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THE MULTIPLICATIVE GENUS ASSOCIATED WITH THE FORMAL GROUP LAW $(x+y-2axy)/(1-(a^2+b^2)xy)$

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1. Introduction

The complex cobordism group $MU^*(CP^\infty)$ is isomorphic to the ring of formal power series $MU^*[[x]]$, where $x=e_{MU}(\eta)$ is the Euler class of the tautological line bundle over the infinite complex projective space CP^∞. Since $MU^*(CP^\infty\times CP^\infty)\cong MU^*[[x_1,x_2]]$, $x_1=e_{MU}(\eta\hat{\otimes}1)$ and $x_2=e_{MU}(1\hat{\otimes}\eta)$, we can write

$$e_{MU}(\eta\hat{\otimes}\eta)=\sum a_{ij}^U x_1^i x_2^j.$$

The formal power series induces a formal group law over MU^*

$$F_{MU}(x,y)=\sum a_{ij}^U x^i y^j.$$

The complex cobordism ring MU^* with the formal group law F_{MU} is isomorphic to Lazard's ring with the universal formal group law [8]. Given any formal group law $F(x,y)$ over a commutative ring R, there is a ring homomorphism $\varphi:MU^*\to R$ which is called a multiplicative genus. In this paper we study the multiplicative genus $\varphi_{a,b}$: $MU^*\to Q$ associated with the formal group law

$$F(x,y)=\frac{x+y-2axy}{1-(a^2+b^2)xy}$$

which is related to the following formal power series, called the logarithm for $F(x,y)$,

$$l(z)=\int_0^z \frac{1}{1-2ax+(a^2+b^2)x^2}\,dx,$$

which satisfies $l(F(x,y))=l(x)+l(y)$. The characteristic power series $Q(z)=z/l^{-1}(z)$ (cf. [3]) for the multiplicative genus is given by

$$Q(z)=\frac{z(b+a\tan bz)}{\tan bz}\,.$$

The cobordism classes of Milnor manifolds

$$H_{ij}=\{([z_0,z_1,\cdots,z_i],\ [w_0,w_2,\cdots,w_j])\ |z_0w_0+z_1w_1+\cdots+z_iw_i=0\}$$
$$\subset CP^i\times CP^j,$$

where $i\leq j$, and the complex projective spaces CP^n generate MU^*. Let $H(x,y)=\Sigma[H_{ij}]x^iy^j$, and

$$\log_{MU}(z)=\frac{[CP^n]}{n+1}\,z^{n+1},$$

which is the logarithm for $F_{MU}(x,y)=\Sigma\, a^U_{i,j}x^iy^j$. Then relations on a^U_{ij}, $[H_{ij}]$ and $[CP^n]$ are given by the following [2]:

$$H(x,y)=\frac{d\log_{MU}(x)}{dx}\,\frac{d\log_{MU}(y)}{dy}\,F_{MU}(x,y).$$

We use the relations to calculate the multiplicative genus $\varphi_{a,b}$: $MU^*\to Q$ associated with the above formal group law for Milnor manifolds. The main theorem of this paper is the following.

Theorem 1.1. *Let $\varphi_{a,b}$: $MU^*\to Q$ be the multiplicative genus associated with the formal group law $F(x,y)=(x+y-2axy)/(1-(a^2+b^2)xy)$. The values of $\varphi_{a,b}$ for the Milnor manifolds $H_{s,k}$, $s\leq k$, are as follows:*

$$\varphi_{a,b}([H_{s,k}])=\left(\frac{\alpha^{s+1}-\beta^{s+1}}{\alpha-\beta}\right)\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right),$$

where $\alpha=a+b\sqrt{-1}$ and $\beta=a-b\sqrt{-1}$.

The paper is organized as follows. In Section 2 we study multiplicative idempotent natural transformations over the cobordism cohomology $MU^*(-)\otimes Q$ which induce multiplicative genera. In Section 3 we investigate the multiplicative genus $\varphi_{a,b}$: $MU^*\to Q$ and we give a proof of Theorem 1.1. In Section 4 we discuss multiplicative genus related to the logarithm given by the integral of $1/\{$a polynomial$\}$.

