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Osaka J. Math.
31 (1994), 663–728

THE ASYMPTOTIC EXPANSION OF THE FUNDAMENTAL SOLUTION FOR PARABOLIC INITIAL-BOUNDARY VALUE PROBLEMS AND ITS APPLICATION

Dedicated to Professor Hiroki Tanabe for his 60th birthday

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(Received October 22, 1992)

0. Introduction

Let M be a smooth compact Riemannian manifold of dimension n with smooth boundary Γ. In this paper we consider parabolic initial-boundary value problems as follow:

$$\begin{cases} (\frac{\partial}{\partial t}+P)u(t,x)=0 & \text{in } (0,T)\times M, \\ Bu(t,x)=0 & \text{on } (0,T)\times\Gamma, \\ u(0,x)=m(x) & \text{in } M, \end{cases}$$

where $P=-\Delta+h$ with a smooth vector field h on M of complex coefficients. The boundary operator B which we consider in this paper is related to one of the following conditions with smooth coefficients.

($\mathscr{D}$) the Dirichlet condition,

($\mathscr{N}$) the Neumann condition,

($\mathscr{R}$) the Robin's condition,

($\mathscr{O}$) the Oblique condition with parabolic condition, that is, $B=-\frac{\partial}{\partial n}+b(x,D)$ with the outer unit normal vector field $\frac{\partial}{\partial n}$ and a vector field $b(x,D)$ satisfying (3.2) in §3

and

($\mathscr{S}$) the Singular boundary condition $B=-a(x)\frac{\partial}{\partial n}+\mathrm{b}(x)$ with the following assumption (*) (See (3.3) for more general cases including that B may depend on t.)

$(*) a(x) \geq 0, \quad b(x) < 0 \quad \text{when } a(x) = 0.$

We note that ($\mathscr{S}$) is not a parabolic boundary value problem in the sense of [1].

For each one of the above boundary conditions we construct an asymptotic expansion of the fumdamental solution by means of the calculus of the pseudo-differential operators. This asymptotic expansion leads us both to the construction of the fundamental solution and to the asymptotic behavior of $T_t(\mathscr{B}) = (4\pi t)^{n/2} \sum_{j=1}^{\infty} \exp(-t\lambda_j)$ when t tends to 0, where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of elliptic (subelliptic in case ($\mathscr{S}$)) problem (P,B), if the boundary operator B is independent of t. In this paper the asymptotic expansion of the fundamental solution can be represented directly by functions $p(x,\xi)$ and $b(t,x,\xi)$ which are symbols of P and B. This fact is also applicable to the proof of the Gauss-Bonnet-Chern theorem for a manifold with boundary. About this problem we discuss in the forthcoming paper [7].

The construction of the fundamental solution for the general parabolic boundary problems was staudied in [1]. Roughly speaking, there are two methods of its construction applicable to get the behavior of $T_t(\mathscr{B})$ directly. The one method is to use the fundmantal solution for the Cauchy problem on M', the double of M. This method is adapted to the problem ($\mathscr{D}$) and ($\mathscr{N}$) by MeKean-Singer [10]. They extended P to an operator P' defined in M'. In this case they miss the smoothess of the coefficients of the operator P' even if P has smooth coefficients. The other is to reduce the construction of the fundamental solution to the construction of the Green operator of the boundary valeu problem (P,B), using the Laplace transformation. One we solve the Direchlet problem, construction of the Green operator of the boundary value problem (P,B) can be reduced to solving an equation of pseudo-differntial opeators on Γ. This method was apdapted by P.C. Greiner [4] and he calculated $T_t(\mathscr{D})$ in case of M is a bounded domain in $\boldsymbol{R}^2$.

For the singular boundary value problem ($\mathscr{S}$), we give some commets. S. Ito [5] constructed the funcamental solution in case $b(t,x) = a(t,x) - 1$. Y. Kannai [9] showed the existence of the solution of ($\mathscr{S}$) under the compatibility condition for the initial data $m(x)$. K. Taira [15] obtains the existence of the fundamental solution by operator theory. About the condtion (*), S. Mizohata [11] showed that the assumption (*) is necessary for $\mathbf{H}^{\infty}$ well-posedness of the problem. K.

Taira [14] has shown that the main term of $T_t(\mathscr{S})$ is $|M|$.

The Green operator for an elliptic boundary value probme (P,B) is obtained by the integration of the fundamental solution $\int_0^T E(t)e^{-\lambda t}dt$ for any positive constant T and some positive constant λ. For example, singularities of the kernel of the Green operator can be studied by this method (cf. D. Fujiwara [3], R.T. Seely [12]).

Although we treat, in this paper, operators acting on functions on M, we can apply our method to a parabolic system whose principal symbol is diagonal.

In §1 we present main theorems of this paper. The reviews of both the theory for pseudo-differential operators and construction of the fundmanetal slutions of the Cauchy prblem are stated in §2. The construction of the asymptotic exampansion of the fundamental solution for intial-boundary value problem in $\boldsymbol{R}^n_+$ are discussed in §3. Section 4 is devoted to the construction of an asymptotic expansion of the Poisson operator in $\boldsymbol{R}^n_+$. In §5 we discuss $\mathbf{L}^p$ theory for our operator. In §6 we construct the fundamental solution $E(t)$. In §7 applications to the behavior of $T_t(\mathscr{B})$ are treated.

1. Main theorems

Let P be a strongly elliptic differential opertor of the second order on M, that is, $P=-\Delta+\mathrm{h}$, where h is a vector field on M with complex coefficients. The purpose of this paper is constructing the fundamental solution for the boundary value problem $(\mathscr{B})$ as stated in Introduction.

We say that an operator $E(t)$ is the fundmamental solution for $(\mathscr{B})$ it $E(t)$ satisfies

$$(\mathscr{B})\qquad \begin{cases} LE(t)=0 & \text{in } (0,T)\times M, \\ BE(t)=0 & \text{on } (0,T)\times \Gamma, \\ E(0)=I & \text{in } M, \end{cases}$$

where B is one of operators stated in Introduction. For the construction of the fundamental solution we have:

Theorem I (The existence of the solution). *We can construct the fundamental solution $E(t)$ for $(\mathscr{B})$ such that for any $1<p<\infty$ and $m\in\mathbf{L}^p(M)$ $u(t)=E(t)m$ belongs to $C([0,T];\ \mathbf{L}^p(M))$ and $\cap_s\mathbf{H}^s_p(M)$ for $t>0$, satisfying $u(t)\to m\in\mathbf{L}^p$ as $t\to 0$.*

Corollary. *For any* $m \in C(M)$ *there exists a solution* $u(t,x) \in C^{\infty}((0,T)\times M)$ *of* $(\mathscr{B})$ *with*

$$\lim_{t\to 0} u(t,x) = m(x), \qquad x \in M.$$

Owing to the precise calculus of the asymptotic expansion of the fundamental solution $E(t)$, we get the folowing theorem.

Theorem II. *For the problem* $(\mathscr{D})$, $(\mathscr{N})$, $(\mathscr{R})$ *and* $(\mathscr{O})$ *we have the following expansion* $T_t(\mathscr{B}) = \sum_{j=0}^{\infty} C_j(\mathscr{B}) t^{\frac{j}{2}}$ *as* $t \to 0$:

For any boundary problem $(\mathscr{B})$ *as stated above, we have*

$$C_0(\mathscr{B}) = |M|, \tag{0}$$

where $|M|$ *means the volume of* M *induced by the Riemannian metric* g. *The second terms* $C_1(\mathscr{B})$ *are*

$$\tag{1} \begin{cases} C_1(\mathscr{D}) = -\dfrac{\sqrt{\pi}}{2}|\Gamma|, \\[2ex] C_1(\mathscr{N}) = \dfrac{\sqrt{\pi}}{2}|\Gamma|, \\[2ex] C_1(\mathscr{R}) = \dfrac{\sqrt{\pi}}{2}|\Gamma|, \\[2ex] C_1(\mathscr{O}) = \sqrt{\pi}\displaystyle\int_{\Gamma}\Big(\frac{1}{\sqrt{1+\|d_1\|^2-\|d_2\|^2+2<d_1,d_2>i}}-\frac{1}{2}\Big)dS, \end{cases}$$

where d_1 *and* d_2 *are real vector fields on* Γ *such that* $b(x,D) = d_1 + d_2$ *and* $\|d\|$ *means the norm of a vector field* d *induced by the metric of* Γ. *The third terms* $C_2(\mathscr{B})$ *are given by*

$$\tag{2} \begin{cases} C_2(\mathscr{D}) = \displaystyle\int_M \Big(\frac{K}{3}-\frac{\|h\|^2}{4}\Big)dV - \int_{\Gamma}\frac{J}{6}\,dS, \\[2ex] C_2(\mathscr{N}) = C_2(\mathscr{D}) + \displaystyle\int_{\Gamma} \text{flux}\ h\ dS, \\[2ex] C_2(\mathscr{R}) = C_2(\mathscr{N}) + 2\displaystyle\int_{\Gamma} b\ dS, \end{cases}$$

where K is the scalar curvature and J is the mean curvature. For the singular problem we have

$$T_t(\mathscr{S})=|M|+\frac{\sqrt{\pi t}}{2}(|\Gamma_1|-|\Gamma_0|)+o(t^{\frac{1}{2}}) \tag{3}$$

under the assumption $|\Gamma_0|>0$, *where*

$$\Gamma_0=\{x\in\Gamma; a(x)=0\},\ \Gamma_1=\Gamma\backslash\Gamma_0.$$

REMARK. If the vector field $b(t,x,D)$ has real coefficients, we have

$$C_1(\mathscr{D})<C_1(\mathscr{O})\leq C_1(\mathscr{N}).$$

Moreover $C_1(\mathscr{O})=C_1(\mathscr{N})$ holds if and only if b vanishes everywhere.

We remark that L. Smith [13] and T.P. Branson-P.B. Gilkey [2] computed $C_3(\mathscr{D})$, $C_4(\mathscr{D})$, $C_3(\mathscr{N})$, $C_4(\mathscr{N})$, $C_3(\mathscr{R})$, $C_4(\mathscr{R})$ by different methods.

2. Pseudo-differential operators and the fundamental solution for the Cauchy problem

We introduce some notations on pseudo-differantial operators.

DEFINITION 1. For a symbol of pseudo-differential operators $p(x,\xi)\in S^m_{\rho,\delta}(\boldsymbol{R}^n)=S^m_{\rho,\delta}$ $(0\leq\delta\leq\rho\leq 1, \delta<1)$, we define the seminorms $|p|_l^{(m)}$ $(l=0,1,2,\cdots,)$ by

$$|p|_l^{(m)}=\max_{|\alpha|+|\beta|\leq l}\ \sup_{(x,\xi)\in\boldsymbol{R}^n\times\boldsymbol{R}^n}\{|p^{(\alpha)}_{(\beta)}(x,\xi)|<\xi>^{-m+\rho|\alpha|-\delta|\beta|}\}.$$

We denote a pseudo-differential operator by the capital P of which symbol is $p(x,\xi)$. For a symbol $p(t;x,\xi)\in C(S^m_{\rho,\delta})$ we define a pseudo-differential operator with parameter t by

$$P(t)u(x)=P(t;x,D)u(x)=Os-(2\pi)^{-n}\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}e^{i(x-y)\cdot\xi}p(t;x;\xi)u(y)dyd\xi.$$

DEFINITION 2. Let $p\circ q$ denote the symbol of product operator $p(x,D)q(x,D)$. So we have

$$p\circ q(x,\xi)=Os-(2\pi)^{-n}\int_{\boldsymbol{R}^n}\int_{\boldsymbol{R}^n}e^{-iy\cdot\eta}p(x,\xi+\eta)q(x+y,\xi)dyd\eta.$$

The basic theorems for the symbol of multi product of pseudo-differential operators are as follow.

Theorem A. *If p_j belong to $S^{m(j)}_{\rho,\delta}$ $(j=1,\cdots,\nu)$, then $p_1\circ\cdots\circ p_\nu=p$ belongs to $S^m_{\rho,\delta}$ $(m=\sum_{j=1}^{\nu} m(j))$ and satisfies the following estimate for any l.*

$$|p|_l^{(m)}\leq C^\nu\prod_{j=1}^{\nu}|p_j|_{l+l_0}^{(m(j))},$$

where C and l_0 are constants independent of ν.

Theorem B. *Let $p\in S^{m_1}_{\rho,\delta}$ and $q\in S^{m_2}_{\rho,\delta}$. Then for any integer N we have an expansion*

$$p\circ q=\sum_{j=0}^{N-1}s_j(p,q)+r_N(p,q),$$

where

$$s_j(p,q)=\sum_{|\alpha|=j}\frac{1}{\alpha!}\Big(\frac{\partial}{\partial\xi}\Big)^\alpha p(x,\xi)D_x^\alpha q(x,\xi)\in S^{m-(\rho-\delta)j}_{\rho,\delta}$$

and $r_N(p,q)\in S^{m-(\rho-\delta)N}_{\rho,\delta}$ has the estimate

$$|r_N|_l^{(m-(\rho-\delta)N)}\leq C\sum_{|\alpha|=N}|p^{(\alpha)}|_{l+l_0}^{(m_1-\rho|\alpha|)}|q_{(\alpha)}|_{l+l_0}^{(m_2+\delta|\alpha|)}.$$

We review the construction of the fundamental solution $U(t)$

$$\begin{cases} LU=\Big(\dfrac{d}{dt}+P\Big)U(t)=0 & \text{in } (0,T)\times \boldsymbol{R}^n,\\ U(0)=I & \text{on } \boldsymbol{R}^n,\end{cases}$$

for the Cauchy problem on $\boldsymbol{R}^n$ according to Tsutsumi [16]. Here P is a strongly elliptic differential operator of second order defined on $\boldsymbol{R}^n$ of which symbol is $p(x,\xi)$. Let $p(x,\xi)=p_2(x,\xi)+p_1(x,\xi)+p_0(x,\xi)$, where $p_j(x,\xi)$ are homogeneous of order j with respect to ξ.

Theorem C. *The fundamental solution $U(t)$ is constructed as a pseudo-differential operator of a symbol $u(t)$ belonging to $S^0_{1,0}$ with parameter t. Moreover $u(t)$ has the following expansion for any N:*

$$u(t)-\sum_{j=0}^{N-1} u_j(t) \text{ belongs to } S^{-N}_{1,0},$$

$$u_0(t)=\exp(-p_2 t),\ u_j(t)=f_j(t)u_0(t)\in S^{-j}_{1,0},$$

where $f_j(t)$ are polynomials with respect to ξ and t, satisfying the equation $k-2l=-j$, where k is the degree of ξ and l is that of t.

The sketch of the proof of Theorem C is the following.
$\{f_j(t;x,\xi)\}_{j\geq 1}$ are obtained as the solution of the following ordinary differential opeators with parameter (x,ξ).

$$(2.1)\qquad \begin{cases} \dfrac{df_j}{dt}u_0+\displaystyle\sum_{k+l+m=j,k\geq 0,m<j} s_k(p_{2-l},f_m u_0)=0, & t>0, \\ f_j|_{t=0}=0. \end{cases}$$

In fact, for example, we have

$$(2.2)\qquad \begin{cases} f_1=-p_1 t+\dfrac{t^2}{2}s_1(p_2,p_2), \\ f_2=-p_0 t+\dfrac{t^2}{2}\{(p_1)^2+s_1(p_1,p_2)+s_1(p_2,p_1)+s_2(p_2,p_2)\} \\ \qquad +\dfrac{t^3}{6}\{\displaystyle\sum_{j,k=1}^{n}(\frac{\partial}{\partial x_j})p_2(\frac{\partial}{\partial x_k})p_2(\frac{\partial}{\partial \xi_j})(\frac{\partial}{\partial \xi_k})p_2-s_1(p_2,s_1(p_2,p_2)) \\ \qquad -3p_1 s_1(p_2,p_2)\}+\dfrac{t^4}{8}\{s_1(p_2,p_2)\}. \end{cases}$$

For any $N\geq 1$, $\sum_{j=0}^{N-1}u_j=g_N$ satisfies according to (2.1)

$$\begin{cases} \dfrac{dg_N}{dt}+p\circ g_N=r_N, \\ g_N|_{t=0}=1, \end{cases}$$

where r_N belongs to $C(S_{1,0}^{-N+2})$ and satisfies

$$|r_{N(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta} t^l <\xi>^{-N+2+2l-|\alpha|} \tag{2.3}$$

for any $l \leq \frac{N}{2}-2$. The symbol of the fundamental solution is obtained as the solution of the form

$$u(t) = g_N(t) + \int_0^t g_N(t-s) \circ \varphi(s) ds, \tag{2.4}$$

where $\varphi(t)$ is the solution of

$$r_N(t) + \varphi(t) + \int_0^t r_N(t-s) \circ \varphi(s) ds = 0. \tag{2.5}$$

For solving (2.5) we apply the estimate of the symbol of multi-product of pseudo-differential operators in $S_{\rho,\delta}^0$ stated in Theorem A. Then we obtain the solution $\varphi(t)$ in $S_{1,0}^{-N+2}$. Also we have the estimate by (2.3)

$$|\varphi_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta} t^l <\xi>^{-N+2+2l-|\alpha|} \tag{2.6}$$

for any $l \leq \frac{N}{2}-2$. Thus we have $u(t) - g_N(t) \in S_{1,0}^{-N+2}$. Also we have by (2.4), (2.6) and Theorem A

$$|\{u(t) - g_N(t)\}_{(\beta)}^{(\alpha)}| \leq C_{\alpha,\beta} t^{l+1} <\xi>^{-N+2+2l-|\alpha|} \tag{2.7}$$

for any $l \leq \frac{N}{2}-2$. Nothing N is any number, we get Theorem C.

q.e.d.

The kernel of $U(t) = u(t;x,D)$ is given by the integral

$$U(t,x,y) = (2\pi)^{-n} \int_{\mathbf{R}^n} u(t;x,\xi) e^{i(x-y)\cdot\xi} d\xi = u^\natural(t;x,x-y).$$

For $u^\natural(t;x,z)$ we have the following expansion for any $N \geq 1$

$$u^\natural(t;x,z) = \sum_{j=0}^{N-1} u_j^\natural(t;x,z) + k_N(t;x,z),$$

where $u_j^\natural(t;x,z) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{iz\cdot\xi} u_j(t;x,\xi) d\xi$ and $k_N(t;x,z)$ have the following estimates for some positive contant δ

$$|u_j^\natural(t;x,z)|\le Ct^{-\frac{n}{2}+\frac{j}{2}}e^{-\delta\frac{|z|^2}{4t}} \qquad (j=0,1,\cdots,N-1),$$

$$u_j^\natural(t;x,0)=0 \qquad j=\text{odd},$$

$$|k_N(t;x,z)|\le Ct^{-\frac{n}{2}+\frac{N}{2}},$$

where we use (2.7) and the fact that N in Theorem C may be taken any number. So we have the expansion

$$U(t;x,x)=u^\natural(t;x,0)\sim\sum_{j=0}^{\infty}t^{-\frac{n}{2}+j}C_j(x),$$

where

$$C_j(x)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}u_{2j}(1;x,\xi)d\xi=u_{2j}^\natural(1;x,0).$$

3. Construction of an asymptotic expansion of the fundamental solution on $\boldsymbol{R}^n_+$

In this section we construct an asymptotic expansion of the fundamental solution $E(t)$ of the following problem in $I\times\boldsymbol{R}^n_+$:

$$(L,B)\qquad\begin{cases}(\frac{d}{dt}+P)u(t)=0 & \text{in } I\times\boldsymbol{R}^n_+,\\ Bu(t)=0 & \text{on } I\times\boldsymbol{R}^{n-1}\times\{x_n=0\},\\ \lim_{t\to 0}u(t)=m(x) & \text{in } \boldsymbol{R}^n_+.\end{cases}$$

We use the following notations. $I=(0,T)$, $\boldsymbol{R}^n_+=\{x=(x',x_n)\colon x'\in\boldsymbol{R}^{n-1}, x_n>0\}$, P is the similar operator defined in §2 and the boundary operator B is one of operators introduced in §0.

If we assume $E(t)=U(t)+V(t)$, where $U(t)$ is the fundamental solution for the Cauchy problem in $\boldsymbol{R}^n$, $V(t)$ must satisfy

$$\begin{cases} (\frac{d}{dt}+P)V(t)=0 & \text{in } I\times \boldsymbol{R}^n_+, \\ BV(t)=-BU(t) & \text{on } \mathrm{I}\times \boldsymbol{R}^{n-1}\times\{x_n=0\}, \\ \lim_{t\to 0} V(t)=0 & \text{in } \boldsymbol{R}^n_+. \end{cases}$$

We assume the principal symbol $p_2(x,\xi)$ of P satisfies for some positive constant α

$$\text{(3.1)} \qquad \begin{cases} p_2(x',0,\xi',\xi_n)=\xi_n^2+\beta(x',\xi), \\ \beta(x',\xi')\geq\alpha|\xi'|^2. \end{cases}$$

In this section we consider the following boundary operator B.

$$B=\text{identity},\ (\frac{\partial}{\partial x_n}),\ (\frac{\partial}{\partial x_n})+b(t,x),\ (\frac{\partial}{\partial x_n})+b(t,x',D').$$

The symbol $b(t,x',\xi')$ of $b(t,x',D')$ satisfies

$$\text{(3.2)} \qquad \mathrm{Re}\{\beta(x',\xi')-(b(t,x',\xi'))^2\}\geq C|\xi'|^2$$

for some positive constant C for any $t\in I$.

