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A DOMAIN MONOTONICITY PROPERTY OF THE NEUMANN HEAT KERNEL

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1. Introduction

Throughout this paper $p^{\Omega}(t, x, y)$ denotes the Neumann heat kernel of a bounded euclidean domain $\Omega\subset \boldsymbol{R}^d$ with the Neumann boundary condition. By definition, $p^{\Omega}(t, x, y)$ is the fundamental solution of the heat operator $L=\partial/\partial t-\Delta/2$ (Δ is the Laplace operator) with the Neumann boundary condition, i.e., for fixed $x\in\Omega$, it is a function in (t, y) satisfying the equation

$$\begin{cases} Lp^{\Omega}(t, x, y) = 0\,, & (t, y)\in \boldsymbol{R}^{+}\times\Omega\,, \\ \dfrac{\partial}{\partial n}p^{\Omega}(t, x, y) = 0\,, & (t, y)\in \boldsymbol{R}^{+}\times\partial\Omega\,, \\ p^{\Omega}(0, x, y) = \delta_x(y)\,, & y\in\Omega\,. \end{cases}$$

Physically $p^{\Omega}(t, x, y)$ represents the temperautre distribution in Ω at time t and point y if a heat source of total capacity one is present at point x at time 0 with the assumption that the boundary $\partial\Omega$ is impervious to heat conduction (adiabatic boundary). From this interpretation of the Neumann heat kernel it was conjectured (see Chavel [2] and Kendall [7]) that if Ω is a smooth convex domain and D is another smooth domain containing Ω, then for all $(t, x, y)\in (0, \infty)\times\Omega\times\Omega$

$$p^{\Omega}(t, x, y)\geq p^{D}(t, x, y)\,.$$

While the counterpart of this conjecture for the Dirichlet heat kernel (absorbing boundary condition) is obvious by the maximum principle (without assuming the convexity of the smaller domain Ω), the conjecture stated above was recently proved to be false without further hypotheses on the domains Ω and D, see Bass and Burdzy [1].

The convexity of the inner domian is necessary because of the following basic asymptotic relation:

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$$\lim_{t\to 0} t \log p(t, x, y) = -\frac{d^{\Omega}(x, y)^2}{2},$$

where $d^{\Omega}(x, y)$ is the distance of the two points x, y restricted in Ω, i.e., the infimum of the lengths of smooth paths inside Ω connecting x and y. It also follows from the above limiting relation that if Ω is strictly contained inside D, then the inequality in the conjecture holds for all pairs of points in Ω and sufficienly small times. A much more complicated argument shows that the inequality holds for x, y in the interior of Ω and for all times smaller than a fixed constant which depends on the distances of the two points to the boundary $\partial\Omega$ (see Carmona and Zheng [3]). One also knows that the inequality holds for all t if there exists a ball B such that $\Omega \subset B \subset D$ and centered at either x or y (see Kendall [7]). Save for some simple cases, e.g., when both Ω and D are multi-dimensional cubes with corresponding parallel sides, the conjecture is proved in its full generality only in the case where Ω is a convex domain contained in a half space D (see Kendall [2] and Davies [4]).

In the present article using a new probabilistic approach we prove an interesting special case of the above conjecture.

Theorem. *Suppose that D is a parallelopiped centered at the origin O and that Ω is a smooth convex domain contained in D such that everywhere on the boundary $\partial\Omega$ its inward normal vector field points towards the center respect to D (the precise meaning of this assumption is given below). Then for a pair of points x, y in Ω such that one of them is the center of Ω, and every $t>0$ we have*

$$p^{\Omega}(t, x, y) \geq p^{D}(t, x, y)\,. \tag{1}$$

We will first discuss the one-dimensional case in Section 2. Although the we know that the conjecture holds in its complete generality in the one-dimensional case, it is the method in Section 2 that we will generalize to handle the higher dimensional case stated in the theorem. The complete proof of the main theorem is carried out in Section 3.

