Z=0$, $(C_{q,i}^2)_Z=-2$. We note that $\sigma_q\simeq C_q+C'_q$ is impossible. In fact, if so, then since $P\cap\triangle=\{2q\}$, there are no reducible members in $|L|_Z$ other than σ_q. However if $Z\simeq\boldsymbol{F}_0$, then there are two reducible members in $|L|_Z$, while if $Z\simeq\boldsymbol{F}_2$, then there is a unique reducible member in $|L|_Z$, which however consists of 3 irreducible components. Hence $\sigma_q\simeq C_q+C'_q$ is impossible. Therefore $\sigma_q\simeq C_q+C_{q,1}+\cdots+C'_q$, whence $Z\simeq\boldsymbol{F}_2$, $\sigma_q\simeq C_q+C_{q,1}+C'_q$, which completes the proof of the second part of (4.3.5.1). (4.3.5.2) is proved similarly. q.e.d.

Lemma 4.3.6. *A general* $Z\simeq\boldsymbol{F}_0$, and $\triangle$ is a smooth quadric hypersurface in R^*.

Proof. First we note that $K_Z\simeq -2L_Z$ for general $Z=Z(P)$. Let F

be a fiber of the ruling of Z. Then $(LF)_X=(L_ZF)_Z=-(K_ZF)_Z/2=1$. This shows that $\operatorname{Pic} X/\operatorname{Tor}(\operatorname{Pic} X)\simeq \boldsymbol{Z}L$.

Now we prove that $\triangle$ is an irreducible hypersurface of degree two. Assume that $\triangle$ contains a hyperplane $\triangle'$. By Claim 4.3.5, $\sigma_q\simeq C_q+C'_q$ or $\sigma_q\simeq C_{q,1}+C'_q$ for any $q\in\triangle'\setminus\operatorname{Sing}\triangle$, where we may assume that $p_0\in C_q$, $p_\infty\in C'_q$. Note that $h(C_q\setminus\{p_0\})=\{q\}$, $h(C'_q)=l^*_q$. Let

$$G:=\overline{\bigcup_{q\in\Delta'\setminus\operatorname{Sing}\Delta} C_q},\qquad G':=\overline{\bigcup_{q\in\Delta'\setminus\operatorname{Sing}\Delta} C'_q}.$$

Then G and G' are (mutually distinct) divisors of X. In fact, if $C_q\cap C_s\neq\{p_0\}$ for q, $s\in\triangle'\setminus\operatorname{Sing}\triangle$, then $C_q=C_s$ by $LC_q=LC_s=1$ so that $q=s$. Hence $\dim G=n-1$. Similarly $\dim G'=n-1$. Meanwhile if for general $q\in\triangle'\setminus\operatorname{Sing}\triangle$, there exists $s\in\triangle'\setminus\operatorname{Sing}\triangle$ such that $C_q\cap C'_s\neq\emptyset$, then $C_q\cap\sigma_s$ contains at least two points, whence $C_q\subset\sigma_s$ by $LC_q=1$. Hence $C_q=C_s$, whence $q=s$. Therefore $G\cap G'$ is at most $(n-2)$-dimensional. Since L generates $\operatorname{Pic} X$/torsions, we have $c_1(G)=ac_1(L)$ and $c_1(G')=a'c_1(L)$ for some positive integers a and a'. However $G+G'$ is a subset of $D:=\cup_{q\in\Delta'}\sigma_q$, which is a member of $|L|$. It follows that $a+a'\leq 1$, a contradiction. Consequently $\triangle$ is an irreducible hypersurface of degree two. In particular a general $Z\simeq\boldsymbol{F}_0$ by Claim 4.3.4. (There is another proof of $Z\simeq\boldsymbol{F}_0$ due to Fujiki.)

Next we prove that $\triangle$ is smooth. Assume that $\triangle$ is singular. By (4.3.5.1) we have $\sigma_q\simeq C_q+C_{q,1}+C'_q$ with $LC_{q,1}=0$ for any $q\in\operatorname{Sing}\triangle$. Conversely, given a rational curve C with $LC=0$, we have a unique point q of $\operatorname{Sing}\triangle$ such that $C=C_{q,1}$. In fact, since $p_0\notin C$, it suffices to set $q:=h(C)$. It follows from Claim 4.3.5 that $C\subset\sigma_q$, $C=C_{q,1}$ and $q\in\operatorname{Sing}\triangle$. Therefore C moves on X in an at most $(n-4)$-dimensional family. On the other hand $\chi(C,N_{C/X})=n-3$, whence by [9, Proposition 3], there exists an at least $(n-3)$-dimensional family of displacements $C(t)$ of C in X, a contradiction. q.e.d.

Claim 4.3.7. *$\hat{X}\simeq Q_\Delta(W)$, the monoidal transform of W with smooth center $\triangle$.*

Proof. By Claim 4.3.5, $\sigma_q\simeq C_q+C'_q$ for any $q\in\triangle$, where $p_0\in C_q$ and $p_\infty\in C'_q$. Let $\hat{C}_q:=\varphi^{-1}(C_q)$, $\hat{\triangle}:=\hat{h}^{-1}(\triangle)_{\mathrm{red}}$, $M:=\varphi^*L-E$ and $J(\hat{h}):=K_{\hat{X}}-\hat{h}^*K_W$. Then $\hat{\triangle}=\cup_{q\in\Delta}\hat{C}_q$ is a unique $\hat{h}$-exceptional divisor. Hence $J(\hat{h})=r\hat{\triangle}$ for a positive integer r.

First we prove that $(\hat{\triangle}\hat{C}_q)_{\hat{X}}=-1$. Since a general member of $|L|$ intersects C_q $(q\in\triangle)$ tranversally at p_0, we have $M\hat{C}_q=0$ and $E\hat{C}_q=1$, whence $M=\hat{h}^*O_W(1)$. We also have

$$J(\hat{h})=\varphi^*K_X+(n-1)E+(n+1)M=M-E=\varphi^*L-2E,$$

whence $r(\hat{\triangle}\hat{C}_q)_{\hat{X}}=(LC_q)_X-2(E\hat{C}_q)_{\hat{X}}=-1$. Hence $r=1$ and $(\hat{\triangle}\hat{C}_q)_{\hat{X}}=-1$.

Next we prove that $\hat{\triangle}$ is smooth. Let $q\in\triangle$. Then we can choose general $(n-2)$-hyperplanes $H_1,\cdots,H_{n-2}$ of W such that $\triangle\cap H_1\cap\cdots\cap H_{n-2}=q+s$ for some s ($\neq q$). Let $P:=H_1\cap\cdots\cap H_{n-2}$. Then $Z(P)\simeq \boldsymbol{F}_0$ and $Z(P)\cap\varphi(\hat{\triangle})=C_q\cup C_s$. Let M_i be the proper transform of H_i. It follows that $\hat{\triangle}\cap M_1\cap\cdots\cap M_{n-2}=\hat{C}_q$ along $\hat{C}_q\backslash E$. Since $\hat{C}_q$ is smooth, so is $\hat{\triangle}$ along $\hat{C}_q\backslash E$ for any q. Therefore $\hat{\triangle}\backslash E$ is smooth. Meanwhile $\hat{\triangle}\cap E\simeq\triangle$, whence $\hat{\triangle}$ is smooth along $\hat{\triangle}\cap E$. Hence $\hat{\triangle}$ is smooth everywhere. Note that $\varphi(\hat{\triangle})\in|L|$.