The above inequality (3.2) coincides with the assumption that a boundary problem $(L.B)$ is parabolic in the sense of [1] for the oblique condition $(\mathscr{O})$. We consider also

$$B=a(t,x')(\frac{\partial}{\partial x_n})+b(t,x'),$$

where $a(t,x')$ and $b(t,x')$ satisfy

$$\text{(3.3)} \qquad \begin{cases} b(t,x)\neq 0 & \text{if} \quad a(t,x')=0, \\ |\arg\frac{a}{b}|\geq\frac{\pi}{4}+\varepsilon \text{ in a neighourhood of } \{(t,x')\colon a(t,x')=0\}, \end{cases}$$

for some positive constant ε. Y. Kannai studied the existence of the solution under the above condition in [9].

In §3–1 we will discuss the construction of the asymptotic expansion of $V(t)$ for $(\mathscr{D})$, $(\mathscr{N})$, $(\mathscr{R})$ and $(\mathscr{O})$ under the restriction that $b(t,x',\xi')$ is indepednent of t. We treat in §3–2 the general case. $V(t)$ for $(\mathscr{S})$ will be constructed in §3–3.

3–1. Asymptotic expansion of $V(t)$ **for** $(\mathscr{D})$, $(\mathscr{N})$, $(\mathscr{R})$ **and** $(\mathscr{O})$. We introduce new symbol classes $\mathscr{F}_s$, $\mathscr{F}_s'$ as follow.

DEFINITION 3. (1) $\mathscr{F}_s$ is the set of all finite sum of the following functions

$$\{t^d(x_n)^l r(x',\xi',\xi_n);\ \text{nonnegative integers } l,\ d,\ r\in S_{1,0}^{s+2d+l}(\boldsymbol{R}^n)\},$$

where $r(x',\xi)$ is a polynomial with respect to ξ.
(2) $\mathscr{F}_s'$ is the set of all finite sum of the following functions

$$\{t^d(x_n)^l r(x,\xi',\xi_n);\ \text{nonnegative integers } l,\ d,\ r\in S_{1,0}^{s+2d+l}(\boldsymbol{R}^n)\},$$

where $r(x,\xi)$ is a polynomial with respect to ξ.

DEFNIITION 4. We define $f^*=f^*(t,x',\xi)=f(t,x',0,\xi)$ for a function $f(t,x,\xi)$ defined on $\boldsymbol{R}^{2n+1}$.

DEFINITION 5. For a function $\varphi(x',x_n)$ defined on $\boldsymbol{R}^n_+$ we define

$$\varphi^-(x',x_n)=\begin{cases}0, & \text{if } x_n>0;\\ \varphi(x',-x_n), & \text{otherwise.}\end{cases} \tag{1}$$

(2) We also use the notation $\varphi^+(x',x_n)$ if we extend the function $\varphi(x',x_n)$ on $\boldsymbol{R}^n_+$ such that

$$\varphi^+(x',x_n)=\begin{cases}\varphi(x',x_n), & \text{if } x_n\geq 0;\\ 0, & \text{othrewise.}\end{cases}$$

DEFINITION 6. Let $\{q_j\}_{j\leq 2}$ be defined as

$$q_2=p_2(x',0,\xi',\xi_n)=p_2^*,$$

$$q_{2-j}=\sum_{l+k=j,0\leq k\leq 2}((\frac{\partial}{\partial x_n})^l p_{2-k})^*\frac{x_n^l}{l!},\quad j\geq 1.$$

Then we have for any N

$$p=\sum_{j=2}^{-N+1} q_j+q'_{-N}$$

with $q_j\in\mathscr{F}_j$ and $q'_{-N}\in\mathscr{F}'_{-N}$.

DEFNITION 7. For a pair (j,k) of integer j and nonpositive integer k we define functions $\{\tilde{w}_{j,k}(t,\omega;\mathrm{b})\}_{j,k}$ as follow:

$$w_{0,0}(t,\xi_n)=\exp(-t\xi_n^2),$$

$$w_{j,0}(t,\xi_n)=(i\xi_n)^j w_{0,0}(t,\xi_n),\ j\geq 0,$$

$$\tilde{w}_{j,0}(t,\omega)=(2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\omega\cdot\xi_n}w_{j,0}(t,\xi_n)d\xi_n,\ j\geq 0,$$

$$\tilde{w}_{j,0}(t,\omega;b)=-\frac{1}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{t}}\right)^{j+1}\int_0^\infty e^{-(\sigma+\frac{\omega}{2\sqrt{t}})^2}\frac{(-\sigma)^{-j-1}}{(-j-1)!}d\sigma,\ j\leq -1,$$

for $k\leq -1$ $\tilde{w}_{j,k}(t,\omega;\mathrm{b})$

$$=\begin{cases} -\dfrac{1}{\sqrt{\pi}}\left(\dfrac{1}{2\sqrt{t}}\right)^{j+k+1}\displaystyle\int_0^\infty e^{-(\sigma+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\frac{(-\sigma)^{-k-1}}{(-k-1)!}h_j\left(\sigma+\frac{\omega}{2\sqrt{\mathrm{t}}}\right)d\sigma, & \text{if } j\geq 0; \\ \dfrac{1}{\sqrt{\pi}}\left(\dfrac{1}{2\sqrt{t}}\right)^{j+k+1}\displaystyle\int_0^\infty\frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau\int_0^\infty e^{-(\sigma+\tau+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\frac{(-\sigma)^{-k-1}}{(-k-1)!}d\sigma, & \text{if } j\leq -1, \end{cases}$$

where $h_j(\sigma)=\{(\frac{\partial}{\partial\sigma})^j e^{-\sigma^2}\}e^{\sigma^2}$. We define an integral operator $\mathrm{W}_{j,k}(t;b)$ with parameters (t,b) for a function $\varphi(y_n)$ defined on $\boldsymbol{R}^1_+$ as follows.

$$\begin{aligned}(W_{j,k}(t;b)\varphi)(x_n)&=(W_{j,k}(b)\varphi)(t,x_n)\\ &=\int_0^\infty \tilde{w}_{j,k}(t,x_n+y_n;b)\varphi(y_n)dy_n\\ &=\int_{-\infty}^\infty \tilde{w}_{j,k}(t,x_n+y_n;b)\varphi^+(y_n)dy_n\\ &=\int_{-\infty}^\infty \tilde{w}_{j,k}(t,x_n-y_n;b)\varphi^-(y_n)dy_n.\end{aligned}$$

We have proposition for this series of operators $\{W_{j,k}(t;b)\}_{j,k}$.

Proposition 1. (1) *For $j\geq 0$, we have*

$$(W_{j,0}(t)\varphi)(x_n)=(2\pi)^{-n}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{i(x_n-y_n)\cdot\xi_n}w_{j,0}(t,\xi_n)\varphi^-(y_n)dy_n d\xi_n,\ j\geq 0.$$

(2) *If $t>0$ or $x_n>0$, the kernel $\tilde{w}_{j,k}(t,x_n+y_n;b)$ of $W_{j,k}(t,b)$ is smooth with the estimate*

$$(3.4)\qquad |\tilde{w}_{j,k}(t,\omega;b)|\leq C\Big(\frac{1}{\sqrt{t}}\Big)^{j+k+1}e^{-\frac{\delta\omega^2}{4t}+b^2t(1+\varepsilon)}$$

for any positive ε and $0<\delta<1$. Also $W_{j,k}(t,b)$ are bounded operators on $\mathbf{L}^p(\boldsymbol{R}^1_+)(1<p<\infty)$ with norm

$$(3.5)\qquad \|W_{j,k}(t;b)\|\leq C\Big(\frac{1}{\sqrt{t}}\Big)^{j+k}e^{b^2t(1+\varepsilon)}.$$

(3) *The operators $W_{j,k}(t;b)$ satisfy the following equations*:

$$(3.6)\qquad \Big\{\frac{\partial}{\partial t}-\Big(\frac{\partial}{\partial x_n}\Big)^2\Big\}W_{j,k}(t;b)=0 \quad in\ I\times\bar{\boldsymbol{R}}^1_+,$$

$$(3.7)\qquad \Big(\frac{\partial}{\partial x_n}+b\Big)W_{j,k}(t;b)=W_{j,k+1}(t;b) \quad in\ I\times\bar{\boldsymbol{R}}^1_+,\ (k\leq -1)$$

$$(3.8)\qquad \frac{\partial}{\partial x_n}W_{j,k}(t;b)=W_{j+1,k}(t;b) \quad in\ I\times\bar{\boldsymbol{R}}^1_+,$$

$$(3.9)\qquad \lim_{t\to+0}(W_{j,k}(t;b)\varphi)(x_n)=0 \quad in\ x_n>0,$$

for $\varphi\in C(\bar{\boldsymbol{R}}^1_+)$.

REMARK 1. By (1) of the above Proposition we have

$$W_{j,0}\varphi(t,x_n)=w_{j,0}(t;x_n,D_n)\varphi^-,\qquad j\geq 0,$$

where $w_{j,0}(t;x_n,D_n)$ means a pseudo-differential operator with symbol $w_{j,0}(t,\xi_n)$.

REMARK 2. In case $(\mathscr{N})$ and $(\mathscr{D})$ we use only $\{W_{j,0}\}$ ($W_{j,k}=W_{j+k,0}$ if $b=0$).

Proof. (1) and (3.4) are trivial by the definitions. (3.5) holds by the following fact

$$\int_0^\infty |\tilde{w}_{j,k}(t,\omega;b)|d\omega \leq C(\frac{1}{\sqrt{t}})^{j+k}e^{b^2t(1+\varepsilon)}.$$

We have by the equation (1) and Definition 7

$$\frac{\partial}{\partial x_n}W_{j,0}=W_{j+1,0}, \qquad \{\frac{\partial}{\partial t}-(\frac{\partial}{\partial x_n})^2\}W_{j,0}=0$$

and

$$\lim_{t\to 0} W_{j,0}\varphi(x_n)=(\frac{\partial}{\partial x_n})^j\varphi^-(x_n)=0 \qquad \text{for } x_n>0$$

hold for $j\geq 0$. In case j is negative, we get (3.8) for $k=0$ by the following equation

$$\frac{\partial}{\partial\omega}\tilde{w}_{j,0}=\tilde{w}_{j+1,0} \qquad \text{for } \omega\geq 0.$$

(3.8) for $k\leq -1$ is proved in the same way by

$$\frac{\partial}{\partial\omega}\tilde{w}_{j,k}=\tilde{w}_{j+1,k} \qquad \text{for } \omega\geq 0.$$

For $j\leq -1$ and $k\leq -1$ we have

$$\frac{\partial}{\partial\omega}\tilde{w}_{j,k}=-b\tilde{w}_{j,k}+\tilde{w}_{j,k+1} \qquad \text{for } \omega\geq 0.$$

Taking derivatives of the above equation with respect to x_n, we get (3.7) for any j, k. It is clear the following equality holds

$$(3.10) \qquad W_{j,k}(t;b)=W_{j-1,k+1}(t;b)-bW_{j-1,k}(t;b) \qquad \text{for } k\leq -1$$

by (3.7) and (3.8). We shall prove (3.6) in case $j\leq -3$ and $k\leq -2$. Other cases can be obtained by differentiating (3.6). The following equation holds for $j\leq 3$ and $k\leq -2$.

$$\tilde{w}_{j,k}=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau\int_{\frac{\omega}{2\sqrt{t}}}^\infty e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t}\sigma)^{-k-1}}{(-k-1)!}d\sigma.$$

So we have

$$\frac{\partial}{\partial t}\tilde{w}_{j,k}=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau\int_{\frac{\omega}{2\sqrt{t}}}^\infty$$

$$\times[-\frac{\tau}{2t}\partial_\tau\{e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}$$

$$+\frac{\sigma}{2t}\partial_\sigma\{e^{2b\sqrt{t}\sigma}\}e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}$$

$$-\frac{\sigma}{\sqrt{t}}e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-2}}{(-k-2)!}]d\sigma.$$

On the other hand we have

$$\int_{\frac{\omega}{2\sqrt{t}}}^\infty\frac{\sigma}{2t}\partial_\sigma\{e^{2b\sqrt{t}\sigma}\}e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}$$

$$=\int_{\frac{\omega}{2\sqrt{t}}}^\infty[\frac{-1}{2t}e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}$$

$$+\frac{\sigma}{\sqrt{t}}e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-2}}{(-k-2)!}$$

$$-\frac{\sigma}{\sqrt{t}}\partial_\tau\{e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}]d\sigma.$$

Hence we have

$$\frac{\partial}{\partial t}\tilde{w}_{j,k}=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau\int_{\frac{\omega}{2\sqrt{t}}}^\infty$$

$$\times[-(\frac{\tau}{2t}+\frac{\sigma}{\sqrt{t}})\partial_\tau\{e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}$$

$$-\frac{1}{2t}e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}]d\sigma$$

$$=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{(-\tau)^{-j-2}}{(-j-2)!}d\tau\int_{\frac{\omega}{2\sqrt{t}}}^\infty$$

$$\times [-(\frac{\tau}{2t}+\frac{\sigma}{\sqrt{t}})e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}]d\sigma$$

$$=\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{(-\tau)^{-j-2}}{(-j-2)!}d\tau\int_{\frac{\omega}{2\sqrt{t}}}^\infty$$

$$\times\,\partial_\tau\{e^{-(\sigma+\frac{\tau}{2\sqrt{t}})^2+2b\sqrt{t}\sigma-b\omega}\}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}d\sigma.$$

So we get

$$\frac{\partial}{\partial t}\tilde{w}_{j,k}==\tilde{w}_{j+2,k}.$$

Owing to (3.10), it is sufficient to show (3.9) only for $j\leq -1$ and $k\leq -1$. If $j\leq -1, k\leq -1$, we have

$$\tilde{w}_{j,k}=-\frac{1}{\sqrt{\pi}(2\sqrt{t})^j}\int_0^\infty\frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau\int_{\frac{\omega}{2\sqrt{t}}}^\infty e^{-\sigma^2+2b\sqrt{t}\sigma-b\omega}\frac{(\omega-2\sqrt{t\sigma})^{-k-1}}{(-k-1)!}d\sigma.$$

So we have

$$\tilde{w}_{j,k}\to 0 \text{ as } t\to 0$$

for $\omega>0$. Then (3.9) holds. q.e.d.

Proposition 2. *We have for any* $k\leq 0$

$$\frac{\partial}{\partial b}\tilde{w}_{j,k}(t,\omega;b)=k\tilde{w}_{j,k-1}(t,\omega;b).$$

Proof. We have the following equation for $k\leq -1$.

$$\frac{\partial}{\partial b}\tilde{w}_{0,k}(t,\omega;b)=-\frac{1}{\sqrt{\pi}}(\frac{1}{2\sqrt{t}})^{k+1}\int_0^\infty e^{-(\sigma+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\frac{(-\sigma)^{-k-1}}{(-k-1)!}2\sqrt{t}\sigma d\sigma$$

$$=k\tilde{w}_{0,k-1}$$

The assertion can be shown by the same way for other cases. q.e.d.

DEFINITION 8. $\mathscr{H}_s$ is the set of all finite sum of the following functions

$$\hat{\mathscr{H}}_s=\{g(t,x_n,y_n)=t^d(x_n)^l\tilde{w}_{j,k}(t,x_n+y_n;b);\quad d,l,j,k\in\mathbf{Z},d\geq 0,l\geq 0,k\leq 0,j+k-l-2d\leq s\}.$$

For a symbol $g(t,x_n,y_n)=t^d(x_n)^l\tilde{w}_{j,k}(t,x_n+y_n;b)\in\hat{\mathscr{H}}_s$ we define an operator as follows:

$$(G(t)\varphi)(x_n)=t^d(x_n)^l(W_{j,k}(t;b)\varphi)(x_n).$$

We state Proposition 3, which is the key idea in this section. Let $B_0=\frac{\partial}{\partial x_n}+b$ or $B_0=$ identity.

Proposition 3. (1) *For any* $g\in\hat{\mathscr{H}}_s$ *we have* $v\in\hat{\mathscr{H}}_{s-2}$ *such that*

$$\begin{cases}(\frac{\partial}{\partial t}-(\frac{\partial}{\partial x_n})^2)V(t)=G(t) & \text{in } I\times\{x_n>0\},\\ B_0V(t)|_{x_n=0}=0 & \text{in } I.\end{cases}$$

(2) *For any* $h\in\hat{\mathscr{H}}_{s-1}$ *we have* $v\in\hat{\mathscr{H}}_{s-2}$ ($v\in\hat{\mathscr{H}}_{s-1}$ *if* $B_0=$ identity) *such that*

$$\begin{cases}(\frac{\partial}{\partial t}-(\frac{\partial}{\partial x_n})^2)V(t)=0 & \text{in } I\times\{x_n>0\},\\ B_0V(t)|_{x_n=0}=H(t) & \text{in } I.\end{cases}$$

Proof. Set $L_0=\frac{\partial}{\partial t}-(\frac{\partial}{\partial x_n})^2$. It is sufficient to prove (1) for g such that

$$g=t^d\frac{(x_n)^l}{l!}\tilde{w}_{j,k}(t,x_n+y_n;b).$$

(Step-1). $d=0,l=0$. In this case, the following $v=v(t)$ is a solution for (1).

$$v(t)=-\frac{1}{2}x_n\tilde{w}_{j-1,k}(t,x_n+y_n;b)+\frac{1}{2}\tilde{w}_{j-1,k-1}(t,x_n+y_n;b)$$

If $B_0=$ identity, the second term of the above equation is dropped.

(Step-2). $d=0,l\geq 1$. Set

$$v_1=-\frac{(x_n)^{l+1}}{2(l+1)!}\tilde{w}_{j-1,k}.$$

Then $V_1(t)$ satisfies

$$\begin{cases} L_0 V_1(t) = G(t) + G_1(t) & \text{in} \quad I \times \{x_n > 0\}, \\ B_0 V_1(t)|_{x_n=0} = 0 & \text{in} \quad I, \end{cases}$$

where $g_1 = \frac{(x_n)^{l-1}}{2(l-1)!}\tilde{w}_{j-1,k}$. So we can reduce to (Step-1) by the induction with respect to l.

(Step-3). $d \geq 1$. Set

$$v_2 = t^d v_1,$$

where v_1 is the solution of

$$\begin{cases} L_0 V_1(t) = G_1(t) & \text{in} \quad I \times \{x_n > 0\}, \\ B_0 V_1(t)|_{x_n=0} = 0 & \text{in} \quad I, \end{cases}$$

which is obtained by (Step-2) with $g_1 = \frac{(x_n)^l}{l!}\tilde{w}_{j,k}$. Then $V_2(t)$ satisfies

$$\begin{cases} L_0 V_2(t) = dt^{d-1} V_1(t) + G(t) & \text{in} \quad I \times \{x_n > 0\}, \\ B_0 V_2(t)|_{x_n=0} = 0 & \text{in} \quad I. \end{cases}$$

So, by the induction with respect to d we can reduce to (Step-2). It is clear that v belongs to $\hat{\mathscr{H}}_{s-2}$ in any case.
For the proof of (2) we set $h = t^d \tilde{w}_{j,k}$.

(Step-1). $d=0$. If $B_0 = \frac{\partial}{\partial x_n} + b$, It is clear that $v = \tilde{w}_{j,k-1}$ is the solution by Proposition 1. If $B_0 =$ identity, $v = \tilde{w}_{j,k}$ is the solution.

(Step-2). $d \geq 1$. Set $v_1 = t^d \tilde{v}$, where $\tilde{v} \in \mathscr{H}_{j+k-1}$ ($\tilde{v} \in \mathscr{H}_{j+k}$ if $B =$ identity) is the solutioin of

$$\begin{cases} L_0 \tilde{V}(t) = 0 & \text{in} \quad I \times \{x_n > 0\}, \\ B_0 \tilde{V}(t)|_{x_n=0} = W_{j,k} & \text{in} \quad I, \end{cases}$$

which is obtained by (Step-1). Then

$$\begin{cases} L_0 V_1(t) = G_1(t) & \text{in} \quad I \times \{x_n > 0\}, \\ B_0 V_1(t)|_{x_n=0} = H(t) & \text{in} \quad I, \end{cases}$$

where $g_1(t) = dt^{d-1}\tilde{v}$. By (1) we get $v_2 \in \hat{\mathscr{H}}_{s-2}$ ($v_2 \in \hat{\mathscr{H}}_{s-1}$) such that

$$\begin{cases} L_0 V_2(t) = -G_1(t) & \text{in} \quad I \times \{x_n > 0\}, \\ B_0 V_2(t)|_{x_n=0} = 0 & \text{in} \quad I. \end{cases}$$

Then $v=v_1+v_2$ in the solution of (2). q.e.d.

We discuss only the case ($\mathscr{O}$). For other cases, in the following argument, we take $b(t,x')$ instead of $b(t,x',\xi')$ in case ($\mathscr{R}$). In case ($\mathscr{N}$) and ($\mathscr{D}$), we take $b=0$. In these cases we use only $\{W_{j,0}\}$ as Remark at the end of Proposition 1.

