Negoro, A.
Osaka J. Math.
31 (1994), 189–214

STABLE-LIKE PROCESSES: CONSTRUCTION OF THE TRANSITION DENSITY AND THE BEHAVIOR OF SAMPLE PATHS NEAR $t=0$

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(Received August 5, 1992)

Introduction

Let $X=(X_t, \boldsymbol{P}_x; x\in\boldsymbol{R}^d)$ be a d-dimensional pure jump type Markov process associated with the operator $-(-\Delta)^{\alpha(x)/2}(0<\alpha(x)<2)$. Following Bass [1], we call it the stable-like process with exponent $\alpha(x)$. Under a mild regularity condition for $\alpha(x)$, the process is first constructed by Bass [1] and next by Tsuchiya [12]: Bass has done it by showing the uniqueness of solutions to the martingale problem for the operator and Tsuchiya by showing the pathwise uniqueness of solutions to a stochastic differential equation associated with the operator.

In this paper, we will show the existence of a transition density and local Hölder conditions for sample paths of the process X with smooth exponent $\alpha(x)$. For this aim, we want to adapt the theory of pseudo-differential operators to the operator $-(-\Delta)^{-\alpha(x)/2}$, but its symbol $-|\xi|^{\alpha(x)}$ is not smooth. Hence we consider the operator L_Φ which is obtained from $-(-\Delta)^{\alpha(x)/2}$ by cutting off the support of its integral kernel (i.e. Lévy measure) with a positive smooth function Φ (see Section 1 for the precise definition of L_Φ). There exists a pure jump type Markov process X_Φ associated with L_Φ in the same sense as the above. Since L_Φ can be regarded as a pseudo-differential operator of variable order, we introduce a class of such operators and provide the fundamental theorem for algebra and asymptotic expansion formula of their symbols. Next we prove that L_Φ satisfies the (H)-condition (see [7] p.83 for the (H)-condition). These facts allow us to construct a fundamental solution, in the sense of pseudo-differential operators, to the initial-value problem for the equation $\partial_t-L_\Phi=0$. Furthermore, we show that this fundamental solution has a smooth kernel and this gives a transition density of X_Φ. Using a localization argument, we see that X also has a transition density. Finally, using certain estimates for the symbol of the fundamental solution and expanding the method of Khintchine [6] and Blumenthal and Getoor [3], we obtain the local Hölder conditions for sample paths of X; this result is a natural extension of that of

[3] in the case of symmetric stable processes.

Pseudo-differential operators of variable order are treated by Unterberger and Bokobza [14], [15], Unterberger [13], Višik and Eskin [16], [17], Beasuzamy [2] and Leopold [9] [10], etc. They, however, do not treat the initial-value problem for evolution equations with respect to such operators.

Section 1 is devoted to construction of a fundamental solution $E(\cdot)$ to the initial-value probelm for $\partial_t - L_\Phi = 0$ (Theorem 1.3). It implies the existence of a transition density of X_Φ (Theorem 1.6) and also implies the existence of a transition density of X (Theorem 1.7). The (H)-condition follows from Theorem 1.1, which is a key result for the construction of the fundamental solution.

In Section 2, we prove local Hölder conditions for sample paths of X (Theorem 2.1). Lemma 2.1 is an extension of a fundamental result of Khintchine [6]. Lemma 2.2 gives a relation between the symbol of the fundamental solution $E(\cdot)$ and the characteristic function of a random variable used in [3].

I express my gratitude to Professor M. Tsuchiya for valuable discussions and the guidance on the topic of this paper. In particular, the proof of Theorem 1.1 was accomplished with his aid. To Professor K. Kikuchi, I also express may appreciation for his useful advice and helpful conversations on the theory of pseudo-differential operators.

1. Constrcution of the transition density

We begin with introducing some notations. For $n=0, 1, 2, \cdots, \infty$, $\boldsymbol{C}_b^n(\boldsymbol{R}^d)$ is the space of real-valued n times differentiable functions which are defined on $\boldsymbol{R}^d$ and have bounded continuous derivatives up to order n. $\boldsymbol{C}_0^\infty(\boldsymbol{R}^d)$ is the subspace of $\boldsymbol{C}_b^\infty(\boldsymbol{R}^d)$ consisting of those functions with compact support. $\mathcal{S}$ or $\mathcal{S}(\boldsymbol{R}^d)$ denotes the Schwartz class on $\boldsymbol{R}^d$. $\boldsymbol{C}_b^{1,2}([0, \infty)\times\boldsymbol{R}^d)$ denotes the space of real-valued functions on $[0, \infty)\times\boldsymbol{R}^d$ which together with first-derivative in time variable and first two-derivatives in space variables are bounded and continuous. For a bounded function $\alpha(x)$ on $\boldsymbol{R}^d$, set

$$\bar{\alpha} = \sup_{x\in\boldsymbol{R}^d} \alpha(x) \quad \text{and} \quad \underline{\alpha} = \inf_{x\in\boldsymbol{R}^d} \alpha(x)\,.$$

Let Ω be the space of $\boldsymbol{R}^d$-valued càdlàg functions ω on $[0, \infty)$ and let $X_t\colon \Omega\to\boldsymbol{R}^d$ be the function defined by $X_t(\omega)=\omega(t)$. Let $\mathcal{F}_t$ be the σ-field generated by $\{X_s, s\leq t\}$ and $\mathcal{F}=\mathcal{F}_\infty$. Given a positive kernel $\nu(x, dy)$ on $\boldsymbol{R}^d\times(\boldsymbol{R}^d\setminus\{0\})$ satisfying $\int_{\boldsymbol{R}^d\setminus\{0\}}(|y|^2\wedge 1)\,\nu(x, dy)<\infty$, we define the operator L on $\boldsymbol{C}_b^2(\boldsymbol{R}^d)$ by

$$Lf(x) = \int_{\boldsymbol{R}^d\setminus\{0\}} \{f(x+y)-f(x)-\nabla f(x)\cdot y 1_{\{|y|\leq 1\}}(y)\}\ \nu(x, dy)\,,$$

where $z\cdot y$ is the scalar product in $\boldsymbol{R}^d$, ∇ is the gradient operator and $1_E(\cdot)$ the indicator function of a set E. We say that a probability measure $\boldsymbol{P}$ on $(\Omega, \mathcal{F})$ is a solution to the martingale problem for the operator L starting at x if $\boldsymbol{P}(X_0=x)=1$ and, for every $f\in \boldsymbol{C}_b^{1,2}([0,\infty)\times\boldsymbol{R}^d)$,

$$f(t, X_t)-f(0, X_0)-\int_0^t (\partial_u+L)f(u, X_u)\,du$$

is a $\boldsymbol{P}$-martingale with respect to the filtration $\{\mathcal{F}_t\}$.

In this paper, we will focus our attention on the following type of kernels:

$$\nu(x, dy) = w_{\alpha(x)}|y|^{-(d+\alpha(x))}\,dy\,,$$

where $\alpha(x)$ is of $\boldsymbol{C}_b^\infty(\boldsymbol{R}^d)$ with $0<\underline{\alpha}\le\alpha(x)\le\overline{\alpha}<2$, and $w_{\alpha(x)}$ is defined through the Lévy-Khintchine formula

$$|\xi|^{\alpha(x)} = \int_{\boldsymbol{R}^d\setminus\{0\}} \{1-\cos\xi\cdot y\}\, w_{\alpha(x)}\,|y|^{-(d+\alpha(x))}\,dy\,.$$

We note that $w_{\alpha(x)}$ is a positive function of $\boldsymbol{C}_b^\infty(\boldsymbol{R}^d)$. Then the operator L can be regarded as a pseudo-differential operator with symbol $-|\xi|^{\alpha(x)}$; hence, in the following, we will denote the operator L by $-(-\Delta)^{\alpha(x)/2}$. By a result of Bass [1] or Tsuchiya [12], for each starting point, there exists a unique solution to the martingale problem for the opeartor $-(-\Delta)^{\alpha(x)/2}$. Therefore, the family of solutions to the martingale problem defines a Markov process on $\boldsymbol{R}^d$, and it is called the stable-like process with exponent $\alpha(x)$.

The purpose of this section is to show the existence of a transition density of the process. To conclude this, we consider the kernel ν_Φ defined by

$$\nu_\Phi(x, dy) = w_{\alpha(x)}|y|^{-(d+\alpha(x))}\,\Phi(|y|)\,dy\,,$$

where Φ is a function of $\boldsymbol{C}_b^\infty([0,\infty))$ satisfying the conditions:
(1) $0\le\Phi\le1$ on $[0,\infty)$,
(2) there exists a real number $r_0>0$ such that $\Phi(t)=1$ for any $t\in[0, r_0]$,
(3) $\Phi(t)=0$ for any $t\in[1,\infty)$.
Let L_Φ denote the operator corresponding to this kernel. Then the uniqueness of solutions to the martingale problem for L_Φ also holds and hence there exists a unique Markov process X_Φ associated with L_Φ in the same sense as the above (cf. [12]). At first, we will construct a transition density of this Markov process and obtain some estimates for the density. Then, using them, we show the existence of a transition density of the original stable-like process.

Now, the operator L_Φ can be regarded as a pseudo-differential operator with symbol p_Φ:

$$p_\Phi(x, \xi) = \int_{\boldsymbol{R}^d\setminus\{0\}} \{\exp(i\xi\cdot y)-1-i\xi\cdot y\}\,\frac{w_{\alpha(x)}\,\Phi(|y|)}{|y|^{d+\alpha(x)}}\,dy\,. \tag{1.1}$$

To adapt the theory of pseudo-differential operators for L_Φ, we start to discuss some properties of the function p_Φ. For a multi-index $n=(n_1, n_2, \cdots, n_d)$, let $\partial_\xi^n=\partial^{n_1}/\partial\xi_1^{n_1}\cdots\partial^{n_d}/\partial\xi_d^{n_d}$ and $D_x^n=(-i)^{|n|}\partial_x^n$, where $|n|=n_1+n_2+\cdots+n_d$.

Theorem 1.1. (1) *p_Φ is of $C_b^\infty(\boldsymbol{R}^d\times\boldsymbol{R}^d)$.*
(2) *For any multi-indices m and n, there exists a constant $C_{m,n}>0$ such that for any $(x,\xi)\in\boldsymbol{R}^d\times\boldsymbol{R}^d$*

$$(1.2)\qquad |\partial_\xi^n D_x^m p_\Phi(x,\xi)|\le C_{m,n}(|\xi|\vee 1)^{(\alpha(x)-|n|)}(1+\log(|\xi|\vee 1))^{|m|}.$$

(3) *There exist constants $R>0$ and $C_0>0$ such that for any $x\in\boldsymbol{R}^d$ and $|\xi|>R$*

$$(1.3)\qquad |p_\Phi(x,\xi)|\ge C_0|\xi|^{\alpha(x)}.$$

REMARK. If we set $C'_{m,n}=C_{m,n}/C_0$, then

$$(1.4)\qquad \left|\frac{\partial_\xi^n D_x^m p_\Phi(x,\xi)}{p_\Phi(x,\xi)}\right|\le C'_{m,n}|\xi|^{-|n|}\{1+\log(|\xi|\vee 1)\}^{|m|},$$

for any $x\in\boldsymbol{R}^d$ and $|\xi|>R$. This implies the (H)-condition.