Since $\hat{C}_q\cap\hat{C}_s=\emptyset$ for $q\neq s$ $(q,s\in\triangle)$, this shows that $\hat{X}\simeq Q_\Delta(W)$. q.e.d.

(4.3.8) COMPLETION OF THE PROOF OF LEMMA 4.3. By Claim 4.3.2 and Claim 4.3.7, X is recovered from W ($\simeq \boldsymbol{P}^n$), R^* and $\triangle$ as follows. By Claim 4.3.7, $\hat{X}$ is the monoidal transform of W with $\triangle$ center. Then E ($\simeq\boldsymbol{P}^{n-1}$) is a proper transform of R^* with $N_{E/\hat{X}}\simeq O_E(-1)$. In fact,

$$\begin{aligned} N_{E/\hat{X}}\simeq E_E &\simeq (M-J(h))_E \\ &\simeq (\hat{h}_{|E})^*(O_{R^*}(1)-\triangle) \\ &\simeq O_E(-1). \end{aligned}$$

Consequently we obtain X from $\hat{X}$ by blowing down E to a smooth point p_0 of X. Obviously X thus obtained is unique up to isomorphism. Hence we have $X\simeq\boldsymbol{Q}^n$, whence $h^0(X,L)=n+2$, a contradiction. This completes the proof of Lemma4.3. q.e.d.

Lemma 4.4. $\deg h=2$ *is impossible.*

We prove Lemma 4.4 in Claims 4.4.1–4.4.4. We use the same notation as in (4.1) and Lemmas 4.2–4.3.

We assume $\deg h=2$. We first prove

Claim 4.4.1. *Let l^* be a line of W ($\simeq \boldsymbol{P}^n$) not contained in R^*. Then*

(4.4.1.1) $\sharp(l^*\cap R^*)=1$ *or* 2.

(4.4.1.2) $\hat{l}(l^*):=\hat{h}^{-1}(l^*)$ *is a connected subset of* $\hat{X}$.

(4.4.1.3) $\sigma(l^*):=\varphi(\hat{l}(l^*))$ *is a connected subset of X disjoint from B.*

Proof. First we prove (4.4.1.2). Let σ_0 be a connected component of $\sigma(l^*)$ mapped onto l^*, and $\hat{\sigma}_0 := \varphi^{-1}(\sigma_0)$. Then σ_0 is of type (2.1.1) or (2.1.4) by Lemma 4.2.

If l^* is general, then $\sigma(l^*)$ is irreducible outside B by Lemma 4.2, whence σ_0 is of type (2.1.1). That is, σ_0 is a rational curve C with $LC=2$. Hence by Lemma 4.2 $\sigma_0 \cap B = \emptyset$ and $\sigma_0 \cap B^* = p_1 + p_2$ for some points p_i because $\deg(h_{|C}) = \deg h = 2$. Note that $\hat{\sigma}_0 \simeq \sigma_0$ because $\sigma_0 \cap B = \emptyset$. Let $q_i := h(p_i)$. Since $\deg \hat{h} = 2$, $\hat{h}^{-1}(q_i)$ is a connected subset of $\hat{l}(l^*)$. Since $\hat{l}(l^*) = \hat{\sigma}_0 \cup \hat{h}^{-1}(q_1) \cup \hat{h}^{-1}(q_2)$, $\hat{l}(l^*)$ is connected. Since $\hat{l}(l^*)$ is connected for general l^*, it is connected for any l^*. This proves (4.4.1.2).

Next we prove (4.4.1.3). By (4.4.1.2) $\sigma(l^*)$ is connected. Hence $\sigma(l^*)$ is of type (2.1.1) or (2.1.4). If $\sigma(l^*)$ is of type (2.1.1), then it is a smooth rational curve C with $LC=2$. Since $\deg h_{|C} = 2$, we have $C \cap B = \emptyset$ by Claim 4.2.2. If $\sigma(l^*)$ is of type (2.1.4), then $\sigma(l^*) \simeq C_0 + \cdots + C_m$ with $LC_0 = LC_m = 1$, $LC_i = 0$ $(1 \le i \le m-1)$. We prove $\sigma(l^*) \cap B = \emptyset$. If $\sigma(l^*) \cap B \neq \emptyset$, then we may assume $C_m \cap B \neq \emptyset$. Then $h(C_j)$ $(1 \le j \le m-1)$ and $h(C_m \backslash B)$ are one and the same point of W, while $h_{|C_0}$ is unramified on $C_0 \backslash C_1$ by $LC_0 = 1$. Since $l^* \not\subset R^*$, this contradicts $\deg h = 2$. Therefore $\sigma(l^*) \cap B = \emptyset$. This proves (4.4.1.3).

Finally we prove (4.4.1.1). If σ_0 is of type (2.1.1), then $l^* \cap R^*$ is two points by the above proof. If σ_0 is of type (2.1.4), then $l^* \cap R^*$ is one point. q.e.d.

Claim 4.4.2. *$B = \emptyset$, $\hat{B} = \emptyset$ and R^* is an irreducible hyperquadric of W.*

Proof. First we prove $B = \emptyset$. Assume $B \neq \emptyset$, whence $\hat{B} \neq \emptyset$. Then there is a line l^* not contained in R^* such that $\hat{l}(l^*) \cap \hat{B} \neq \emptyset$. Hence $\sigma(l^*) \cap B \neq \emptyset$, which contradicts (4.4.1.3). Hence $B = \emptyset$. Therefore $X \simeq \hat{X}$.

Next we prove that R^* is a hypersurface of degree two in W. Choose a general line l^* of W intersecting R^*. Then $l(l^*)$ contains a rational curve C with $LC=2$ by the assumpion. By Lemma 4.2 we have $B \cap C = \emptyset$, $\dim |L|_C = 1$. It follows that there exist exactly two points $p_i \in C$ $(i=1,2)$ such that $h_{|C}$ is unramified on $C \backslash \{p_1, p_2\}$. Note that $p_i \notin B$. Let $q_i := h(p_i)$. Then $C \cap B^* = p_1 + p_2$ and $l^* \cap R^* = q_1 + q_2$. Hence R^* contains a hypersurface R_0^* of degree two in W. If $R^* \neq R_0^*$, then there exists a line l^* not contained in R^* but intersecting R^* at (at least) 3 distinct points. This contradicts (4.4.1.1).

Finally we prove that R^* is irreducible. Assume the contrary. Then $\dim \operatorname{Sing} R^* = n-2$. We prove that $C := h^{-1}(q)$ is a rational curve with $LC=0$ for any $q \in \operatorname{Sing} R^*$. For this purpose we choose a general point $q_\infty \in W \backslash R^*$ and a line l^* connecting q_∞ and q with $l^* \cap R^* = \{2q\}$. Moreover we choose a general plane P containing l^* so that $Z(P)$ is a smooth

surface. Choose point p_∞ with $q_\infty = h(p_\infty)$. In the same manner as in the proof of Claim 4.3.5, $Z(P) \simeq \boldsymbol{F}_2$. We have a unique reducible curve σ in $|L_Z|$ with $h(\sigma) = l^*$ passing through p_∞, whence $\sigma(l^*) = \sigma \simeq C_0 + C_1 + C_2$ with $LC_i = 1$ $(i=0,2)$, $LC_1 = 0$. Therefore $C_1 = C = h^{-1}(q)$. Let $G := h^{-1}(\operatorname{Sing} R^*)$. Obviously G is a divisor of X with $h_*(G) = 0$, which contradicts $\operatorname{Pic} X/\text{torsions} \simeq \boldsymbol{Z}L$. This proves the irreducibility of R^*.
q.e.d.