Definition 9. We set $\mathscr{H}_s$ the set of all finite sum of the following functions

$$\{g(t,x',x_n,\xi',y_n)=t^d(x_n)^l q(x',\xi')\tilde{w}_{j,k}(t,x_n+y_n;b(x',\xi'))e^{-\beta(x',\xi')t};$$
$$d,l,j,k\in \boldsymbol{Z}, d\geq 0, l\geq 0, k\leq 0,$$
$q(x',\xi')$ is a polynomial with respect to ξ' and
$q\in S^m_{1,0}(\boldsymbol{R}^{n-1})$ with $m=s+2d+l-j-k\}$.

Remark 3. Set

$$\hat{u}_j=(2\pi)^{-1}\int_{\boldsymbol{R}^1} e^{i(x_n+y_n)\cdot\xi_n}(u_j)^*(t;x',\xi',\xi_n)d\xi_n,$$

where u_j is obtained in Theorem C. Then we have the following facts.

$$\hat{u}_0=\tilde{w}_{0,0}(t,x_n+y_n)e^{-\beta(x',\xi')t}\in\mathscr{H}_0, \qquad \hat{u}_j\in\mathscr{H}_{-j}.$$

Lemma 1. *For the boundary conditions ($\mathscr{D}$), ($\mathscr{N}$), ($\mathscr{R}$), or ($\mathscr{O}$) with parabaolic condition, $g\in\mathscr{H}_s$ has the following estimat for $x_n\geq 0$ and $y_n\geq 0$.*

$$|g|\leq C\Big(\frac{1}{\sqrt{t}}\Big)^{s+1}\exp\Big(-\delta\frac{(x_n+y_n)^2}{4t}-c_0|\xi'|^2t\Big) \tag{3.11}$$

for any $0\leq\delta\leq 1$ and some positive constant c_0. Also we have

$$\begin{cases} \displaystyle\int_0^\infty |g(t,x',x_n,\xi',y_n)|dx_n\leq C\Big(\frac{1}{\sqrt{t}}\Big)^s\exp(-c_0|\xi'|^2t), \\ \displaystyle\int_0^\infty |g(t,x',x_n,\xi',y_n)|dy_n\leq C\Big(\frac{1}{\sqrt{t}}\Big)^s\exp(-c_0|\xi'|^2t). \end{cases} \tag{3.12}$$

Proof. ($\mathscr{O}$) with parabolic ondition means that

$$\mathrm{Re}\{\beta(x',\xi')-(b(t;x',\xi'))^2\}\geq C|\xi'|^2 \tag{3.13}$$

holds for some positive constant C. By (3.4), (3.13) and $x_n\leq x_n+y_n$ if $x_n\geq 0$ and $y_n\geq 0$, we get (3.11). (3.12) holds because of (3.5). q.e.d.

REMARK 4. By (3.11) if $t>0$ or $x_n>0$, $g\in\mathscr{H}_s$ belongs to $S^{-\infty}_{1,0}(\boldsymbol{R}^{n-1}_{x',\xi'})$.

We get the following proposition by Proposition 1 and Proposition 2.

Proposition 4. *Let g belong to* $\mathscr{H}_s$. *Then we have*:

(1) $(\frac{\partial}{\partial\xi'})^\alpha(\frac{\partial}{\partial x'})^\beta g\in\mathscr{H}_{s-|\alpha|}$ *with the estimate*

$$|(\frac{\partial}{\partial\xi'})^\alpha(\frac{\partial}{\partial x'})^\beta g|$$

$$\leq C_{\alpha,\beta}\min(|\xi'|^{-|\alpha|},\sqrt{t}^{|\alpha|})(\frac{1}{\sqrt{t}})^{s+1}\exp(-\delta\frac{(x_n+y_n)^2}{4t}-c_0|\xi'|^2t).$$

(2) $\frac{\partial}{\partial t}g\in\mathscr{H}_{s+2}$.

(3) $\frac{\partial}{\partial x_n}g,\frac{\partial}{\partial y_n}g\in\mathscr{H}_{s+1}$.

(4) *If* $r\in\mathscr{F}_j$, *rg belongs to* $\mathscr{H}_{s+j}$.

DEFINITION 10. For a symbol $g(t,x',x_n,\xi',y_n)\in\mathscr{H}_s$

$$g(t,x',x_n,\xi',y_n)=t^d(x_n)^l q(x',\xi')\tilde{w}_{j,k}(t,x_n+y_n;b(x',\xi'))e^{-\beta(x',\xi')t}$$

we define an integral-pseudodifferential operator as follows.

$$\begin{aligned}(G\varphi)(t,x',x_n)&=(G(t)\varphi)(x',x_n)\\&=\int_0^\infty g(t,x',x_n,D',y_n)\varphi(\cdot,y_n)dy_n\\&=(2\pi)^{-n+1}\int_{\boldsymbol{R}^{n-1}}\int_{\boldsymbol{R}^{n-1}}e^{i(x'-y')\cdot\xi'}t^d(x_n)^l\\&\quad\times[W_{j,k}(t;b(x',\xi'))\varphi(y',\cdot)](x_n)q(x',\xi')e^{-\beta(x',\xi')t}dy'd\xi'\\&=(2\pi)^{-n+1}\int_{\boldsymbol{R}^{n-1}}\int_{\boldsymbol{R}^{n-1}}e^{i(x'-y')\cdot\xi'}[G(t;x',\xi')\varphi(y',\cdot)](x_n)dy'd\xi',\end{aligned}$$

for $\varphi \in C(\boldsymbol{R}^1_+, S(\boldsymbol{R}^{n-1}))$, where

$$[G(t;x',\xi')\varphi(y',\cdot)](x_n)=t^d(x_n)^l[W_{j,k}(t;b(x',\xi'))\varphi(y',\cdot)](x_n)q(x',\xi')e^{-\beta(x',\xi')t}.$$

REMARK 5. The kernel $\tilde{g}(t,x',x_n,y',y_n)$ of an operator G is given by

$$\tilde{g}(t,x',x_n,y',y_n)=(2\pi)^{-n+1}\int_{\boldsymbol{R}^{n-1}} e^{i(x'-y')\cdot\xi'}g(t,x',x_n,\xi',y_n)d\xi'.$$

Owing to Lemma 1 and proposition 4 we get the following lemma for the kernel $\tilde{g}(t,x',x_n,y',y_n)$ of an operator G with symbol $g(t,x',x_n,\xi',y_n)$.

Lemma 2. *Let $g\in\mathscr{H}_s$. Then we have*

(1) $$|(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial x_n})^{\alpha_n}(\frac{\partial}{\partial y'})^{\beta}(\frac{\partial}{\partial y_n})^{\beta_n}\tilde{g}(t,x',x_n,y',y_n)|$$
$$\leq C(\frac{1}{\sqrt{t}})^{s+n+|\alpha|+|\beta|+|\alpha_n|+|\beta_n|}\exp(-\delta\frac{(x_n+y_n)^2}{4t})$$

for any $0<\delta<1$.

(2) *If $N>n-1$, the kernel k_N of the operator $G\Lambda^{-N}$ satisfies*

$$|k_N(t,x',x_n,y',y_n)|\leq C(\frac{1}{\sqrt{t}})^{s+1},$$

where Λ is the pseudo-differential operator with symbol $<\xi'>$.

Proof. (1) is clear by Proposition 4 and Lemma 1. Set $h=g<\xi'>^{-N}$. Then the symbol of operator $G\Lambda^{-N}$ coinsides with h. The following estimate holds by Lemma 1.

$$|h|\leq C(\frac{1}{\sqrt{t}})^{s+1}<\xi'>^{-N}.$$

Then k_N satisfies (2) if $N>n-1$. q.e.d.

For the well-posedness of the operator G on $\mathbf{L}^p(\boldsymbol{R}^n_+)$, we will discuss in §4.

DEFINITION 11. Let $r\in\mathscr{F}_{s_1}$, $g\in\mathscr{H}_{s_2}$. $r\circ g$ denotes the symbol of a product operator $r(t,x,D)G$.

Theorem 1 (Product formula). *Let* $r\in\mathscr{F}_{s_1}$, $g\in\mathscr{H}_{s_2}$. *Then we have*

$$r\circ g=\sum_{j=0}^{\infty}\Sigma_j(r,g),\ \Sigma_j(r,g)\in\mathscr{H}_{s_1+s_2-j},$$

where,

$$\Sigma_j(r,g)=\sum_{\alpha\geq 0}(-i)^{\alpha}\frac{1}{\alpha!}\hat{s}_j\Big(\Big(\frac{\partial}{\partial\xi_n}\Big)^{\alpha}r,\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha}g\Big)$$

with

$$\hat{s}_j(r,g)=\sum_{|\alpha|=j}(-i)^{|\alpha|}\frac{1}{\alpha!}\Big(\frac{\partial}{\partial\xi'}\Big)^{\alpha}r\Big(\frac{\partial}{\partial x'}\Big)^{\alpha}g.$$

REMARK 6. $\Sigma_j(r,g)=0$ for large j because r is a polynomial of ξ.

Proof. Owing to Proposition 4, we have

$$\Big(\frac{\partial}{\partial\xi'}\Big)^{\alpha}\Big(\frac{\partial}{\partial\xi_n}\Big)^{\alpha_n}r\in\mathscr{F}_{s_1-|\alpha|-\alpha_n},\qquad \Big(\frac{\partial}{\partial x'}\Big)^{\alpha}\Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}r\in\mathscr{F}_{s_2+\alpha_n}.$$

So we get the assertion. q.e.d.

DEFINITION 12. Fix a positive integer N. Set

$$\hat{q}=\sum_{j=2}^{-N+2}q_j,$$

where $\{q_j\}$ are functions introduced in definition 6.

Theorem 2. (1) *For any* $g(t)\in\mathscr{H}_s$ *and* $h(t)\in\mathscr{H}_{s-1}$ *there exists* $v(t)\in\mathscr{H}_{s-2}$ *such that*

$$\begin{cases}\Big(\dfrac{\partial}{\partial t}+\hat{q}\Big)\circ v(t)=g(t)\ \bmod\mathscr{H}_{s-1} & \text{in } I\times\boldsymbol{R}^n_+,\\ (i\xi_n+b(x',\xi'))\circ v(t)|_{x_n=0}=h(t)\ \bmod\mathscr{H}_{s-2} & \text{in } I\times\boldsymbol{R}^{n-1}.\end{cases}$$

(2) *For any* $g(t)\in\mathscr{H}_s$ *and* $h(t)\in\mathscr{H}_{s-2}$ *there exists* $v(t)\in\mathscr{H}_{s-2}$ *such that*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v(t)=g(t) \mod \mathscr{H}_{s-1} & in\ I\times \boldsymbol{R}^n_+, \\ v(t)|_{x_n=0}=h(t) \mod \mathscr{H}_{s-3} & in\ I\times \boldsymbol{R}^{n-1}. \end{cases}$$

Proof. We get the assertion by Theorem 1 and Proposition 3. q.e.d.

Corollary. (1) *For any* $\tilde{N}$, *and* $g(t)\in\mathscr{H}_s$ *and* $h(t)\in\mathscr{H}_{s-1}$ *there exists* $v(t)\in\mathscr{H}_{s-2}$ $(v(t)=\Sigma_{j=0}^{\kappa}w_j(t),\ w_j(t)\in\mathscr{H}_{s-2-j})$ *such that*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v(t)=g(t) \mod \mathscr{H}_{s-\tilde{N}} & in\ I\times \boldsymbol{R}^n_+, \\ (i\xi_n+b(x',\xi'))\circ v(t)|_{x_n=0}=h((t) \mod \mathscr{H}_{s-\tilde{N}-1} & in\ I\times \boldsymbol{R}^{n-1}. \end{cases}$$

(2) *For any* $\tilde{N}$, *any* $g(t)\in\mathscr{H}_s$ *and* $h(t)\in\mathscr{H}_{s-2}$ *there exists* $v(t)\in\mathscr{H}_{s-2}$ $(v(t)=\sum_{j=0}^{\kappa}w_j(t),\ w_j(t)\in\mathscr{H}_{s-2-j})$ *such that*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v(t)=g(t) \mod \mathscr{H}_{s-\tilde{N}} & in\ I\times \boldsymbol{R}^n_+, \\ v(t)|_{x_n=0}=h(t) \mod \mathscr{H}_{s-\tilde{N}-2} & in\ I\times \boldsymbol{R}^{n-1}. \end{cases}$$

Proposition 5. *Let* $r(X,D)$ *be a pseudo-differential operator with symbol* $r(x,\xi)\in S^{-\infty}$. *Then for* $\varphi(\cdot,x_n)\in C(\boldsymbol{R}^1_+;\mathscr{S}(\boldsymbol{R}^{n-1}))$, *we have*

$$\begin{aligned} &r(x',x_n,D',D_n)\varphi^+|_{x_n=0} \\ &=r(x',0,D',-D_n)\varphi^-|_{x_n=0} \\ &=[(2\pi)^{-1}\int_{-\infty}^{\infty}\int_0^{\infty}e^{i(x_n+y_n)\xi_n}r(x',0,D',-\xi_n)\varphi(\cdot,y_n)dy_nd\xi_n]_{x_n=0} \end{aligned}$$

Proof. We note that the trace is well-defined by the boundedness theorem for pseudo-differatnial operator. We get the assertion by the following equalities:

$$\begin{aligned} &r(x',x_n,D',D_n)\varphi^+|_{x_n=0} \\ &=(2\pi)^{-n}\int_{\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}}\int_{-\infty}^{\infty}\int_0^{\infty}e^{i(x'-y')\cdot\xi'-iy_n\xi_n}r(x',0,\xi',\xi_n)\varphi(y',y_n)dy_nd\xi_ndy'd\xi'. \\ &=(2\pi)^{-n}\int_{\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{0}e^{i(x'-y')\cdot\xi'-iz_n\xi_n} \end{aligned}$$

$$\times r(x',0,\xi',-\xi_n)\varphi(y',-z_n)dz_nd\xi_ndy'd\xi'.$$

$$=[(2\pi)^{-n}\int_{R^{n-1}\times R^{n-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{0}e^{i(x'-y')\cdot\xi'+i(x_n-z_n)\xi_n}$$
$$\times r(x',0,\xi',-\xi_n)\varphi(y',-z_n)dz_nd\xi_ndy'd\xi']_{x_n=0}.$$

q.e.d.

The fundamental solution for the Cauchy problem $U(t)$ with symbol $u(t)$ has the following property owing to Theorem C. "$BU(t)$ is also the psedudo-differential opertator with symbol $S^{-\infty}$ if t>0". In other word, the kernel of $BU(t)$ is smooth if $t>0$. So we can apply the above proposition for the symbol of $BU(t)$.
Fix a positive number N in Definition 12. Set $y_N(t)=u(t)-\sum_{j=0}^{N+n+2}u_j(t)$. Then $y_N(t)$ belongs to $S_{1,0}^{-N-n-3}$ by Theorem C. Also choosing $l=\frac{(N-n)}{2}-1$, we have

$$|y_N(t)_{(\beta)}^{(\alpha)}|\leq C_{\alpha,\beta}\sqrt{t}^{N-n}<\xi>^{-2n-3-|\alpha|}$$

by (2.7). By the above estimate, $h_N(t)=(i\xi_n+b)\circ y_N|_{x_n=0}\in S_{1,0}^{-N-n-2}$ holds the following estimate

$$|h_N(t)_{(\beta)}^{(\alpha)}|\leq C_{\alpha,\beta}\sqrt{t}^{N-n}<\xi>^{-2n-2-|\alpha|}. \tag{3.14}$$

On the other hand we have

$$[(i\xi_n+b)\circ\sum_{j=0}^{N+n+2}u_j]|_{x_n=0}=\sum_{j=0}^{\tilde{N}}g_j(t,x',\xi)u_0^*,$$

for some $\tilde{N}$ with $g_j(t,x',\xi)\in\mathscr{F}_{-j+1}$. So we obtain the following Lemma 3.

Lemma 3. *It holds that*

$$BU(t)\varphi^+|_{x_n=0}=\sum_{j=0}^{\tilde{N}}g_j(t,x',D',-D_n)W_{0,0}\varphi|_{x_n=0}+F_N\varphi,$$

where

$$F_N\varphi=(2\pi)^{-1}\int_{-\infty}^{\infty}\int_{0}^{\infty}e^{iy_n\xi_n}h_N(t,x',D',-\xi_n)\varphi(\cdot,y_n)dy_nd\xi_n.$$

Note that $g_j\tilde{w}_{0,0}\in\mathscr{H}_{-j+1}$ and apply Corollary of Theorem 2 with

$g(t)=0$, $h(t)=-\Sigma_{j=0}^{\tilde{N}} g_j(t,x',\xi',-\xi_n)\tilde{w}_{0,0}$. Then we get $v_N(t)\in\mathscr{H}_0$ ($v_N(t)=\Sigma_{j=0}^{\kappa} w_j(t)$, $w_j\in\mathscr{H}_{-j}$) such that

$$\begin{cases} (\dfrac{\partial}{\partial t}+\hat{q})\circ v_N(t)=0 \mod \mathscr{H}_{-N+1} & \text{in } I\times \boldsymbol{R}_+^n, \\ (i\xi_n+b(x',\xi'))\circ v_N(t)|_{x_n=0}=-\displaystyle\sum_{j=0}^{\tilde{N}} g_j(t,x',\xi',-\xi_n)\tilde{w}_{00} \mod \mathscr{H}_{-N} & \\ & \text{in } I\times \boldsymbol{R}^{n-1}. \end{cases}$$

Then we have the following theorem for any boundary condition B and for any N, owing to $p-\hat{q}\in\mathscr{F}'_{-N+1}$.

Theorem 3. *Set* $E_N(t)=U(t)+V_N(t)$. *Then* $E(t)$ *satisfies*

$$\begin{cases} LE_N(t)=G_N(t) \mod \mathscr{H}_{-N+1} & \textit{in} \quad I\times \boldsymbol{R}_+^n, \\ BE_N(t)|_{x_n=0}=F_N \mod \mathscr{H}_{-N} & \textit{in} \quad I\times \boldsymbol{R}^{n-1} \end{cases}$$

with $G_N(t)=(P-\hat{Q})V_N(t)$. *Moreover*

$$\lim_{t\to 0} E_N(t)\varphi(x',x_n)=\varphi(x',x_n), \qquad x_n>0$$

for $\varphi\in C(\boldsymbol{R}_+^n)$. *The kernel* $\tilde{g}_N$ *of* G_N *satisfies*

$$|(\frac{\partial}{\partial x})^\alpha(\frac{\partial}{\partial y})^\beta \tilde{g}_N|\le C_{\alpha,\beta}(\frac{1}{\sqrt{t}})^{-N+n+1+|\alpha|+|\beta|}.$$

F_N *has a kernel* $\tilde{f}_N$ *such that*

$$|(\frac{\partial}{\partial x'})^\alpha(\frac{\partial}{\partial y'})^\beta \tilde{f}_N|\le C_{\alpha,\beta}(\frac{1}{\sqrt{t}})^{-N+n}, \qquad |\alpha+\beta|\le n+1.$$

3–2. In case $b(t,x',\xi')$ depends on t. Set

$$\tilde{y}_{j,k}(\sigma,\omega;t)=\tilde{w}_{j,k}(\sigma,\omega;b(t,x',\xi')). \tag{3.15}$$

We define the integral operator $\{Y_{j,k}(\sigma;t)\}$ for a function $\varphi(y_n)$ with a kernel $y_{j,k}(\sigma,x_n+y_n;t)$ as follows.

$$\begin{aligned}(Y_{j,k}(\sigma;t)\varphi)(x_n)&=(Y_{j,k}(t)\varphi)(\sigma,x_n)\\ &=\int_0^\infty \tilde{y}_{j,k}(\sigma,x_n+y_n;t)\varphi(y_n)dy_n.\end{aligned}$$

Then $Y_{j,k}(\sigma;t)$ satisfies

$$
(3.16)\quad \begin{cases}
(\dfrac{\partial}{\partial\sigma}-(\dfrac{\partial}{\partial x_n})^2)Y_{j,k}(\sigma;t)=0 \quad \text{in } I\times\bar{\boldsymbol{R}}^1_+, \\
(\dfrac{\partial}{\partial x_n}+b(t,x',\xi'))Y_{j,k}(\sigma;t)=Y_{j,k+1} \quad \text{in } I\times\bar{\boldsymbol{R}}^1_+,\ (k\leq -1), \\
\qquad\qquad \dfrac{\partial}{\partial x_n}Y_{j,k}(\sigma;t)=Y_{j+1,k}(\sigma;t) \quad \text{in } I\times\bar{\boldsymbol{R}}^1_+, \\
\lim\limits_{\sigma\to+0}(Y_{j,k}(\sigma;t)\varphi)(x_n)=0 \quad \text{in } x_n>0,
\end{cases}
$$

for $\varphi\in C(\bar{\boldsymbol{R}}^1_+)$.