2. The decomposition of $MU^*\otimes Q$ and the multiplicative genus

Let $p(t_1,\cdots,t_n)$ be a symmetric polynomial, and let

$$p(t_1,\cdots,t_n)=P(\sigma_1,\cdots,\sigma_n)$$

where σ_i is the i-th elementary symmetric polynomial. For a complex vector bundle ξ over X with $\dim_C\xi=n$ we have

$$P(c_1(\xi),\cdots,c_n(\xi))\in MU^*(X)$$

where $c_i(\xi)$ is the i-th complex cobordism Chern class, and

$$S_P(\xi)=\Phi_\xi(P(c_1(\xi),\cdots,c_n(\xi)))\in \widetilde{MU}^*(T(\xi))$$

where $T(\xi)$ is the Thom complex of ξ and Φ_ξ is the Thom isomorphism. Let

$$\alpha=\{h\}\in MU^k(X)=\lim_{n\to\infty}[S^{2n-k}X^+,\ MU(n)]_0,$$

$$h\colon\ S^{2n-k}X^+\to MU(n)=T(\gamma^n),$$

where $\gamma^n\to BU(n)$ is the universal complex vector bundle over $BU(n)$. The complex cobordism cohomology operation S_P: $MU^*(X)\to MU^*(X)$ is given by

$$S_P(\alpha)=\sigma^{k-2n}h^*(S_P(\gamma^n))$$

where σ^{k-2n} denotes the $(k-2n)$-fold suspension isomorphism. For any set $\omega=(i_1,\cdots,i_q)$ of positive integers, denote $S_\omega(t)$ the smallest symmetric function of variable t_j, $1\leq j\leq n$, which contains the monomial $t_1^{i_1}\cdots t_q^{i_q}$ and write

$$S_\omega(t)=P_\omega(\sigma_1,\cdots,\sigma_n).$$

Then we have the Landweber-Novikov operation $S_\omega(\alpha)=S_{P_\omega}(\alpha)$ (cf. [6] and [4]).

Given a formal power series

$$\hat{f}(x)=1+v_1x+v_2x^2+\cdots$$
$$f(x)=x\hat{f}(x)$$

where $v_i\in MU^{-2i}$, we have a symmetric polynomial

$$\hat{f}(t_1)\cdots\hat{f}(t_n)=P_f(\sigma_1,\cdots,\sigma_n).$$

The multiplicative natural operation S_f: $MU^*(X)\to MU^*(X)$ given by

$$S_f(\alpha)=S_{P_f}(\alpha)$$

satisfies

(1) $S_f(g^*(\alpha))=g^*(S_f(\alpha))$, *for any map* g: $Y\to X$

(2) $S_f(\alpha\beta)=S_f(\alpha)S_f(\beta)$

(3) $S_f(c_1(\gamma^1))=f(c_1(\gamma^1))$.

If $v_i\in MU^*\otimes Q$ then we similarly obtain the multiplicative natural operation S_f: $MU^*(-)\otimes Q\to MU^*(-)\otimes Q$. We now define

$$\mathrm{mog}^f_{MU}(x)=\sum \frac{S_f([CP^n])}{n+1}\, x^{n+1}\in MU^*\hat{\otimes} Q[[x]] \text{ (cf.[10]).}$$

Since the logarithm $\log_{MU}(x)$ for the formal group law $F_{MU}(x,y)$ uniquely exists,

$$S_f(\log_{MU}(x))=\log_{MU}(x)$$

and for $x=c_1(\gamma^1)$,

$$\mathrm{mog}^f_{MU}(f(x))=\log_{MU}(x).$$

$MU^*\otimes Q$ is the polynomial ring over Q generated by $\{[CP^1], [CP^2],\cdots, [CP^n],\cdots\}$. Given a formal power series

$$g(x)=x+\frac{b_1}{2}x^2+\cdots+\frac{b_n}{n+1}\, x^{n+1}+\cdots$$

in $MU^*\otimes Q[[x]]$, $b_i\in MU^{-2i}\otimes Q$, we can consider the ring homomorphism

$$\psi_g\colon\ MU^*\otimes Q\to MU^*\otimes Q,\ \psi_g([CP^n])=b_n.$$

The formal power series $g(x)$ is said to be *projective* if ψ_g is the projection, namely $\psi_g\psi_g=\psi_g$.