Claim 4.4.3. *R^* is smooth.*

Proof. By Claim 4.4.2, we have Hessian-rank $R^* \geq 3$. Assume $4 \leq$ Hessian-rank $R^* \leq n$ so that $\dim \operatorname{Sing} R^* \leq n-4$. Then by the proof of Claim 4.4.2, $C = h^{-1}(q)$ $(q \in \operatorname{Sing} R^*)$ is a rational curve with $LC = 0$, whence $\chi(N_{C/X}) = n-3 > \dim \operatorname{Sing} R^*$ by Lemma 2.1. Then we derive a contradiction as in the proof of Claim 4.3.4. Therefore R^* is smooth or Hessian-rank $R^* = 3$.

Let $r(w) = 0$ be the equation defining R^* in W. h^*R^* is a divisor with multiplicity 2 above a generic point of R^*. let $h^*R^* = 2A + A'$ for some effective divisors A $(\neq 0)$ and A' with $h_*(A') = 0$. Since $\operatorname{Pic} X/\text{torsions} \simeq \boldsymbol{Z}L \simeq \boldsymbol{Z}h^*O_W(1)$, and $h^*(R^*) \in |2L|$, we have $A \in |L|$ and $A' = 0$. Hence $h^*(R^*)$ is a divisor of X with multiplicity 2, whence we have an element $\psi(x) \in H^0(X, L)$ such that $(h^*r)(x) = \psi(x)^2$. Let $\boldsymbol{H}$ be the total space of the hyperplane bundle $O_W(1)$ on W with fiber coordinate ζ, Y a hypersurface of $\boldsymbol{H}$ defined by $\zeta^2 = r(w)$. Then using $\psi(x)$, we can define a natural morphism g, compatible with h, of X onto Y by $g^*\zeta = \psi(x)$. If Hessian-rank $R^* = 3$, Y is isomorphic to a hyperquadric of $\boldsymbol{P}^{n+1}$ with Hessian-rank $Y = 4$, whence it has a reducible hyperplane section. This contradicts $\operatorname{Pic} X/\text{torisons} \simeq \boldsymbol{Z}L$. Consequently R^* is smooth. q.e.d.

(4.4.4) COMPLETION OF THE PROOFS OF LEMMA 4.4 AND PROPOSITION 3.1. By Claim 4.4.3, R^* is a smooth hyperquadric. With the notation in Claim 4.4.3, $Y \simeq \boldsymbol{Q}^n$, whence we have a birational morphism g of X onto $\boldsymbol{Q}^n$. Since $K_X \simeq -nL \simeq -ng^*\pi^*O_W(1) \simeq g^*(K_{\boldsymbol{Q}^n})$, g is an isomorphism. Therefore $h^0(X, L) = n+2$, which contradicts our assumption $h^0(X, L) = n+1$. Thus we complete the proof of Lemma 4.4, hence of Proposition 3.1.
q.e.d.

5. Moishezon fourfolds homeomorphic to Q_C^4

The purpose of this section is to prove:

Theorem 5.1. *Let X be a Moishezon 4-fold homeomorphic to $\boldsymbol{Q}^4$,* and L a line bundle on X with $L^4=2$. Assume that $h^0(X,L)\geq 5$. *Then* $X\simeq \boldsymbol{Q}^4$.

Our proof of Theorem 5.1 is completed in (5.7).

Lemma 5.2. *Under the assumptions in Theorem* 5.1, *let D and D' be distinct members of $|L|$, τ the scheme-theoretic complete intersection $D\cap D'$. Then we have*

(5.2.1) $\operatorname{Pic} X\simeq \boldsymbol{Z}L,\quad K_X\simeq -4L,$

(5.2.2) $H^p(X,-qL)=0\quad (p=0,q\geq 1,$ *or* $1\leq p\leq 3, 0\leq q\leq 4,$ *or* $p=4, q\leq 3)$

(5.2.3) $H^p(D,-qL_D)=0\quad (p=0,q\geq 1,$ *or* $p=1,2, 0\leq q\leq 3,$ *or* $p=3, q\leq 2)$

(5.2.4) $H^p(\tau,-qL_\tau)=0\quad (p=0,q=1,2,$ *or* $p=1, 0\leq q\leq 2,$ *or* $p=2, q=0,1)$

(5.2.5) $H^0(X,O_X)\simeq H^0(D,O_D)\simeq H^0(\tau,O_\tau)\simeq \boldsymbol{C},$

(5.2.6) $|L|_D=|L_D|$ *and* $|L|_\tau=|L_\tau|$.

Proof. The proof of (5.2.1) is similar to [15]. The vanishing of $H^p(X,-qL)$ for $p\neq 2$ is proved in the same way as in [15]. Since X is homemorphic to $\boldsymbol{Q}^4$, we have

$$\chi(X,-qL)=\chi(\boldsymbol{Q}^4,O_{\boldsymbol{Q}^4}(-q))=(q-1)(q-2)^2(q-3)/12$$

for any q in view of (5.2.1). This proves the vanishing of $H^2(X,-qL)$ for $0\leq q\leq 4$. The remaining assertions are easy. q.e.d.

Lemma 5.3. *Under the assumptions in Therem* 5.1, *let* $B:=\operatorname{Bs}|L|$ *be the scheme-theoretic base locus of $|L|$, h: $X\to \boldsymbol{P}^m$ a rational map associated with $|L|$, and W the closure of $h(X\backslash B)$, where $m=h^0(X,L)-1$. Then* $\dim W\geq 3$.

Proof. Let $d=\deg W$. Then $d\geq m+1-\dim W$. If $\dim W=1$, then $d=1$, $m=1$ by $\operatorname{Pic} X\simeq \boldsymbol{Z}L$, which contradicts $m\geq 2$. Therefore we assume $\dim W=2$ to derive a contradiction. So $d=\deg W\geq m-1\geq 2$.

Then by choosing general D and $D'\in|L|$, we have reduced irreducible components Z_i $(1\leq i\leq dr)$ of $\tau:=D\cap D'$ outside B for some positive interger r, where r is the number of irreducible components of a genaral fiber of h. Each Z_i is nonsingular outside B by Bertini's theorem. Let $Z=Z_1$, and let ν: $Y\to Z$ be the normalization of Z, f: $S\to Y$ the minimal resolution of $Y, g=\nu\cdot f$. Then as in the proof of Lemma 3.2, there exist effective Cartier divisors E and G on S with no components in common such that the canonical sheaf of S is given by $K_S=O_S(-2g^*L-E-G)$, where the base locus $\mathrm{Bs}\,g^*|L|$ contains $supp(E+G)$ if D and D' are sufficiently general. Since $h^0(X,L)\geq 3$ and $Z\not\subset B$, g^*L is effective. By $P_m(S)=0$, $S\simeq \boldsymbol{P}^2$ or S is ruled. The proof of Lemma 5.3 is now divided into CASES 1–1, 1–2 and CASE 2.

CASE 1. Assume $S\simeq \boldsymbol{P}^2$. Then we have $G=0$ and $S\simeq Y$. Let $H\in g^*|L|$. Then $K_S=-2H-E$. Since $K_{\boldsymbol{P}^2}$ is indivisible by 2, we have $E\neq 0$ and $H=E\in|O_{\boldsymbol{P}^2}(1)|$ in view of $E_{\mathrm{red}}\subset H_{\mathrm{red}}$. Hence $g^*(D'')$ $(=H)$ is independent of the choice of $D''\in|L|$. Moreover $g_{|E}$, the restriction of g to E, is generically one to one because $(Lg_*(E))=(g^*(L)E)=1$.