We now make precise the assumption on the convex domain Ω in the theorem above. Let the center of the parallelopiped be the origin O of the coordinates. Let e_i, $i=1, \cdots, d$, be d unit vectors which span the parallelopiped D. A vector x can be uniquely written as a linear combination of these vectors:

$$x = \sum_{i=1}^{d} F_i(x)\, e_i\,,$$

where F_i, $i=1, \cdots, d$ are linear functions without constant terms. For example, if D is a multi-dimensional cube with axes parallel to the coordinate axes, then we may take $F_i(x)=x_i$, the ith coordinate of x. The parallelopiped is bounded by $2d$ hyperplanes $F_i=\pm S_i$, $i=1, \cdots, d$, where S_i are some positive constants.

Let $n^{\Omega}(x)$ be the inward unit normal vector of Ω at a point $x\in\partial\Omega$. Then we say that $n^{\Omega}(x)$ points towards the center with respect to D if $F_i(x)$ and $F_i(n^{\Omega}(x))$ have the opposite sign for all $i=1, \cdots, d$. A typical case we have in mind is when Ω is an ellipsoid and D is a multi-dimensional cube both with axes parallel to the coordinate axes

Since $p^{\Omega}(t, x, y)$ and $p^{D}(t, x, y)$ are symmetric in (x, y), in the course of our proof we may assume without loss of generality that y is the center of D. By the continuous dependence of $p^{D}(t, x, y)$ on the domain D, we may also assume for convenience that D strictly contains Ω and the origin O is an interior point of Ω.

2. One-dimensional case

In this section we prove a path domination result (see (2) below) stronger than the Neumann heat kernel comparison in (1). This path domination result is what we will need in the next section.

Let $\Omega=[a_1, a_2]$ be an interval containing the origin and $D=[-S, S]$ a larger interval containing Ω. The argument below in fact works as long as D contains a symmetric interval of the form $[-S, S]$ which in turn contains Ω. Let x be an arbitrary point in Ω. Let $B=\{B_t, t\geq 0\}$ be a one-dimensional Brownian motion starting from zero. We will use this Brownian motion to define a reflecting Brownian motion X^{Ω} in Ω and a reflecting Brownian motion X^{D} in D both starting from a point $x\in\Omega$ such that

$$\forall t\geq 0, \quad |X_t^{\Omega}|\leq |X_t^{D}|. \tag{2}$$

This will imply the result about the heat kernels p^{Ω} and p^{D}. Indeed, (2) implies that for all sufficiently small ε and all $t\geq 0$

$$P_x[|X_t^{\Omega}|\leq\varepsilon]\geq P_x[|X_t^{D}|\leq\varepsilon].$$

In terms of the heat kernels, this means that

$$\int_{[-\varepsilon,\varepsilon]} p^{\Omega}(t, x, y)\, dy\geq\int_{[-\varepsilon,\varepsilon]} p^{D}(t, x, y)\, dy.$$

Dividing by 2ε and taking the limit as $\varepsilon\to 0$, we obtain the desired inequality (1).

To achieve the pathwise domination in (2), we first define X^{Ω} by the well-known Skorohod equation

$$X_t^{\Omega} = x+B_t+L_t^{\Omega,1}-L_t^{\Omega,2}, \tag{3}$$

where $L^{\Omega,i}$ are nondecreasing continuous processes starting from 0 which increases only when X^{Ω} is at a_i, $i=1, 2$. Note that given B_t as a continuous function, the Skorohod equation can be solved deterministically for $\{X_t^{\Omega}, L_t^{\Omega,1}, L_t^{\Omega,2}\}$ (see Ikeda and Watanabe [6], p. 119–120).

The reflecting Brownian motion X^D will also be defined by a Skorohod equation

$$X_t^D = x+W_t+L_t^{D,1}-L_t^{D,2}\,, \tag{4}$$

but with a different Brownian motion W. Our task is to relate the two Brownian motions B and W in such way that the pathwise inequality (2) holds. In the following construction of W, we may regard B and W as deterministic continuous functions since the construction itself has nothing to do with probability.