Proposition 2.1. *Suppose that* $g(x)(\in MU^*\otimes Q[[x]])$ *is projective. Let*

$$f(x)=g^{-1}(\log_{MU}(x))=x+v_1x^2+v_2x^3+\cdots+v_nx^{n+1}+\cdots.$$

Then the multiplicative operation S_f: $MU^*(-)\otimes Q\to MU^*(-)\otimes Q$ *satisfies the following properties.*

(1) $S_f(S_f(c_1(\gamma^1)))=S_f(c_1(\gamma^1))$

(2) $S_f(v_n)=0$, *for any* $n\geq 1$.

Proof. Since $g(f(x))=\log_{MU}(x)=\mathrm{mog}^f_{MU}(f(x))$ for $x=c_1(\gamma^1)$, we have

$S_f([CP^n])=b_n$. Since the operation S_f on $MU^*\otimes Q$ coincides with ψ_g and $g(x)$ is projective, we get $S_f(S_f([CP^n]))=S_f([CP^n])$. Apply S_f to $\mathrm{mog}^f_{MU}(S_f(x))=\log_{MU}(x)$, $x=c_1(\gamma^1)$, to obtain

$$\mathrm{mog}^f_{MU}(S_f(S_f(x)))=\mathrm{mog}^f_{MU}(S_f(x)).$$

Hence (1) follows. Apply S_f to $f(x)=x+v_1x^2+v_2x^3+\cdots+v_nx^{n+1}+\cdots$, $x=c_1(\gamma^1)$, to get

$$f(x)=S_f(S_f(x))=f(x)+S_f(v_1)(f(x))^2+S_f(v_2)(f(x))^3+\cdots$$

and $S_f(v_i)=0$, $i>0$. $\square$

Suppose that $g(x)(\in MU^*\otimes Q[[x]])$ is projective. Put

$$f(x)=g^{-1}(\log_{MU}(x))$$

and

$$f(x)=x+v_1x^2+v_2x^3+\cdots+v_nx^{n+1}+\cdots,\quad v_i\in MU^{-2i}\otimes Q.$$

Then we have a natural transformation

$$\varepsilon_f\colon MU^*(X)\otimes Q\to MU^*(X)\otimes Q,$$
$$\varepsilon_f(\alpha)=\sum v_1^{r_1}\cdots v_k^{r_k}S_{(\underbrace{1,\cdots,1}_{r_1},\cdots,\underbrace{k,\cdots,k}_{r_k})}(\alpha).$$

Theorem 2.2. *The natural transformation ε_f satisfies the following*:

(1) *ε_f is multiplicative.*

(2) *$\varepsilon_f\varepsilon_f=\varepsilon_f$.*

(3) *$\varepsilon_f(MU^*(-)\otimes Q)$ is a generalized cohomology.*

Proof. Let

$$S^{r_1,\cdots,r_k}(\alpha)=S_{(\underbrace{1,\cdots,1}_{r_1},\cdots,\underbrace{k,\cdots,k}_{r_k})}(\alpha)$$

We then have

$$S^R(\alpha\beta)=\sum_{R'+R''=R}S^{R'}(\alpha)S^{R''}(\beta),$$

which completes the proof of (1). We can see that $\varepsilon_f(x)=f(x)$, for

$x=c_1(\gamma^1)$, and so $\varepsilon_f=S_f$. Therefore (2) is an immediate result of Proposition 3.1(2) and $\varepsilon_f(MU^*(-)\otimes Q)$ is a direct summand of $MU^*(-)\otimes Q$. (3) follows from (2). $\square$

EXAMPLE. (1) (Brown-Peterson) The formal power series

$$g(x)=\sum_{n=0}^{\infty}\frac{[CP^{p^n-1}]}{p^n}x^{p^n}$$

is projective. Since the coefficients of $f(x)=g^{-1}(\log_{MU}(x))$ exist in $MU^*\otimes Z_{(p)}$ (cf. [1] and [10]), we have a natural idempotent operation

$$\varepsilon_f\colon MU^*(-)\otimes Z_{(p)}\to MU^*(-)\otimes Z_{(p)}$$

and the Brown-Peterson cohomology $BP^*(-)=\varepsilon_f(MU^*(-)\otimes Z_{(p)})$.