Since the coefficient of E in $-K_S$ relevant to $\mathrm{Sing}\,\tau$ is equal to 1, there exists by Theorem 1.14 a Zariski open subset V of Z with $E\cap v^{-1}(V)\neq\emptyset$ such that

$$e(Q_V,E_U)+e(Q''_V,E_U)-e(Q'_V,E_U)=1.$$

where $U:=v^{-1}(V)$ and $E_U=E\cap U$. By Theorem 1.11, (1.12) and (1.13), we have $e(Q_V,E_U)=1$, $e(Q''_V,E_U)=e(Q'_V,E_U)=0$. See Appendix to section one for the detail. Let p be any point of $E\cap U$. By Lemma 1.6, there exists a local parameter system x,y,z and w at p and another irreducible component Z^* of τ at p such that

$$I_{\tau,p}=(xy,z),\qquad I_{Z,p}=(x,z),\qquad I_{Z^*,p}=(y,z).$$

Hence Z is smooth along $E\cap U$. It follows that Sing Z is finite for general D and D'.

There are two subcases $Z^*\subset B$ or $Z^*\not\subset B$. Let $\bar{E}:=g(E)_{\mathrm{red}}$.

CASE 1–1. Assume $Z^*\subset B$. Let l be a line $(\neq E)$ on S, $C:=g(l)_{\mathrm{red}}$. Hence $g_*(l)=aC$ for an integer $a\geq 1$. Then $a(LC)_X=(g^*(L)l)_S=1$, whence $a=1$ and l is mapped generically isomorphically onto C by g. Take a general $D''\in|L|$. For any point $q\in Z\cap D''$, there exists a line l on S such that $q\in g(l)$. Therefore any general $D''\in|L|$ is smooth along $Z\cap D''$ by $(D''g_*(l))_X=1$. Hence by choosing D sufficiently general, D is also smooth along Z. Therefore Z is a Cartier divisor of D, so that Z is Gorenstein everywhere. Since $\mathrm{Sing}\,Z$ is finite, this implies that Z is normal. Hence $S\simeq Y\simeq Z\simeq \boldsymbol{P}^2$, $Z\cap D''\simeq\bar{E}\simeq E\simeq \boldsymbol{P}^1$ and $O_Z(L)\simeq O_Z(\bar{E})$.

Since $\bar{E}\subset B$, we have $\bar{E}\simeq B\cap Z$. Hence B is a smooth Cartier divisor of D along $\bar{E}$. The surfaces B and Z intersect transversally along $\bar{E}$.

Claim 5.3.1. *$I_BO_\tau\simeq O_Z(-\bar{E})$ along $\bar{E}$.*

Proof. Let $I_{\bar{E}/Z}$ be the ideal of O_Z defining $\bar{E}$. We may assume $O_X \simeq \boldsymbol{C}\{x,y,z,w\}$, $I_\tau=(xy,z)$, $I_Z=(x,z)$, $I_B=(y,z)$. Then $I_{\bar{E}/Z}=I_B O_Z \simeq y\boldsymbol{C}\{y,w\}$ by $\bar{E}\simeq B\cap Z$. Thus we see

$$I_B O_\tau \simeq y\boldsymbol{C}\{x,y,w\}/(xy) \simeq y\boldsymbol{C}\{y,w\} \simeq I_{\bar{E}/Z} \simeq O_Z(-\bar{E}). \qquad \text{q.e.d.}$$

CASE 1–1 (Continued). We also see that

$$1=(HE)_S=(L_Z B)_Z=(\bar{E}B)_Z=((Z+B)BZ)_X=(\bar{E}^2)_B+(\bar{E}^2)_Z=(\bar{E}^2)_B+1,$$

whence $(\bar{E}^2)_B=0$. Hence the unique irreducible component B_0 of B intersecting Z is a (possibly singular) ruled surface with $\bar{E}$ a general fiber. Moreover Z intersects the irreducible components of τ other than Z and B_0 in (at most) finitely many points outside $\bar{E}$. This is true for any Z_i. Since τ is Gorenstein, this implies that Z meets no irreducible components of τ other than B_0. Therefore for general D and D', we have $Z_i \simeq \boldsymbol{P}^2$, $Z_i \cap Z_j=\emptyset$ $(i\neq j)$ and $\bar{E}_i := Z_i \cap B \simeq \boldsymbol{P}^1$ for $1\leq i\leq dr$. Hence we have,

$$H^0(I_B L_\tau) \supset \bigoplus_{i=1}^{dr} H^0(Z_i, I_{\bar{E}_i/Z_i} O_{Z_i}(L)) \simeq \bigoplus_{i=1}^{dr} H^0(Z_i, O_{Z_i}),$$

which shows $m-1=h^0(L_\tau)=h^0(I_B L_\tau)\geq dr\geq (m-1)r$. Hence $r=1$, $d=m-1\geq 3$. Then by [5, Theorem 1], W is a cone over a smooth variety of minimal degree. In this case, W is either the Veronese surface in $\boldsymbol{P}^5$ with $d=4$ or the cone over a normal rational curve of degree $m-1$ in $\boldsymbol{P}^{m-1}$ with $d=m-1$. In either case, there is a reducible or a nonreduced hyperplane section of W, which contradicts $\operatorname{Pic} X \simeq \boldsymbol{Z}L$. See also [7, 5.3.11].

CASE 1–2. Assume $Z^* \not\subset B$. We may assume that $Z^* \simeq \boldsymbol{P}^2$ by choosing general D and D'. By the same argument as in CASE 1–1, Z and Z^* intersect transversally along $\bar{E}$, where $Z\cap Z^* \simeq \bar{E} \simeq \boldsymbol{P}^1$.

Let σ be the sum of all the primary components of τ other than Z and Z^*. Then $\sigma\cap(Z\cup Z^*)$ is finite. This implies that $\sigma=\emptyset$ and $\tau=Z\cup Z^*$ because τ is Gorenstein and connected by (5.2.5). Thus we have an exact sequence of O_τ-modules,

$$0\to O_\tau \to O_Z \oplus O_{Z^*} \to O_{\bar{E}} \to 0.$$

It follows from the exact sequence that $h^0(X,L)=h^0(\tau,L_\tau)+2=6$, $B=\operatorname{Bs}|L_\tau|=\emptyset$. This contradicts $\bar{E}\subset B$ by the general choice of D and D'.

CASE 2. Assume that S is a ruled surface. We let $\pi: S \to T$ be the ruling, F ($\simeq \boldsymbol{P}^1$) a general fiber of π. Let $H \in g^*|L|$, and let M (resp. N) be the movable part (resp. the fixed part) of H in $g^*|L|$. Since $F \simeq \boldsymbol{P}^1$, we have

$$-2 = K_S F + F^2 = K_S F = -2HF - (E+G)F.$$

Since $E_{\mathrm{red}} + G_{\mathrm{red}} \subset H_{\mathrm{red}}$, we have $HF=1$, $EF=GF=0$. Therefore there exists a unique irreducible component Γ of H with $\Gamma F=1$ and $\Gamma \not\subset E+G$. Since dim $W=2$, we have $M=0$ and $\Gamma \subset N$. Any general $D'' \in |L|$ is smooth generically along $g(\Gamma)$ (or at $g(\Gamma)$ if $g(\Gamma)$ is a point) by $(D'' g_*(F))_X = (HF)_S = 1$. Assume that $g(\Gamma)$ is a curve on X. Then any irreducible component Z_i of τ contains $g(\Gamma)$ because $g(\Gamma)$ is a curve component of B by $\Gamma \subset N$. However since $\Gamma \not\subset E$, τ is smooth at a generic point of $g(\Gamma)$. This shows $d=1$, which contradicts $d \geq m-1 \geq 2$. Therefore $g(\Gamma)$ is a point. We note that this can happen if the connected component of the g-exceptional set containing Γ corresponds to one of Du Val singularities.