Hence $Z_{j,k}(t,s)=Y_{j,k}(t-s;t)$ satisfies

$$
(3.17)\quad \begin{cases}
(\dfrac{\partial}{\partial t}-(\dfrac{\partial}{\partial x_n})^2)Z_{j,k}(t,s)=kZ_{j,k-1}(t,s)\dfrac{\partial}{\partial t}b(t,x',\xi') \quad \text{in } I_s\times\bar{\boldsymbol{R}}^1_+, \\
(\dfrac{\partial}{\partial x_n}+b(t,x',\xi'))Z_{j,k}(t,s)=Z_{j,k+1}(t,s) \quad \text{in } I_s\times\bar{\boldsymbol{R}}^1_+,\ (k\leq -1), \\
\qquad\qquad \dfrac{\partial}{\partial x_n}Z_{j,k}(t,s)=Z_{j+1,k}(t,s) \quad \text{in } I_s\times\bar{\boldsymbol{R}}^1_+, \\
\lim\limits_{t\to+s}(Z_{j,k}(t,s)\varphi)(x_n)=0 \quad \text{in } x_n>0
\end{cases}
$$

for $\varphi\in C(\boldsymbol{R}^n_+)$ by Proposition 2 and (3.16), where $I_s=(s,T+s)$.

DEFINITION 9'. Set $\mathscr{H}_s(\sigma;t)$ the set of all finite sum of the functions of the following form

$$
(3.18)\quad \begin{aligned}
\{g(\sigma,x',x_n,\xi',y_n;t)&=\sigma^d(x_n)^l q(t,x',\xi')\tilde{y}_{j,k}(\sigma,x_n+y_n;t)e^{-\beta(x',\xi')\sigma}; \\
&d,l,k,j\in\mathbf{Z}, d\geq 0, l\geq 0, k\leq 0, \\
&q(t,x',\xi') \text{ is a polynomial with respect to } \xi',
\end{aligned}
$$

$(\dfrac{\partial}{\partial t})^r q$ belongs to $S^m_{1,0}$, for any r with parameter t with $m=s+2d+l-j-k\}$.

In this section we use $\mathscr{K}_s(t,s)=\mathscr{H}_s(t-s;t)$ instead of $\mathscr{H}_s$ in the previous section and operators $G(\sigma;t)$ defined by functions $g\in\mathscr{H}_s(\sigma;t)$ in the similar way of §3–1. We can discuss the similar argument in §3–1 for $\mathscr{H}_s(\sigma;t)$. For example $g\in\mathscr{H}_s(\sigma;t)$ satisfies

$$(3.19)\quad |(\frac{\partial}{\partial\xi'})^{\alpha}(\frac{\partial}{\partial x'})^{\beta}g(\sigma;t)|$$
$$\leq C_{\alpha,\beta}\min(|\xi'|^{-|\alpha|},\sqrt{\sigma}^{|\alpha|})(\frac{1}{\sqrt{\sigma}})^{s+1}\exp(-\delta\frac{(x_n+y_n)^2}{4\sigma}-c_0|\xi'|^2\sigma)$$

for any $0<\delta<1$. Let $\tilde{g}$ be the kernel of $G(\sigma;t)$. Then

$$(3.20)\qquad |(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial y})^{\beta}\tilde{g}|\leq C_{\alpha,\beta}(\frac{1}{\sqrt{\sigma}})^{s+n+|\alpha|+|\beta|}\exp(-\delta\frac{(x_n+y_n)^2}{4\sigma})$$

for any $0<\delta<1$. We repeat the same argument using (3.17) instead of (3.6)~(3.9). Then we obtain

Theorem 4. *For any N we have $v_N(t,s)\in\mathscr{K}_0(t,s)$ such that $E_N(t,s)\varphi=U(t-s)\varphi^{+}+V_N(t,s)\varphi$ satisfies*

$$\begin{cases} LE_N(t,s)=G_N(t,s) \mod \mathscr{K}_{-N+1} & in\quad I_s\times \boldsymbol{R}^n_+,\\ B(t)E_N(t,s)|_{x_n=0}=F_N(t,s) \mod \mathscr{K}_{-N} & in\quad I_s\times \boldsymbol{R}^{n-1}\end{cases}$$

and

$$\lim_{t\to s}(E_N(t,s)\varphi)(x',x_n)=\varphi(x',x_n)\qquad x_n>0,$$

with G_N(t,s) and $F_N(t,s)$ whose kernels $\tilde{g}_N(t,s)$ and $\tilde{f}_N(t,s)$ satiasfy

$$|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial y})^{\beta}\tilde{g}_N(t,s)|\leq C(\frac{1}{\sqrt{t-s}})^{-N+n+1+|\alpha|+|\beta|},$$
$$|(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial y'})^{\beta}\tilde{f}_N(t,s)|\leq C(\frac{1}{\sqrt{t-s}})^{-N+n},\qquad |\alpha|+|\beta|\leq n.$$

Proposition 6. *Let φ and ψ be smooth functions. If* supp $\varphi\cap$ supp ψ $=\emptyset$ *and $g\in\mathscr{H}_s(\sigma;t)$, then $\varphi(x)G\psi(y)$ is a smoothing operator, that is for any α, β and N we have*

$$|(\frac{\partial}{\partial x})^{\alpha}(\frac{\partial}{\partial y})^{\beta}\varphi(x)\tilde{g}\psi(y)|\leq C\sigma^N,$$

where $\tilde{g}(x,y)$ *is the kernel of* G.

Proof. By Proposition 4 we have $(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial x_n})^{\alpha_n} g \in \mathscr{H}_{s+|\alpha_n|}(\sigma; t)$ and owing to Lemma 1 we have

$$|(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial x_n})^{\alpha_n} g| \le C(\frac{1}{\sqrt{\sigma}})^{s+1+|\alpha_n|}\exp(\frac{-\delta(x_n+y_n)^2}{4\sigma}-c_0|\xi'|^2\sigma).$$

Let $x \in \operatorname{supp}\varphi$, $y \in \operatorname{supp}\psi$. Then $x' \neq y'$ or $x_n \neq y_n$. If $x' \neq y'$, then the pseudo-local property for pseudo-differential operator leads to the above estimate. If $x_n \neq y_n$, then it is clear that $x_n \neq 0$ or $y_n \neq 0$. Assume $x_n \ge \varepsilon$, then we have

$$\exp(-\delta\frac{x_n^2}{4\sigma}) \le \frac{\sigma^M}{\varepsilon^{2M}}(\frac{x_n^2}{\sigma})^M \exp(-\delta\frac{x_n^2}{4\sigma}) \le C_M \sigma^M \exp(-\tilde{\delta}\frac{x_n^2}{4\sigma})$$

for any M and $\tilde{\delta} < \delta$. So we get the assertion. q.e.d.

3–3. Asymptotic Expansion of $V(t)$ **for** $(\mathscr{S})$. We assume that $a(t,x') = a(x'), b(t,x') = b(x')$ and satisfy $(*)$ in §0. Other cases we shall discuss at the end of this section.

We substitute the following function $\tilde{w}_{j,k}(t,\omega;a,b)$ for $\tilde{w}_{j,k}(t,\omega;b)$ in Definition 7 for $k \le -1$. Set for $k \le -1$

$$\tilde{w}_{j,k}(t,\omega;a,b)$$

$$= \begin{cases} -\dfrac{1}{\sqrt{\pi}}(\dfrac{1}{2\sqrt{t}})^{j+k+1}\displaystyle\int_0^\infty e^{-(a\sigma+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\dfrac{(-\sigma)^{-k-1}}{(-k-1)!}h_j(a\sigma+\dfrac{\omega}{2\sqrt{t}})d\sigma, & \text{if } j \ge 0, \\ \dfrac{1}{\sqrt{\pi}}(\dfrac{1}{2\sqrt{t}})^{j+k+1}\displaystyle\int_0^\infty \dfrac{(-\tau)^{-j-1}}{(-j-1)!}d\tau \int_0^\infty e^{-(a\sigma+\tau+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\dfrac{(-\sigma)^{-k-1}}{(-k-1)!}d\sigma, & \text{if } j \le -1, \end{cases}$$

where $h_j(\sigma) = \{(\frac{\partial}{\partial\sigma})^j e^{-\sigma^2}\}e^{\sigma^2}$.

We will give some remarks and proposition for $\tilde{w}_{j,k}(t,\omega;a,b)$. Note that $\tilde{w}_{j,k} = b^k \tilde{w}_{j,0}$ if $a = 0$. The condition $(*)$ leads the well-posedness of the definition of $\tilde{w}_{j,k}$. An operator $W_{j,k}$ definied by a symbol $\tilde{w}_{j,k}(t,\omega;a,b)$, in this section, satisfies (3.6), (3.8), (3.9) and (3.7)′ instead of (3.7).

$$(3.7)' \qquad (a\frac{\partial}{\partial x_n}+b)W_{j,k} = W_{j,k+1}.$$

Proposition 2′. *Assume a and b are constants. Then it hold that*

$$\frac{\partial}{\partial a}\tilde{w}_{j,k}(t,\omega;a,b)=k\tilde{w}_{j+1,k-1}(t,\omega;a,b), \qquad k\leq 0,$$

$$\frac{\partial}{\partial b}\tilde{w}_{j,k}(t,\omega;a,b)=k\tilde{w}_{j,k-1}(t,\omega;a,b), \qquad k\leq 0.$$

Proof. It is sufficient to prove for $j\leq -2, k\leq -1$. We can prove other cases by differentiating obtained equation for small j and k. For $j\leq -2$, $k\leq -1$, we have

$$\frac{\partial}{\partial a}\tilde{w}_{j,k}(t,\omega;a,b)$$
$$=\frac{1}{\sqrt{\pi}}\Big(\frac{1}{2\sqrt{t}}\Big)^{j+k+1}\int_0^\infty\frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau\int_0^\infty \sigma\partial_\tau\{e^{-(a\sigma+\tau+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\}\frac{(-\sigma)^{-k-1}}{(-k-1)!}d\sigma$$
$$=\frac{1}{\sqrt{\pi}}\Big(\frac{1}{2\sqrt{t}}\Big)^{j+k+1}\int_0^\infty\frac{(-\tau)^{-j-2}}{(-j-2)!}d\tau\int_0^\infty e^{-(a\sigma+\tau+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}\sigma\frac{(-\sigma)^{-k-1}}{(-k-1)!}d\sigma$$
$$=k\tilde{w}_{j+1,k-1}(t,\omega;a,b).$$

We can get the second equation easily. q.e.d.

DEFINITION 8′. Let $\hat{\mathscr{H}}_s$ be the set of all finite sum of the functions of the following form

$$\{g(t,x_n,y_n)=t^d(x_n)^l a^\alpha\tilde{w}_{j,k}(t,x_n+y_n;a,b);$$
$$d,l,j,k\in\mathbf{Z},d\geq 0,l\geq 0,k\leq 0,\alpha\geq 0,\ j-l-2d+\max(k,-\alpha)\leq s\}.$$

Proposition 3′. *For any $g\in\hat{\mathscr{H}}_s$ and $h\in\hat{\mathscr{H}}_{s-2}$ we have $v\in\hat{\mathscr{H}}_{s-2}$ such that*

$$\begin{cases}\Big(\dfrac{\partial}{\partial t}-\Big(\dfrac{\partial}{\partial x_n}\Big)^2\Big)V(t)=G(t) & \text{in } I\times\{x_n>0\},\\[2ex] \Big(a\dfrac{\partial}{\partial x_n}+b\Big)V(t)|_{x_n=0}=H(t) & \text{in } I.\end{cases}$$

Proof. We may assume $g=\frac{(-4t)^d(-2x_n)^l}{d!\quad l!}\tilde{w}_{j,k}\in\tilde{\mathscr{H}}_s$ and $h=0$. In other cases we can reduce to this case by the similar method as Proposition 3. For the above g the following v of class $\tilde{\mathscr{H}}_{s-2}$ is the solution

$$v=\frac{1}{4}\sum_{s=0}^{d}\frac{(-4t)^{d-s}}{(d-s)!}\sum_{\mu=0}^{s+1}\sum_{0\le\nu\le l+s+1}C_{l,s,\mu,\nu}(2a)^{\mu}\frac{(-2x_n)^{l+s+1-\nu}}{(l+s+1-\nu)!}\tilde{w}_{j-s-\nu+\mu-1,k-\mu},$$

where $C_{l.s.\mu,\nu}$ are constants depending on l,s,μ,ν. In fact $C_{l,s,0,\nu}={}_{s+\nu}C_s-{}_{s+\nu}C_{s+l+1}$, $C_{l,s,\mu,\nu}={}_{s+\nu-\mu}C_{s+l}-{}_{s+\nu-\mu}C_{s+l+1}$, $(\mu\ge 1)$ where we use ${}_sC_q=0$ if $s<q$. q.e.d.

We need another function space in this case.

DEFINITION 9″. $\mathscr{H}_s$ is the set of all finite sum of the functions of the followig form

$$\{g(t,x',x_n,\xi',y_n)=t^d(x_n)^l q(x',\xi')a^{\alpha_0}\prod_{i=1}^{n}A_i^{\alpha_i}\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))e^{-\beta(x',\xi')t};$$

$d,l,j,k\in\mathbf{Z},d\ge 0,l\ge 0,k\le 0,\alpha_i\ge 0,$

$q(x',\xi')$ is a polynomial with respect to ξ',

q belongs to $S^m_{1,0}$ with $m=s+2d+l-j-\max\{k,-\alpha_0-\frac{1}{2}\sum_{j=1}^{n}\alpha_j\}\}$,

where $A_j=\frac{\partial}{\partial x_j}a$.

REMARK 7. For any $j\in\mathbf{Z}$ we have

$$a\tilde{w}_{j,k}(t,x_n;a,b)=\tilde{w}_{j-1,k+1}-b\tilde{w}_{j-1,k},\qquad k\le -1.$$

So we may choose $\alpha_0=0$ in the above definition. Repeating the similar argument of §3–1, we have

Lemma 1′. *$g\in\mathscr{H}_s$ has the following estimate*

$$|g|\le C\Big(\frac{1}{\sqrt{t}}\Big)^{s+1}\exp\Big(-\delta\frac{(x_n+y_n)^2}{4t}-c_0|\xi'|^2t\Big),$$

for any $0<\delta<1$.

Proof. By the nonnegativity of a we have $|A_j|\le Ca^{\frac{1}{2}}$. Then it is sufficient to show

$$|(x_n)^l a^\alpha \tilde{w}_{j,k}| \leq C(\frac{1}{\sqrt{t}})^{s+1}\exp(-\delta\frac{(x_n+y_n)^2}{4t}-c_0|\xi'|^2 t),$$

for $k\leq -1$, $\alpha\in \boldsymbol{R}_+$, where $s=-l+j+\max(k,-\alpha)$. In case $j\leq -1$ we have

$$a^\alpha \tilde{w}_{j,k}=\frac{1}{\sqrt{\pi}}(\frac{1}{2\sqrt{t}})^{j+k+1}\int_0^\infty \frac{(-\tau)^{-j-1}}{(-j-1)!}d\tau$$

$$\times\int_0^\infty \frac{(\frac{-\mu}{a})^{-k-1}}{(-k-1)!}a^{\alpha-1}e^{-(\mu+\tau+\frac{x_n+y_n}{2\sqrt{t}})^2+2\frac{b}{a}\sqrt{t}\mu}d\mu.$$

We note that

$$(\frac{\mu}{a})^{-k-1}a^{\alpha-1}\leq\begin{cases} C\mu^{-k-1} & \text{if} \quad k+\alpha\geq 0; \\ (\frac{\mu|b|\sqrt{t}}{a})^{-k-\alpha}\mu^{\alpha-1}(\sqrt{t})^{k+\alpha}, & \text{otherwise.}\end{cases}$$

Then we get the assertion. q.e.d.

Proposition 4'. *Let g belong to $\mathscr{H}_s$. Then we have:*

(1) $(\frac{\partial}{\partial\xi'})^\alpha(\frac{\partial}{\partial x'})^\beta g\in\mathscr{H}_{s-|\alpha|+\frac{|\beta|}{2}}$ *with the estimate*

$$|(\frac{\partial}{\partial\xi'})^\alpha(\frac{\partial}{\partial x'})^\beta g|$$

$$\leq C_{\alpha,\beta}\min(|\xi'|^{-|\alpha|},\sqrt{t}^{|\alpha|})(\frac{1}{\sqrt{t}})^{s+1+\frac{|\beta|}{2}}\exp(-\delta\frac{(x_n+y_n)^2}{4t}-c_0|\xi'|^2 t).$$

(2) $\frac{\partial}{\partial t}g\in\mathscr{H}_{s+2}$.

(3) $\frac{\partial}{\partial x_n}g$, $\frac{\partial}{\partial y_n}g\in\mathscr{H}_{s+1}$.

(4) *If $r\in\mathscr{F}_j$, rg belongs to $\mathscr{H}_{s+j}$.*

Proof. It is sufficient to prove (1) for $|\alpha|+|\beta|=1$. In order to prove the statement for $|\beta|=1$, we may assume $g\in\mathscr{H}_s$ of the following form

$$g=\prod_{i=1}^n A_i^{\alpha_i}\tilde{w}_{j,k}(t,x_n+y_n;\, a(x'),b(x'))e^{-\beta(x',\xi')t}.$$

Then we have

$$\frac{\partial}{\partial x_l}g=\sum_{p=1}^{n}A_p^{\alpha_p-1}\prod_{i=1,i\neq p}^{n}A_i^{\alpha_i}\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))e^{-\beta(x',\xi')t}$$

$$+\prod_{i=1}^{n}A_i^{\alpha_i}\frac{\partial}{\partial x_l}\{\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))\}e^{-\beta(x',\xi')t}$$

$$+\prod_{i=1}^{n}A_i^{\alpha_i}\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))(-\frac{\partial}{\partial x_l}\beta)te^{-\beta(x',\xi')t}$$

$$=h_1+h_2+h_3.$$

We easily see that $h_1\in\mathscr{H}_{s+\frac{1}{2}}$ and $h_3\in\mathscr{H}_s$. For h_2 we note that

$$\frac{\partial}{\partial x_l}\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))$$

$$=\frac{\partial}{\partial a}\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))A_l+\frac{\partial}{\partial b}\tilde{w}_{j,k}(t,x_n+y_n;a(x'),b(x'))\frac{\partial}{\partial x_l}b$$

$$=k\tilde{w}_{j+1,k-1}(t,,x_n+y_n;a(x'),b(x'))A_l+k\tilde{w}_{j,k-1}(t,x_n+y_n;a(x'),b(x'))\frac{\partial}{\partial x_l}b$$

by Proposition 2′. So we get that h_2 belongs to $\mathscr{H}_{s'}$, where $s'=j+1+\max\{k-1,-\frac{1}{2}\sum_{j=1}^{n}\alpha_j-\frac{1}{2}\}$. By the fact $s'\leq s+\frac{1}{2}$ we get the assertion. It is easy to prove the asertion for $|\alpha|=1$. (2)~(4) are gotten by (3.6) and (3.8). q.e.d.

Owing to Lemma 1′ and proposition 4′ we get the following lemma for the kernel $\tilde{g}(t,x',x_n,y',y_n)$ of operator G by the same way as Lemma 2.

Lemma 2′. (1) *Assume a symbol g belong to $\mathscr{H}_s$. Then we have*

$$|(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial x_n})^{\alpha_n}(\frac{\partial}{\partial y'})^{\beta}(\frac{\partial}{\partial y_n})^{\beta_n}\tilde{g}(t,x',x_n,y',y_n)|$$
$$\leq C(\frac{1}{\sqrt{t}})^{s+n+|\alpha|+|\beta|+|\alpha_n|+|\beta_n|}\exp(-\delta\frac{(x_n+y_n)^2}{4t})$$

for any $0<\delta<1$.

(2) *If $N>n-1$, the kernel k_N of the operator $G\Lambda^{-N}$ satisfies*

$$|k_N(t,x',x_n,y',y_n)| \le C\Big(\frac{1}{\sqrt{t}}\Big)^{s+1},$$

where Λ *is the pseudo-differential operator with symbol* $<\xi'>$.

For the well-posedness of the operator G on $\mathbf{L}^p(\boldsymbol{R}^n_+)$, we will discuss in §4.

Theorem 1′ (Product formula).

$$r \circ g = \sum_{j=0}^{\infty} \Sigma_j(r,g), \quad \Sigma_j(r,g) \in \mathscr{H}_{s_1+s_2-\frac{j}{2}},$$

with the same notation of Theorem 1.

Theorem 2′. *For any* $g(t)\in\mathscr{H}_s$ *and* $h(t)\in\mathscr{H}_{s-2}$ *there exists* $v(t)\in\mathscr{H}_{s-2}$ *such that*

$$\begin{cases} \Big(\dfrac{\partial}{\partial t}+q\Big)\circ v(t)=g(t) \mod \mathscr{H}_{s-\frac{1}{2}} & \text{in } I\times\boldsymbol{R}^n_+, \\ (a(x')i\xi_n+b(x'))\circ v(t)|_{x_n=0}=h(t) & \text{in } I\times\boldsymbol{R}^{n-1}. \end{cases}$$

Remark 8. In this case we note that

$$(ai\xi_n+b)\circ v=\Sigma_0(ai\xi_n+b,v)=a\Sigma_0(i\xi_n,v)+bv$$

because $a(x')$ and $b(x')$ are independent of ξ'.