Proof of Theorem 1.1. In the proof, C denotes different positive constants. Let $\boldsymbol{S}^{d-1}$ be the unit sphere of $\boldsymbol{R}^d$ and s be the uniform measure on $\boldsymbol{S}^{d-1}$. Since s is invariant under rotation, we have

$$(1.5)\qquad p_\Phi(x,\xi)=\int_0^1\int_{S^{d-1}}(\cos r\theta\cdot\xi-1)\frac{w_{\alpha(x)}\Phi(r)}{r^{1+\alpha(x)}}dr\,s(d\theta);$$

hence

$$(1.6)\qquad \partial_x^m p_\Phi(x,\xi)=\sum_{k=0}^{|m|}a_k(x)\int_0^1\frac{(\log r)^k\Phi(r)}{r^{1+\alpha(x)}}dr\int_{S^{d-1}}(\cos r\theta\cdot\xi-1)\,s(d\theta),$$

where the function $a_k(x)$ is a linear combination of derivatives up to order k of $\alpha(x)$ and $w_{\alpha(x)}$. Then $a_k(x)$ $(k=1,2,\cdots)$ are of $C_b^\infty(\boldsymbol{R}^d)$. Hence, to obtain the estimate for $\partial_x^m p_\Phi$, it is sufficient to evaluate the following integral:

$$I_k=\int_0^1\frac{(\log r)^k\Phi(r)}{r^{1+\alpha(x)}}dr\int_{S^{d-1}}(\cos r\theta\cdot\xi-1)\,s(d\theta).$$

For $|\xi|\le 1$, noting $|\cos r\theta\cdot\xi-1|\le\frac{1}{2}r^2$, we see that

$$|I_k|\le\frac{1}{2}s(\boldsymbol{S}^{d-1})\int_0^1 r^{1-\alpha(x)}(\log r)^k dr<\infty.$$

When $|\xi|>1$, putting $q=r|\xi|$ and $\tilde{\xi}=\xi/|\xi|$, we can rewrite I_k as follows:

$$\begin{aligned}I_k&=\int_0^{|\xi|}\frac{|\xi|^{\alpha(x)}(\log q-\log|\xi|)^k\Phi(q/|\xi|)}{q^{1+\alpha(x)}}dq\int_{S^{d-1}}(\cos q\theta\cdot\tilde{\xi}-1)\,s(d\theta)\\&=|\xi|^{\alpha(x)}\sum_{j=1}^{k}\binom{k}{j}(-\log|\xi|)^{k-j}\int_0^{|\xi|}\int_{S^{d-1}}(\log q)^j\,\Phi(q/|\xi|)\end{aligned}$$

$$\times \frac{1}{q^{1+\alpha(x)}} (\cos q\theta \cdot \tilde{\xi} - 1)\, dq\, s(d\theta)\,.$$

Since

$$\begin{aligned}&\left| \int_0^{|\xi|} \frac{\Phi(q/|\xi|)(\log q)^j}{q^{1+\alpha(x)}} (\cos q\theta\cdot\tilde{\xi}-1)\, dq \right| \\ &\quad \le \frac{1}{2}\int_0^1 \frac{\Phi(q/|\xi|)|\log q|^j}{q^{1+\alpha(x)}} q^2\, dq + 2\int_1^\infty \frac{\Phi(q/|\xi|)(\log q)^j}{q^{1+\alpha(x)}}\, dq < \infty\,, \\ &\quad |I_k| \le C(|\xi|^{\alpha(x)} \vee 1)(1+\log(|\xi|\vee 1))^k\,.\end{aligned}$$

Hence, we have

$$|\partial_x^m p_\Phi(x,\xi)| \le C(|\xi|^{\alpha(x)}\vee 1)(1+\log(|\xi|\vee 1))^{|m|}\,.$$

From (1.6), it follows that for any $m=(m_1, m_2, \cdots, m_d)$, $n=(n_1, n_2, \cdots, n_d)$ ($|n|\ge 1$) and $(x,\xi)\in \boldsymbol{R}^d\times\boldsymbol{R}^d$

$$\begin{aligned}&\partial_\xi^n \partial_x^m p_\Phi(x,\xi) \\ &= \sum_{k=1}^{|m|} a_k(x) \int_0^1 \frac{(\log r)^k \Phi(r)}{r^{1+\alpha(x)-|n|}}\, dr \int_{S^d} \exp(ir\theta\cdot\xi)(i\theta_1)^{n_1}\cdots(i\theta_d)^{n_d} s(d\theta)\,.\end{aligned}$$

Therefore, we will estimate the integral:

$$K_{n,k} = \int_0^1 \frac{(\log r)^k \Phi(r)}{r^{1+\alpha(x)-|n|}} \int_{S^{d-1}} \exp(ir\theta\cdot\xi)(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d} s(d\theta)\,.$$

If $|\xi|\le 1$ and $n\ge 2$, then we immediately see that

$$|K_{n,k}| \le \frac{k!\, s(S^{d-1})}{(|n|-\bar{\alpha})^{k+1}} < \infty\,.$$

When $|n|=1$ and $|\xi|\le 1$, noting

$$\int_{S^{d-1}} (i\theta_j)\, s(d\theta) = 0\,,$$

we have

$$\begin{aligned}|K_{n,k}| &\le \left| \int_{S^d} \{\exp(ir\theta\cdot\xi)-1\}\, i\theta_j\, s(d\theta) \right| \int_0^1 \frac{\Phi(r)(-\log r)^k}{r^{\bar{\alpha}}}\, dr \\ &\le s(S^{d-1}) \int_0^1 r^{1-\bar{\alpha}}(-\log r)^k\, dr < \infty\,.\end{aligned}$$

Next, we consider the case when $|\xi|>1$. We rewrite $K_{n,k}$ in the form:

$$\begin{aligned}K_{n,k} &= |\xi|^{\alpha(x)-|n|} \sum_{j=0}^k \binom{k}{j} (-\log|\xi|)^{k-j} \int_0^{|\xi|} \frac{(\log q)^j \Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\, dq \\ &\quad \times \int_{S^d} \exp(iq\theta\cdot\tilde{\xi})(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d} s(d\theta).\end{aligned}$$

We will evaluate the integral

$$\tilde{K}_{n,j} = \int_0^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\, dq \times \int_{S^d} \exp(iq\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\, s(d\theta)\,.$$

We divide $\tilde{K}_{n,j}$ into two parts $\tilde{K}^{(1)}_{n,j}$ and $\tilde{K}^{(2)}_{n,j}$:

$$\tilde{K}^{(1)}_{n,j} = \int_0^{1} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\, dq \times \int_{S^{d-1}} \exp(iq\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\, s(d\theta)$$

and

$$\tilde{K}^{(2)}_{n,j} = \int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}}\, dq \times \int_{S^{d-1}} \exp(iq\theta\cdot\xi)\,(i\theta_1)^{n_1}(i\theta_2)^{n_2}\cdots(i\theta_d)^{n_d}\, s(d\theta)\,.$$

Adopting the same method as in estimating of $K_{n,k}$ for $|\xi|\leq 1$, we can show that

$$|\tilde{K}^{(1)}_{n,j}|<\infty \quad \text{if} \quad |n|\geq 1\,.$$

Now, let $\eta=q\xi$. Then

$$\tilde{K}^{(2)}_{n,j} = \int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{1+\alpha(x)-|n|}} \left\{\partial^n_\eta \int_{S^{d-1}} \exp(i\eta\cdot\theta)\, s(d\theta)\,|_{\eta=q\xi}\right\} dq\,. \tag{1.7}$$

To estimate $\tilde{K}^{(2)}_{n,j}$, we use the following result of Jones ([5] p.9):

$$\int_{S^{d-1}} \exp(i\eta\cdot\theta)\, s(d\theta) = \omega_d \frac{2^\nu\,\Gamma(\nu+1)}{|\eta|^\nu} J_\nu(|\eta|)\,, \tag{1.8}$$

where $\omega_d=2\sqrt{\pi^d}/\Gamma(d/2)$ and J_ν is the Bessel function of index $\nu=(d-2)/2$. Let

$$F_h(\eta) = (\eta/2)^{-(\nu+h)} J_{\nu+h}(|h|) = \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{2p}\,p!\,\Gamma(\nu+p+h+1)} |\eta|^{2p}\,.$$

Taking the $|n|$-th derivative of both the sides of (1.8), we have the equation

$$\partial^n_\eta \int_{S^{d-1}} \exp(i\eta\cdot\theta)\, s(d\theta) = \sum_{l}^{[n/2]} C_l\, \eta_1^{n_1-2l_1}\, \eta_2^{n-2l_2}\cdots\eta_d^{n_d-2l_d}\, F_{\nu+|n|-|l|}(\eta),$$

where $i=(l_1, l_2, \cdots, l_d)$, $n=(n_1, n_2, \cdots, n_d)$, $[n/2]=([n_1/2], [n_2/2], \cdots[n_d/2])$ and $[\cdot]$ is Gauss' symbol, C_l is a constant depending on only l; hence

$$\partial^n_\eta \int_{S^{d-1}} \exp(i\eta\cdot\theta)\, s(d\theta) \tag{1.9}$$

$$= \sum_{l}^{[n/2]} C_l \left(\frac{|\eta|}{2}\right)^{-(\nu+h)} J_{\nu+|n|-|l|}(|\eta|)\, \eta_1^{n_1-2l_1}\, \eta_2^{n_2-2l_2} \cdots \eta_d^{n_d-2l_d}\,.$$

From (1.7) and (1.9), it follows that

$$\tilde{K}_{n,j}^{(2)} = \int_1^{|\xi|} \sum_{l}^{[n/2]} b_l(\xi) \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+1+\nu+2|l|-|n|}} J_{\nu+|n|-|l|}(q)\, dq\,,$$

where $b_l(\xi)$ denotes a polynomial of ξ. Therefore, we have to estimate the integral

$$\int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+1+2|l|+\nu-|n|}} J_{\nu+|n|-|l|}(q)\, dq\,. \tag{1.10}$$