Let $p: = g(\Gamma)$ and $q: = f(\Gamma)$.

Claim 5.3.2. $q \notin f(E)$.

Proof. Since $g(\Gamma)$ is a point, $H\Gamma = (g^*L)\Gamma = 0$ and $\Gamma^2 < 0$. It follows that $K_S\Gamma + \Gamma^2 = \Gamma^2 - (E+G)\Gamma \leq -1$, whence $K_S\Gamma = E\Gamma = G\Gamma = 0$, $\Gamma^2 = -2$, and $\Gamma \simeq \boldsymbol{P}^1$ in view of the minimality of the resolution f. Assume $q \in f(E)$. Then there is a sequence N_i $(1 \leq i \leq s)$ of irreducible components of N with $q = f(N_i)$ such that $N_1\Gamma > 0$, $N_{i-1}N_i > 0$ and $N_s E > 0$. Then since $E\Gamma = G\Gamma = 0$, N_1 is not contained in $E+G$. Hence $K_S N_1 = HN_1 = EN_1 = GN_1 = 0$, $N_1^2 = -2$ in the same manner as for Γ. Therefore $s \geq 2$ and $N_2 \not\subset E + G$. Repeating the same arguments as above, we see that $K_S N_i = HN_i = EN_i = GN_i = 0$, $N_i^2 = -2$ for any i. This contradicts $N_s E > 0$. Therefore $q \notin f(E)$. q.e.d.

Claim 5.3.3. $p \notin g(E)$.

Proof. Assume the contrary. Let V be a sufficiently small open neighborhood of $f^{-1}(q)$ in S. Note that $\Gamma \subset V$. Since $f(V) \setminus \{q\}$ is disjoint from $f(E)$, $g(V) \setminus \{p\}$ is disjoint from $g(E)$. Therefore the germ $(g(V), p)$ is a locally irreducible component of (τ, p) which intersects the other locally irreducible components at the point p only. This shows that $\tau \setminus \{p\}$ has at least two local connected components, which contradicts that τ is Gorenstein. Hence $p \notin g(E)$. q.e.d.

(5.3.4) COMPLETION OF THE PROOF OF LEMMA 5.3. Claim 5.3.3 shows that the point p ($\in g(N) \subset B$) is isolated in B, whence any irreducible component Z_i of τ passes through p. Since $d \geq m-1 \geq 2$, there is another component Z_2 of τ with $p \in Z_2$. Since $p \notin g(E)$, Sing $\tau_{\mathrm{red}}(\subset B)$ is isolated at p, whence $Z_1 \cap Z_2$ is isolated at p. However since τ is Gorenstein, $Z_1 \cap Z_2$ has a curve component at p, a contradiction. This completes the proof of Lemma 5.3. q.e.d.

Lemma 5.4. *If* $\dim W=3$, *then* $W \simeq \boldsymbol{Q}^3$, *a smooth hyperquadric in* $\boldsymbol{P}^4$.

Proof. Since $\dim W=3, \tau:=D \cap D'$ is irreducible nonsingular outside B for general D, $D' \in |L|$ by Bertini's theorem. Let Z be the unique irreducible component of τ outside B, g: $S \rightarrow Z$ the minimal resolution of the normalization of Z. We see that $S \not\simeq \boldsymbol{P}^2$. In fact, if $S \simeq \boldsymbol{P}^2$, then $g^*|L|$ has no movable components by the same arguments as in the proof of Lemma 3.2 CASE 1, whence $\dim W \leq 2$, a contradiction. Hence S is a ruled surface with $F \simeq \boldsymbol{P}^1$ a general fiber. Under the same notation as before, $H:=g^*(D'') \in g^*|L|$ has an irreducible component Γ with $F\Gamma=1$. We see $EF=GF=0$ and $\Gamma \not\subset E+G$. If the movable part M of H contains Γ, then $M=\Gamma$ by $HF=\Gamma F=1$, which shows that $d=\deg W=1 \geq m-2 \geq 2$, a contradiction. Therefore the fixed part N of H contains Γ.

Since $HF=NF=\Gamma F=1$, the movable part M satisfies $MF=0$ so that $M^2=0$ and that M is a union of general fibers F_i of the ruling, $M=F_1+\cdots+F_d$. Let $C_i=g(F_i)_{\mathrm{red}}$, and $\bar{M}=C_1+\cdots+C_d$. Then $g_*(F_i)=C_i$ and $(LC_i)=1$ for any i by $HF_i=1$. We note that $\bar{M}$ is the movable part of the intersection $l:=D \cap D' \cap D''$. The image $C_i=g(F_i)$ is a rational curve intersecting $g(\Gamma)$ (passing through $g(\Gamma)$ if $g(\Gamma)$ is a point) with $(LC_i)_X=(HF_i)_S=1$, whence both C_i and D'' are smooth at $C_i \cap g(\Gamma)$. Since $g_{|S \backslash N}$ is an isomorphism, $C_i \backslash g(\Gamma)$ is smooth. Hence C_i is a smooth rational curve.

Assume that $g(\Gamma)$ is a point. Then in the same manner as in the proofs of Claims 5.3.2 and 5.3.3 we see that $(\Gamma^2)_S=-2$ and $p:=g(\Gamma) \notin g(E)$. Hence p is isolated in B. Therefore (2.1.3) is possible and $d=2$. Moreover p is the point where C_1 and C_2 intersect. By a suitable coordinate system at p, we have

$$I_{l,p}=(x,y,zw), \qquad I_{C,p}=(x,y,z), \qquad I_{B,p}=(x,y,z,w).$$

Therefore if we choose general $\tau:=D \cap D'$, then we may assume $I_{\tau,p}=(x+\alpha zw, y+\beta zw)$ for some $\alpha, \beta \in O_{X,p}$, whence by rechoosing x,y modulo zw, we may assume $I_{\tau,p}=(x,y)$. Therefore τ is smooth at p, whence $\tau \simeq Z$ at p and Z is smooth at p. This contradicts that Γ is contracted into a singular point of Y by f. Thus $g(\Gamma)$ is a curve.