Corollary. *For any* $\tilde{N}$, *any* $g(t)\in\mathscr{H}_s$ *and* $h(t)\in\mathscr{H}_{s-2}$ *there exists* $v(t)\in\mathscr{H}_{s-2}$ $(v(t)=\Sigma_{j=0}^{\kappa}w_j(t),\ w_j(t)\in\mathscr{H}_{s-2-\frac{j}{2}})$ *such that*

$$\begin{cases} \Big(\dfrac{\partial}{\partial t}+\hat{q}\Big)\circ v(t)=g(t) \mod \mathscr{H}_{s-\tilde{N}} & \text{in } I\times\boldsymbol{R}^n_+, \\ (a(x')i\xi_n+b(x'))\circ v(t)|_{x_n=0}=h(t) \mod \mathscr{H}_{s-\tilde{N}-1} & \text{in } I\times\boldsymbol{R}^{n-1}. \end{cases}$$

If $a(t,x')$ or $b(t,x')$ depends on t, we introduce symbols $\tilde{y}_{j,k}(\sigma,\omega;t)=\tilde{w}_{j,k}(\sigma,\omega;a(t,x'),b(t,x'))$ and repeat the similar argument in §3–2. In this case, the operator $Z_{j,k}(t,s)=Y_{j,k}(t-s;t)$ satisfies (3.18) of which the first equation replaced by

$$(\frac{\partial}{\partial t}-(\frac{\partial}{\partial x_n})^2)Z_{j,k}(t,s)=kZ_{j+1,k-1}(t,s)\frac{\partial}{\partial t}a(t,x')+kZ_{j,k-1}(t,s)\frac{\partial}{\partial t}b(t,x').$$

So Theorem 4 holds for $(\mathscr{S})$.

We note that in the above arguemt the following estimate is not necessary.

$$|\frac{\partial}{\partial t}a|\leq Ca^{\frac{1}{2}}.$$

Now we consider the case that $a(t,x')$ and $b(t,x')$ are complex valued function satisfying (3.3). In this case we replace the integral domain $[0,\infty)$ in the definition of $\tilde{w}_{j,k}$ by the following line Λ.

$$\Lambda=\{re^{i(\theta-\arg a)}:\ 0\leq r<\infty\},$$

where θ is chosen as

$$\cos(\theta-\arg(\frac{a}{b}))<0, \qquad |\theta|<\frac{\pi}{4}.$$

For example the definition of $\tilde{w}_{0,k}(t,\omega;a,b)$ is defined by

$$\tilde{w}_{0,k}=\begin{cases} -\frac{1}{\sqrt{\pi}}(\frac{1}{2\sqrt{t}})^{k+1}\int_\Lambda\frac{(-\sigma)^{-k-1}}{(-k-1)!}e^{-(a\sigma+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma}d\sigma, & \text{if } a(t,x')\neq 0; \\ b^k\tilde{w}_{0,0} & \text{if } a(t,x')=0. \end{cases}$$

4. Construction of an asymptotic expansion of the Poisson operator

We discuss the construction of an asymptotic expansion of the Poisson operator with respect to $(\mathscr{O})$ in this section. The similar arguments can be repeated for other boundary conditions.

Proposition 7. *Let $g(\sigma;t)$ belong to $\mathscr{H}_s(\sigma;t)$. If $s<1$, the following operator has the limit*

$$\lim_{x_n\to+0}\int_0^t g(t-\sigma,x',x_n,D',0;t)h(\sigma,\cdot)d\sigma$$

for $h(t,x)\in C((0,T);\mathscr{S}(\boldsymbol{R}^{n-1}))$.

Proof. By (3.19) we have

$$|(\frac{\partial}{\partial \xi'})^{\alpha}(\frac{\partial}{\partial x'})^{\beta} g(\sigma, x', x_n, \xi', 0; t)|$$
$$\leq C_{\alpha,\beta} <\xi'>^{-|\alpha|}(\frac{1}{\sqrt{\sigma}})^{s+1} \exp(-\delta\frac{x_n^2}{4\sigma} - c_0|\xi'|^2\sigma) \qquad (0<\delta<1).$$

For $x_n>0$ the above operator is well-defined for any s and smooth with respect to x'. If $s<1$, the operators is well-defined even in $x_n \geq 0$. q.e.d.

For the special case of $s=1$, we have

Proposition 8. (1) *If* $t>0$, *then we have*

$$\lim_{x_n \to 0} \int_0^t \tilde{w}_{1,0}(\sigma, x_n) h(t-\sigma) d\sigma = -\frac{1}{2} h(t)$$

for $h \in C((0, T))$.

(2) *We have*

$$\int_0^t \tilde{w}_{1,0}(\sigma, x_n) h(t-\sigma) d\sigma = h(t) \frac{1}{\sqrt{\pi}} \int_{\infty}^{\frac{x_n}{2\sqrt{t}}} \exp(-\sigma^2) d\sigma$$
$$- \int_0^t \tilde{w}_{1,0}(\sigma, x_n) \sigma \{ \int_0^1 h(t-\theta\sigma) d\theta \} d\sigma$$

for $h \in C^1((0, T))$.

Proof. We can write

$$\int_0^t \tilde{w}_{1,0}(\sigma, x_n) h(t-\sigma) d\sigma = -\int_0^t \frac{x_n}{4\sqrt{\pi}\sqrt{\sigma^3}} \exp(-\frac{x_n^2}{4\sigma}) h(t-\sigma) d\sigma.$$

Set $\mu = \frac{x_n}{2\sqrt{\sigma}}$. Then

$$\int_0^t \tilde{w}_{1,0}(\sigma, x_n) h(t-\sigma) d\sigma = \frac{1}{\sqrt{\pi}} \int_{\infty}^{\frac{x_n}{2\sqrt{t}}} \exp(-\mu^2) h(t - \frac{x_n^2}{4\mu^2}) d\mu.$$

Hence when x_n tends to 0, this tends to $\frac{1}{\sqrt{\pi}} \int_{\infty}^{0} \exp(-\sigma^2) d\sigma h(t) = -\frac{1}{2} h(t)$.

q.e.d.

Corollary 1. *Let* $g(t,x',x_n,\xi',y_n)=\tilde{w}_{1,0}(t,x_n+y_n)e^{-\beta(x',\xi')t}$. *Then*

$$\lim_{x_n\to 0}\int_0^t g(t-\sigma,x',x_n,D',0)h(\sigma,\cdot)d\sigma=-\frac{1}{2}h(t,x') \qquad t>0,$$

for $h\in C((0,T));\mathscr{S}(\mathbf{R}^{n-1}))$.

Corollary 2. *Let* $\varphi(t,s)$ *be a* C^1 *function satisfying the following inequalities for a positive constant* M

$$|\varphi(t,s)|\leq C(t-s)^M, \qquad |\frac{\partial}{\partial t}\varphi(t,s)|\leq C(t,s)^{M-1}.$$

Then the following estimate

$$|\int_s^t \tilde{w}_{1,0}(t-\sigma,x_n)\varphi(\sigma,s)d\sigma|\leq C(t-s)^M$$

holds.

Proof. Apply Proposition 8 (2) for $\varphi(\sigma,s)$. Then we have

$$\int_s^t \tilde{w}_{1,0}(t-\sigma,x_n)\varphi(\sigma,s)d\sigma=\int_0^{t-s}\tilde{w}_{1,0}(\sigma,x_n)\varphi(t-\sigma,s)d\sigma$$

$$=\varphi(t,s)\frac{1}{\sqrt{\pi}}\int_\infty^{\frac{x_n}{2\sqrt{t-s}}}\exp(-\sigma^2)d\sigma-\int_0^{t-s}\tilde{w}_{1,0}(\sigma,x_n)\sigma\{\int_0^1\frac{\partial}{\partial t}\varphi(t-\theta\sigma,s)d\theta\}d\sigma.$$

We get the assertion by the assumption for φ and the following facts $\tilde{w}_{1,0}(\sigma,x_n)\sigma$ is bounded and $|t-\sigma-s|\leq|t-\theta\sigma-s|\leq|t-s|$, for $0\leq\theta\leq 1$.

q.e.d.

Theorem 5. *Let* N *be any integer.*

(1) *We can find* $v_B\in\mathscr{K}_0(t,s)$ *for* B *related to* $(\mathscr{N})$, $(\mathscr{R})$ *and* $(\mathscr{O})$ *such that*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v_B=s_N & in\ I_s\times \boldsymbol{R}^n_+, \\ B\circ v_B+2\tilde{w}_{1,0}(t-s,x_n+y_n)e^{-\beta(t-s)}|_{x_n=0}=r_N & in\ I_s\times \boldsymbol{R}^{n-1}, \end{cases}$$

with $s_N\in\mathscr{K}_{-N+1}(t,s)$ *and* $r_N\in\mathscr{K}_{-N}(t,s)$.

(2) *We can find* $v_B\in\mathscr{K}_1(t,s)$ *for* B *related to* $(\mathscr{D})$ *and* $(\mathscr{S})$ *such that*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v_B=s_N & in\ I_s\times \boldsymbol{R}^n_+, \\ B\circ v_B+2\tilde{w}_{1,0}(t-s,x_n+y_n)e^{-\beta(t-s)}|_{x_n=0}=r_N & in\ I_s\times \boldsymbol{R}^{n-1}, \end{cases}$$

with $s_N\in\mathscr{K}_{-N+1}(t,s)$ *and* $r_N\in\mathscr{K}_{-N}(t,s)$

Proof. In any case the main term of $v_B(t,s)$ is $-2\tilde{w}_{1,-1}(t-s,x_n+y_n$: $b(t,x',\xi'))e^{-\beta(t-s)}$. Apply Theorem 2 or Theorem 2′. we get the assertion. q.e.d.

DEFINITION 13. For a function $h\in C((0,T);\mathscr{S}(\boldsymbol{R}^{n-1}))$ we set

$$(Z_Bh)(t.s)=\int_s^t v_B(t-\sigma,x',x_n,D',0;t)h(\sigma,\cdot)d\sigma.$$

Proposition 9. *For* $x_n>0,(Z_Bh)(t,s)$ *is well-defined and*

$$\begin{cases} L(Z_Bh)(t,s)=(Sh)(t,s) & in\ I_s\times\boldsymbol{R}^n_+, \\ \lim_{x_n\to 0} B(t)(Z_Bh)(t,s)=h(t)+(Rh)(t,s) & in\ I_s\times\boldsymbol{R}^{n-1}, \\ \lim_{t\to s}(Z_Bh)(t,s)=0 & in\ \boldsymbol{R}^n_+, \end{cases}$$

where S *and* R *are integral operators of the form*

$$(Sh)(t,s)=\int_s^t s(t,\sigma,x_n)h(\sigma)d\sigma, \qquad (Rh)(t,s)=\int_s^t r(t,\sigma)h(\sigma)d\sigma$$

with smoothing kernels in the sense

$$|(\frac{\partial}{\partial x'})^\alpha(\frac{\partial}{\partial y'})^\beta s(t,s,x_n)|\le C_{\alpha,\beta}(\frac{1}{\sqrt{t-s}})^{-N+n+1+|\alpha|+|\beta|}\exp(-\frac{\delta x_n^2}{4(t-s)}),$$

$$|(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial y'})^{\beta} r(t,s)| \leq C_{\alpha,\beta}(\frac{1}{\sqrt{t-s}})^{-N+n+|\alpha|+|\beta|}.$$

Proof. By the definition of Z_B we have

$$\begin{aligned} L(Z_B h)(t,s) &= \lim_{s\to t} v_B(t-s,x',x_n,D',0;t)h(s,\cdot) \\ &\quad + \int_s^t (\frac{\partial}{\partial t}+\hat{Q})V_B(t-\sigma;t)h(\sigma,\cdot)d\sigma + \int_s^t (P-\hat{Q})V_B(t-\sigma;t)h(\sigma,\cdot)d\sigma \\ &= \int_s^t S_N(t-\sigma;t)h(\sigma,\cdot)d\sigma + \int_s^t (P-\hat{Q})V_B(t-\sigma;t)h(\sigma,\cdot)d\sigma, \end{aligned}$$

where we used that $\lim_{s\to t} V_B(t-s;t)f=0$ at $x_n>0$ for any continuous function f. By the facts that $r_N(\sigma;t)\in\mathscr{H}_{-N}(\sigma;t)$, $s_N(\sigma;t)\in\mathscr{H}_{-N+1}(\sigma;t)$, $v_B(\sigma;t)\in\mathscr{H}_0(\sigma;t), P-\hat{Q}\in\mathscr{F}'_{-N+1}$ and (3.20), we get the first part of the assertion. From Theorem 5 it holds that

$$\begin{aligned} B(t)(Z_B h)(t,s) &= \int_s^t B(t)v_B(t-\sigma,x',x_n,D',0;t)h(\sigma,\cdot)d\sigma \\ &= -2\int_s^t W_{1,0}(t-\sigma,x_n)e^{-(t-\sigma)\beta(x',D')}h(\sigma,\cdot)d\sigma \\ &\quad + \int_s^t r_N(t-\sigma,x',x_n,D',0;t)h(\sigma,\cdot)d\sigma. \end{aligned}$$

By Proposition 7, Proposition 8 and the above equation we get

$$\lim_{x_n\to 0} B(t)(Z_B h)(t,s) = h(t) + \int_s^t r_N(t-\sigma,x',0,D',0;t)h(\sigma,\cdot)d\sigma.$$

q.e.d.

5. $\mathbf{L}^p(\boldsymbol{R}^n_+)$ boundedness of operators of $\mathscr{H}_0$

In this section we shall show that

Proposition 10. *Let $g(\sigma;t)$ belong to $\mathscr{H}_s(\sigma;t)$. Then an operator $G(\sigma;t)$ corresponded to $g(\sigma;t)$ is a bounded operator on $\mathbf{L}^p(\boldsymbol{R}^n_+)$ for $1<p<\infty$ if $\sigma>0$ or $s\leq 0$. Moreover we have the estimate*

$$\|G(\sigma;t)\|\leq C\left(\frac{1}{\sqrt{\sigma}}\right)^{s}\qquad (0\leq\sigma\leq T).$$

Theorem 6. *For operators $U(t)$ constructed by Theorem C and $V_N(t,s)$ constructed in Theorem 4, we have*

$$\lim_{t\to 0}U(t)\varphi=\varphi\quad in\ \mathbf{L}^p(\boldsymbol{R}^n_+)$$

and

$$\lim_{t\to 0}V_N(t,0)\varphi=0\quad in\ \mathbf{L}^p(\boldsymbol{R}^n_+)$$

for any $\varphi\in\mathbf{L}^p(\boldsymbol{R}^n_+)$ and for any integer N.

For the proof of Proposition 10 and Theorem 6 we prepare the following lemma and propositions.

Lemma 4. *Let $q(x',v,\xi',w)$ satisfy*

$$|\left(\frac{\partial}{\partial\xi'}\right)^{\alpha}\left(\frac{\partial}{\partial x'}\right)^{\beta}q|\leq C_{\alpha,\beta}<\xi'>^{-|\alpha|+\delta|\beta|}H(v,w),$$

where $H(v,w)$ satisfies for an interval J in $\boldsymbol{R}$

$$(5.1)\qquad \int_J H(v,w)dv\leq C_0,\qquad \int_J H(v,w)dw\leq C_0.$$

Then $\int_J q(x',v,D',w)\varphi(\cdot,w)dw$ defined by

$$(2\pi)^{-n+1}\int_J\int_{\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}}e^{i(x'-y')\cdot\xi'}q(x',v,\xi',w)\varphi(y',w)dy'd\xi'dw$$

is a bounded operator on $\mathbf{L}^p(\boldsymbol{R}^{n-1}\times J)$ for $1<p<\infty$ with some constant C

$$\left\|\int_J q(x',v,D',w)\varphi(\cdot,w)dw\right\|_{L^p(\boldsymbol{R}^{n-1}\times J)}\leq CC_0\|\varphi\|_{L^p(\boldsymbol{R}^{n-1}\times J)}.$$

Proof. Set

$$u(x',v)=\int_J q(x',v,D',w)\varphi(\cdot,w)dw$$

$$=(2\pi)^{-n+1}\int_J\int_{R^{n-1}\times R^{n-1}} e^{i(x'-y')\cdot\xi'}q(x',v,\xi',w)\varphi(y',w)dy'd\xi'dw.$$

Then the boundedness of pseudo-differential operators of class $S^0_{1,\delta}(\boldsymbol{R}^{n-1})$ on $\mathbf{L}^p(\boldsymbol{R}^{n-1})$ indicates that there exist l and $\tilde{C}$ such that

$$\|u(\cdot,v)\|_{L^p(\boldsymbol{R}^{n-1})}\leq\tilde{C}\int_J|q(\cdot,v,\cdot,w)|_l^{(0)}\|\varphi(\cdot,w)\|_{L^p(\boldsymbol{R}^{n-1})}dw. \tag{5.2}$$

By the assumption we have

$$|q(\cdot,v,\cdot,w)|_l^{(0)}\leq C_lH(v,w),$$

where $C_l=\max_{|\alpha|+|\beta|\leq l}C_{\alpha,\beta}$. So the Hausdorff-Young theorem concludes to

$$\int_J\{\int_J|q(\cdot,v,\cdot,w)|_l^{(0)}\|\varphi(\cdot,w)\|_{L^p(\boldsymbol{R}^{n-1})}dw\}^p dv\leq C_l^pC_0^p\|\varphi\|^p_{L^p(\boldsymbol{R}^{n-1}\times J)}. \tag{5.3}$$

By (5.2) and (5.3) we get the assertion, taking $C=\tilde{C}C_l$. q.e.d.

Proof of Proposition 10. For the operators corresponding to ($\mathscr{D}$), ($\mathscr{N}$), ($\mathscr{O}$) and ($\mathscr{R}$) we can apply Proposition 7, taking

$$H(v,w)=\left(\frac{1}{\sqrt{\sigma}}\right)^{s+1}\exp\left(-\delta\frac{(v+w)^2}{4\sigma}\right),\qquad J=(0,\infty).$$

Then we get the assertion. But in case ($\mathscr{S}$) we can not apply the above argument to Proposition 4'-(1). In case ($\mathscr{S}$) we have the following estimate for $g(\sigma;t)$.

$$\left|\left(\frac{\partial}{\partial\xi'}\right)^\alpha\left(\frac{\partial}{\partial x'}\right)^\beta g\right|\leq C_{\alpha,\beta}\left(\frac{1}{|\xi'|+\frac{1}{\sqrt{\sigma}}}\right)^\alpha\left(\frac{1}{\sqrt{\sigma}}\right)^{s+1+\frac{|\beta|}{2}}\exp\left(-\delta\frac{(x_n+y_n)^2}{4\sigma}-c_0|\xi'|^2\sigma\right).$$

Now let $\psi(x)$ be a smooth funcion such that

$$\psi(r)=\begin{cases}1, & \text{if } |r|<1;\\ 0, & \text{if } |r|>2.\end{cases}$$

Set

$$g(\sigma;t)=\psi(|\xi'|\sqrt{\sigma})g(\sigma;t)+(1-\psi(|\xi'|\sqrt{\sigma}))g(\sigma;t)=g_1+g_2.$$

Then $g_2(\sigma;t)$ satisfies the assumption of Lemma 4 with $\delta=\frac{1}{2}$. On the other hand, $g_1(\sigma,x',x_n,D',y_n;t)$ has a kernel $\tilde{g}_1(\sigma,x',x_n,y',y_n;t)$ defined below

$$\begin{aligned}\tilde{g}_1(\sigma,x',x_n,y',y_n;t)&=(2\pi)^{-(n-1)}\int_{\boldsymbol{R}^{n-1}}\psi(|\xi'|\sqrt{\sigma})g(\sigma,x',x_n,y',y_n;t)e^{i(x'-y')\cdot\xi'}d\xi'\\&=(2\pi)^{-(n-1)}\int_{\boldsymbol{R}^{n-1}}e^{i(x'-y')\cdot\xi'}g(\sigma,x',x_n,y',y_n;t)\\&\quad\times\{1+(-\Delta_{\xi'})^N\sigma^{-N}\}\varphi(|\xi'|\sqrt{\sigma})(1+\sigma^{-N}|x'-y'|^{2N})^{-1}d\xi',\end{aligned}$$

$N>\frac{n}{2}$. So we have

$$\begin{aligned}|\tilde{g}_1(\sigma,x',x_n,y',y_n;t)|&\leq C\Big(\frac{1}{\sqrt{\sigma}}\Big)^{s+1}\exp\Big(-\delta\frac{(x_n+y_n)^2}{4\sigma}\Big)\\&\quad\times F\Big(\frac{|x'-y'|}{\sqrt{\sigma}}\Big)\mathrm{vol}\Big(\{\xi';|\xi'|\leq\frac{2}{\sqrt{\sigma}}\}\Big),\end{aligned}$$

where $F(z)=(1+|z|^{2N})^{-1}$. Then

$$\begin{aligned}&\int_{\boldsymbol{R}^{n-1}}|\tilde{g}_1(\sigma,x',x_n,y',y_n;t)|dx',\ \int_{\boldsymbol{R}^{n-1}}|\tilde{g}_1(\sigma,x',x_n,y',y_n;t)|dy'\\&\leq C\Big(\frac{1}{\sqrt{\sigma}}\Big)^{s+1}\exp\Big\{-\delta\frac{(x_n+y_n)^2}{4\sigma}\Big\}.\end{aligned}$$

Then we are able to apply Proposition 11 below and get the assertion. q.e.d.