Using the asymptotic expansion formula for Bessel functions (cf. [4] p.230), we obtain

$$\begin{aligned}
&\int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+1+\nu+2|l|-|n|}} J_{|n|+\nu-|l|}(q)\, dq \\
&= \frac{(2/\pi)^{1/2}}{\Gamma(\nu+|n|-|l|+1/2)} \sum_{k=0}^{N-1} \binom{\nu+|n|-|l|+1/2}{k} \frac{\Gamma(\nu+|n|-|l|+k+1/2)}{2^k} \\
&\times \int_1^{|\xi|} \frac{(-1)^{k/2}(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+3/2+k+\nu+2|l|-|n|}} \begin{Bmatrix} \cos\{q-(\nu+|n|-|l|)\pi/2-\pi/4\} \\ \sin\{q-(\nu+|n|-|l|)\pi/2-\pi/4\} \end{Bmatrix} dq \\
&+ \int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+3/2+p+\nu+2|l|-|n|}} O(q^{-p-1/2})\, dq\,.
\end{aligned}$$

If N is a sufficiently large integer,

$$\int_1^{\infty} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+3/2+N+\nu+2|l|-|n|}} O(q^{-p-1/2})\, dq < \infty\,.$$

Thus, it is sufficient to prove the boundedness of the integrals:

$$\int_1^{|\xi|} \frac{(\log q)^j\, \Phi(q/|\xi|)}{q^{\alpha(x)+s}} \begin{Bmatrix} \cos(q+c\pi) \\ \sin(q+c\pi) \end{Bmatrix} dq \quad (j=0, 1, \cdots, k)\,. \tag{1.11}$$

Repeating the integration by parts and using the property $\Phi^{(l)}(1)=0$ ($l=0,1,2,\cdot$), we see that the integrals of the type (1.11) are represented by a linear combination of the following formula:

$$\begin{aligned}
&\pm(\alpha(x)+s)\cdots(\alpha(x)+s+u-1) \frac{1}{|\xi|^v} \int_1^{|\xi|} \frac{\Phi^{(v)}(q/|\xi|)(\log q)^j}{q^{\alpha(x)+s+u}} \begin{Bmatrix} \cos(q+c\pi) \\ \sin(q+c\pi) \end{Bmatrix} dq \\
&+c\cos(q+c\pi)\ (\text{or } c\sin(q+c\pi)) \quad (j, u, v = 0, 1, 2, \cdots)\,.
\end{aligned}$$

Therefore, it is enough to show the boundedness of the integral with the form:

$$\int_1^{|\xi|} \frac{\Phi^{(v)}(q/|\xi|)(\log q)^j}{q^{\underline{\alpha}+v+s+u}}\, dq\ ;$$

it is easily verified by the use of the integration by parts. Consequently, we prove the assertions (1) and (2). Next, we show the assertion (3). From (1.8), we see that

$$\begin{aligned} |p_\Phi(x,\xi)| &= |\xi|^{\alpha(x)} w_{\alpha(x)} \int_0^{|\xi|} \frac{\Phi(q/|\xi|)}{q^{1+\alpha(x)}}\, dq \int_{S^{d-1}} \{1-\exp(iq\theta\cdot\bar{\xi})\}\, s(d\theta) \\ &= |\xi|^{\alpha(x)} w_{\alpha(x)}\, \omega_d \int_0^{|\xi|} \frac{\Phi(q/|\xi|)}{q^{1+\alpha(x)}} \{1-\Gamma(\nu+1)\sum_{p=0}^{\infty} \frac{(-1)^p}{2^{2p}\, p!\,\Gamma(\nu+p+1)}\, q^{2p}\}\, dq \\ &= |\xi|^{\alpha(x)} w_{\alpha(x)}\, \omega_d\, \Gamma(\nu+1) \int_0^{|\xi|} \frac{\Phi(q/|\xi|)}{q^{1+\alpha(x)}} \Big\{\frac{q^2}{2^2\,\Gamma(\nu+2)} \\ &\quad - \sum_{p=2}^{\infty} \frac{(-1)^p\, q^{2p}}{2^{2p}\, p!\,\Gamma(\nu+p+1)}\Big\}\, dq\,. \end{aligned}$$

The convergence radius of the power series $\sum_{p=2}^{\infty}(-1)^p q^{2p}/2^{2p} p!\,\Gamma(\nu+p+1)$ is infinite and it is equal to zero at $q=0$. Hence, there is a sufficiently small number $q_0>0$ such that, for any $q\in[0, q_0]$,

$$\frac{q^2}{2^2\,\Gamma(\nu+2)} - \sum_{p=2}^{\infty} \frac{(-1)^{p-1}\, q^{2p}}{2^{2p}\, p!\,\Gamma(\nu+p+1)} > \frac{q^2}{2^3\,\Gamma(\nu+2)}\,.$$

Therefore,

$$|p_\Phi(x,\xi)| \geq |\xi|^{\alpha(x)} w_{\alpha(x)} \frac{\omega_d\,\Gamma(\nu+1)}{2^3\,\Gamma(\nu+2)} \int_0^{q_0} q^{1-\alpha(x)}\, dq \quad \text{for any } \xi \text{ with } |\xi|>R=\frac{q_0}{r_0}\ ;$$

hence the assertion (3) is verified. Consequently Theorem 1.1 is proved.

Since L_Φ can be regarded as a pseudo-differential operator of variable order, extending the theory for pseudo-differential operator of constant order, we prepare a general theory for such operators of variable order in the following. In what follows, for simplicity, we let

$$p_{(m)}^{(n)}(x,\xi) = \partial_\xi^n D_x^m\, p(x,\xi)$$

and, in particular,

$$p^{(l)}(x,\xi) = p_{(0)}^{(l)}(x,\xi) \quad \text{and} \quad p_{(l)}(x,\xi) = p_{(l)}^{(0)}(x,\xi)\,.$$

DEFINITION 1.1. *Let ζ be a bounded function on $\boldsymbol{R}^d$.*
(1) *We say that a function $p(x,\xi)$ of $C^\infty(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ is a symbol of the class $S_{\rho,\delta}^{\zeta}(0\leq\delta\leq\rho\leq 1,\ \delta<1)$, if for any multi-indices m and n, there exists a constant $C_{m,n}$ such that*

$$(1.12) \qquad |p_{(m)}^{(n)}(x,\xi)| \leq C_{m,n} \langle\xi\rangle^{\zeta(x)+\delta|m|-\rho|n|}$$

for any $(x, \xi) \in \boldsymbol{R}^d \times \boldsymbol{R}^d$, *where* $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$. *We set*

$$S^{-\infty} = \bigcap_{-\infty<\theta<\infty} S^{\theta}_{\rho,\delta} \quad and \quad S^{\infty}_{\rho,\delta} = \bigcup_{-\infty<\theta<\infty} S^{\theta}_{\rho,\delta}. \tag{1.13}$$

(2) *We say that a linear operator* $P: \mathcal{S}(\boldsymbol{R}^d)\to\mathcal{S}(\boldsymbol{R}^d)$ *is a pseudo-differential operator with symbol* $p(x, \xi)$ *of class* $S_{\rho,\delta}$, *if* Pu *can be represented by*

$$Pu(x) = \int \exp(ix\cdot\xi)\, p(x, \xi)\, \hat{u}(\xi)\, \check{d}\xi \quad for \quad u\in\mathcal{S}(\boldsymbol{R}^d), \tag{1.14}$$

where $\check{d}\xi=(1/2\pi)^d\, d\xi$, *and* $\hat{u}$ *is the Fourier transform of* u. *In this case, we write* $P=p(x, D_x)\in S^{\zeta}_{\rho,\delta}$, *and we also denote the symbol* $p(x, \xi)$ *of* P *by* $\sigma(P)(x, \xi)$. *Moreover the semi-norms* $|p|^{\zeta}_k$ $(k=1, 2, \cdots)$ *are defined by*

$$|p|^{\zeta}_k = \max_{|m+n|\leq k}\ \sup_{(x,\xi)\in\boldsymbol{R}\times\boldsymbol{R}^d} \{|p^{(n)}_{(m)}(x, \xi)|\langle\xi\rangle^{-(\zeta(x)+\delta|m|-\rho|n|)}\}.$$

DEFINITION 1.2. (1) *We say that a function* $a(\eta, y)$ *of* $C^{\infty}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ *belongs to the class* $\mathcal{A}^{\theta}_{\delta,\kappa}$ $(-\infty<\theta<\infty,\ 0\leq\delta<1,\ 0\leq\kappa)$, *if for any multi-indices* m *and* n, *there exists a canstant* $C_{m,n}$ *such that*

$$|\partial^m_\eta\, \partial^n_y\, a(\eta, y)| \leq C_{m,n}\langle\eta\rangle^{\theta+\delta|n|}\langle y\rangle^{\kappa}.$$

We set

$$\mathcal{A} = \bigcup_{0\leq\delta<1}\ \bigcup_{-\infty<\theta<\infty}\ \bigcup_{\kappa\geq0} \mathcal{A}^{\theta}_{\delta,\kappa}.$$

(3) *For an element* $a(\eta, y)$ *of* $\mathcal{A}$, *we define the oscillatory integral* $Os[e^{-iy\cdot\eta}\, a]$ *by*

$$\begin{aligned} Os[e^{iy\cdot\eta}\, a] &= Os\text{-}\iint \exp(-i\eta\cdot y)\, a(\eta, y)\, \check{d}\eta\, dy \\ &= \lim_{\varepsilon\to0}\iint \exp(-i\eta\cdot y)\, \chi(\varepsilon\eta, \varepsilon y)\, a(\eta, y)\, \check{d}\eta\, dy, \end{aligned}$$

where $\chi\in\mathcal{S}(\boldsymbol{R}^d\times\boldsymbol{R}^d)$ *and* $\chi(0, 0)=1$.

Theorem 1.2. *Assume that* $0\leq\delta<\rho\leq1$.
(1) *Let* $\zeta_j (j=1, 2)$ *be a bounded function on* $\boldsymbol{R}^d$ *and* $P_j=p_j(x, D_x)\in S^{\zeta_j}_{\rho,\delta} (j=1, 2)$. *Then* $P=P_1\cdot P_2$ *belongs to* $S^{\zeta_1+\zeta_2}_{\rho,\delta}$ *with symbol* $p(x, \xi)$:

$$p(x, \xi) = Os\text{-}\iint \exp(-i\eta\cdot y)\, p_1(x, \xi+\eta)\, p_2(x+y, \xi)\, \check{d}\eta\, dy \tag{1.15}$$

and it has the asymptotic expansion formula:

$$p(x, \xi) - \sum_{|l|<N} \frac{1}{l!}\, p_1^{(l)}(x, \xi)\, p_{2(l)}(x, \xi) \in S^{\zeta_1+\zeta_2-N(\rho-\delta)}_{\rho,\delta} \tag{1.16}$$

for any integer $N\geq1$.
(2) *Let* $P=p(x, D_x)\in S^{\zeta}_{\rho,\delta}$. *We define* P^* *by*

$$(Pu, v) = (u, P^*v) \quad for \quad u, v\in\mathcal{S}(\boldsymbol{R}^d).$$

Then $P^*(x, D_x)$ *is a pseudo-differential operator of the class* $S^{\zeta}_{\rho,\delta}$ *and its symbol* $p^*(x, \xi)$ *is given by*

$$p^*(x, \xi) = Os-\iint \exp(-i\eta\cdot y)\overline{p(x+y, \xi+\eta)}\, \bar{d}\eta\, dy\,,$$

and it has the asymptotic expansion formula:

$$p^*(x, \xi) - \sum_{|l|<N} \frac{(-1)^{|l|}}{l!} \overline{p^{(l)}_{(l)}(x, \xi)} \in S^{\zeta-N(\rho-\delta)}_{\rho,\delta} \tag{1.17}$$

for any integer $N \geq 1$.