Initially we let $W_t=B_t$ and define X_t^D by the Skorohod equation (4). These definitions of W_t and X_t^D are in force until the first time $t=\tau_1$ that $X_t^D=-X_t^\Omega \neq 0$. After τ_1, we let the increment of W to be the negative of the increment of B, i.e., we define $t\geq\tau_1$

$$W_t = W_{\tau_1}+B_{\tau_1}-B_t\,.$$

We then define X_t^D by the Skorohod equation (4) until the first time $t=\tau_2$ that $X_t^D=X_t^\Omega\neq 0$. We now let the increment of W to be the same as that of B, i.e., for $t\geq\tau_2$

$$W_t = W_{\tau_2}+B_t-B_{\tau_2}\,.$$

For $t\geq\tau_2$, X_t^D is determined by (4) until the first time $t=\tau_3$ that $X_t^D=-X_t^\Omega\neq 0$ and set for $t\geq\tau_3$

$$W_t = W_{\tau_3}+B_{\tau_3}-B_t\,.$$

In general, for $t\in[\tau_n, \tau_{n+1}]$ we define

$$W_t = W_{\tau_n}+(-1)^n\,[B_t-B_{\tau_n}]\,,$$

and define τ_{n+1} to be the first time t after τ_n such that $X_t^D=(-1)^n\,X_t^\Omega\neq 0$. Let $\tau_\infty=\lim_{n\to\infty}\tau_n$. We thus have defined W_t and X_t^D for all time $t<\tau_\infty$. Note that τ_n, $n=1, 2, \cdots$ is a sequence of strictly increasing stopping times. Clearly, W is a Brownian motion. Therefore the process X^D, which satisfies the Skorohod equation (4) is indeed a reflecting Brownian motion on D. We claim that $\tau_\infty=\infty$ and that Inequality (2) holds.

Let us look at a fixed time interval $[\tau_n, \tau_{n+1}]$, $n=0, 1, 2, \cdots$, with the convention that $\tau_0=0$. We first prove Inequality (2) on this interval by contradiction. We know that (2) holds at time $t=\tau_n$. Suppose that (2) does not hold for all t in $[\tau_n, \tau_{n+1}]$. Let τ be the infimum of those time points in the interval for which (2) does not hold. Then $\tau<\tau_{n+1}$. We have $X_\tau^D=\pm X_\tau^\Omega$ and $W_t-W_\tau=\pm[B_t-B_\tau]$ on the interval $[\tau, \tau_{n+1}]$. In both equalities the positive sign holds when n is even, otherwise the negative sign holds. Therefore we have in either case

(5) $$|X_\tau^\Omega+B_t-B_\tau|=|X_\tau^D+W_t-W_\tau|.$$

The two reflecting Brownian motions cannot be at 0 at time τ, because otherwise for a positive amount of time after τ, both processes are away from the boundary $\partial\Omega$ and the boundary local times in (3) and (4) do not increase. Thus from (3), (4) and (5) we see that $|X_t^\Omega|=|X_t^D|$ for a positive duration of time after τ, which contradicts the definition of τ. Now we assume that $X_\tau^\Omega>0$. In this case, X_t^Ω will be positive for a positive duration of time after τ. Thus $L^{\Omega,1}$ does not increase. Also since X_τ^D is at a positive distance away from the boundary ∂D, the boundary local times $L^{D,i}$, $i=1,2$ do not increase for a positive duration of time after τ. It follows from (3), (4) and (5) that for a positive duration of time after τ, we have

$$\begin{aligned}|X_t^\Omega| &= X_t^\Omega\\ &= X_\tau^\Omega+B_t-B_\tau-[L_t^{\Omega,1}-L_\tau^{\Omega,1}]\\ &\leq X_\tau^\Omega+B_t-B_\tau\\ &\leq |X_\tau^D+W_t-W_\tau|\\ &= |X_t^D|.\end{aligned}$$

This again contradicts the definition of τ. The case $X_\tau^\Omega<0$ can be discussed similarly. We therefore have proved that Inequality (2) holds throughout the interval $[\tau_n, \tau_{n+1}]$. This implies immediately that (2) holds for all time $t<\tau_\infty$.