Proposition 11. *Let $r(x',v,y',w)$ satisfy*

$$\int_{\boldsymbol{R}^{n-1}}|r(x',v,y',w)|dx'\leq H(v,w),$$

$$\int_{\boldsymbol{R}^{n-1}}|r(x',v,y',w)|dy'\leq H(v,w),$$

with $H(v,w)$ *satisfying* (5.1). *Then an operator* $(\mathscr{R}\varphi)(x',v)$ *defined by* $(\mathscr{R}\varphi)(x',v)=\int_J\int_{\mathbf{R}^{n-1}}r(x',v,y',w)\varphi(y',w)dy'dw$ *is a bounded operator on* $\mathbf{L}^p(\mathbf{R}^{n-1}\times J)$ *for* $1<p<\infty$.

For the proof of Theorem 6 we prepare

Proposition 12. *The fundamental solution* $U(t)$ *constructed in Theorem C satisfies*

$$(1) \qquad U(t)\varphi^+\to\varphi \quad \text{in } \mathbf{L}^p(\mathbf{R}^n_+)$$

as t *tends to* 0.

(2) *Set* $v=\{\tilde{w}_{0,0}(t,x_n,+y_n)-2b(t,x')\tilde{w}_{0,-1}(t,x_n+y_n;a(t,x'),b(t,x'))\}e^{-\beta t}$ *or* $v=\{\tilde{w}_{0,0}(t,x_n,+y_n)-2b(t,x',\xi')\tilde{w}_{0,-1}(t,x_n+y_n;b(t,x',\xi'))\}e^{-\beta t}$. *Then*

$$V(t)\varphi\to 0 \qquad \text{in } \mathbf{L}^p(\mathbf{R}^n_+)$$

as t *tends to* 0.

Proof. The fundamental solution $U(t)$ for the Cauchy problem is a pseudodifferential operator of which symbol has the following expansion by Theorem C.

$$u(t)=u_0(t)+u_1(t)+u_2(t)+\cdots+u_j(t)+\cdots,$$

where $u_j(t;x,\xi)=f_j(t;x,\xi)\exp(-p_2(x,\xi)t)$. These functions $f_j(t;x,\xi)$ are polymonials with respect to ξ and t, satisfying the equation $k-2l=-j$, where k is the degree of ξ and l is that of t. The operator $u_j(t;x,D)$ has kernel

$$\tilde{u}_j(t;x,x-y)=(2\pi)^{-n}\int_{\mathbf{R}^n}u_j(t;x,\xi)e^{i(x-y)\cdot\xi}d\xi$$

$$=K_j\Big(t;x,\frac{y-x}{\sqrt{t}}\Big),$$

where $K_j(t;x,z)$ satisfies

$$\int_{\mathbf{R}^n}|K_j(t;x,z)|dz\leq C\sqrt{t}^j.$$

It is well-known that pseudo-differential operators of class $S^0_{1,0}$ are

$\mathbf{L}^p(\boldsymbol{R}^n)$-bounded for $1<p<\infty$. The symbol u_0 convergences to 0 in the weak sense, that is, $\lim_{t\to 0} u_0(t;x,\xi)=0$ for $\{\xi;|\xi|\leq B\}$. This indicates that

$$\lim_{t\to 0} U_0(t)\chi=\chi \qquad \text{in } \mathbf{L}^p(\boldsymbol{R}^n)$$

for a bounded continuous function χ defined on $\boldsymbol{R}^n$. We have

$$\lim_{t\to 0} U_j(t)\chi=0 \qquad \text{in } \mathbf{L}^p(\boldsymbol{R}^n) \qquad (j\geq 1)$$

by the similar methods of Proposition 10. Then we get

$$\lim_{t\to 0} U(t)\chi=\chi \qquad \text{in } \mathbf{L}^p(\boldsymbol{R}^n).$$

We have the assertion (1) for a function $\varphi\in\mathbf{L}^p(\boldsymbol{R}^n_+)$, applying the above arguments for φ^+.

(2) Set $v_1=w_{0,0}e^{-\beta t}$. Then $V_1(t)\varphi=U_0(t)\varphi^-$ by the following equality given in Remark 1.

$$W_{0,0}\varphi(t,x_n)=w_{0,0}(t;x_n,D_n)\varphi^-.$$

By (1) we have

$$\lim_{t\to 0} V_1(t)\varphi=\varphi^- \qquad \text{in } \mathbf{L}^p(\boldsymbol{R}^n).$$

So we have

$$\lim_{t\to 0} V_1(t)\varphi=0 \qquad \text{in } L^p(\boldsymbol{R}^n_+).$$

Set $v_2=v-v_1$. In case $(\mathscr{D})$, $(\mathscr{N})$, $(\mathscr{R})$, v_2 belongs to $\mathscr{H}_{-1}$. Hence we get

$$\lim_{t\to 0} V_2(t)\varphi=0 \qquad \text{in } L^p(\boldsymbol{R}^n_+)$$

by Proposition 10. It is nessesary to consider only cases $(\mathscr{O})$ and $(\mathscr{S})$. We can write the operator $V_2(t)$ corresponding to a symbol $v_2(t)$ as follows.

$$V_2(t)\varphi(x_n)=\int_0^\infty v_2(t,x',x_n,D',y_n)\varphi(\cdot,y_n)dy_n.$$

We extend the operator $V_2(t)$ as an integral-pseudodifferential operator $V_3(t)$ on $\mathbf{L}^p(\boldsymbol{R}^n)$ of symbol $v_3(t,x',x_n,\xi',y_n)$ which is defined as

$$V_3(t)f=\int_{-\infty}^{\infty} v_3(t,x',x_n,D',y_n)f(\cdot,y_n)dy_n,$$

where

$$v_3(t,x',x_n,\xi',y_n)=\begin{cases} v_2(t,x',x_n,\xi',y_n), & \text{if} \quad x_n+y_n\geq 0; \\ 0, & \text{otherwise.}\end{cases}$$

Then for $x_n\geq 0$ we have

$$V_2(t)\varphi(x_n)=V_3(t)\varphi^+(x_n). \tag{5.4}$$

Assume that

$$\lim_{t\to 0} V_3(t)\psi=\tilde{\psi} \qquad \text{in } \mathbf{L}^p(\boldsymbol{R}^n), \tag{5.5}$$

where

$$\tilde{\psi}(x',x_n)=0$$

for $(\mathscr{O})$, or

$$\tilde{\psi}(x',x_n)=\begin{cases} -\psi(x',-x_n), & \text{if} \quad a(0,x')=0; \\ 0, & \text{otherwise.}\end{cases} \tag{5.6}$$

for $(\mathscr{S})$. Then by (5.4) it is clear that $V_2(t)\varphi\to 0$ in $\mathbf{L}^p(\boldsymbol{R}^n_+)$.

For the proof of (5.5), repeating the same argument of Proposition 10, we have $\mathbf{L}^p(\boldsymbol{R}^n)$ boundedness for $V_3(t)$. So it is sufficient to prove (5.5) for smooth functions. Set for case $(\mathscr{O})$

$$V_3(t)\psi=\int_0^{\infty}\int_0^{\infty}\frac{4\sqrt{t}b(t,x',D')}{\sqrt{\pi}}e^{-(\sigma+\mu)^2+2b\sqrt{t}\sigma-\beta(x',D')t}d\sigma\psi(\cdot,-x_n+2\sqrt{t}\mu)d\mu.$$

Then we have $V_3(t)\psi-v_4(t,x',D')\psi(\cdot,-x_n)$ converges to 0 in $L^p(\boldsymbol{R}^n)$ as $t\to 0$, where

$$\begin{aligned} v_4&=\int_0^{\infty}\int_0^{\infty}\frac{4\sqrt{t}b(t,x',\xi')}{\sqrt{\pi}}e^{-(\sigma+\mu)^2+2b\sqrt{t}\sigma}d\sigma d\mu e^{-\beta(x',\xi')t} \\ &=\int_0^{\infty}2\sqrt{t}\{e^{-(\sigma^2+2b\sqrt{t}\sigma}-e^{-\sigma^2}\}d\sigma e^{-\beta(x',\xi')t}. \end{aligned}$$

On the other hand $v_4(t,x',D')\psi(\cdot,-x_n)$ converges to 0 in $\mathbf{L}^p(\boldsymbol{R}^n)$ as $t\to 0$, where we use the fact

$\int_0^\infty\{e^{-\sigma^2+2b\sqrt{t}\sigma}-e^{-\sigma^2}\}d\sigma e^{-\beta(x',\xi')t}$ weakly onverges to 0 in $S^0_{1,0}$ as $t\to 0$.

Set for case $(\mathscr{S})$

$$V_3(t)\psi=\int_0^\infty\int_0^\infty\frac{4\sqrt{t}b(t,x')}{\sqrt{\pi}}e^{-(a(t,x')\sigma+\mu)^2+2b\sqrt{t}\sigma-\beta(x',D')t}d\sigma\psi(\cdot,-x_n+2\sqrt{t}\mu)d\mu.$$

Then we have $V_3(t)\psi-v_5(t,x',D')\psi(\cdot,-x_n)$ converges to 0 in $\mathbf{L}^p(\boldsymbol{R}^n)$ as $t\to 0$, where

$$v_5=\int_0^\infty\int_0^\infty\frac{4\sqrt{t}b(t,x')}{\sqrt{\pi}}e^{-(a(x',\xi')\sigma+\mu)^2+2b\sqrt{t}\sigma}d\sigma d\mu e^{-\beta(x',\xi')t}$$
$$=\begin{cases}\int_0^\infty 2\sqrt{\pi}\{e^{-\sigma^2+2b\sqrt{t}\sigma}-e^{-\sigma^2}\}d\sigma e^{-\beta(x',\xi')t}, & \text{if } a(x',0)\neq 0;\\ -e^{-\beta(x',\xi')t}, & \text{otherwise.}\end{cases}$$

On the other hand $v_5(t,x',D')\psi(\cdot,-x_n)$ converges to $\tilde{\psi}$ defined as (5.6) in $L^p(\boldsymbol{R}^n)$ as $t\to 0$. q.e.d.

Proof of Theorem 6. The symbol of $V_N(t,0)$ is obtained by

$$v_N=(\tilde{w}_{0,0}-2b(t)\tilde{w}_{0,-1})e^{-\beta t}+v',$$

with $v'\in\mathscr{H}_{-1}$ or $v'\in\mathscr{H}_{-\frac{1}{2}}$ (for the problem $(\mathscr{S})$). By Proposition 10 and Proposition 12 we get the assertion. q.e.d.

Set an integral operator $(\mathscr{I}_g h)(t)$ of the following form

$$(\mathscr{I}_g h)(t)=\int_0^t g(t-\sigma,x',x_n,D',0;t)h(\sigma,\cdot)d\sigma.$$

By the same method of Proposition 10 we have the following Lemma 5. In this case, we apply Lemma 4 taking $H(v,w)=(\frac{1}{\sqrt{v-w}})^{s+1}e^{-\frac{x_n^2}{4(v-w)}}$.

Lemma 5. *Let $g(\sigma;t)$ belong to $\mathscr{H}_s(\sigma;t)$. Then $(\mathscr{I}_g h)(t)$ is a bounded opertor on $\mathbf{L}^p(\boldsymbol{R}^{n-1}\times(0,T))$, if $x_n>0$ or $s\leq 1$. Moreover we have the estimate*

$$\|\mathscr{I}_g h(t)\| \leq \begin{cases} C x_n^{(-s+1)}\|h\|, & \textit{if} \quad s>1; \\ C\|h\|, & \text{otherwise.} \end{cases}$$

Theorem 7. *If v_B is the symbol which is constructed in Theorem* 5, *we have*

$$BZ_B h(t,0) \to h(t) \qquad \textit{in } \mathbf{L}^p(\boldsymbol{R}^{n-1}\times(0,T))$$

as $x_n \to 0$.

Proof. Noting $Z_B = \mathscr{I}_{v_B}$, we obtain the assertion by Corollary 1 of Proposition 8 and the above Lemma 5.

6. Global construction of the fundamental solution and the proof of Theorem I

Let $\{\Omega_\mu\}_{\mu\in\mathscr{M}}$ be a finite open covering of M. Let $\mathscr{N}$ be a subset of $\mathscr{M}$ such that $\Omega_\mu(\mu\in\mathscr{N})$ are diffeomorphic to domains $\tilde{\Omega}_\mu$ in $\bar{\boldsymbol{R}}^n_+$, with the property $\Gamma\cap\tilde{\Omega}_\mu(\mu\in\mathscr{N})$ are diffeomorphic to domains in $\{(x',x_n); x_n=0\}$ and $\mathrm{dis}(\Omega_\mu,\Gamma)>\delta\geq 0$ for $\mu\in\mathscr{M}\backslash\mathscr{N}$. Let $\{\varphi_\mu\}_{\mu\in\mathscr{M}}$ be a partition of unity subordinate to the covering $\{\Omega_\mu\}_{\mu\in\mathscr{M}}$ and let $\{\psi_\mu\}_{\mu\in\mathscr{M}}$ be $C_0^\infty(\Omega_\mu)$ functions such that $\psi_\mu=1$ on supp φ_μ.

In each local patch $(\Omega_\mu)_{\mu\in\mathscr{M}}$ the problem is reducecd to the following form.

(1)For $\mu\in\mathscr{N}$

$$(L_\mu, B_\mu) \qquad \begin{cases} (\dfrac{\partial}{\partial t}+P_\mu)u_\mu=0 & \text{in} \quad I_s\times\boldsymbol{R}^n_+, \\ B_\mu u_\mu|_{x_n=0}=0 & \text{in} \quad I_s\times\boldsymbol{R}^{n-1}, \\ u_\mu|_{t=s}=m_\mu(x) & \text{in} \quad \boldsymbol{R}^n_+. \end{cases}$$

(2)For $\mu\in\mathscr{M}\backslash\mathscr{N}$

$$(L_\mu) \qquad \begin{cases} (\dfrac{\partial}{\partial t}+P_\mu)u_\mu=0 & \text{in} \quad I_s\times\boldsymbol{R}^n, \\ u_\mu|_{t=s}=m_\mu(x) & \text{in} \quad \boldsymbol{R}^n, \end{cases}$$

where $P_\mu=P$ on Ω_μ, $B_\mu=B$ on $\Omega_\mu\cap\Gamma, m_\mu=\varphi_\mu m$.

By the assumption P_μ can be extended to be strongly elliptic in $\boldsymbol{R}^n$. Choosing a covering $\{\Omega_\mu\}_{\mu\in\mathscr{M}}$ sufficiently small, we can assume that

P_μ satisfies the assumption (3.1).

Let $U^\mu(t)(\mu\in\mathscr{M}\backslash\mathscr{N})$ be the fundamental solution for the problem (L_μ) which is consructed in §2. $E_N^\mu(t,s)(\mu\in\mathscr{N})$ be the approximate solution for (L_μ, B_μ) constructed in §3, that is,

$$\begin{cases} L_\mu E_N^\mu(t,s)-G^\mu(t,s)\in\mathscr{K}_{-N+1}(t,s), \\ B_\mu E_N^\mu(t,s)-F^\mu(t,s)\in\mathscr{K}_{-N}(t,s), \\ E_N^\mu(t,t)=I. \end{cases}$$

By Theorem 4 $G^\mu(t,s)$ and $F^\mu(t,s)$ are smoothing operators with kernels $\tilde{g}^\mu(t,s)$, $\tilde{f}^\mu(t,s)$ which saisfy

$$(6.1)\qquad |(\frac{\partial}{\partial x})^\alpha(\frac{\partial}{\partial y})^\beta\tilde{g}^\mu(t,s)|\le C_{\alpha,\beta}(\frac{1}{\sqrt{t-s}})^{-N+n+1+|\alpha|+|\beta|},$$

$$(6.2)\qquad |(\frac{\partial}{\partial x'})^\alpha(\frac{\partial}{\partial y'})^\beta(\frac{\partial}{\partial y_n})^{\beta_n}\tilde{f}^\mu(t,s)|\le C_{\alpha,\beta}(\frac{1}{\sqrt{t-s}})^{-N+n+|\alpha|+|\beta|+|\beta_n|}.$$

Set

$$E_N(t,s)=\sum_{\mu\in\mathscr{N}}\psi_\mu E_N^\mu(t,s)\varphi_\mu+\sum_{\mu\in\mathscr{M}\backslash\mathscr{N}}\psi_\mu U^\mu(t-s)\varphi_\mu.$$

Then

$$\begin{aligned} LE_N(t,s)&=\sum_{\mu\in\mathscr{N}}\{\psi_\mu LE_N^\mu(t,s)\varphi_\mu+[L,\psi_\mu]E_N^\mu(t,s)\varphi_\mu\} \\ &\quad+\sum_{\mu\in\mathscr{M}\backslash\mathscr{N}}\{\psi_\mu LU^\mu(t-s)\varphi_\mu+[L,\psi_\mu]U^\mu(t-s)\varphi_\mu\} \\ &=\sum_{\mu\in\mathscr{N}}\{\psi_\mu G^\mu(t,s)\varphi_\mu+[P,\psi_\mu]E_N^\mu(t,s)\varphi_\mu\} \\ &\quad+\sum_{\mu\in\mathscr{M}\backslash\mathscr{N}}\{[P,\psi_\mu]U^\mu(t-s)\varphi_\mu\}, \end{aligned}$$

$$\begin{aligned} B(t)E_N(t,s)|_\Gamma&=\sum_{\mu\in\mathscr{N}}\{\psi_\mu B_\mu(t)E_N^\mu(t,s)\varphi_\mu+[B_\mu(t),\psi_\mu]E_N^\mu(t,s)\varphi_\mu\}|_\Gamma \\ &=\sum_{\mu\in\mathscr{N}}\{\psi_\mu F^\mu(t,s)\varphi_\mu+[B_\mu(t),\psi_\mu]E_N^\mu(t,s)\varphi_\mu\}|_\Gamma, \end{aligned}$$

$$E_N(t,t)=\sum_{\mu\in\mathscr{N}}\psi_\mu E_N^\mu(t,t)\varphi_\mu+\sum_{\mu\in\mathscr{M}\backslash\mathscr{N}}\psi_\mu U^\mu(t-t)\varphi_\mu$$

$$=I.$$

Hence we have

Proposition 13. *For any fixed N, $E_N(t,s)$ defined above satisfies*

$$\begin{cases} LE_N(t,s)=G(t,s), \\ B(t)E_N(t,s)=F(t,s), \\ E_N(t,t)=I, \end{cases}$$

where $G(t,s)$ and $F(t,s)$ are operators whose kernels $\tilde{g}(t,s)$ and $\tilde{f}(t,s)$ satisfy (6.1) *and* (6.2) *respectively.*

Proof. $\mathrm{supp}[P_\mu,\psi_\mu]\cap\mathrm{supp}\varphi_\mu=\emptyset$, $\mathrm{supp}[B_\mu,\psi_\mu]\cap\mathrm{supp}\varphi_\mu=\emptyset$ by the definition of ψ_μ. Owing to the above fact and the pseudo-local property of $\mathscr{H}_s(\sigma;t)$ and $S^m_{1,0}$, (6.1) and (6.2) hold for $\tilde{g}(t,s)$ and $\tilde{f}(t,s)$ respectively. q.e.d.

On the other hand in §4 we construct the approximate Poisson operator Z^μ_B in $\boldsymbol{R}^n_+$ for any $\mu\in\mathscr{N}$ such that

$$(Z^\mu_B(t,s)h)(x',x_n)=\int_s^t v_{B_\mu}(t-\sigma,x',x_n,D',0;t)h(\sigma,\cdot)d\sigma$$

satisfies

$$\begin{cases} L_\mu(Z^\mu_B(t,s)h)=S^\mu(t,s)h \qquad \text{in } I_s\times\boldsymbol{R}^n_+ \\ \lim\limits_{x_n\to 0} B_\mu(t)(Z^\mu_B(t,s)h)=h(t)+R^\mu(t,s)h \qquad \text{in } I_s\times\boldsymbol{R}^{n-1} \\ \lim\limits_{t\to s}(Z^\mu_B(t,s)h)=0 \qquad \text{in } \boldsymbol{R}^n_+, \end{cases}$$

where $S^\mu(t,s)$ and $R^\mu(t,s)$ are integral operators of the form

$$(S^\mu(t,s)h)(x',x_n)=\int_s^t\int_{\boldsymbol{R}^{n-1}} s^\mu(t,\sigma,x_n;x',y')h(\sigma,y')dy'd\sigma,$$

$$(R^\mu(t,s)h)(x')=\int_s^t\int_{\boldsymbol{R}^{n-1}} r^\mu(t,\sigma;x',y')h(\sigma,y')dy'd\sigma$$

with smoothing kernels in the sense

$$(6.3)\quad |(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial x_n})^{\alpha_n}(\frac{\partial}{\partial y'})^{\beta}s^{\mu}(t,s,x_n;x',y')|\le C_{\alpha,\alpha_n,\beta}(\frac{1}{\sqrt{t-s}})^{-N+n+1+|\alpha|+|\alpha_n|+|\beta|},$$

$$(6.4)\quad |(\frac{\partial}{\partial x'})^{\alpha}(\frac{\partial}{\partial y'})^{\beta}r^{\mu}(t,s;x',y')|\le C_{\alpha,\beta}(\frac{1}{\sqrt{t-s}})^{-N+n+|\alpha|+|\beta|}.$$

Set $Z_B(t,s)=\sum_{\mu\in\mathcal{N}}\psi_\mu Z_B^{\mu}(t,s)\varphi_\mu$. By the similar argument to $E_N(t,s)$, we get that $Z_B(t,s)$ satisfies the following equations

$$\begin{cases} LZ_B(t,s)=S(t,s) & \text{in } I_s\times M, \\ B(t)Z_B(t,s)=I+R(t,s) & \text{in } I_s\times\Gamma, \\ \lim\limits_{t\to s}Z_B(t,s)=0 & \text{in } M, \end{cases}$$

where opeators $S(t,s)$ and $R(t,s)$ have kernels $\tilde{s}(t,s)$ and $\tilde{r}(t,s)$ satisfying (6.3) and (6.4), respectivily.