Proof. By Theorem 3.1 in Chap. 2 of [7], we obtain that

$$p(x, \xi) - \sum_{|l|<N} \frac{1}{l!} p_1^{(l)}(x, \xi)\, p_{2(l)}(x, \xi) \in S^{\bar{\zeta}_1+\bar{\zeta}_2-N(\rho-\delta)}_{\rho,\delta}\,. \tag{1.18}$$

Moreover, noting that, when $|l|=0$, $p_1(x, \xi)\, p_2(x, \xi)$ is the symbol with variable order $\zeta_1(x)+\zeta_2(x)$ and, when $|l|\geqq 1$, the order of $p_1^{(l)}(x, \xi)\, p_{2(l)}(x, \xi)$ is $\zeta_1(x)+\zeta_2(x)-|l|(\rho-\delta)$, we have

$$p \in S^{\zeta_1+\zeta_2}_{\rho,\delta}\,.$$

Therefore the assertion (1) holds. In the same way as the above, we can verify the assertion (2).

DEFINITION 1.3. *We say that a sequence* $\{p_k\}_{k\geq 1}$ *of* $S^{\zeta}_{\rho,\delta}$ *converges weakly to* $p \in S^{\zeta}_{\rho,\delta}$ *as* $k\to\infty$ *if, for each* $h\geq 1$, *there is a constant* M_h *such that* $|p|^{\zeta}_h < M_h$, *and, for any multi-indices m and n, we have*

$$p^{(n)}_{k(m)} \to p^{(n)}_{(m)} \quad \text{as } k\to\infty \text{ on } \boldsymbol{R}^d\times\boldsymbol{R}^d\,. \tag{1.19}$$

DEFINITION 1.4. *Let* $\boldsymbol{I}$ *be an interval of* $\boldsymbol{R}^1$ *and* $\boldsymbol{V}$ *be a Fréchet space. For a mapping* ϕ: $\boldsymbol{I}\to\phi(t)\in \boldsymbol{V}$, *we write* $\phi\in\mathcal{B}^{|m|}(\boldsymbol{I}, \boldsymbol{V})$ *if* ϕ *is* $|m|$*-times continuously differentiable in* $\boldsymbol{I}$ *in the topology of* $\boldsymbol{V}$ *and each derivative* $D^l_t\phi$ *is bounded* $(|l|\leq|m|)$.

From Theorem 1.1, we see that L_Φ is a pseudo-differential operator of the class $S^{\alpha}_{1,\delta}$, where δ is any positive number less than 1. Now we will construct a fundamental solution in the sense of pseudo-differential operators to the initial-value peoblem for the evolution equation with respect to L_Φ:

$$\begin{aligned} \{\partial_t - L_\Phi\}\, u &= f \quad \text{in} \quad (0, T)\,, \\ \lim_{t\to 0} u(t) &= \phi \quad \text{in} \quad L_2(\boldsymbol{R}^d)\,. \end{aligned} \tag{1.20}$$

By virtue of Theorems 1.1 and 1.2, we can adapt the argument used in the proof of Theorem 2.1 in Section 2 of Chap. 8 in [8] to the proof of the next theorem.

Theerem 1.3. *There exists a fundamental solution $E(\cdot)$ to the initial-value problem for the evolution equation* (1.20) *such that it satisfies the following conditions: for each $T>0$,*

(1)

(1.21) $$E(t)=e(t,x,D_x)\in\mathscr{B}^0((0,T];S^0_{1,\delta})\cap\mathscr{B}^1((0,T];S^{\alpha}_{1,\delta})$$

and, for any $t_0\in(0,T)$,

(1.22) $$E(t)\in\mathscr{B}^1([t_0,T];S^{-\infty})\equiv\bigcap_{-\infty<\kappa<\infty}\mathscr{B}^1([t_0,T];S^{\kappa}_{1,\delta});$$

(2) *for any $t\in(0,T)$,*

(1.23) $$(\partial_t-L_\Phi)\,E(t)=0;$$

(3)

(1.24) $$e(t,x,\xi)\to 1 \text{ in } S^0_{1,\delta} \text{ weakly as } t\to 0\,;$$

(4)

(1.25) $$r_0(t,x,\xi)\equiv e(t,x,\xi)-\exp(tp_\Phi(x,\xi))\to 0$$
$$\text{in } S^{-(1-\delta)}_{1,\delta} \text{ weakly as } t\to 0$$

and

(1.26) $$r_0(t,x,\xi)/t\in\mathscr{B}^0((0,T];S^{\alpha-(1-\delta)}_{1,\delta})\,.$$

Proof. Let $e_0(t,x,\xi)=\exp(tp_\Phi(x,\xi))$. Then this function satisfies the equation:

(1.27) $$\{\partial_t-p_\Phi(x,\xi)\}\,e_0(t,x,\xi)=0$$
$$e_0(0,x,\xi)=1\,.$$

Furthermore, for any multi-indices m and n,

(1.28) $$\partial^n_\xi D^m_x\,e_0(t,x,\xi)=\sum_{k=1}^{|m+n|}t^k((p_\Phi)_k)^{(n)}_{(m)}(x,\xi)\,e_0(t,x,\xi)\,,$$

where

$$((p_\Phi)_k)^{(n)}_{(m)}=\sum C^{n^1,n^2,\cdots,n^k}_{m^1,m^2,\cdots,m^k}\,p_{\Phi}{}^{(n^1)}_{(m^1)}(x,\xi)\,p_{\Phi}{}^{(n^2)}_{(m^2)}(x,\xi)\cdots p_{\Phi}{}^{(n^k)}_{(m^k)}(x,\xi)$$

and the summation is taken over multi-indices m^j and n^j $(j=1,2,\cdots,k)$ such that $\sum_{j=1}^k m^j=m$, $\sum_{j=1}^k n^j=n$ and $C^{n^1,n^2,\cdots,n^k}_{m^1,m^1,\cdots,m^k}$ denotes a constant depending only on m^j and n^j $(j=1,2,\cdots,k)$. From (1.3), there exists a constant $C_1>0$ such that for any $(x,\xi)\in\boldsymbol{R}^d\times\boldsymbol{R}^d$

$$|p_\Phi(x,\xi)|>C_0\langle\xi\rangle^{\alpha(x)}-C_1$$

Therefore, putting $C=\exp(-TC_1)$, we have, for any $(t, x, \xi)\in(0, T]\times \boldsymbol{R}^d\times \boldsymbol{R}^d$,

$$e_0(t, x, \xi)\leq C\exp(-tC_0\langle\xi\rangle^{\alpha(x)})\,. \tag{1.29}$$

Since $(t\langle\xi\rangle^{\alpha(x)})^k\exp(-tC_0\langle\xi\rangle^{\alpha(x)})$ is bounded in (t, x, ξ) of $(0, T]\times \boldsymbol{R}^d\times \boldsymbol{R}^d$, there exists a constant $C'_{m,n}$ such that

$$|\partial_\xi^n D_x^m e_0(t, x, \xi)|\leq C'_{m,n}\langle\xi\rangle^{-|n|+\delta|m|} \tag{1.30}$$

for any $(t, x, \xi)\in(0, T]\times \boldsymbol{R}^d\times \boldsymbol{R}^d$. Hence

$$\begin{aligned}(1.31)\qquad &|\partial_\xi^n D_x^m \partial_t e_0(t, x, \xi)|\\ &\leq\sum_{k=0}^{|m+n|} C_{0,m,n,k}\, t^k\langle\xi\rangle^{(k+1)\alpha(x)-|n|+\delta|m|}\exp(-tC_0\langle\xi\rangle^{\alpha(x)})\end{aligned}$$

for any $(t, x, \xi)\in(0, T]\times \boldsymbol{R}^d\times \boldsymbol{R}^d$, where $C_{0,m,n,k}$ is a constant depending only on m, n, and k. These estimates (1.30) and (1.31) yield that

$$e_0\in\mathscr{B}^0((0, T]; S^0_{1,\delta})\cap\mathscr{B}^1((0, T]; S^\alpha_{1,\delta})\,,$$

and it is clear that $e_0\to 0$ weakly as $t\to 0$.

We can define $\{e_j(t)\}_{j=1}^\infty$ and $\{q_j(t)\}_{j=1}^\infty$ $(0\leq t\leq T)$ inductively by

$$q_j(t)=\sum_{k=0}^{j-1}\sum_{|\alpha|+k=j}\frac{1}{n!}p_\Phi^{(n)}(x, \xi)\, e_{k(n)}(t, x, \xi)\quad (j\geq 1) \tag{1.32}$$

and

$$\begin{aligned}(1.33)\qquad \{\partial_t-p_\Phi(x, \xi)\}\, e_j(t, x, \xi) &= q_j(t, x, \xi)\\ e_j(0, x, \xi) &= 0\quad (j\geq 1)\,.\end{aligned}$$

Then the solution $e_j(t, x, \xi)$ of (1.33) has the form:

$$e_j(t, x, \xi)=e_0(t, x, \xi)\int_0^t\frac{q_j(s, x, \xi)}{e_0(s, x, \xi)}ds\,. \tag{1.34}$$