Finally to prove that $\tau_\infty=\infty$, we first show that X_t^Ω has to reach both zero and the boundary $\partial\Omega$ during the time interval $[\tau_n, \tau_{n+1}]$. We assume that $X_{\tau_n}^\Omega=X_{\tau_n}^D$ (the other possibility $X=-X_{\tau_n}^D$ can be handled similarly). At time τ_{n+1} we have $X_{\tau_{n+1}}^\Omega=-X_{\tau_{n+1}}^D\neq 0$, i.e., the two processes are on the different sides of the origin. Since as proved before X_t^D is always farther from the origin than X_t^Ω, this cannot happen unless X_t^Ω is at the origin sometime. Now if X_t^Ω does not reach the boundary $\partial\Omega$, then the boundary local times in (3) do not increase. By (3) and (4) we have $X_t^\Omega=X_t^D$ until X_t^D reaches the boundary ∂D. Clearly then X_t^D will not reach the boundary at all and we always have $X_t^\Omega=X_t^D$. Therefore X_t^Ω and X_t^D cannot be on the different sides of the origin. This contradiction shows that X_t^Ω has to reach the boundary sometime between τ_n and τ_{n+1}.

Now the fact that X_t^Ω has to hit both zero and the boundary $\partial\Omega$ during the interval $[\tau_n, \tau_{n+1}]$ shows that the oscillation of X^Ω is at least $\min\{|a_1|, |a_2|\}>0$. By the continuity of X_t^Ω we must have $\tau_\infty=\lim_{n\to\infty}\tau_n=\infty$.

The proof of one-dimensional case is completed.

The idea of the above proof is called coupling, namely we couple the two reflecting Brownian motions X^Ω and X^D so that they behave collectively in a certain way, which in this case is the pathwise domination (2).

3. Proof of the main theorem

Let B be a d-dimensional Brownian motion (d is the dimension of the space) starting from the origin and $x \in \Omega$. As in the one-dimensional case, we will define a new Brownian motion W from B. A multidimensional version of the Skorohod equation holds (see Hsu [5], Chapter 2, or Lions and Sznitman [8]), and we have two reflecting Brownian motions X^{Ω} and X^{D} on Ω and D satisfying the Skorohod equations

$$X_t^{\Omega} = x + B_t + \int_0^t n^{\Omega}(X_s^{\Omega})\, dL_s^{\Omega}\,, \tag{6}$$

$$X_t^{D} = x + W_t + \int_0^t n^{D}(X_s^{D})\, dL_s^{D}\,. \tag{7}$$

Here in (6), $n^{\Omega}(x)$ is the inward unit normal vector at $x \in \partial\Omega$ and L^{Ω} is a non-decreasing process starting from zero which increases only when $X_t^{\Omega} \in \partial\Omega$. The same description holds for the second equation (7).

Let e_i, F_i be defined as in Section 1. We have

$$B_t = \sum_{i=1}^{d} F_i(B_t)\, e_i\,,$$

where $F_i(B_t)$, $i=1, \cdots, d$, are scaled (by constant) Brownian motions. Note that these Brownian motions are not necessarily independent.