(2) (Ochanine) The formal power series

$$g(x)=\int_0^x\frac{1}{\sqrt{1-2[CP^2]x^2+(3[CP^2]^2-2[CP^4])x^4}}\,dx$$

is projective. The multiplicative idempotent operation ε_f for $f(x)=g^{-1}(\log_{MU}(x))$ gives rise to a generalized cohomology $h^*(-)=\varepsilon_f(MU^*(-)\otimes Q)$ with $h^*(a\ point)=Q[[CP^2],\ [CP^4]]$ and the multiplicative genus

$$\varphi=\varepsilon_f\colon\ MU^*\to Q[[CP^2],[CP^4]].$$

The Ochanine genus Φ: $MU^*\to Q[\varepsilon,\delta]$(cf.[7] and [5]) is the multiplicative genus associated with the formal group law $F(x,y)=l^{-1}(l(x)+l(y))$ with

$$l(x)=\int_0^x\frac{1}{\sqrt{1-2\delta x^2+\varepsilon x^4}}\,dx,$$

and $\Phi([CP^2])=\delta$ and $\Phi([H_{3,2}])=\varepsilon$ (cf. Proposition 3 of [7]). Thus the Ochanine multiplicative genus is represented as the following composite.

$$\begin{tikzcd} MU^* \arrow[rr, "\Phi"] \arrow[dr, "\varepsilon_f"'] & & Q[\varepsilon,\delta] \\ & Q[[CP^2],[CP^4]] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Here ψ is the ring homomorphism defined by

$$\psi([CP^2])=\delta \text{ and } \psi([CP^4])=(3\delta^2-\varepsilon)/2.$$

(3) The formal power series

$$g(x)=\int_0^x \frac{1}{1-[CP^1]x+[CP^1]^2x^2}\,dx$$

is projective. The multiplicative idempotent operation ε_f for $f(x)=g^{-1}(\log_{MU}(x))$ induces the multiplicative genus ε_f: $MU^*\to Q[[CP^1]]$. The values for complex projective spaces are as follows.

$$\varepsilon_f([CP^{3n}])=(-1)^n[CP^1]^{3n},\ \varepsilon_f([CP^{3n+1}])=(-1)^n[CP^1]^{3n+1},$$
$$\varepsilon_f([CP^{3n+2}])=0.$$

3. The genus associated with $(x+y-2axy)/(1-(a^2+b^2)xy)$

Let $\varphi_{a,b}$: $MU^*\to Q$ be the multiplicative genus associated with the formal group law $F(x,y)=(x+y-2axy)/(1-(a^2+b^2)xy)$ for rational numbers a and b. The logarithm of $\varphi_{a,b}$ is

$$l(x)=\int_0^x \frac{1}{1-2ax+(a^2+b^2)x^2}\,dx.$$

Consider the formal power series

$$g(x)=\int_0^x \frac{1}{1-[CP]x+([\mathrm{CP}^1]^2-[CP^2])x^2}\,dx$$

which is projective. By Theorem 2.2 it induces a multiplicative idempotent operation ε_f for $f(x)=g^{-1}(\log_{MU}(x))$ which gives a generalized cohomology $h^*(-)=\varepsilon_f(MU^*(-)\otimes Q)$ with $h^*(a\ point)=Q[[CP^1],[CP^2]]$ and the multiplicative genus

$$\varphi=\varepsilon_f\colon\ MU^*\to Q[[CP^1],[CP^2]].$$

The multiplicative genus $\varphi_{a,b}$ is represented as the following composite.

$$\begin{tikzcd}[column sep=small] MU^* \arrow[rr, "\varphi_{a,b}"] \arrow[dr, "\varepsilon_f"'] & & Q \\ & Q[[CP^1],[CP^2]] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Here ψ is the ring homomorphism defined by $\psi([CP^1])=2a$ and $\psi([CP^2])=3a^2-b^2$.