Then since $(LC_i)=1$, a general $D''' \in |L|$ is smooth along a sufficiently small Zariski open subset V of $g(\Gamma)$, and C_i intersects general $D''' \in |L|$ transversally at a point of V. We also see that Z is smooth along V because $\Gamma \not\subset E$. Moreover $\tau \simeq Z$ along V, whence $\tau \simeq Z$ over a smooth Zariski open subset U of Z containing both V and $\bar{M}$. Then we have a natural exact sequence

$$0 \to O_\tau \to O_\tau(\bar{M}) \to \bigoplus_{i=1}^{d} O_{C_i}(\bar{M})(\simeq \bigoplus_{i=1}^{d} O_{C_i}) \to 0.$$

Since $h^1(\tau, O_\tau)=0$ in (5.2.4), we infer from the above sequence that $h^0(\tau, O_\tau(\bar{M}))=d+1$. Hence we have

$$m-1=h^0(\tau, L_\tau) \geq h^0(\tau, \bar{M})=d+1 \geq m-2+1.$$

Hence $d=m-2 \geq 2$. If W is not a smooth hyperquadric $\boldsymbol{Q}^3$, W is a cone over a smooth variety of minimal degree d with a reducible or nonreduced hyperplane section [5, Theorem 1]. This contradicts $\operatorname{Pic} X \simeq \boldsymbol{Z}L$. Hence $W \simeq \boldsymbol{Q}^3$. q.e.d.

Lemma 5.5. *Any line on $\boldsymbol{Q}^3$ is algebraically equivalent to each other.*

Proof. I learned this proof from I. Shimada. Let p be a point of $\boldsymbol{Q}^3$. Then those lines on $\boldsymbol{Q}^3$ passing through p are parametrized by a smooth conic in $\boldsymbol{P}(T^*_{\boldsymbol{Q}^3,p})$. Therefore the Hilbert scheme of lines on $\boldsymbol{Q}^3$ is dominated by a smooth conic bundle over $\boldsymbol{Q}^3$ in $\boldsymbol{P}(T^*_{\boldsymbol{Q}^3})$. Hence it is irreducible. See also [4]. q.e.d.

Lemma 5.6. $\dim W=4$.

Proof. STEP 1. Assume $\dim W=3$. Then $W \simeq \boldsymbol{Q}^3$ and $h^0(X,L)=h^0(\tau, L_\tau)+2=5$ by Lemma 5.4. Let H be a general member of $|O_W(1)|$, $\hat{D}:=\hat{h}^*H$ and $D:=\varphi_*(\hat{D}) \in |L|$. Then $\operatorname{Sing} D \subset B:=\operatorname{Bs}|L|$ and $\operatorname{Sing} \hat{D} \subset \hat{B}:=\varphi^{-1}(B)$ by Bertini's theorem.

Let Q, Q' and Q'' be general members of $|O_W(1)|$. Then the complete intersection $Q \cap Q' \cap Q''$ consists of two distinct points p_1 and p_2. Since $Q \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$, we have two lines l_i on Q with $(l_1 l_2)_Q=1$, $(l_i)^2_Q=0$. There exist (general) members $f_i \in |O_Q(l_i)|$ such that $p_i \in f_i$ and $p_i \notin f_j$ $(i \neq j)$. By choosing new Q' and Q'' if necessary, we may assume that $Q \cap Q' \cap Q''=\{p_1, p_2\}$, $Q \cap Q'$ is an irreducible rational curve $(\in |O_Q(l_1+l_2)|)$ on Q, while

$Q \cap Q'' = f_1 + f_2$. Let $D := \varphi(\hat{h}^*(Q))$, $D' := \varphi(\hat{h}^*(Q'))$ and $D'' := \varphi(\hat{h}^*(Q''))$. Let $\tau := D \cap D'$ and $\sigma := D \cap D''$.

STEP 2. By Bertini's theorem we have a unique irreducible component Z of τ outside B. Let $g\colon S \to Z$ be the minimal resolution of the normalization of Z. By the proof of Lemma 5.4 S is a ruled surface. Under the same notation as in the proof of Lemma 5.4 there is an irreducible component Γ of the fixed part N of $g^*|L|$ with $(\Gamma F)_S = 1$ where F is a general fiber of the ruling of S. By the proof of Lemma 5.4, $g(\Gamma)$ is a curve, along which $\tau \simeq Z$ and Z is smooth generically. Moreover the movable part M of $g^*|L|$ consists of a pair of smooth rational curves F_1 and F_2. The complete intersection $l := D \cap D' \cap D''$ is therefore a union of smooth rational curves $C_i := g(F_i)$ $(i = 1, 2)$ outside B, where C_i intersect $g(\Gamma)$ transversally at distinct points as was given in (2.1.2). Moreover the proof of Lemma 5.4 shows that the linear system $|M|$ on τ, hence $|L_\tau|$ separates C_1 and C_2, that is, $h(C_1 \backslash B) \neq h(C_2 \backslash B)$. It follows that $\hat{h}^{-1}(w) \simeq \boldsymbol{P}^1$ for any general $w \in W$.

STEP 3. Let $\hat{Z}_i := \hat{h}^{-1}(f_i)$, $Z_i := \varphi(\hat{Z}_i)_{\mathrm{red}}$. Then since $\hat{h}^{-1}(w)$ is irreducible, Z_i is an irreducible component of σ outside B. Since $\mathrm{Bs}\,|O_Q(l_i)| = \emptyset$, $\hat{Z}_i$ is smooth outside $\mathrm{Sing}\,\hat{D}$, whence smooth outside $\hat{B}$. As $X \backslash B \simeq \hat{X} \backslash \hat{B}$, we have $Z_i \backslash B \simeq \hat{Z}_i \backslash \hat{B}$, whence Z_i is smooth outside B. Moreover $Z_1 \cap Z_2 \backslash B \simeq \hat{Z}_1 \cap \hat{Z}_2 \backslash \hat{B} \simeq \hat{h}^{-1}(f_1 \cap f_2) \backslash \hat{B}$. Hence $Z_1 \cap Z_2$ is a smooth rational curve $C := \varphi(\hat{h}^{-1}(f_1 \cap f_2))$ outside B, along which Z_1 and Z_2 intersect transversally.

Let $\nu_i\colon Y_i \to Z_i$ be the normalization of Z_i, $f_i\colon S_i \to Y_i$ the minimal resolution of Y_i, $g_i = \nu_i \cdot f_i$. Then as in the proof of Lemma 3.2, there exist effective Cartier divisors E_i and G_i on S_i with no components in common such that the canonical sheaf of S_i is given by $K_{S_i} = O_{S_i}(-2g_i^*L - E_i - G_i)$. Let A_i be a unique smooth rational curve on S_i such that $g_i(A_i) = C$. Then since $Z_1 \cap Z_2 \simeq C$ generically along C, we have by Theorem 1.14

$$A_i \subset E_i, \qquad A_i \not\subset E'_i := E_i - A_i, \qquad E'_{i,\mathrm{red}} + G_{i,\mathrm{red}} \subset g_i^{-1}(B)_{\mathrm{red}}$$

if f_1 and f_2 are sufficiently general. See also the proof of Lemma 5.3.

Let M_i (resp. N_i) be the movable part (resp. the fixed part) of g_i^*D'. Then since $\varphi^*L \simeq \hat{h}^*O_W(1)$ on $\hat{X} \backslash \hat{B}$, $L_{Z_i} \simeq ((\varphi_{|\hat{Z}_i \backslash \hat{B}})^{-1})^* \hat{h}^* O_Q(f_1 + f_2) \simeq O_{Z_i}(A_i)$ on $Z_i \backslash B$. Hence we have $A_i \in |M_i|$. It follows that $-K_{S_i} = 3A_i + 2N_i + E'_i + G_i$. Hence S_i is either $\boldsymbol{P}^2$ or ruled.