Our task is to define the Brownian motion W in terms of B so that the following path domination relation holds with probability one:

$$\forall t \geq 0, \quad \text{and} \quad i = 1, \cdots, d: \qquad |F_i(X_t^{\Omega})| \leq |F_i(X_t^{D})|\,. \tag{8}$$

The main theorem follows from (8) immediately. Indeed, let

$$\Gamma_\varepsilon = \{y : \max_{1 \leq i \leq d} |F_i(y)| \leq \varepsilon\}\,.$$

From the above inequality, we have for sufficiently small ε,

$$P_x[X_t^{\Omega} \in \Gamma_\varepsilon] \geq P_x[X_t^{D} \in \Gamma_\varepsilon]\,.$$

In terms of the Neumann heat kernels, this is just

$$\int_{\Gamma_\varepsilon} p^{\Omega}(t, x, y)\, dy \geq \int_{\Gamma_\varepsilon} p^{D}(t, x, y)\, dy\,.$$

Dividing by the volume $|\Gamma_\varepsilon|$ of Γ_ε and letting ε go to zero, we obtain the desired inequality (1). Thus it is enough to establish (8).

The procedure we use is similar to the one in Section 2 for the one-dimensional case. To start with we let $W_t = B_t$ and define X_t^{D} by the Skorohod equation (7). Let $t = \tau_1$ be the first time that $F_i(X_t^{D}) = -F_i(X_t^{\Omega}) \neq 0$ for some i. Let I_1 be those indices i such that the above event occurs at time τ_1. We reverse the directions of the increments of the components $F_i(B_t)$, $i\in I_1$, namely we define for $t\geq\tau_1$,

$$W_t = W_{\tau_1}+\sum_{i\in I_1}[F(B_{\tau_1})-F(B_t)]\,e_i+\sum_{i\in I_1^c}[F(B_t)-F(B_{\tau_1})]\,e_i\,.$$

Define X_t^D by the Skorohod equation (7) until the first time $i=\tau_2$ a such that either $F_i(X_t^D)=F_i(X_t^\Omega)\neq0$ for some $i\in I_1$ or $F_i(X_t^D)=-F_i(X_t^\Omega)\neq0$ for some $i\in I_1^c$. For the indices i such that either of the above event occurs, we reverse the direction of the increments of the components $F_i(B_t)$. This means the following. Let I_{11} be the set of indices for which the first event occurs at τ_2 and I_{12} be the set of indices for which the second event occurs at τ_2. Let $I_2=(I_1\backslash I_{11})\cup I_{12}$. Then we set for $t\geq\tau_2$

$$W_t = W_{\tau_2}+\sum_{i\in I_2}[F_i(B_{\tau_2})-F(B_t)]\,e_i+\sum_{i\in I_2^c}[F_i(B_t)-F(B_{\tau_2})]\,e_i\,.$$

This procedure can be carried out indefinitely. Let $\tau_\infty=\lim_{n\to\infty}\tau_n$. We thus have defined the processes W_t and X_t^D for all time $t<\tau_\infty$. Note as before that τ_n, $n=1, 2, \cdots$ is a sequence of strictly increasing stopping times. Clearly, W is a Brownian motion. Therefore the process X^D, which satisfies the Skorohod equation (7), is a reflecting Borwnian motion on D. It remains to show that $\tau_\infty=\infty$ and (8) holds almost surely.

As before we prove by contradiction that (8) holds for all intervals $[\tau_n, \tau_{n+1}]$. Suppose that this is not true and let $[\tau_{n_0}, \tau_{n_0+1}]$ be the first interval for which (8) does not hold. For each index i, let σ_i be the infimum of those time points in the interval for which (8) fails for index $i(\sigma_i=\tau_{n_0+1}$ if such time points do not exist.) Let $\tau=\min\{\sigma_i, 1\leq i\leq d\}$. Then $\tau<\tau_{n_0+1}$. Let j be an index such that $\tau=\sigma_j$. By the Skorohod equations (6) and (7) we have

$$F_j(X_t^\Omega) = F_j(X_\tau^\Omega)+F_j(B_t)-F_j(B_\tau)+\int_\tau^t F_j[n^\Omega(X_s^\Omega)]\,dL_s^\Omega\,, \tag{9}$$

$$F_j(X_t^D) = F_j(X_\tau^D)+F_j(W_t)-F_j(W_\tau)+\int_\tau^t F_j[n^D(X_s^D)]\,dL_s^D\,. \tag{10}$$