The characteristic power series for the multiplicative genus $\varphi_{a,b}$ is given by $Q(z)=z/l^{-1}(z)$. Since

$$l(x)=\frac{1}{b}\left(\arctan\left(\frac{a^2+b^2}{b}(x-\frac{a}{a^2+b^2})\right)+\arctan\frac{a}{b}\right),$$

it follows that

$$Q(z)=\frac{z(b+a\tan bz)}{\tan bz}.$$

For rational numbers a and b, put

$$h(x,y)=\frac{x+y-2axy}{1-(a^2+b^2)xy}\cdot\frac{1}{1-2ax+(a^2+b^2)x^2}\cdot\frac{1}{1-2ay+(a^2+b^2)y^2}.$$

Then

$$(1-(a^2+b^2)xy)(1-2ax+(a^2+b^2)x^2)(1-2ay+(a^2+b^2)y^2)h(x,y)$$
$$=x+y-2axy$$

and for $k\geq 3$

$$\frac{\partial^k h}{\partial y^k}(x,0)-k(2a+(a^2+b^2)x)\frac{\partial^{k-1}h}{\partial y^{k-1}}(x,0)$$
$$+k(k-1)(a^2+b^2)(1+2ax)\frac{\partial^{k-2}h}{\partial y^{k-2}}(x,0)$$
$$-k(k-1)(k-2)(a^2+b^2)^2x\frac{\partial^{k-3}h}{\partial y^{k-3}}(x,0)=0.$$

Proposition 3.1. *Let $\alpha=a+b\sqrt{-1}$ and $\beta=a-b\sqrt{-1}$. Then*

$$\frac{\partial^k h}{\partial y^k}(x,0)=\frac{k!}{(1-\alpha x)(1-\beta x)}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}+\alpha^k\beta^k x^{k+1}\right).$$

Proof. Put

$$P_k=\frac{\partial^k h}{\partial y^k}(x,0)$$

$$Q_k=P_k-2akP_{k-1}+k(k-1)(a^2+b^2)P_{k-2}.$$

We then have

$$Q_k-k(a^2+b^2)xQ_{k-1}=0$$

and

$$Q_k=\frac{k!}{2}(a^2+b^2)^{k-2}x^{k-2}Q_2.$$

Thus it follows that

$$P_k-2akP_{k-1}+k(k-1)(a^2+b^2)P_{k-2}=\frac{k!}{2}(a^2+b^2)^{k-2}x^{k-2}Q_2.$$

Let $\alpha=a+b\sqrt{-1}$, $\beta=a-b\sqrt{-1}$ and $R_k=P_k-k\alpha P_{k-1}$. Then we get

$$R_k-k\beta R_{k-1}=(\alpha\beta x)^{k-2}\frac{k!}{2}Q_2$$

and

$$R_k-\beta^{k-2}\frac{k!}{2}R_2=\frac{k!}{2}\beta^{k-2}(P_2-2(\alpha+\beta)P_1+2\alpha\beta P_0)\sum_{i=1}^{k-2}(ax)^i.$$

Note that

$$P_0=\frac{x}{(1-\alpha x)(1-\beta x)}$$

$$P_1=\frac{1+\alpha\beta x^2}{(1-\alpha x)(1-\beta x)}$$

$$P_2=\frac{2(\alpha+\beta)+2\alpha^2\beta^2x^3}{(1-\alpha x)(1-\beta x)}.$$

Then we get

$$R^k=k!\beta^{k-1}\frac{1-(\alpha x)^k(1-\beta x)}{(1-\alpha x)(1-\beta x)}$$

and

$$P_k = \frac{k!}{(1-\alpha x)(1-\beta x)} \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} + \alpha^k \beta^k x^{k+1} \right). \qquad \square$$

We utilize Proposition 3.1 to obtain Theorem 1.1.

Proof of Theorem 1.1.

The $h(x,y)$ is described as

$$h(x,y) = h(x,0) + \frac{\partial h}{\partial y}(x,0)y + \frac{1}{2!} \frac{\partial^2 h}{\partial y^2}(x,0)y^2 + \cdots + \frac{1}{k!} \frac{\partial^k h}{\partial y^k}(x,0)y^k + \cdots.$$

From Proposition 3.1 it follows that

$$\frac{1}{k!} \frac{\partial^k h}{\partial y^k}(x,0) = \frac{\alpha^k - \beta^k}{\alpha - \beta} \sum_{s=0}^{\infty} \left(\sum_{i+j=s} \alpha^i \beta^j \right) x^s + \alpha^k \beta^k \sum_{s=0}^{\infty} \left(\sum_{i+j=s} \alpha^i \beta^j \right) x^{s+k+1}$$

and the coefficient of $x^s y^k$ is

$$\frac{\alpha^k - \beta^k}{\alpha - \beta} \sum_{i+j=s} \alpha^i \beta^j = \left(\frac{\alpha^k - \beta^k}{\alpha - \beta} \right) \left(\frac{\alpha^{s+1} - \beta^{s+1}}{\alpha - \beta} \right). \qquad \square$$

REMARK. When $b=0$ in this theorem,

$$\varphi_{a,0}([H_{s,k}]) = (s+1)ka^{s+k-1}.$$

Especially $\varphi_{1,0}$ corresponds to the Euler characteristic and $\varphi_{0,b}$ is the Ochanine multiplicative genus of case $\delta = -b^2$ and $\varepsilon = b^4$.