We will show the following estimate:

$$|e_{j(m)}^{(n)}(t, x, \xi)|\leq\begin{cases} C_{j,m,n}\langle\xi\rangle^{-j(1-\delta)-|n|+\delta|m|}\\ C'_{j,m,n}\, t\langle\xi\rangle^{\alpha(x)-j(1-\delta)-|n|+\delta|m|}\end{cases} \tag{1.35}$$

for any $(t, x, \xi)\in(0, T]\times \boldsymbol{R}^d\times \boldsymbol{R}^d (j\geq 1)$, where $C_{j,m,n}$ and $C'_{j,m,n}$ are constants depending only on j, m and n. In fact, assume that the inequality

$$\begin{aligned}(1.36)\qquad &\left|\left(\frac{q_j(t, x, \xi)}{e_0(t, x, \xi)}\right)_{(m)}^{(n)}\right|\\ &\leq\tilde{C}_{j,m,n}\langle\xi\rangle^{\alpha(x)}\sum_{k=1}^{2j-1}(t\langle\xi\rangle^{\alpha(x)})^k\langle\xi\rangle^{-j(1-\delta)-|n|+\delta|m|}\\ &((t, x, \xi)\in(0, T]\times \boldsymbol{R}^d\times \boldsymbol{R}^d)\end{aligned}$$

holds for $j \leq j_0 - 1$. Then, combining (1.34) with (1.36), we have

$$(1.37) \qquad \left| \left(\frac{e_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)} \right)^{(n)}_{(m)} \right| \leq C_{j_0-1,m,n} \sum_{k=2}^{2(j_0-1)} (t\langle\xi\rangle^{\alpha(x)})^k \langle\xi\rangle^{-(j_0-1)(1-\delta)-|n|+\delta|m|}$$

for any $(t, x, \xi) \in (0, T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. Note that

$$(1.38) \qquad \begin{aligned} &\left| \left(\frac{q_{j_0}(s, x, \xi)}{e_0(s, x, \xi)} \right)^{(n)}_{(m)} \right| \\ &\leq \sum_{|l|=1} \left| \left(\frac{p_\Phi^{(l)}(x, \xi)\, e_{j_0-1(l)}(t, x, \xi)}{e_0(t, x, \xi)} \right)^{(n)}_{(m)} \right| \\ &+ \tilde{C}_{j_0,m,n} \sum_{|l|=1} \left| \left(\frac{(q_{j_0-1}(t, x, \xi))^{(l)}_{(l)}}{e_0(t, x, \xi)} \right)^{(n)}_{(m)} \right| \\ &\leq \sum_{|l|=1} \left| \left(p_\Phi^{(l)}(x, \xi) \left(\frac{e_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)} \right)_{(l)} \right)^{(n)}_{(m)} \right| \\ &+ \sum_{|l|=1} \left| \left(t p_\Phi^{(l)}(x, \xi)\, p_{\Phi(l)}(x, \xi) \frac{e_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)} \right)^{(n)}_{(m)} \right| \\ &+ \tilde{C}_{j_0,m,n} \sum_{|l|=1} \left| \left(t p(x, \xi)_{\Phi(l)} \frac{q_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)} \right)^{(n+1)}_{(m)} \right| \\ &+ \tilde{C}_{j_0,m,n} \left| \left(\frac{q_{j_0-1}(t, x, \xi)}{e_0(t, x, \xi)} \right)^{(n+1)}_{(m+1)} \right| \end{aligned}$$

for any $(t, x, \xi) \in (0, T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. Then, from (1.34), we see that the inequality (1.36) holds for $j = j_0$. Thus, by induction, it holds for any $j \geq 0$. Hence, from (1.29), (1.34) and (1.38) for $j = j_0$, we see that the first inequality of (1.35) holds when $j = j_0$. Moreover, writing $(t\langle\xi\rangle^{\alpha(x)})^k = (t\langle\xi\rangle^{\alpha(x)})\,(t\langle\xi\rangle^{\alpha(x)})^{k-1}$ and using a similar argument to the above, we obtain the second inequality of (1.35). This means that

$$(1.39) \qquad e_j(t, x, \xi) \in \mathscr{B}^0([0, T]; S_{1,\delta}^{-j(1-\delta)}) \cap \mathscr{B}^1([0, T]; S_{1,\delta}^{\alpha-j(1-\delta)}).$$

Next, put $E_j(t) = e_j(t, x, D_x)$ $(j \geq 0)$. Then, by Theorem 1.2, we can write

$$(1.40) \qquad \begin{aligned} &\sigma(L_\Phi E_j(t))(x, \xi) \\ &= p_\Phi(x, \xi)\, e_j(t, x, \xi) + \sum_{0<|l|<N-j} \frac{1}{l!} p_\Phi^{(l)}(x, \xi)\, e_{j(l)}(t, x, \xi) \\ &+ r_{N,j}(t, x, \xi) \quad (j = 0, 1, 2, \cdots N-1). \end{aligned}$$

From Theorem 1.1 and 1.2, the first inequality of (1.35) and (1.40), we find that

$$(1.41) \qquad r_{N,j}(t) \in \mathscr{B}^0((0, T]; S_{1,\delta}^{\alpha-N(1-\delta)}) \quad j = 1, 2, \cdots.$$

Similarly, replacing the first inequality of (1.35) by the second one of (1.35), we have

$$r_{N,j}(t)/t \in \mathscr{B}^0((0, T]; S^{2\alpha-N(1-\delta)}_{1,\delta}) \quad j = 1, 2, \cdots . \tag{1.42}$$

From the above discussion, we have a sequence $\{e_j\}_{j=0}^{\infty}$ of symbols satisfying $e_j \in S^{j(1-\delta)}_{1,\delta}$. Therefore, we can construct an operator

$$\tilde{E}(t) = \tilde{e}(t, x, D_x) \in S^0_{1,\delta} \tag{1.43}$$

with an analogous argument used in Theorem A.1 of [8] (p.238–239). Indeed, let ψ be a function of $C_0^\infty((0, \infty))$ with

$$0 \leq \psi(t) \leq 1, \quad \psi(t) = 0 \ (0 < t \leq 1) \quad \text{and} \quad \psi(t) = 1 \quad (t \geq 2).$$

Putting $\psi_j(\xi) = \psi(\varepsilon_j |\xi|)$ $(j = 1, 2, \cdots)$ for any sequence $\{\varepsilon_j\}_{j\geq 1}$ of positive numbers, we have the estimate

$$|\partial_\xi^n D_x^m (e_j(t, x, \xi)\, \psi_j(\xi))| \leq \begin{cases} C_{j,m,n} \langle\xi\rangle^{-j(1-\delta)+\delta|m|-|n|} \\ C_{j,m,n}\, \varepsilon_j \langle\xi\rangle^{-j(1-\delta)+\delta|m|-|n|+1} \end{cases}$$

for any $(t, x, \xi) \in (0, T] \times R^d \times R^d$ and any multi-indices m and n. Now, we inductively choose the sequence $\{\varepsilon_j\}_{j\geq 1}$ satisfying

$$0 < \varepsilon_j \leq 2^{-j} (\max_{|m+n| \leq j} (C_{j,m,n}))^{-1}$$

and

$$1 > \varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_n > \cdots \to 0,$$

and define the symbol $\tilde{e}$ by

$$\tilde{e}(t, x, \xi) = e_0(t, x, \xi) + \sum_{j=1}^{\infty} e_j(t, x, \xi)\, \psi_j(\xi)$$

for any $(t, x, \xi) \in (0, T] \times R^d \times R^d$. Then the symbol $\tilde{e}$ satisfies the following properties:

(i)

$$\tilde{e}(t, x, \xi) - \sum_{j=0}^{N-1} e_j(t, x, \xi) \in \mathscr{B}^0((0, T]; S^{-N(1-\delta)}_{1,\delta}) \cap \mathscr{B}^1((0, T]; S^{\alpha-N(1-\delta)}_{1,\delta}), \tag{1.44}$$

(ii)

$$\tilde{e}(t) \to 1 \text{ and } \tilde{e}(t) - \sum_{j=0}^{N-1} e_j(t) \to 0 \text{ weakly in } S^0_{1,\delta} \tag{1.45}$$

as $t \to 0$ for any $N \geq 1$ (see [8] in detail). Let $R(t) = (\partial_t - L_\Phi) \tilde{E}(t)$. For any positive integer N, we rewrite $R(t)$ in the form

$$R(t) = (\partial_t - L_\Phi) (\sum_{j=0}^{N-1} E_j(t)) + (\partial_t - L_\Phi) (\tilde{E}(t) - \sum_{j=0}^{N-1} E_j(t)). \tag{1.46}$$

Then from Theorem 1.2 and (1.44), we see that, for any positive integer N,

$$(\partial_t - L_\Phi)(\widetilde{E}(t) - \sum_{j=0}^{N-1} E_j(t)) \in \mathscr{B}^0((0, T]; S_{1,\delta}^{\alpha - N(1-\delta)}). \tag{1.47}$$

Moreover, it follows from (1.32), (1.33) and (1.40) that

$$\begin{aligned} \sigma((\partial_t - L_\Phi)(\sum_{j=0}^{N-1} E_j(t)))(x, \xi) &= \sum_{j=0}^{N-1} (\partial_t - p_\Phi(x, \xi))\, e_j(t, x, \xi) \\ &\quad - \sum_{j=1}^{N-1} \sum_{|l|+k=j, k<j} \frac{1}{l!} p_\Phi^{(l)}(x, \xi)\, e_{k(l)}(t, x, \xi) - \sum_{i=0}^{N-1} r_{N,j}(t, x, \xi) \\ &= -\sum_{j=0}^{N-1} r_{N,j}(t, x, \xi) \end{aligned} \tag{1.48}$$

for any positive integer N and $(t, x, \xi) \in (0, T] \times \boldsymbol{R}^d \times \boldsymbol{R}^d$. Therefore, (1.41) and (1.42) yield that

$$\begin{aligned} (\partial_t - L_\Phi)(\sum_{j=0}^{N-1} E_j(t)) &\in \mathscr{B}^0((0, T]; S_{1,\delta}^{\alpha - N(1-\delta)}) \\ &\cap \mathscr{B}^1((0, T]; S_{1,\delta}^{2\alpha - N(1-\delta)}). \end{aligned} \tag{1.49}$$

Hence, it follows from (1.47) and (1.49) that

$$R(t) \in \mathscr{B}^0((0, T]; S^{-\infty}). \tag{1.50}$$

Now, let $\{W_\nu(t)\}_{\nu \geq 1}$ be a sequence of operators defined by

$$W_1(t) = -R(t)$$

and

$$W_\nu(t) = \int_0^t W_1(t-s)\, W_{\nu-1}(s)\, ds.$$

Then, using the same method as in the proof of Theorem 2.1 in Chap. 8 of [8], we see that

$$\sigma(W(t))(x, \xi) = \sum_{\nu=1}^{\infty} \sigma(W_\nu(t))(x, \xi)$$

converges in the topology of $\mathscr{B}^0((0, T]; S^{-\infty})$. If we set

$$E(t) = \widetilde{E}(t) + \int_0^t \widetilde{E}(t-s)\, W(s)\, ds, \tag{1.51}$$

then we have

$$(\partial_t - L_\Phi)\, E(t) = R(t) + W(t) + \int_0^t R(t-s)\, W(s)\, ds = 0$$

for any $t \in (0, T]$. We get (1.21) from (1.44) and (1.50). The relations (1.24) and (1.25) follow from (1.45) and (1.51). Moreover, with the same argument as in Theorem 2.1 in Chap. 8 of [8], we see that, for any positive number $t_0 \in (0, T]$,

$$e_j(t) \in \mathscr{B}^1([t_0, T]; \boldsymbol{S}^{-\infty}) \quad j=1, 2, \cdots .$$

The proof of Theorem 1.3 is complete.