Now we have the following facts;

(i) By the hypotheses on the domains Ω and D the increment of the boundary local time term in (9) always have the opposite sign as $F_j(X_t^\Omega)$;

(ii) $|F_j(X_t^\Omega)|<S_j$;

(iii) $F_j(X_\tau^\Omega)=\pm F_j(X_\tau^D)$ and $F_j(W_t)-F_j(W_\tau)=\pm[F_j(B_t)-F_j(W_\tau)]$ for all t in $[\tau, \tau_{n_0+1}]$. Therefore, similarly to (5) we have

$$|F_j(X_\tau^\Omega)+F_j(B_t)-F_j(B_\tau)| = |F_j(X_t^D)+F_j(W_t)-F_j(W_\tau)|\,.$$

(iv) Since $F_j(e_i)=0$ if $i\neq j$ the boundary local time term in (10) increases or decreases only when $F_j(X_t^D)=\pm S_j$. In fact, $F_i(X_t^D)$ is a scaled (by constant) reflecting Brownian motion on $[-S_j, S_j]$.

Thus we are in exactly the same situation as in the one-dimensional case of the last section. By the argument there, we see that $|F_j(X_t^\Omega)| \leq |F_j(X_t^D)|$ for a positive duration of time after τ. This contradicts the definitions of τ and j and proves the desired inequalilty (8).

Finally we have to show that $\tau_\infty = \infty$ almost surely. We first show that if τ_∞ is finite then $X_{\tau_\infty}^\Omega$ must be in the union $A = \bigcup_{i=1}^d (\partial\Omega \cap \{F_i = 0\})$. Let us suppose otherwise, i.e., for some positive constants λ_0 and $\mathcal{E}_0$, the path X_t^Ω will be at least λ_0 distance away from the set A and all time $t \in [\tau_\infty - \mathcal{E}_0, \tau_\infty)$. Then there is a positive constant c_0 such that if $X_t^\Omega \in \partial\Omega$ for some $t \in [\tau_\infty - \mathcal{E}_0, \tau_\infty)$ then $|F_i(X_t^\Omega)| \geq c_0$ for all $i = 1, \cdots, d$. Let n_0 be a large positive integer such that $\tau_{n_0 d} \geq \tau_\infty - \mathcal{E}_0$. Let $n \geq n_0$. Clearly there exists at least one index j such that

$$F_i(X_{\tau_k}^D) = \pm F_j(X_{\tau_k}^\Omega) \quad \text{and} \quad F_j(X_{\tau_l}^D) = \mp F_j(X_{\tau_l}^\Omega)$$

for some k and l between nd and $(n+1)d$ (inclusive). Thus by an argument similar to the one used in the last section we see that $F_j(X_t^\Omega)$ must reach zero some time between τ_k and τ_l and X_t^Ω must reach the boundary at some time (say t_0) between τ_k and τ_l. This implies that $|F_j(X_{t_0}^\Omega)| \geq c_0$. It follows that for each $n \geq n_0$ the oscillation of at least one component $F_i(X_t^\Omega)$ in the integral $[\tau_{nd}, \tau_{(n+1)d}]$ is at least c_0. By the continuity of X_t^Ω this implies that $\tau_\infty = \infty$, a contradiction. Thus $\tau_\infty < \infty$ implies that $X_{\tau_\infty}^\Omega \in A$. This is as far as we can go without probability!

To finish the proof, we note that A, being a union of finitely many compact and smooth manifolds of dimension $d-2$, is a polar set for the reflecting Brownian motion X^Ω. Thus with probability one X^Ω never reaches A. It follows from this observation and what we have shown in the preceding paragraph that with probability one $\tau_\infty = \infty$. This completes the proof of the main theorem.

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