Let ξ be the canonical line bundle over CP^2 and let $P(m,n)$ denote the projective space bundle associated with $\xi^m \oplus \xi^n$.

Proposition 3.2.

$$[P(m,n)] = -\frac{(m-n)^2}{3}[H_{2,2}] + \frac{(m-n)^2 + 3}{3}[CP^2][CP^1]$$

Proof. Put

$$[P(m,n)] = x[H_{2,2}] + y[CP^2][CP^1] + z[CP^1]^3.$$

We can determine x, y and z by the facts that

$S_{(3)}(P(m,n))=2(m-n)^2,\ S_{(2,1)}(P(m,n))=6,\ S_{(1,1,1)}(P(m,n))=6,$

$S_{(3)}(H_{2,2})=-6,\ S_{(2,1)}(H_{2,2})=6,\ S_{(1,1,1)}(H_{2,2})=6,$

$S_{(3)}(CP^2\times CP^1)=0,\ S_{(2,1)}(CP^2\times CP^1)=6,\ S_{(1,1,1)}(CP^2\times CP^1)=6,$

$S_{(3)}((CP^1)^3)=0,\ S_{(2,1)}((CP^1)^3)=0,\ S_{(1,1,1)}((CP^1)^3)=8.$ □

To the formal group law $F(x,y)=(x+y-2axy)/(1-(a^2+b^2)xy)$ we apply

$$a=\frac{\delta}{2} \quad \text{and} \quad b=\frac{\sqrt{3}\delta}{2}.$$

The logarithm is

$$l(x)=\int_0^x \frac{1}{1-\delta x+\delta^2 x^2}\,dx.$$

Let φ_δ: $MU^*\to Q$ denote the multiplicative genus associated with the formal group law $F(x,y)=(x+y-\delta xy)/(1-\delta^2 xy)$. Then we have the following.

Theorem 3.3. *The ideal in $MU^*\otimes Q$ consisting of α with $\varphi_\delta(\alpha)=0$ for any δ is generated by cobordism classes of fibre bundles over CP^2.*

Proof. Let $S_{(n)}(M)$ denote the characteristic number of M corresponding to the symmetric polynomial Σt_i^n. The characteristic number of the Milnor manifold (cf.[9]) is

$$S_{(n+m-1)}(H_{n,m})=-\binom{n+m}{n},\ 2\le n\le m.$$

We take a generating set $\{[CP^1],[CP^2],[H_{2,j}]|j\ge 2\}$ of the ring $MU^*\otimes Q$. We use Theorem 1.1 to get $\varphi_\delta([CP^1])=\delta$, $\varphi_\delta([CP^2])=0$ and $\varphi_\delta([H_{2,j}])=0$, $j\ge 2$. Therefore the kernel of φ_δ is generated by $\{[CP^2],[H_{2,j}],\ j\ge 2\}$. If $j>2$, $H_{2,j}$ is a fibre bundle over CP^2 with the fibre CP^{j-1}. By using Proposition 3.3 we see that $[H_{2,2}]$ belongs to the ideal generated by cobordism classes of fibre bundles over CP^2. □

4. Genera cancelling $[H_{2,j}]$, $j>n$

We discuss the multiplicative genus associated with the formal group law $F(x,y)=l^{-1}(l(x)+l(y))$ with the logarithm $l(x)$ given by the integral of $1/\{$a polynomial$\}$ which is a generalization of the logarithm for the multiplicative genus φ_δ in Section 3. For rational numbers $\delta_1,\delta_2,\cdots,\delta_n$, we consider a formal power series

$$l(z)=\int_0^z \frac{1}{1+\delta_1 x+\delta_1^2 x^2+\delta_2 x^3+\cdots+\delta_n x^{n+1}}\,dx$$

and denote the multiplicative genus associated with the formal group law $F(x,y)=l^{-1}(l(x)+l(y))$ by

$$\varphi_{\delta_1,\cdots,\delta_n}\colon MU^*\to Q.$$

We then have the following

Proposition 4.1.