Let $\boldsymbol{H}_s(-\infty<s<\infty)$ be the Sobolev space with the norm $\|\cdot\|_s$ (see [7] p.116 for the definition). Then, using the $\boldsymbol{L}_2$-boundedness theorem (cf. [7], Chap. 2, Theorem 4.1), we have

Theorem 1.4. *Let ζ be a bounded function on $\boldsymbol{R}^d$ and $P=p(x, D_x) \in \boldsymbol{S}^{\zeta}_{\rho,\delta}(\delta<\rho)$. Then, for any $s \in \boldsymbol{R}$, P defines a continuous mapping $P: \boldsymbol{H}_{s+\bar{\zeta}} \rightarrow \boldsymbol{H}_s$ and there exist an integer k and a constant C such that*

$$\|Pu\|_s \leq C\,|p|^{\zeta}_k\,\|u\|_{s+\bar{\zeta}} \quad for \quad u \in \boldsymbol{H}_{s+\bar{\zeta}}\,. \tag{1.52}$$

It is well-known that if κ and s are real numbers and $p_j \rightarrow p$ in $\boldsymbol{S}^{\kappa}_{\rho,\delta}$ weakly as $j \rightarrow \infty$, then

$$p_j(X, D_x)\,u \rightarrow p(X, D_x)\,u \text{ in } \boldsymbol{H}_s \text{ as } j \rightarrow \infty \quad \text{for} \quad u \in \boldsymbol{H}_{s+\kappa} \tag{1.53}$$

(cf. [7] p.157). Immediately, from Theorem 1.3, Theorem 1.4, and (1.53), we get the following theorem.

Theorem 1.5. *Let $E(\cdot)$ be the same one as in Theorem* 1.3 *and let s be any real number. Then, for $\phi \in \boldsymbol{H}_s$, $u(\cdot)=E(\cdot)\phi$ belongs to $\mathscr{B}^0([0, T]; \boldsymbol{H}_s) \cap \mathscr{B}^1((0, T]; \boldsymbol{H}_{s-\bar{\alpha}})$ for each $T>0$ and is a solution to the initial-value problem for the evolution equation* (1.20).

Now, we state the main theorems in this paper.

Theorem 1.6. *Let $e(t, x, \xi)$ be the symbol of the fundamental solution $E(t)$ given by Theorem* 1.3. *Then, the function defied by*

$$K(t, x, y) = \int \exp(i(x-y)\cdot\xi)\, e(t, x, \xi)\, \check{d}\xi \tag{1.54}$$

($t \in (0, \infty)$, $x, y \in \boldsymbol{R}^d$) is a transition density of the Markov process X_Φ.

Proof. Let $\phi \in C_0^\infty(\boldsymbol{R}^d)$ and $u(t, x)=E(t)\,\phi(x)$. Then $u(t)$, $\partial_t u(t)$ and $L_\Phi\phi$ belong to $\mathcal{S}$. From Theorem 1.3, Theorem 1.5 and (1.53), we see that, for any $s \in \boldsymbol{R}$,

$$\lim_{t\to 0} u(t) = \phi \quad \text{in} \quad \boldsymbol{H}_s\,,$$

$$\partial_t u(t)\,|_{t=0} = \lim_{t\to 0} \partial_t u(t) = \lim_{t\to 0} L_\Phi u(t) = L_\Phi \phi \quad \text{in} \quad \boldsymbol{H}_{s-\bar{\alpha}}\,.$$

Noting that for any multi-index m and any real number $s > |m| + d/2$

$$|\partial_x^m u(t,x) - \partial_x^m \phi(x)| \\ \leq \left| \int \langle \xi \rangle^{-2(s-|m|)} \check{d}\xi \right|^{1/2} \|u(t)-\phi\|_s ,$$

we have $\partial_x^m u(t) \to \partial_x^m \phi$ uniformly on $\boldsymbol{R}^d$ as $t \to 0$. Similarly, we have $\partial_t u(t) \to L_\Phi \phi$ uniformly on $\boldsymbol{R}^d$ as $t \to 0$. These facts imply that $u \in \boldsymbol{C}_b^{1,2}([0,T] \times \boldsymbol{R}^d)$. Put $f(s,x) = u(t-s,x)$ $(0 \leq s \leq t)$. Then, $f \in \boldsymbol{C}_b^{1,2}([0,t] \times \boldsymbol{R}^d)$ and f satisfies

$$\begin{cases} \partial_s f(s,x) = -L_\Phi f(s,x) & (0 \leq s < t) \\ f(t,x) = \phi(x) . \end{cases} \tag{1.55}$$

Let $\boldsymbol{P}_x$ be a solution to the martingale problem for L_Φ starting at x. Then

$$f(t,X_t) - f(0,x) = \int_0^t \{\partial_s f(s,X_s) \\ + L_\Phi f(s,X_s)\}\, ds + \text{a } \boldsymbol{P}_x\text{-martingale.} \tag{1.56}$$

Using (1.55) and (1.56), we have

$$u(t,x) = \boldsymbol{E}_x[\phi(X_t)] . \tag{1.57}$$

On the other hand, from Theorems 1.3 and 3.3 in Chap. 2 of [7], it follows that

$$u(t,x) = \int_{\boldsymbol{R}^d} K(t,x,y)\, \phi(y)\, dy \quad \text{for} \quad t>0 \text{ and } x \in \boldsymbol{R}^d . \tag{1.58}$$

Since (1.57) and (1.58) hold for any $\phi \in \boldsymbol{C}_0^\infty(\boldsymbol{R}^d)$, we see that the function $K(t,x,y)$ $(t>0,\ x,y \in \boldsymbol{R}^d)$ is a transition density of the Markov process X_Φ.

Theorem 1.7. *Let $\{\boldsymbol{P}(t,x,\Gamma);\ t \geq 0,\ x \in \boldsymbol{R}^d,\ \Gamma \in \mathcal{B}(\boldsymbol{R}^d)\}$ be the transition function of the stable-like process with exponent $\alpha(x)$. Then, for each $(t,x) \in (0,\infty) \times \boldsymbol{R}^d$, $\boldsymbol{P}(t,x,dy)$ has a density with respect to Legesgue measure.*

Proof. We first show that the short time behavior of the process X coincides with that of the process X_Φ. Using polar decomposition, we rewrite ν and ν_Φ in the following forms:

$$\nu(x;dy) = 1_{(0,r_0]}(r) \frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\, dr\, s(d\theta) + 1_{(r_0,\infty)}(r) \frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\, dr\, s(d\theta)$$

and

$$\nu_\Phi(x,dy) = 1_{(0,r_0]}(r) \frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\, dr\, s(d\theta) + 1_{(r_0,\infty)}(r) \frac{w_{\alpha(x)}\, \Phi(r)}{r^{1+\alpha(x)}}\, dr\, s(d\theta) ,$$

where r_0 is the same constant as in the definition of the cut-off function Φ. We set

$$G_1(x;\lambda)=\int_\lambda^{r_0}\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\quad(\lambda>0)\,,$$

$$G_2(x;\lambda)=\int_\lambda^\infty g(x)^{-1}\frac{w_{\alpha(x)}}{r^{1+\alpha(x)}}\,dr\quad(\lambda>r_0)$$

and

$$G_{\Phi,2}(x;\lambda)=\int_\lambda^\infty g_\Phi(x)^{-1}\frac{w_{\alpha(x)}\,\Phi(r)}{r^{1+\alpha(x)}}\,dr\quad(\lambda>r_0)\,,$$

where $g(x)=\int_{r_0}^\infty w_{\alpha(x)}/r^{1+\alpha(x)}\,dr$ and $g_\Phi(x)=\int_{r_0}^\infty w_{\alpha(x)}\,\Phi(r)/r^{1+\alpha(x)}\,dr$. In the following, $\hat{G}(x,\cdot)$ denotes the right continuous inverse function of $G(x,\cdot)$, that is,

$$\hat{G}(x,l)=\inf\{\lambda>0\colon G(x,\lambda)\le l\}\,.$$

Let

$$\boldsymbol{U}_1=(0,\infty)\times\boldsymbol{S}^{d-1},\quad \boldsymbol{U}_2=(-1,0)\times\boldsymbol{S}^{d-1}\quad\text{and}\quad \boldsymbol{U}=\boldsymbol{U}_1\cup\boldsymbol{U}_2\,.$$

We denote a generic element of $\boldsymbol{U}$ as $u=(l,\theta)$. Now, let $\{p(t)\}$ be a stationary Poisson point process defined on a probability space $(\Omega,\mathcal{F},\boldsymbol{P})$ with values in $\boldsymbol{U}$ and the characteristic measure $n(du)=dl\,s(d\theta)$. $N_p(ds\times du)$ denotes the counting measure defined by $\{p(t)\}$ and $\tilde{N}_p(ds\times du)=N_p(ds\times du)-dsn(du)$. If we set $a(x,u)=a(x,l)=\hat{G}_1(x,l)$, $b(x,u)=b(x,l)=g(x)\,\hat{G}_2(x,l+1)$ and $b_\Phi(x,u)=b_\Phi(x,l)=g_\Phi(x)\,\hat{G}_{\Phi,2}(x,l+1)$, then the processes X and X_Φ starting at x are respectively realized as solutions of the stochastic differential equations with jumps:

$$\begin{aligned}X(t)&=x+\int_0^t\int_{U_1}a(X(s_-),u)\,\tilde{N}_p(ds\times du)\\&\quad+\int_0^t\int_{U_2}b(X(s_-),u)\,N_p(ds\times du)\,,\\X_\Phi(t)&=x+\int_0^t\int_{U_1}a(X_\Phi(s_-),u)\,\tilde{N}_p(ds\times du)\\&\quad+\int_0^t\int_{U_2}b_\Phi(X_\Phi(s_-),u)\,N_p(ds\times du)\,.\end{aligned}$$

Since the coefficient $a(x,u)$ satisfies the Lipschitz condition with respect to the measure $n(du)$ (see [12]), they have unique solutions in the pathwise sense. For specifying the starting point u of the processes, we denote them by $X(t,x)$ and $X_\Phi(t,x)$, respectively. Let $\sigma=\inf\{t>0\colon N_p((0,t]\times\boldsymbol{U}_2)=1\}$. Then for $t<\sigma$

$$X(t)=x+\int_0^t\int_{U_1}a(X_\Phi(s_-),u)\,\tilde{N}_p(ds\times du)$$

and

$$X_\Phi(t)=x+\int_0^t\int_{U_1}a(X_\Phi(s^-),u)\,\tilde{N}_p(ds\times du)\,,$$

because, for $A_1\subset\boldsymbol{U}_1$ and $A_2\subset\boldsymbol{U}_2$, the Poisson processes $N_p((0,t]\times A_1)$ and