Proposition 14. *We can construct an operator $\bar{Z}_B$ of the form $(\bar{Z}_B(t,s)h)=\int_s^t\bar{v}_B(t,\sigma)h(\sigma)d\sigma$ such that*

$$\begin{cases} L\bar{Z}_B(t,s)=S_1(t,s) & \text{in } I_s\times M, \\ B(t)\bar{Z}_B(t,s)=I & \text{in } I_s\times\Gamma, \\ \lim\limits_{t\to s}\bar{Z}_B(t,s)=0 & \text{in } M, \end{cases}$$

with $S_1(t,s)$ of which kernel $\tilde{s}_1(t,s)$ satisfies (6.3).

Proof. Let $\varphi(t,s)$ be the solution of the equation

$$r(t,s)+\varphi(t,s)+\int_s^t r(t,\sigma)\cdot\varphi(\sigma,s)d\sigma=0,$$

where $r(t,\sigma)\cdot\varphi(\sigma,s)$ means that

$$(r(t,\sigma)\cdot\varphi(\sigma,s))(x',z')=\int_\Gamma r(t,\sigma;x',y')\varphi(\sigma,s;y',z')dy'.$$

Then $\varphi(t,s)$ also satisfies (6.4). Set

$$v_B(t,s)=\sum_{\mu\in\mathcal{N}}\psi_\mu v_{B_\mu}(t-s,x',x_n,D',0;t)\varphi_\mu.$$

Then $Z_B(t,s)h=\int_s^t v_B(t,\sigma)h(\sigma)d\sigma$ by the definition. Let $\bar{v}_B$ be the solution of

$$\bar{v}_B(t,s)=v_B(t,s)+\int_s^t v_B(t,\sigma)\cdot\varphi(\sigma,s)d\sigma.$$

Then we have

$$\bar{Z}_B(t,s)h=Z_B(t,s)h_1,$$

where $h_1(t)=h(t)+\int_s^t\varphi(t,\mu)\cdot h(\mu)d\mu$. So we obtain the following equation:

$$\begin{aligned}L\bar{Z}_B(t,s)h&=S(t,s)h_1\\&=S(t,s)h+\int_s^t\tilde{s}(t,\sigma)\cdot(\int_s^\sigma\varphi(\sigma,\mu)\cdot h(\mu)d\mu)d\sigma\\&=S(t,s)h+\int_s^t(\int_\mu^t\tilde{s}(t,\sigma)\cdot\varphi(\sigma,\mu)d\sigma)\cdot h(\mu)d\mu\\&=S_1(t,s)h.\end{aligned}$$

The kernel $\tilde{s}_1(t,s)$ of an operator $S_1(t,s)$ is given by

$$\tilde{s}_1(t,s)=\tilde{s}(t,s)+\int_s^t\tilde{s}(t,\sigma)\cdot\varphi(\sigma,s)d\sigma. \tag{6.5}$$

So $\tilde{s}_1(t,s)$ also satisfies (6.3). On the other hand on Γ we have

$$\begin{aligned}B(t)\bar{Z}_B(t,s)h&=h_1(t)+R(t,s)h_1\\&=h(t)+\int_s^t\varphi(t,\mu)\cdot h(\mu)d\mu\\&\quad+\int_s^t r(t,\sigma)\cdot(h(\sigma)+\int_s^\sigma\varphi(\sigma,\mu)\cdot h(\mu)d\mu)d\sigma\\&=h(t)+\int_s^t(r(t,\sigma)+\varphi(t,\sigma)+\int_\sigma^t r(t,\mu)\cdot\varphi(\mu,\sigma)d\mu)\cdot h(\sigma)d\sigma\end{aligned}$$

$$=h(t).$$

The last equation follows by the definition of $\varphi(t,s)$.

q.e.d.

Proof of Theorem I. Let $E_{N,\infty}(t,s)=E_N(t,s)-\bar{Z}_B(t,s)\tilde{f}(\cdot,s)$. Then

$$\begin{cases} LE_{N,\infty}(t,s)=G(t,s)-S_1(t,s)\tilde{f}(\cdot,s)=G_1(t,s) & \text{in } I_s\times M, \\ B(t)E_{N,\infty}(t,s)=0 \quad \text{in } I_s\times\Gamma, & \\ \lim\limits_{t\to s}E_{N,\infty}(t,s)=I \quad \text{in } M, & \end{cases}$$

where $G_1(t,s)$ has the kernel $\tilde{g}_1(t,s)$ defined by

$$\tilde{g}_1(t,s)=\tilde{g}(t,s)-\int_s^t \tilde{s}_1(t,\sigma)\cdot\tilde{f}(\sigma,s)d\sigma. \tag{6.6}$$

So $\tilde{g}_1(t,s)$ also satisfies (6.1). Let $\psi(t,s)$ be the solution of the followig equation

$$\tilde{g}_1(t,s)+\psi(t,s)+\int_s^t \tilde{g}_1(t,\sigma)\odot\psi(\sigma,s)d\sigma=0,$$

where $\tilde{g}_1(t,\sigma)\odot\psi(\sigma,s)$ means that

$$(\tilde{g}_1(t,\sigma)\odot\psi(\sigma,s))(x,z)=\int_M \tilde{g}_1(t,\sigma;x,y)\psi(\sigma,s;y,z)dy.$$

Then the following $\tilde{e}(t,s)$

$$\tilde{e}(t,s)=e_{N,\infty}(t,s)+\int_s^t e_{N,\infty}(t,\sigma)\odot\psi(\sigma,s)d\sigma \tag{6.7}$$

is the kernel of the fundamental solution. In fact it is easy to show the kernel of $L\tilde{E}(t,s)$ coincides with $\tilde{g}_1(t,s)+\psi(t,s)+\int_s^t\tilde{g}_1(t,\sigma)\odot\psi(\sigma,s)d\sigma$, which is equal to 0 by the definition of $\tilde{g}_1(t,s)$. Now $\psi(t,\sigma)$ also satisfies (6.1) because $\tilde{g}_1(t,\sigma)$ satisfies (6.1). By the definition of $E_{N,\infty}(t,s)$ it holds

$$\tilde{e}_{N,\infty}(t,s)=\tilde{e}_N(t,s)-\int_s^t \bar{v}_B(t,\sigma)\cdot\tilde{f}(\sigma,s)d\sigma. \tag{6.8}$$

We note also that

$$|\tilde{e}(t.s)-\tilde{e}_N(t,s)|\leq C\sqrt{t-s}^{N-n-N_0},$$

if we prove the following Lemma 6. $E_N(t,s)$ is $\boldsymbol{L}^p(M)$-bounded by Proposition 10. So $E(t,s)$ is also $\boldsymbol{L}^p(M)$-bounded. Moreover we have $\lim_{t\to s}E(t,s)m=m$ in $\boldsymbol{L}^p(M)$ by Theorem 6. q.e.d.

Corollary. *The Poisson operator is obtained of the form* $Z(t,s)h=\int_s^t z(t,\sigma)h(\sigma)d\sigma$, *where*

$$z(t,s)=\bar{v}_B(t,s)-\int_s^t e(t,\sigma)\odot\tilde{s}_1(\sigma,s)d\sigma.$$

Lemma 6. *If* $\psi(\sigma,s)$ *satisfy* (6.1) *or* (6.3), *then*

$$|\int_s^t \tilde{e}_N(t,\sigma)\odot\psi(\sigma,s)d\sigma|\leq C(\sqrt{t-s})^{N-n-N_0}, \tag{6.9}$$

$$|\int_s^t \tilde{e}_{N,\infty}(t,\sigma)\odot\psi(\sigma,s)d\sigma|\leq C(\sqrt{t-s})^{N-n-N_0}. \tag{6.10}$$

If $\tilde{r}(t,s)$ *satisfy* (6.2) *or* (6.4), *then*

$$|\int_s^t v_B(t,\sigma)\cdot\tilde{r}(\sigma,s)d\sigma|\leq C(\sqrt{t-s})^{N-n-N_0}, \tag{6.11}$$

where N_0 is a fixed integer such that $N_0>n-1$.

Proof. Owing to that the symbol e_N^μ of E_N^μ belongs to $\mathscr{H}_0$, we have

$$|\text{kernel } of\ (E_N^\mu(t,\sigma)\Lambda^{-N_0})|\leq C\frac{1}{\sqrt{t-\sigma}}$$

for $N_0>n-1$ by Lemma 2. By the assumption we have

$$|\text{kernel } of\ (\Lambda^{N_0}\Psi(\sigma,s))|\leq C(\frac{1}{\sqrt{\sigma-s}})^{-N+n+N_0+1}.$$

So (6.9) holds. (6.10) is clear by the fact that $\int_s^t \tilde{f}(\mu,\sigma)\odot\psi(\sigma,s)d\sigma$ satisfies (6.2) and by the following equation.

$$\int_s^t (\tilde{e}_{N,\infty},-\tilde{e}_N)(t,\sigma)\odot\psi(\sigma,s)d\sigma$$

$$=-\int_s^t\{\int_\sigma^t \bar{v}_B(t,\mu)\cdot\tilde{f}(\mu,\sigma)d\mu\}\odot\psi(\sigma,s)d\sigma$$

$$=-\int_s^t \bar{v}_B(t,\mu)\cdot\{\int_s^\mu \tilde{f}(\mu,\sigma)\odot\psi(\sigma,s)d\sigma\}d\mu.$$

For the proof of (6.11) we devide into two cases.

1^0. For $(\mathscr{O}),(\mathscr{N}),(\mathscr{R})$.

It is cleat that v_{B_μ} belongs to $\mathscr{H}_0$. So we have

$$|v_{B_\mu}(t,\sigma)|\leq C\frac{1}{\sqrt{t-\sigma}}$$

and also we get by Lemma 2

$$|\text{kernel } of\ (V_{B_\mu}\Lambda^{-N_0})|\leq C\frac{1}{\sqrt{t-\sigma}}$$

for $N_0>n-1$. We also get

$$|\text{kernel } of\ (\Lambda^{N_0}R(\sigma,s))|\leq C(\frac{1}{\sqrt{\sigma-s}})^{-N+n+N_0} \tag{6.12}$$

by (6.2). So we get

$$|\int_s^t \bar{v}_B(t,\sigma)\cdot r(\sigma,s)d\sigma|\leq C(\sqrt{t-s})^{N-n-N_0+1}.$$

2^0. For $(\mathscr{D})$ and $(\mathscr{S})$.

It is clear v_{B_μ} belongs to $\mathscr{H}_1$. We apply Proposition 15 below and (6.12) to the main term $\tilde{w}_{1,0}e^{-\beta t}(\tilde{w}_{1,-1}e^{-\beta t})$ of v_{B_μ} for $(\mathscr{D})((\mathscr{S}))$, respectively. Then we get (6.11). q.e.d.

Proposition 15. *Let* $g(t,x',x_n,\xi')=\tilde{w}_{1,-1}(t,x_n)e^{-\beta(x',\xi')t}$ or $g(t,x',x_n,\xi')=\tilde{w}_{1,0}(t,x_n\colon a,b)e^{-\beta(x',\xi')t}$. *Then the operator* $A=\int_s^t g(t-\sigma,x',x_n,D')R(\sigma,s)d\sigma$ *has the kernel* $\tilde{a}$ *which satisfies* $|\tilde{a}|\leq C(\sqrt{t-s})^{N-n-N_0}$ *under the assumption that* $R(\sigma,s)$ *has the kernel* $\tilde{r}(\sigma,s)$ *which satisfies* (6.2) *or* (6.4).

Proof. By the definition of g we have

$$A=\int_s^t \tilde{w}_{1,0}(t-\sigma,x_n)e^{-\beta(x',D')(t-\sigma)}\Lambda(D')^{-N_0}\Lambda(D')^{N_0}R(\sigma,s)d\sigma.$$

Choose $N_0>n-1$. The kernel of $e^{-\beta(x',D')(t-\sigma)}\Lambda(D')^{-N_0}\Lambda(D')^{N_0}R(\sigma,s)$ is estimated by $C(\sqrt{\sigma-s})^{N-n-N_0}$. So we can apply the argument of Corollary 2 of Proposition 8, which completes the proof. q.e.d.

7. Applications to the asymptotic behavior

We calculate $T_t(\mathscr{B})$ for all boundary value problems introduced in §0 and give the proof of Theorem II.

For any fixed point $x^0\in\bar{M}$, choose an open covering as stated in the previous section such that $\{\Omega_\mu\}_\mu$, $x^0\in\Omega_\nu$ and choose a partition of unity $\{\varphi_\mu\}$ subordinate to $\{\Omega_\mu\}$ such that $\varphi_\nu(x^0)=1$, Then we obtain

$$\tilde{e}(t,0;x^0,x^0)-\tilde{e}_N^\nu(t,0;x^0,x^0)=o(t^N)$$

for any N as stated in the proof of Theorem I. If $x^0\notin\Gamma$, the difference of the fundamental solution of the intial-boundary value problem and that of the Cauchy problem is of any power of t. Thus we have

$$\tilde{e}(t,0;x^0,x^0)\sim U^\nu(t;x^0,x^0)=\tilde{u}(t;x^0,0)\sim\sum_{j=0}^{\infty}t^{-\frac{n}{2}+j}C_j(x^0),$$

where

$$C_j(x^0)=(2\pi)^{-n}\int_{R^n}u_{2j}(1;x^0,\xi)d\xi.$$

If $x^0\in\Gamma$, the approximate of the fundamental solution E_N^ν for the intial-boundary value problem (L_ν,B_ν) is obtained in the previous section as $E_N^\nu(t)=U^\nu(t)+V_N^\nu(t,0)$. We have out of Γ

$$\operatorname{tr}V_N^\nu(t,0)\sim o(t^l)\text{ for any }l$$

for any boundary problem considered in this paper owing to Theorem 3, Lemma 2 and Lemma 2′. Also we have the expansion

$$\operatorname{tr} V_N^\nu(t,0) \sim \sum_{j=0}^{\infty} t^{-\frac{n}{2}+\frac{j}{2}} d_j(x')$$

on Γ for $(\mathscr{D})$, $(\mathscr{N})$, $(\mathscr{R})$ and $(\mathscr{O})$ because of Theorem 3 and the definition of $\mathscr{H}_j$.

We will prove in this section that

$$\int_0^\varepsilon \operatorname{tr} V_N^\nu(t,0)dx_n \sim \sum_{j=0}^{\infty} t^{-\frac{n}{2}+\frac{1}{2}+\frac{j}{2}} D_j(x')$$

and calculate $D_0(x'), D_1(x')$ for $(\mathscr{D})$, $(\mathscr{N})$, $(\mathscr{R})$ and $(\mathscr{O})$. We consider the singular problem in 4^0.

1^0 The asymptotic behavior of the trace of the fundamental solution for the Cauchy problem.

Let $U(t)$ be the fundamental solution for the Cauchy problem, that is,

$$\begin{cases} LU = (\dfrac{d}{dt} + P)U(t) = 0 & \text{in} \quad (0,T)\times M, \\ U(0) = I \quad \text{on} \quad M. \end{cases}$$

In a local patch $U(t)$ can be obtained as a pseudo-differential operator with symbol $u(t) = u_0(t) + u_1(t) + u_2(t) + \cdots$, where $u_j(t) = f_j(t)u_0(t)$ are defined as (2.1) and (2.2). If we calculate

$$C_j(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} u_{2j}(1;x,\xi)d\xi,$$

we get

$$\operatorname{tr}(U(t)) \sim \sum_{j=0}^{\infty} t^{-\frac{n}{2}+j} C_j(x).$$

Let g be the Riemannian mertic of M. Set

$$g_{jk} = g(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}), \qquad g^{jk} = (g_{jk})^{-1}.$$

Then the symbol of $P=-(\Delta+h)$ is given by

$$p_2=\sum_{j,k=1}^{n} g^{jk}\xi_j\xi_k,$$

$$p_1=-i\sum_{j=1}^{n}\{\sum_{k=1}^{n}\frac{\partial}{\partial x_k}g^{jk}+\frac{1}{2}\sum_{k=1}^{n}g^{jk}G\frac{\partial}{\partial x_k}G+h_j\}\xi_j,$$

$$p_0=0,$$

where $G=\det(g^{ij})$.

Now we fix a local coordinate such that g^{ij} satisfies the following conditions at a fix point x^0. The first derivatives of g^{ij} vanish at x^0 and $g^{ij}(x^0)=\delta_{ij}$. For simplicity we put $x^0=0$. Then we have by (2.2)

$$\begin{cases} u_0(t,0,\xi)=\exp(-|\xi|^2t),\ u_j(t,0,\xi)=f_j(t,0,\xi)u_0(t,0,\xi) \qquad (j\geq 1), \\ f_1(t,0,\xi)=i\sum_{j=1}^{n}h_j(0)\xi_jt, \\ f_2(t,0,\xi)=-\frac{t^2}{2}\{\sum_{j=1}^{n}(h_j(0)\xi_j)^2+2\sum_{j,l=1}^{n}(\frac{\partial}{\partial x_l}h_j)(0)\xi_l\xi_j+2\sum_{i,j,l=1}^{n}(\frac{\partial^2}{\partial x_l\partial x_i}g^{ij})(0)\xi_l\xi_j \\ \qquad +\sum_{i,j,l=1}^{n}((\frac{\partial}{\partial x_i})^2g^{jl})(0)\xi_j\xi_l+G(0)\sum_{i,j=1}^{n}(\frac{\partial^2}{\partial x_i\partial x_j})G(0)\xi_i\xi_j\} \\ \qquad +\frac{2}{3}t^3\sum_{i,j,l,m=1}^{n}(\frac{\partial^2}{\partial x_i\partial x_j}g^{lm})(0)\xi_i\xi_j\xi_l\xi_m, \end{cases}$$

where $h=\sum_{j=1}^{n}h_j(x)\frac{\partial}{\partial x_j}$. Then we have

$$\begin{cases} (2\pi)^{-n}\int_{R^n}u_0(1;0,\xi)d\xi=(\frac{\Gamma(\frac{1}{2})}{2\pi})^n, \\ (2\pi)^{-n}\int_{R^n}u_{-2}(1;0,\xi)d\xi=(\frac{\Gamma(\frac{1}{2})}{2\pi})^n\{-\frac{\|h\|(0)}{4}-\frac{\operatorname{div}h(0)}{2} \\ \qquad +\frac{1}{6}(\sum_{i,j=1}^{n}((\frac{\partial}{\partial x_i})^2g^{jj})(0)-\sum_{i,j=1}^{n}(\frac{\partial^2}{\partial x_i\partial x_j}g^{ij})(0))\}. \end{cases}$$

Noting the following equation

$$\sum_{i,j=1}^{n}((\frac{\partial}{\partial x_i})^2g^{jj})(0)-\sum_{i,j=1}^{n}(\frac{\partial^2}{\partial x_i\partial x_j}g^{ij})(0)=2K,$$

we get

$$C_0(0)=\left(\frac{\Gamma(\frac{1}{2})}{2\pi}\right)^n,$$

$$C_1(0)=\left(\frac{\Gamma(\frac{1}{2})}{2\pi}\right)^n\left\{\frac{K}{3}-\frac{\|h\|(0)}{4}-\frac{\operatorname{div}h(0)}{2}\right\}.$$

By the fact $\int_M \operatorname{div} h dV=0$, we get the (0) and half part of (2) of Theorem II.

2^0 The asymptotic behavior for Dirichlet and that of Neumann boundary conditions. We calculate the trace of the opeator $V(t)$.