(1) $\varphi_{\delta_1,\cdots,\delta_n}([CP^1])=-\delta_1,\ \varphi_{\delta_1,\cdots,\delta_n}([CP^2])=0,\ \varphi_{\delta_1,\cdots,\delta_n}([H_{2,1}])=\delta_1^2$

(2) $\varphi_{\delta_1,\cdots,\delta_n}([H_{2,j}])=\begin{cases}\dfrac{j+1}{2}\delta_j, & 2\le j\le n\\ 0, & j>n\end{cases}$

Proof. The logarithm $l(x)$ for $F(x,y)$ is described as

$$l(x)=\sum\frac{\varphi_{\delta_1,\cdots,\delta_n}([CP^n])}{n+1}\,x^{n+1}.$$

We obtain (1) by the facts that

$$l(z)=z-\frac{\delta_1}{2}z^2+\textit{higher terms of degree}\ge 4$$

and $[H_{2,1}]=[CP^1]^2$. Consider the formal power series

$$h(x,y)=\sum\varphi_{\delta_1,\cdots,\delta_n}([H_{i,j}])x^i y^j.$$

The Buchstaber formula [2]

$$H(x,y)=(\sum[CP^n]x^n)(\sum[CP^n]y^n)F_{MU}(x,y)$$

implies

$$h(x,y)=l'(x)l'(y)F(x,y)$$
$$=\left(l'(0)+\frac{l''(0)}{1!}x+\cdots+\frac{l^{(n+1)}(0)}{n!}x^n+\cdots\right)\cdot l'(y)\cdot$$

$$\left(F(0,y)+\frac{1}{1!}\frac{\partial F}{\partial x}(0,y)x+\frac{1}{2!}\frac{\partial^2 F}{\partial x^2}(0,y)x^2+\cdots+\frac{1}{n!}\frac{\partial^n F}{\partial x^n}(0,y)x^n+\cdots\right).$$

Comparing the coefficients of x^2, we have

$$\begin{aligned}&h_{2,0}+h_{2,1}y+h_{2,2}y^2+\cdots+h_{2,n}y^n+\cdots\\&=l'(y)\cdot\left(\frac{l'(0)}{2}\frac{\partial^2 F}{\partial x^2}(0,y)+l''(0)\frac{\partial F}{\partial x}(0,y)+\frac{l^{(3)}(0)}{2}F(0,y)\right),\end{aligned}$$

where $h_{i,j}=\varphi_{\delta_1,\cdots,\delta_n}([H_{i,j}])$. Since $l(F(x,y))=l(x)+l(y)$, it follows that

$$l'(F(x,y))\frac{\partial F}{\partial x}=l'(x)$$

and

$$l''(F(x,y))(\frac{\partial F}{\partial x})^2+l'(F(x,y))\frac{\partial^2 F}{\partial x^2}=l''(x).$$

From the facts that $l'(0)=1$, $l''(0)=-\delta_1$ and $l^{(3)}(0)=0$, it follows that

$$\begin{aligned}&h_{2,0}+h_{2,1}y+h_{2,2}y^2+\cdots+h_{2,n}y^n+\cdots\\&=-\delta_1+\delta_1^2y+\frac{3}{2}\delta_2y^2+\cdots+\frac{n+1}{2}\delta_ny^n\end{aligned}$$

and we complete the proof of (2). □

We can take a generating set $\{[CP^1], [CP^2], [H_{2,2}],\cdots,[H_{2,n}],\cdots\}$ for the polynomial ring $MU^*\otimes Q$. Therefore it follows from Proposition 4.1 that

Theorem 4.2. *$[M](\in MU^*\otimes Q)$ belongs to the ideal generated by $\{[CP^2], [H_{2,j}]\ (j>n)\}$ in $MU^*\otimes Q$ if and only if for any $\delta_1,\cdots,\delta_{n-1}$ and $\delta_n(\in Q)$*

$$\varphi_{\delta_1,\cdots,\delta_n}([M])=0.$$

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