CASE 1. Assume $S_1 \simeq \boldsymbol{P}^2$. Since Z_i is algebraically equivalent to each other by Lemma 5.5, we may assume that $S_2 \simeq \boldsymbol{P}^2$. Then we have $A_i \in |O_{S_i}(1)|$ and $N_i = E'_i = G_i = 0$. Hence $S_i \simeq Y_i$. By the argument in the proof of Lemma 5.3 CASES 1–1 and 1–2, we see $D \cap D'' = Z_1 \cup Z_2$, $h^0(X, L) = 6$, which contradicts $h^0(X, L) = 5$.

CASE 2. Assume that S_1 is ruled. By Lemma 5.5, we may assume that S_2 is also ruled. Let π_i: $S_i \to \boldsymbol{P}^1$ be a ruling of S_i with $\tilde{F}_i$ a general fiber. Then we have

$$2 = -K_{S_i}\tilde{F}_i = (3A_i + 2N_i + E'_i + G_i)\tilde{F}_i,$$

whence $A_i\tilde{F}_i = 0$. Hence $A_i \in |\tilde{F}_i|$. There exists a unique irreducible component Γ_i of N_i such that $\Gamma_i\tilde{F}_i = 1$ because $E'_{i,\mathrm{red}} + G_{i,\mathrm{red}} \subset N_i$. Hence we have $E'_i\tilde{F}_i = G_i\tilde{F}_i = 0$.

Let $C'_i := g_i(M_i)$. Then since $l \simeq \sigma \cap D' \simeq \tau \cap D''$, l has C'_1 and C'_2 as irreducible components outside B. Hence we may assume by STEP 2 that $C_i = C'_i$ $(i = 1, 2)$.

STEP 4. Next we prove that $g_i(\Gamma_i)$ is a curve $(i = 1, 2)$. Assume the contrary. Hence $g_1(\Gamma_1)$ is a point, say $p \in B$. By STEP 2 $(LC)_X = 1$ where $C = \varphi(\hat{h}^{-1}(f_1 \cap f_2))$. Hence C passes through a unique point of B, hence through the point p by $(\Gamma_1 A_1)_{S_1} = (\Gamma_1\tilde{F}_1)_{S_1} = 1$. As Z_1 and Z_2 are algebraically equivalent, $g_2(\Gamma_2)$ is also a point of B, which C passes through too. Hence $g_2(\Gamma_2) = p$. It follows that C_1 and C_2 intersect at p, which contradicts that C_1 and C_2 intersect $g(\Gamma)$ transversally at distinct points. Hence $g_i(\Gamma_i)$ is a curve component of B by $\Gamma_i \subset N_i \subset g_i^{-1}(B)$. Since Z_1 and Z_2 are algebraically equivalent, $g_2(\Gamma_2) \subset Z_2$ implies $g_2(\Gamma_2) \subset Z_1$, whence $g_2(\Gamma_2) \subset Z_1 \cap Z_2$. By (2.1.2) and by STEP 2, C_i intersects a unique curve component $g(\Gamma)$ of B, while C_i intersects $g_i(\Gamma_i)$ by $(M_i\Gamma_i)_{S_i} = 1$. Hence $g(\Gamma) = g_1(\Gamma_1) = g_2(\Gamma_2)$. However since $g_i(\Gamma_i) \subset Z_1 \cap Z_2$, we have $\Gamma_1 \subset E'_1$ and $\Gamma_2 \subset E'_2$ by Theorem 1.14, which contradicts $\Gamma_i\tilde{F}_i = 1$, $E'_i\tilde{F}_i = 0$. q.e.d.

(5.7) COMPLETION OF THE PROOF OF THEOREM 5.1. Since $\dim W = 4$, $D_1 \cap D_2 \cap D_3$ is irreducible nonsingular outside B for general D_i by Bertini's theorem. Let C be the unique irreducible component of $D_1 \cap D_2 \cap D_3$ outside B. If $LC = 0$, then C is mapped to a point by the rational map associated with $|pL|$ for any p. Since C sweeps out an open subset of X, this contradicts that X is Moishezon with $b_2(X) = 1$. Hence $LC \geq 1$. In view of Lemma 2.1, $LC = 1$ or 2. If $LC = 1$, then (2.1.2) is possible for D_i general. In this case, $C \cap B \neq \emptyset$, whence C is mapped to a point by the rational map h. This shows that $\dim W \leq 3$, which contradicts Lemma 5.6. Thus (2.1.1) only is possible, so that $LC=2$. Theorem 5.1 now follows from Proposition 3.1. q.e.d.

(5.8) REMARK. It is plausible that Theorem 5.1 is true by assuming only $h^0(X,L)\geq 3$ instead of $h^0(X,L)\geq 5$. We were however unable to prove even that $h^0(X,L)=4$ is impossible. Here we make some comments.

Assume that X is a Moishezon 4-fold homeomorphic to $\boldsymbol{Q}^4$ with $h^0(X,L)=4$. Then the proof in Lemmas 5.2–5.7 fails only at two points in the proofs of Lemmas 5.3 and 5.4. The first point is corrected by a slight modification of the previous proof, while it is difficult to do so for the second.

In what follows we keep the previous notation. First in the proof of Lemma 5.3 CASE 1–1, $m=3$, $d=m-1=2$ and W is a (possibly singular) quadric surface in $\boldsymbol{P}^3$. Then W has a reducible hyperplane section, which contradicts $\operatorname{Pic} X\simeq \boldsymbol{Z}L$.

The second point is in the proof of Lemma 5.4, where $W\simeq\boldsymbol{P}^3$. In view of the proof of Lemma 5.4, S is ruled, $(M+N)F=1$ and $FE=FG=0$. Moreover we see that there are two cases.

CASE 1. $M=\Gamma$, $\Gamma F=1$.

CASE 2. $M=F$, $\Gamma F=1$, $\Gamma\subset N$.

Claim 5.8.1. CASE 1 *is impossible.*

NOTE. We do not know whether CASE 2 is impossible.

Proof. By the proof of Lemma 5.4, we have $K_S=-(2g^*(L)+E+G)=-(2\Gamma+2N+E+G)$. By Lemma 2.1, the movable component $g(\Gamma)$ of the scheme-theoretic complete intersection l of 3 general members of $|L|$ is a smooth rational curve. Therefore $\Gamma\simeq g(\Gamma)\simeq\boldsymbol{P}^1$ and S is a rational ruled surface. We also see that any general mamber of $|L|$ as well as a complete intersection τ of two general members of $|L|$ is smooth along $g(\Gamma)$ by Lemma 2.1. We have

$$2=-(K_S+\Gamma)\Gamma=(\Gamma+2N+E+G)\Gamma.$$

Since Γ is movable, we have $\Gamma^2\geq 0$, whence $0\leq\Gamma^2\leq 2$.

Now we prove $\Gamma^2=0$. If $\Gamma^2=2$, then $N\Gamma=E\Gamma=G\Gamma=0$. Therefore $B\cap g(\Gamma)=\emptyset$. On the other hand, $h(g(\Gamma))$ is one point. However we have

$$2=((\Gamma+N)\Gamma)_S=(g^*(L)\Gamma)_S=(Lg_*(\Gamma))_X=\deg \operatorname{Bs} g^*|L|_\Gamma=0,$$

which is a contradiction. Next we assume $\Gamma^2=1$. Then $N\Gamma=0$, $(E+G)\Gamma=1$, which contradicts $E_{\mathrm{red}}+G_{\mathrm{red}}\leq N_{\mathrm{red}}$. Therefore $\Gamma^2=0$, $N\Gamma=1$ and $E\Gamma=G\Gamma=0$.