$N_p((0, t] \times A_2)$ almost surely do not jump simultaneously. Therefore

$$P(1_{\{t<\sigma\}} X(t, x) = 1_{\{t<\sigma\}} X_\Phi(t, x), t \geq 0) = 1 .$$

We next show the absolute continuity of the transition probability of X. Let $\sigma_0 = 0$ and

$$\sigma_n = \inf \{t > \sigma_{n-1}; N_p(\{t\} \times U_2) = 1\} \quad (n = 1, 2, \cdots) .$$

Then $\sigma_1 = \sigma$ and $P(\sigma_n = t) = 0$ for each $t > 0$. Therefore, for each $t > 0$, $x \in R^d$ and Borel set Γ of R^d,

$$\begin{aligned} P(t, x, \Gamma) &= P(X(t, x) \in \Gamma) \\ &= \sum_{n=0}^{\infty} P(X(t, x) \in \Gamma; \sigma_n \leq t < \sigma_{n+1}) \\ &= \sum_{n=0}^{\infty} P(X(t, x) \in \Gamma; \sigma_n < t < \sigma_{n+1}) \\ &= \sum_{n=0}^{\infty} E[1_{\{\sigma_n < t\}} P(X(t-s, y) \in \Gamma; t-s < \sigma)|_{s=\sigma_n, y=X(\sigma_n, x)}] \\ &= \sum_{n=0}^{\infty} E[1_{\{\sigma_n < t\}} P(X_\Phi(t-s, y) \in \Gamma; t-s < \sigma)|_{s=\sigma_n, y=X(\sigma_n, x)}] . \end{aligned}$$

Hence, if the Lebesgue measure of Γ is equal to zero,

$$P(t, x, \Gamma) = 0$$

for any $t > 0$ and $x \in R^d$; consequently we have the conclusion.

2. The Behavior of Sample Paths near $t=0$

In this section, we investigate the behavior of sample paths of the stable-like process $X = (X(t), P_x)$ with exponent $\alpha(x)$. At first, we state the main result in this section.

Theorem 2.1. *Let x be an arbitrarily fixed point.*
(1) *If $\alpha(x) < \beta$, then*

$$P_x(\lim_{t \to 0} |X(t) - x| / t^{1/\beta} = 0) = 1 . \tag{2.1}$$

(2) *If $\alpha(x) > \beta > 0$, then*

$$P_x(\limsup_{t \to 0} |X(t) - x| / t^{1/\beta} = \infty) = 1 . \tag{2.2}$$

We provide two lemmas for the proof of this theorem. The first lemma is a modification of Khintchine's result [6]. It is obtained only for processes with stationary independent increments. However a stable-like process is not such a process in general. Accordingly we modify Khintchine's result in the following form, where, for simplicity, we restrict the consideration to conservative processes.

Lemma 2.1. *Let $Y=(Y(t), \boldsymbol{P}_x)$ be a standard process on $\boldsymbol{R}^d$ and let h be a non-decreasing positive funciton on $(0, \lambda)$ with $\lim_{t\downarrow 0} h(t)=0$, where λ is a positive number. $U_r(x)$ is the open ball with center x and radius r. $\boldsymbol{P}_c^{U_r(x)}(\cdot)$ $(c>0)$ is the function defined on $(0, \lambda)$ by*

$$\boldsymbol{P}_c^{U_r(x)}(t) = \sup_{y\in U_r(x)} \boldsymbol{P}_y(|Y(t)-y|>ch(t)) . \tag{2.3}$$

Let x_0 be a point of $\boldsymbol{R}^d$. If there exist positive numbers c_0 and r such that

$$\int_0^\lambda \boldsymbol{P}_c^{U_r(x_0)}(t)/t\, dt<\infty \tag{2.4}$$

for any $c\in(0, c_0)$, then

$$\boldsymbol{P}_{x_0}(\lim_{t\to 0} |Y(t)-x_0|/h(t) = 0) = 1 . \tag{2.5}$$

Proof. Let U_j be the open ball with center x_0 and radius $jr/3$ $(j=1, 2, 3)$. It is clear that, for any positive number a and $t_1\in[0, t]$,

$$\begin{aligned}&\boldsymbol{P}_x(|Y(t)-x|>a)\\ &\quad\le \boldsymbol{P}_x(|Y(t_1)-x|>a/2)+\boldsymbol{P}_x(|Y(t)-Y(t_1)|>a/2,\ |Y(t_1)-x|\le a/2) .\end{aligned}$$

By the Markov property of Y, we get

$$\begin{aligned}&\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a)\\ &\quad\le \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_1)-x|>a/2)\\ &\quad+ \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t-t_1)-x|>a/2)\end{aligned} \tag{2.6}$$

for any $a\in(0, r/3)$, $t_1\in[0, t]$ and $j=1, 2$. In particular,

$$\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a)\le 2 \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t/2)-x|>a/2) \tag{2.7}$$

for any $a\in(0, r/3)$ and $j=1, 2$. In the same way as the above, we have, for any $a>0$ and $t_1, t_2, t_3\in[0, t]$ $(t_1<t_2<t_3)$,

$$\begin{aligned}\boldsymbol{P}_x(|Y(t)-x|>a)&\le \boldsymbol{P}_x(|Y(t_1)-x|>a/4)\\ &+\boldsymbol{P}_x(|Y(t_2)-Y(t_1)|>a/4,\ |Y(t_1)-x|\le a/4)\\ &+\boldsymbol{P}_x(|Y(t_3)-Y(t_2)|>a/4,\ |Y(t_2)-x|\le a/2)\\ &+\boldsymbol{P}_x(|Y(t)-Y(t_3)|>a/4,\ |Y(t_3)-x|\le 3a/4) .\end{aligned}$$

Furthermore, using the Markov property again, we obtain

$$\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a) \tag{2.8}$$

$$\leq \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_1)-x|>a/4)$$
$$+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_2-t_1)-x|>a/4)$$
$$+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t_3-t_2)-x|>a/4)$$
$$+\sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t-t_3)-x|>a/4)$$

for any $a\in(0, r/3)$, $t_1, t_2, t_3\in[0, t]$ $(t_1<t_2<t_3)$ and $j=1, 2$, and particularly

(2.9) $$\sup_{x\in U_j} \boldsymbol{P}_x(|Y(t)-x|>a)\leq 4 \sup_{x\in U_{j+1}} \boldsymbol{P}_x(|Y(t/4)-x|>a/4)$$

for any $a\in(0, r/3)$, and $j=1, 2$. Next, we will show that, for any positive number c less than c_0,

(2.10) $$\sup_{x\in U_1} \boldsymbol{P}_x(|Y(t)-x|>ch(t/4))\to 0 \quad \text{as } t\to 0\,.$$

In fact, let $ch(t)/4<r/3$ and $t\in(0, \lambda)$. Then, it follows from (2.6) that

(2.11) $$\boldsymbol{P}^{U_2}_{c/4}(t)=\sup_{x\in U_2} \boldsymbol{P}_x(|Y(t)-x|>\frac{c}{4}h(t))$$
$$\leq \sup_{x\in U_3} \boldsymbol{P}_x(|Y(t_1)-x|>\frac{c}{8}h(t))$$
$$+\sup_{x\in U_2} \boldsymbol{P}_x(|Y(t-t_1)-x|>\frac{c}{8}h(t))$$
$$\leq \sup_{x\in U_3} \boldsymbol{P}_x(|Y(t_1)-x|>\frac{c}{8}h(t_1))$$
$$+\sup_{x\in U_3} \boldsymbol{P}_x(|Y(t-t_1)-x|>\frac{c}{8}h(t-t_1))$$

for any $t_1\in[0, t]$. Hence, if $t\in(0, \lambda)$ and $ch(t)/4<r/3$,

(2.12) $$\boldsymbol{P}^{c/4}_{U_2}(t)\leq \boldsymbol{P}^{U_3}_{c/8}(t_1)+\boldsymbol{P}^{U_3}_{c/8}(t-t_1) \quad \forall t_1\in[0, t]\,.$$

Moreover, if $t\in(0, \lambda)$ and $ch(t)/4<r/3$, then

(2.13) $$\boldsymbol{P}^{U_2}_{c/4}(t)$$
$$=\frac{1}{\log 2}\int_{t/2}^{t} \boldsymbol{P}^{U_2}_{c/4}(t)\frac{ds}{s}\leq \frac{1}{\log 2}\int_{t/2}^{t}\{\boldsymbol{P}^{U_3}_{c/8}(s)+\boldsymbol{P}^{U_3}_{c/8}(t-s)\}\frac{ds}{s}$$
$$\leq \frac{1}{\log 2}\int_{t/2}^{t} \boldsymbol{P}^{U_3}_{c/8}(s)\frac{ds}{s}+\frac{1}{\log 2}\int_{t/2}^{t} \boldsymbol{P}^{U_3}_{c/8}(t-s)\frac{ds}{t-s}$$
$$\leq \frac{1}{\log 2}\int_{0}^{t} \boldsymbol{P}^{U_3}_{c/8}(s)\frac{ds}{s}\,.$$

Thus

$$\sup_{x\in U_1} \boldsymbol{P}_x(|Y(t)-x|>ch(t/4))\leq \frac{4}{\log 2}\int_0^t \boldsymbol{P}^{U_3}_{c/8}(s)\frac{ds}{s}$$

for $t\in(0, \lambda)$ with $ch(t)/4<r/3$. Under the condition (2.4), this means (2.10). Let c and t be positive numbers satisfying $ch(t/4)<r/6$ and $t\in(0, \lambda)$, and let $\sigma_{c,t}$ be the hitting time defined by

$$\sigma_{c,t}=\inf\{s>0: |Y(s)-x_0|>ch(t/4)\} .$$

Then, the strong Markov property of Y yields that

$$\begin{aligned}(2.14)\quad P_{x_0}(|Y(t)-x_0|>\frac{c}{2}h(t/4))&\geq P_{x_0}(\sigma_{c,t}\leq t, |Y(t)-Y(\sigma_{c,t})|\leq\frac{c}{3}h(t/4))\\ &=\int_{\{\sigma_{c,t}\leq t\}}P_y(|Y(t-s)-y|\leq\frac{c}{3}h(t/4))|_{s=\sigma_{c,t},y=Y(\sigma_{c,t})}dP_{x_0}\\ &\geq\int_{\{\sigma_{c,t}\leq t, Y(\sigma_{c,t})\in U_1\}}P_y(|Y(t-s)-y|\leq\frac{c}{3}h(t/4))|_{s=\sigma_{c,t},y=Y(\sigma_{c,t})}dP_{x_0} .\end{aligned}$$