Take a local coordinate as in §6. We consider about the Neumann condition. From Lemma 3 in this case we must solve the following equation asymptotically.

$$(7.1)\qquad \begin{cases} \left(\frac{\partial}{\partial t}+\hat{q}\right)\circ v(t)=0 & \text{in } I\times \boldsymbol{R}^n_+, \\ (i\xi_n)\circ v(t)|_{x_n=0}=k(t,x',\xi',-\xi_n)\tilde{w}_{0,0} & \text{in } I\times \boldsymbol{R}^{n-1}, \end{cases}$$

where $k(t,x',\xi',\xi_n)=k(t,x',\xi)=-i\sum_{j=0}^{N+n+2}(\xi_n\circ u_j)(t,x',0,\xi)(u_0(t,x',0,\xi))^{-1}$. Here we use the asymptotic expansion $u(t)\sim\sum_{j\geq 0}u_j(t)$ $(u_j(t)=f_j(t)u_0(t))$. We will calculate $k(t,x',\xi)$. Set

$$u_j^{\#}(t)=\sum_{k=0}^{j}\left[\left(\frac{\partial}{\partial x_n}\right)^k u_{j-k}(t)\right]^*\frac{x_n^k}{k!}.$$

Then we have

$$u(t)\sim\sum_{j\geq 0}u_j^{\#}(t),$$

where

$$u_j^{\#}(t)=h_j(t,x,\xi',\xi_n)u_0^*(t),\qquad \text{with } h_j\in\mathscr{F}_{-j}.$$

Using the above notations, we have $k(t,x',\xi)=-i\xi_n-i\xi_n h_1^*-(\frac{\partial}{\partial x_n}h_1)^*+k'$ with $k'\in\mathscr{F}_{-1}$. We get specially

$$h_0=1,\qquad h_1=f_1^*-\left(\frac{\partial}{\partial x_n}p_2\right)^*x_n.$$

We will calculate the asymptotic $y(t)$ of $v(t)$ such that $v-y\in\mathscr{H}_{-2}$. Set $w=w_1-v(t)$, where

$$w_1=\{1+f_1^*(t,x,\xi',-\xi_n)+x_nt(\frac{\partial}{\partial x_n}p_2)^*(x',\xi',-\xi_n)\}\tilde{w}_{0,0}\exp(-\beta t). \tag{7.2}$$

Then w must satisfy

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ w(t)=-(\frac{\partial}{\partial t}+\hat{q})\circ w_1(t) & \text{in} \quad I\times\boldsymbol{R}^n_+, \\ (i\xi_n)\circ w(t)|_{x_n=0}=0 \quad \text{in} \quad I\times\boldsymbol{R}^{n-1}. \end{cases} \tag{7.3}$$

The main part of the above equation (7.3) is

$$\begin{cases} \frac{\partial}{\partial t}w+\sum_0(q_2,w)=-\{p_1^*-\bar{p}_1^*+x_n(r_2+\bar{r}_2)\}\tilde{w}_{0,0}e^{-\beta t} & \text{in} \quad I\times\boldsymbol{R}^n_+, \\ (i\xi_n)\circ w(t)|_{x_n=0}=0 \quad \text{in} \quad I\times\boldsymbol{R}^{n-1}, \end{cases} \tag{7.4}$$

where we used the following notations:

$$\bar{p}_1(t,x,\xi)=p_1(t,x,\xi',-\xi_n), \qquad r_2=(\frac{\partial}{\partial x_n}p_2)^*.$$

If the boundary condition is the Dirichlet condition or the Neumann condition, according to the above argument we get the main part of $V(t)$ as follows.

Lemma 7. *Set*

$$k_1=(\bar{f}_1^*+tx_n\bar{r}_2)\tilde{w}_{0,0}e^{-\beta t}\in\mathscr{H}_{-1},$$
$$k_2=\{p_1^*-\bar{p}_1^*+x_n(r_2+\bar{r}_2)\}\tilde{w}_{0,0}e^{-\beta t}\in\mathscr{H}_{-1}.$$

Then we get

(1) *(Dirichlet) $v(t)-y_D(t)$ belongs to $\mathscr{H}_{-2}$ with*

$$y_D=-\tilde{w}_{0,0}e^{-\beta t}-k_1+w_D,$$

where $w_D\in\mathscr{H}_{-1}$ is the solution of the following equations.

$$\begin{cases} \frac{\partial}{\partial t}w_D+\sum_0(q_2,w_D)=k_2 \mod\mathscr{H}_0 & \textit{in} \quad I\times\boldsymbol{R}^n_+, \\ w_D(t)|_{x_n=0}=0 \quad \textit{in} \quad I\times\boldsymbol{R}^{n-1}. \end{cases}$$

(2) *(Neumann)* $v(t)-y_N(t)$ *belongs to* $\mathscr{H}_{-2}$ *with*

$$y_N=\tilde{w}_{0,0}e^{-\beta t}+k_1+w_N,$$

where $w_N\in\mathscr{H}_{-1}$ *is the solution of the follwing equations.*

$$\begin{cases} \dfrac{\partial}{\partial t}w_N+\sum_0(q_2,w_N)=-k_2 \mod \mathscr{H}_0 & \text{in} \quad I\times \boldsymbol{R}^n_+, \\ (i\xi_n)\circ w_N(t)|_{x_n=0}=0 \quad \text{in} \quad I\times\boldsymbol{R}^{n-1}. & \end{cases}$$

We prepare some statement to calculate the trace of V_N for the Dirichlet problem and the Neumann problem.

Lemma 8. *Let* v_+ and v_- *be the solution of the following equation*

$$\begin{cases} \dfrac{\partial}{\partial t}v_+ +\sum_0(q_2,v_+)=\dfrac{(2x_n)^l}{l!}\tilde{w}_{j,0}e^{-\beta t}f(x',\xi') & \text{in} \quad I\times\boldsymbol{R}^n_+, \\ (i\xi_n)\circ v_+(t)|_{x_n=0}=0 \quad \text{in} \quad I\times\boldsymbol{R}^{n-1}, & \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial t}v_- +\sum_0(q_2,v_-)=\dfrac{(2x_n)^l}{l!}\tilde{w}_{j,0}e^{-\beta t}f(x',\xi') & \text{in} \quad I\times\boldsymbol{R}^n_+, \\ v_-(t)|_{x_n=0}=0 \quad \text{in} \quad I\times\boldsymbol{R}^{n-1}. & \end{cases}$$

Then we have

(1)

$$v_\pm=e^{-\beta t}f(x',\xi')\{\sum_{0\le s\le l}C_s\frac{(2x_n)^{l+1-s}}{(l+1-s)!}\tilde{w}_{j-1-s,0}+C^\pm_{l+1}\tilde{w}_{j-l-2,0}\},$$

where

$$C_s=\frac{1}{4}(-1)^{s+1},\qquad C^+_{l+1}=\frac{1}{2}(-1)^l,\qquad C^-_{l+1}=0.$$

(2) *We can calculate* tr $V_\pm(t)$ *corresponding to* $v_\pm$ *as*

$$\int_0^\varepsilon \operatorname{tr} V_\pm(t)dx_n$$
$$\sim\frac{(-1)^jC_\pm(l)}{16\Gamma(\frac{l-j}{2}+2)}t^{1+\frac{l-j}{2}}(2\pi)^{-n+1}\int_{\boldsymbol{R}^{n-1}}e^{-\beta(x',\xi')t}f(x',\xi')d\xi',$$

where

$$C_+(l)=l+3, \qquad C_-(l)=l+1.$$

Here we used Proposition 16 below to obtain (2) of Lemma 7.

Proposition 16. *For any fixed positive consant* ε, *we have*

$$\int_0^\varepsilon \mathrm{tr}[\frac{(2x_n)^l}{l!} W_{j,0} e^{-\beta(x',D')t} f(x',D')] dx_n$$

$$\sim \frac{(-1)^j}{4\Gamma(\frac{l-j}{2}+1)} t^{\frac{l-j}{2}} (2\pi)^{-n+1} \int_{R^{n-1}} e^{-\beta(x',\xi')t} f(x',\xi') d\xi',$$

where

$$\frac{1}{\Gamma(-p+\frac{1}{2})} = \frac{(-1)^p}{\pi} \Gamma(p+\frac{1}{2}) \ (p\geq 0), \quad \frac{1}{\Gamma(p)}=0 \ (p\in Z_-).$$

Corollary. *Let* $g(t)$ *belong to* $\mathscr{H}_j$. *Then*

$$\int_0^\varepsilon \mathrm{tr}\ G(t) dx_n = 0\left(t^{-\frac{j+n-1}{2}}\right).$$

By Theorem 3 and the above Corollary we have

Theorem 8. *We have the following expansion for* $V_N(t)$ *which is constructed in Theorem 3 for the Dirichlet problem and the Neumann problem*

$$\int_0^\varepsilon \mathrm{tr}\ V_N(t,x',x_n) dx_n \sim \sum_{j=0}^\infty t^{-\frac{n}{2}+\frac{1}{2}+\frac{j}{2}} D_j(x'), \quad t\to 0.$$

Thus

$$\int_M \mathrm{tr}\ V_N(t) dV \sim \sum_{j=0}^\infty t^{-\frac{n}{2}+\frac{1}{2}+\frac{j}{2}} \int_\Gamma D_j(x') dS, \quad t\to 0.$$

Let calculate the main term in the above Theorem 8. In a local patch Ω such that $\Omega\cap\Gamma\neq\emptyset$, we choose a local coordinate of Ω as follows.

$$g^{jk}(0)=\delta_{j,k}, \qquad 1\leq j,k\leq n,$$

$$g^{jn}(x',0)=0, \qquad 1\leq j\leq n-1,$$

$$\frac{\partial}{\partial x_j} g^{lm}(0) = 0, \qquad 1 \le j, l, m \le n-1.$$

Set $r_2 = (\frac{\partial}{\partial x_n} p_2)^* = \sum_{i,j=1}^{n} d^{ij}\xi_i\xi_j$. Then the terms in Lemma 7 are calculated as

$$p_1 - \bar{p}_1 = -i\xi_n a_0,$$

where

$$a_0 = d - \tilde{d} + 2h_n$$

with $d = \mathrm{d}^{nn}$, $\tilde{d} = \sum_{i=1}^{n-1} d^{ii}$. So we have

$$k_1 = \{tx_n(\gamma\tilde{w}_{0,0} - d\tilde{w}_{2,0}) - \frac{1}{2} a_0 t\tilde{w}_{1,0} + t^2(\gamma\tilde{w}_{1,0} - d\tilde{w}_{3,0}) + k_1'\}e^{-\beta t},$$

$$k_2 = \{-a_0\tilde{w}_{1,0} + 2x_n(\gamma\tilde{w}_{0,0} - d\tilde{w}_{2,0})\}e^{-\beta t},$$

where $\gamma = \sum_{i,j=1}^{n-1} d^{ij}\xi_i\xi_j$, k_1' is a polynomial of odd degree with respect to ξ'.

By Proposition 16 and Lemma 7 we have

Lemma 9.

(1) *For the kernel* $\tilde{k}(t, x', x_n, y', y_n)$ *of the operator* K_1 *corresponding to the symbol* k_1, *we have*

$$\int_0^\varepsilon \mathrm{tr}\ \tilde{k}(t, 0, x_n, 0, x_n)dx_n \sim \left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\sqrt{t}\left(\frac{a_0}{8\Gamma(\frac{1}{2})} - \frac{d}{4\Gamma(\frac{1}{2})}\right).$$

(2) *For the kernel* $\tilde{w}_D(t, x', x_n, y', y_n)$ *of the operator* W_D *corresponding to the symbol* w_D *defined in Lemma* 7, *we have*

$$\int_0^\varepsilon \mathrm{tr}\ \tilde{w}_D(t, 0, x_n, 0, x_n)dx_n \sim \left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\sqrt{t}\frac{1}{16}\left(\frac{a_0}{\Gamma(\frac{3}{2})} - \frac{2d}{\Gamma(\frac{3}{2})} + \frac{\tilde{d}}{\Gamma(\frac{5}{2})}\right).$$

(3) *For the kernel* $\tilde{w}_N(t, x', x_n, y', y_n)$ *of the operator* W_N *corresponding to the symbol* w_N *defined in Lemma* 7, *we have*

$$\int_0^\varepsilon \mathrm{tr}\ \tilde{w}_N(t, 0, x_n, 0, x_n)dx_n \sim -\left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\sqrt{t}\frac{1}{16}\left(\frac{3a_0}{\Gamma(\frac{3}{2})} - \frac{4d}{\Gamma(\frac{3}{2})} + \frac{2\tilde{d}}{\Gamma(\frac{5}{2})}\right).$$

From Lemma 7 and Lemma 9 we obtain the following theorem.

Theorem 9. *Let $Y_D(t,x',x_n)$ and $Y_N(t,x',x_n)$ be operators corresponding to $y_D(t)$ and $y_N(t)$ which are the main term of the fundamental solutions. Then we have*

$$\int_0^{\varepsilon} \operatorname{tr}\ Y_D(t,x',x_n)dx_n \sim \left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\left(-\frac{1}{4}-\frac{\sqrt{t}}{12\Gamma(\frac{1}{2})}J+0(t)\right),$$

and

$$\int_0^{\varepsilon} \operatorname{tr}\ Y_N(t,x',x_n)\mathrm{d}x_n \sim \left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\left(\frac{1}{4}-\frac{\sqrt{t}}{12\Gamma(\frac{1}{2})}J+\frac{\sqrt{t}}{2\Gamma(\frac{1}{2})}\ \mathrm{flux}\, h+0(t)\right),$$

where J is the mean curvature, that is, $J=-\sum_{i\neq n}\frac{\partial}{\partial x_n}g^{ii}$, flux $h=-h_n$ *in this case.*

3^0. Oblique Problem and Robin's Problem.

For oblique problem the main term of $V(t)$ is

$$v_0(t)=(\tilde{w}_{0,0}-2b\tilde{w}_{0,-1})e^{-\beta t},$$

which belongs to $\mathscr{H}_0$. The main term means that $v(t)-v_0(t)\in\mathscr{H}_{-1}$. We get Theorem II by the following fact and Proposition 17.

$$\int_0^{\varepsilon} \operatorname{tr}[W_{0,0}e^{-\beta(x',D')t}]dx_n \sim \left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\frac{1}{4\sqrt{det\,\beta_0(x')}}\quad (t\to 0),$$

where $\beta(x',\xi')=<\beta_0(x')\xi',\xi'>$.

Proposition 17. *If the symbol $b(x',\xi')$ is defined by $b(x',\xi')=B(x')\cdot\xi'$ with a vector $B(x')$, then we get*

$$\int_0^{\varepsilon} \operatorname{tr}[b(x',D')W_{0,-1}e^{-\beta(x',D')t}]dx_n$$

$$\sim\left(\frac{\Gamma(\frac{1}{2})^{n-1}}{2\pi\sqrt{t}}\right)\frac{1}{4\sqrt{\det\beta_0(x')}}\left(1-\frac{1}{\sqrt{1-<\beta_0(x')^{-1}B,B>}}\right)\quad (t\to 0). \tag{7.5}$$

Remark 9. The inequality $\mathrm{Re}(1-<\beta_0(x')^{-1}B,B>)>0$ holds by the fact that the boundary condition is parabolic.

Proof. By change of variables the left hand side of (7.5) coincide with

$$\left(\frac{1}{2\pi\sqrt{t}}\right)^{n-1}\left(\frac{-1}{\sqrt{\pi}}\right)\int_{R^{n-1}}\int_0^\infty\int_0^\infty (B\cdot\zeta)\exp\{-(\sigma+\omega)^2$$

$$+2\sigma B\cdot\zeta-<\beta_0\zeta,\zeta>\}d\sigma d\omega d\zeta$$

$$=\left(\frac{1}{2\pi\sqrt{t}}\right)^{n-1}\left(\frac{-1}{2\sqrt{\pi}}\right)\int_{R^{n-1}}\int_0^\infty \exp\{-\sigma^2+2\sigma B\cdot\zeta-<\beta_0\zeta,\zeta>\}$$

$$-\exp\{-\sigma^2-<\beta_0\zeta,\zeta>\}d\sigma d\zeta.$$

q.e.d.

In case Robin's problem $b=b(x)$ is independent of ξ'. So we have

$$(i\xi_n+b)\circ u=i\xi_n u+\frac{\partial}{\partial x_n}u+bu.$$

Set $v=w_1+\tilde{w}$, where w_1 is defined by (7.2). Then $\tilde{w}$ must satisfy

$$(7.3)'\quad\begin{cases}(\frac{\partial}{\partial t}+\hat{q})\circ\tilde{w}(t)=-(\frac{\partial}{\partial t}+\hat{q})\circ w_1(t) & \text{in}\quad I\times \boldsymbol{R}^n_+,\\ (i\xi_n+b)\circ\tilde{w}(t)|_{x_n=0}=-2b\tilde{w}_{0,0} & \text{in}\quad I\times\boldsymbol{R}^{n-1}.\end{cases}$$

Set $\tilde{w}=w_2+w_3$, where w_2 and w_3 are solutions of the following equations.

$$(7.4)'\quad\begin{cases}\frac{\partial}{\partial t}w_2+\Sigma_0(q_2,w_2)=-\{q_1-\hat{q}_1+2x_n\bar{r}_2\}\tilde{w}_{0,0}e^{-\beta t} & \text{in}\quad I\times\boldsymbol{R}^n_+,\\ (i\xi_n+b)\circ w_2(t)|_{x_n=0}=0 & \text{in}\quad I\times\boldsymbol{R}^{n-1},\end{cases}$$

$$(7.6)\quad\begin{cases}\frac{\partial}{\partial t}w_3+\Sigma_0(q_2,w_3)=0 & \text{in}\quad I\times\boldsymbol{R}^n_+,\\ (i\xi_n+b)\circ w_3(t)|_{x_n=0}=-2b\tilde{w}_{0,0} & \text{in}\quad I\times\boldsymbol{R}^{n-1}.\end{cases}$$

Repeating the similar argument with that of for Neumann condition, we get w_2 and its trace. For example Lemma 8′ and Proposition 16′ for Robin's problem are as follows.

Lemma 8′. *Let v be the solution of the following equation*

$$\begin{cases} \dfrac{\partial}{\partial t}v+\Sigma_0(q_2,v)=\dfrac{(2x_n)^l}{l!}\tilde{w}_{j,0}e^{-\beta t}f(x',\xi') & \text{in } I\times \boldsymbol{R}^n_+, \\ (i\xi+b)\circ v(t)|_{x_n=0}=0 & \text{in } I\times \boldsymbol{R}^{n-1}. \end{cases}$$

Then

$$v=e^{-\beta t}f(x',\xi')\{\sum_{0\le s\le l}C_s\frac{(2x_n)^{l+1-s}}{(l+1-s)!}\tilde{w}_{j-1-s,0}+\frac{1}{2}(-1)^l\tilde{w}_{j-l-1,-1}\},$$

where C_s are the constants defined in Lemma 8.

Proposition 16′. *For any fixed positive constant* ε, *we have*

$$\int_0^\varepsilon \mathrm{tr}[\frac{(2x_n)^l}{l!}W_{j,-1}e^{-\beta(x',D')t}f(x',D')]dx_n$$

$$\sim\frac{(-1)^{j+1}}{4}(2\pi)^{-n+1}t^{\frac{l-j+1}{2}}\sum_{m=0}^{\infty}\frac{(b\sqrt{t})^m}{\Gamma(\frac{l-j+m+1}{2}+1)}\int_{\boldsymbol{R}^{n-1}}e^{-t\beta(x',\xi')}f(x',\xi')d\xi'.$$

So the main term of the asymptotic behavior of tr $W_{j,-1}$ is the same with that of $W_{j-1,0}$. Hence the main term of the asymptotic behavior of tr W_2 coincides with that of W_N for Neumann problem. On the other hand the solution w_3 of (7.6) is $-2bw_{0,-1}$. Then by Proposition 16′ we have

$$\int_0^\varepsilon \mathrm{tr}\ W_3dx_n\sim\left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{n-1}\frac{b(x')\sqrt{t}}{\sqrt{\pi}\sqrt{\det\beta_0(x')}}$$

as $t\to 0$. Then we get Theorem II.

4^0. Singular boundry problem.

In this case $v=w_0+w_1$, $w_0=(\tilde{w}_{0,0}-2b\tilde{w}_{0,-1})e^{-\beta t}$, $w_1\in\mathscr{H}_{-\frac{1}{2}}$. So we get Theorem II by the following lemma.

Lemma 10. (1) *If* $g(t)\in\mathscr{H}_j$,

$$\mathrm{tr}\ G(t)=0(t^{-\frac{j+n}{2}}).$$

(2) *If* $g(t)\in\mathscr{H}_j$,

$$\int_0^\varepsilon \mathrm{tr}\ G(t)dx_n=0(t^{-\frac{j+n-1}{2}}).$$

(3) *For* W_0 *corresponding to* $w_0=(\tilde{w}_{0,0}-2bw_{0,-1})e^{-\beta t}$ *we have*

$$\lim_{t\to 0}\left(\frac{\Gamma(\frac{1}{2})}{2\pi\sqrt{t}}\right)^{1-n}\int_0^\varepsilon \operatorname{tr}\ W_0 dx_n=\begin{cases}\frac{1}{4\sqrt{\det\beta_0(x')}}, & \text{if}\quad a(x')\neq 0;\\ -\frac{1}{4\sqrt{\det\beta_0(x')}}, & \text{otherwise.}\end{cases}$$

Proof. (1) and (2) are clear by Lemma 2′. (3) is obtained by the following equation.

$$\int_0^\varepsilon \tilde{w}_{0,0}(t,x_n+x_n)-2b\tilde{w}_{0,1}(t,x_n+x_n;a,b)dx_n$$

$$\underset{t\to 0}{\rightarrow}\int_0^\infty \left[\frac{1}{2\pi}\exp(-w^2)+\frac{2b\sqrt{t}}{\sqrt{\pi}}\int_0^\infty \exp\{-(a\sigma+w)^2+2b\sqrt{t}\sigma\}d\sigma\right]dw$$

$$=-\frac{1}{4}+\frac{1}{\sqrt{\pi}}\int_0^\infty \exp(-\mu^2+2\frac{b}{a}\sqrt{t}\mu)d\mu$$

$$\underset{t\to 0}{\rightarrow}\begin{cases}\frac{1}{4}, & \text{if}\quad a(x')\neq 0;\\ -\frac{1}{4}, & \text{otherwise.}\end{cases}$$

q.e.d.

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