Since $\Gamma^2=0$, $F^2=0$ and $\Gamma F=1$, we have a birational morphism η: $S\to \boldsymbol{P}^1\times\boldsymbol{P}^1$ ($\subset\boldsymbol{P}^3$) associated with the linear system $|\Gamma+F|$. The surface S is obtained from $\boldsymbol{P}^1\times\boldsymbol{P}^1$ by repeating blowing-ups. Since $K_S=-(2\Gamma+2N+E+G)$ and since $E_{\mathrm{red}}+G_{\mathrm{red}}\leq N_{\mathrm{red}}$, any irreducible component of $2N+E+G$ has mutilplicity at least two in $-K_S$. Therefore the anticanonical divisor $-K_{\boldsymbol{P}^1\times\boldsymbol{P}^1}$ is an effective divisor with multiple components only. That is, $-K_{\boldsymbol{P}^1\times\boldsymbol{P}^1}=-2(\eta_*(\Gamma)+\eta_*(F))$ for some fiber F with $2\eta_*(F)=\eta_*(2N+E+G)$. Since $\Gamma^2=0$, the centers of blowing-ups are chosen from the outside of $\eta(\Gamma)$ (or its proper transform). Hence the proper transform of the (-1)-curve arising from (any of) the first blowing-up appears in $2N+E+G$ as a component with multiplicity exactly one. This is a contradiction if S is not isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$. It follows that $S\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$, $N=F$ and $E=G=0$, whence $S\simeq Y\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$. Since $E=0$, $S\simeq Z$ outside a finite set Σ of Z.

Moreover Z intersects the other irreducible components of τ in a subset of Σ only. Since τ is Gorenstein and connected by (5.2.5), this implies that $\tau_{\mathrm{red}}\simeq Z$. As τ is Gorenstein and generically reduced along Z, τ is reduced everywhere, so that $\tau\simeq Z$. Therefore Z is Gorenstein and has isolated singularities only, whence Z is normal. Consequently $\tau\simeq Z\simeq S\simeq\boldsymbol{P}^1\times\boldsymbol{P}^1$. Hence $h^0(X,L)=h^0(\tau,L_\tau)+2=6$ by (5.2.6), contradicting the assumption $h^0(X,L)=4$. q.e.d.

NOTATION.

$\boldsymbol{A}^n$	Spec $\boldsymbol{C}[x_1, \cdots, x_n]$
B, Bs$\|L\|$	the scheme-theoretic base locus of $\|L\|$
$B^*, \hat{B}, \hat{B}^*$	(3.5), (4.1)
$c(E)$	the total Chern class $\sum_{i \in \boldsymbol{Z}} c_i(E)$ of a vector bundle E
$c_i(E)$	the i-th Chern class of a vector bundle E
$c_i(X)$	the i-th Chern class of X
$e(Q_C^{(\nu)}, p)$	(1.10), (1.11), [15, (2.6)]
$e(Q_V^{(\nu)}, B_U)$	(1.12), (1.13), [15, (2.A)]
$\boldsymbol{F}_b$	Proj$(O_{\boldsymbol{P}^1}(b) \oplus O_{\boldsymbol{P}^1})$
$\bar{\boldsymbol{F}}_2$	$\boldsymbol{F}_2$ with the unique (-2)-curve contracted,
	isomorphic to a singular quadric surface in $\boldsymbol{P}^3$
$g^*\|L\|$	$\{g^*D \,;\, D \in \|L\|\}$
$h^q(X,F)$	dim $H^q(X,F)$ for a coherent sheaf F
$\hat{l}(l^*), l(l^*)$	(4.1)
$l(Q_C^{(\nu)}, p)$	(1.2), (1.3)
$l_q, \hat{l}_q, l_q^*$	(4.3.2)
$N_{C/X}$	the normal bundle of C in X
O_X, O_S, O_Z	the structure sheaf of X, S, Z respectively
$\hat{O}_X$	the formal completion of O_X
Pic X	$H^1(X, O_X^*)$
$P(l^*)$	(4.3.5)
R^*	(3.5), (4.1)
$S \xrightarrow{\nu} Y \xrightarrow{f} Z$	(3.2)
σ_q	(4.3.2)
$W, X \xrightarrow{h} W$	(3.2), (4.1), (5.3)
$\hat{X} \xrightarrow{\hat{h}} W$	(3.5), (4.1)
$\hat{X} \xrightarrow{\varphi} X$	(3.5), (4.1)
$\chi(X,F)$	$\sum_{q \in \boldsymbol{Z}} (-1)^q h^q(X,F)$
$(\ )_S, (\ )_X$	the intersection numbers on S, X

Bibliography

[1] M. Artin: *On the solutions of analytic equations*, Invent. Math. **5** (1968), 277–291.

[2] A. Altman-S. Kleiman: Introduction to Grothendieck duality theory, Lecture Notes in Math. **146**, Springer Verlag (1970).

[3] E. Brieskorn: *Ein Satz über die komplexen Quadriken*, Math. Ann. **155** (1964), 184–193.

[4] W. Barth-A. Van de Ven: *Fano varieities of lines on hypersurfaces*, Arch. Math. **31** (1978), 94–104.

[5] D. Eisenbud-J. Harris: *On varieties of minimal degree*, Algebraic Geometry in Bowdoin 1985, Proc. Symp. Pure Math. **46** (1987), 3–14.

[6] H. Hironaka: *An example of non-Kaehlerian complex-analytic deformation of Kaehlerian complex structures*, Ann. of Math. **75** (1962), 190–208.

[7] J. Kollar: *Flips, flops, minimal models etc.*, Preprint (1990).

[8] S. Kobayashi-T. Ochiai: *Characterizations of complex projective spaces and hyperquadrics*, J.Math. Kyoto Univ. **13** (1973), 31–47.

[9] S. Mori: *Projective manifolds with ample tangent bundles*, Ann. of Math. **110** (1979), 593–606.

[10] S. Mori: *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math. **116** (1982), 133–176.

[11] J. Morrow: *A survey of some results on complex Kähler manifolds*, Global Analysis, Univ. Tokyo Press and Princeton University Press (1969), 315–324.

[12] I. Nakamura: *Moishezon threefolds homeomorphic to* $\boldsymbol{P}^3$, J. Math. Soc. Japan **39** (1987), 521–535.

[13] I. Nakamura: *Threefolds homeomorphic to a hyperquadric in* $\boldsymbol{P}^4$, Algebraic Geometry and Commutative Algebra in Honor of M. Nagata, Kinokuniya, Tokyo Japan (1987), 379–404.

[14] I. Nakamura: *A subadjunction formula and Moishezon fourfolds nomeomorphic to* $\boldsymbol{P}_C^4$, Proc. Japan Acad. **67A** (1991), 65–67.

[15] I. Nakamura: *On Moishezon manifolds homeomorphic to* $\boldsymbol{P}_C^n$, Jour. Math. Soc. Japan **44** (1992), 667–692.

[16] T. Peternell: *A rigidity theorem for* $\boldsymbol{P}_3(\boldsymbol{C})$, Manuscripta Math. **50** (1985), 397–428.

[17] T. Peternell: *Algebraic structures on certain 3-folds*, Math. Ann. **274** (1986), 133–156.

[18] K. Ueno: Classification theory of algebraic varieties and compact complex spaces, Lectures Notes in Math. **439**, Springer-Verlag, Berlin, Heidelberg, New York (1975).

Depatrment of Mathematics
Hokkaido University
Sapporo, 060, Japan