On the other hand, by virtue of (2.10), we can find a sufficiently small $t>0$ satisfying

$$(2.15)\qquad \inf_{x\in U_1}P_x(|Y(t)-x|\leq\frac{c}{3}h(t))>\frac{1}{2} .$$

Therefore, from (2.14) and (2.15), it follows that for sufficiently small $t>0$

$$(2.16)\qquad P_{x_0}(\sigma_{c,t}\leq t, Y(\sigma_{c,t})\in U_1)\leq 2P_{x_0}(|Y(t)-x_0|>\frac{c}{2}h(t/4)) .$$

Set $\tau=\inf\{s>0: |Y(s)-Y(s_-)|>r/6\}$. Then

$$\begin{aligned}(2.17)\qquad &P_{x_0}(\sigma_{c,t}\leq t<\tau)\\ &\leq P_{x_0}(\sigma_{c,t}\leq t<\tau, Y(\sigma_{c,t})\in U_1)\\ &\leq P_{x_0}(\sigma_{c,t}\leq t, Y(\sigma_{c,t})\in U_1) .\end{aligned}$$

It follows from (2.16) and (2.17) that if $ch(t/4)<r/6$ and t is sufficiently small, then

$$(2.18)\qquad P_{x_0}(\sigma_{c,t}\leq t<\tau)\leq 2P_{x_0}(|Y(t)-x_0|>\frac{c}{2}h(t/4)) .$$

Now, put

$$w_m=P_{x_0}(\sup_{2^{-(m+1)}\leq t\leq 2^{-m}}|Y(t)-x_0|/h(t)>\varepsilon, 2^{-m+1}<\tau) ,$$

where ε is any small positive number. It follows from the increasing property of h that

$$(2.19)\qquad w_m\leq P_{x_0}(\sup_{2^{-(m+1)}\leq t\leq 2^{-m}}|Y(t)-x_0|>\varepsilon h(2^{-(m+1)}), 2^{-m+1}<\tau) .$$

Let m be a sufficiently large integer and choose θ_m as any number greater than 2^{-m}. The relationship (2.19) implies that

$$w_m \leq \boldsymbol{P}_{x_0}(\sup_{0\leq t\leq \theta_m} |Y(t)-x_0|>\varepsilon h(2^{-(m+1)}), 2^{-m+1}<\tau).$$

If $\theta_m \in (2^{-m}, 2^{-m+1})$, then

$$\begin{aligned} w_m &\leq \boldsymbol{P}_{x_0}(\sup_{0\leq t\leq \theta_m} |Y(t)-x_0|>\varepsilon h(\theta_m/4), 2^{-m+1}<\tau) \\ &\leq \boldsymbol{P}_{x_0}(\sigma_{\varepsilon,\theta_m}\leq \theta_m<\tau). \end{aligned} \tag{2.20}$$

Therefore, from (2.9), (2.16), (2.17), (2.18) and (2.20), we have

$$w_m \leq 8\boldsymbol{P}^{U_2}_{\varepsilon/8}(\theta_m/4)$$

for any $\theta_m \in (2^{-m}, 2^{-m+1})$. Let $\theta_m = 2^{-z}$ and integrate both the sides of the last inequality with respect to z from $m-1$ to m. Then, for sufficiently large integer m, we have

$$w_m \leq 8\int_{m-1}^{m} \boldsymbol{P}^{U_2}_{\varepsilon/8}(2^{-z}/4)\, dz = \frac{8}{\log 2}\int_{2^{-(m+2)}}^{2^{-(m+1)}} \boldsymbol{P}^{U_2}_{\varepsilon/8}(u)\frac{dz}{u}.$$

Under the condition (2.4), this relationship implies that the series $\sum w_m$ converges. By virtue of the Borel-Cantelli lemma, this means that

$$\boldsymbol{P}_{x_0}(\limsup_{m\to\infty}\{\sup_{2^{-(m+1)}\leq t\leq 2^{-m}} |Y(t)-x_0|/h(t)>\varepsilon, 2^{-m+1}<\tau\}) = 0. \tag{2.21}$$

Accordingly, for convenience sake, set

$$F_m = \{\sup_{2^{-(m+1)}\leq t\leq 2^{-m}} |Y(t)-x_0|/h(t)>\varepsilon\}, \quad \text{and} \quad G_m = \{\tau>2^{-m+1}\}.$$

Then, noting that

$$\boldsymbol{P}_{x_0}(\liminf_{m\to\infty}(F_m\cap G_m)^c) = \boldsymbol{P}_{x_0}(\cup_{N=0}^{\infty}\{(\cap_{m>N}(F_m^c\cap G_m))\cup(\cap_{m>N}G_m^c)\})$$

and $\boldsymbol{P}_{x_0}(\liminf_{m\to\infty} G_m^c)=0$, from (2.21), we obtain

$$\boldsymbol{P}_{x_0}(\liminf_{m\to\infty} F_m^c)\geq \boldsymbol{P}_{x_0}(\liminf_{m\to\infty} F_m^c\cap G_m) = 1;$$

hence (2.5) holds. The proof is complete.

Lemma 2.2. *Let γ be a positive number. The characteristic function $\phi_t^\gamma(x,\cdot)$ of the random variable $t^{-1/\gamma}(X_\Phi(t)-x)$ admits the representation*

$$\phi_t^\gamma(x,\eta) = e(t,x,t^{-1/\gamma}\eta) \tag{2.22}$$

for any $(t,x,\eta)\in(0,\infty)\times\boldsymbol{R}^d\times\boldsymbol{R}^d$, where $e(t,x,\xi)$ is the symbol of $E(t)$.

Proof. From Theorem 1.6, we get

$$\begin{aligned} &\phi_t^\gamma(x,\eta) \\ &= \int_{\boldsymbol{R}^d}\exp(i\eta\cdot t^{-1/\gamma}(y-x))\,K(t,x,y)\,dy \end{aligned}$$

$$= Os-\iint \exp(-iz\cdot\mu)\, e(t, x, \mu+t^{-1/\gamma}\eta)\, dz d\mu$$

for any $(t, x, \eta)\in(0, \infty)\times \boldsymbol{R}^d\times \boldsymbol{R}^d$. Using the fact that $Os\,[\exp(-iy\cdot\mu)\, a(y)]=a(0)$ for any $a\in\mathcal{A}$, we obtain (2.22).

Proof of Theorem 2.1. As is shown in the proof of Theorem 1.7, the short time behavior of sample paths of the stable-like process X coincides with that of the process X_Φ. Hence we prove the theorem replacing X by X_Φ. At first, we will show (2.1). Choose real numbers ν, κ satisfying $\alpha(x)<\nu<\kappa<\beta$. Let T be a positive number and let g_κ be the continuous density of d-dimensional symmetric stable distribution of index κ, $(0<\kappa\leq 2)$, that is,

$$\exp(-|\xi|^\kappa)=\int_{\boldsymbol{R}^d}\exp(iy\cdot\xi)\, g_\kappa(y)\, dy \quad \text{for} \quad \xi\in\boldsymbol{R}^d\,. \tag{2.23}$$

Set

$$A(t, x)=\int_{\boldsymbol{R}^d}\exp(-|y-x|^\kappa)\, K(t, x, y)\, dy \tag{2.24}$$

for any $(t, x)\in(0, \infty)\times\boldsymbol{R}^d$. From the definition of $K(t, x, y)$, (2.23) and (2.24), we have

$$A(t, x)=\int_{\boldsymbol{R}^d} e(t, x, \xi)\, g_\kappa(\xi)\, d\xi \quad \text{for} \quad \forall(t, x)\in(0, \infty)\times\boldsymbol{R}^d\,.$$

From (4) in the Theorem 1.3, we see that for any $(t, x, \xi)\in(0, T]\times\boldsymbol{R}^d\times\boldsymbol{R}^d$.

$$|1-\exp(tp_\Phi(x, \xi))|/t\leq|p_\Phi(x, \xi)|\leq C\langle\xi\rangle^{\alpha(x)} \tag{2.25}$$

and

$$|r_0(t, x, \xi)|/t\leq C\langle\xi\rangle^{\alpha(x)}\,. \tag{2.26}$$

Put $\mathcal{D}_\nu=\{z: \alpha(z)<\nu\}$. Then, from (2.23), (2.25) and (2.26), we obtain

$$\begin{aligned}&\frac{1}{t}|1-A(t, z)|\\ &\leq C\int_{\boldsymbol{R}^d}\langle\xi\rangle^{\alpha(z)} g_\kappa(\xi)\, d\xi\leq C\int_{\boldsymbol{R}^d}\langle\xi\rangle^{\nu} g_\kappa(\xi)\, d\xi\equiv\Lambda_{\kappa,\nu}<\infty\end{aligned}$$

for any $(t, z)\in(0, T]\times\mathcal{D}_\nu$. Using the same argument as in [3], we have, for sufficiently small δ,

$$\boldsymbol{P}_z(|X_\Phi(t)-z|^\kappa>\delta)\leq\frac{2\Lambda_{\kappa,\nu}\, t}{\delta} \tag{2.27}$$

for any $(t, z)\in[0, T]\times\mathcal{D}_\nu$. Let

$$\boldsymbol{P}_c^{\mathcal{D}_\nu}(t)=\sup_{z\in\mathcal{D}_\nu}\boldsymbol{P}_z(|X_\Phi(t)-z|>ct^{1/\beta})\,. \tag{2.28}$$

Then, by (2.27), the relation (2.28) implies that for sufficiently small $t>0$

$$P_c^{\mathcal{D}_v}(t)\leq 2\Lambda_{\kappa,v}\, c^{-\kappa}\, t^{1-\kappa/\beta}\,.$$

By Lemma 2.1, this means that

$$P_x(\lim_{t\to 0} |X_\Phi(t)-x|/t^{1/\beta}=0)=1 \quad \text{if} \quad \alpha(x)<\beta\,.$$

Therefore, the assertion (2.1) holds. Next, we establish the relation (2.2). Choose γ satisfying $\beta<\gamma<\alpha(x)$. Let $\{\xi_n\}_{n\geq 0}$ be a sequence of points in $\boldsymbol{R}^d$ with $|\xi_n|\to\infty$ as $n\to\infty$. Put $t_n=|\xi_n|^{-\gamma}$, and $\tilde{\xi}_n=\xi_n/|\xi_n|$ $(n=1,2,\cdots)$. Noting that $|\xi_n|^{-\gamma}|p_\Phi(x,\xi_n)|\to\infty$ as $n\to\infty$, from (4) in Theorem 1.3 and Lemma 2.2, we see that

$$\lim_{n\to\infty}\phi^\gamma_{t_n}(x,\tilde{\xi}_n)=0\,. \tag{2.29}$$

Using the same argument as in [3], we also see that (2.29) implies

$$P_x(\limsup_{t\to\infty} |X_\Phi(t,x)-x|/t^{1/\beta}=\infty)=1 \quad \text{if} \quad \beta<\alpha(x)\,.$$

Hence, the assertion (2.2) holds.

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