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BLOCK INDUCTION, NORMAL SUBGROUPS AND CHARACTERS OF HEIGHT ZERO

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Introduction

Let G be a finite group and p a prime. Let (K, R, k) be a p-modular system. Let (π) be the maximal ideal of R. We assume that K contains the $|G|$-th roots of unity and that k is algebraically closed. Let ν be the valuation of K normalized so that $\nu(p)=1$. For an (R-free) RG-module U lying in a block B of G, we define $ht(U)$, the height of U, by $ht(U)=\nu(\mathrm{rank}_R U)-\nu(|G|)+d(B)$, where $d(B)$ is the defect of B. The heights of kG-modules are defined in a similar way, and heights are always nonnegative. In this paper we study indecomposable RG-(or kG-) modules of height zero, especially their behaviors under the block induction. In section 1 we introduce, motivated by Broué [7], the notion of linkage for arbitrary block pairs as a generalization of the one for Brauer pairs, and establish fundamental properties about it. In section 2 we give a condition for a block of a normal subgroup to be induced to the whole group. In section 3 a characterization of RG-(or kG-) modules of height 0 via their vertices and sources is given, which generalizes a result of Knörr [14]. Based on this result it is shown in section 4 that for any irreducible character χ of height 0 in B and any normal subgroup N of G, χ_N contains an irreducible character of height 0. This is well-known when B is weakly regular with respect to N. An answer to the problem of determining which irreducible (Brauer) characters of N appear as irreducible constituents of irreducible (Brauer) characters of height 0 is also obtained (Theorem 4.4). In section 5 a generalization of a theorem of Isaacs and Smith [11] is given. In section 6 an alternative proof of a theorem of Berger and Knörr [1] is given. Throughout this paper an RG-module is assumed to be R-free of finite rank.

1. Block induction and characters of height 0

Throughout this section H is a subgroup of G, and B and b are p-blocks of G and H, respectively.

Let $G_{p'}$ be the set of p-regular elements of G, ZRG the center of RG, and $ZRG_{p'}$ be the R-submodule of ZRG spanned by p-regular conjugacy class sums.

We let

$$Z_0(B) = \{a \in (ZRG_{p'})\, e_B;\ \omega_B(a) \not\equiv 0 \pmod{\pi}\}\ ,$$

where e_B is the block idempotent of B. An element $a \in (ZRG_{p'})\, e_B$ is said to be of *height* 0 ([7]) if $a \in Z_0(B)$. Let s_H be the R-linear map from RG to RH defined by $s_H(x)=x$ if $x \in H$, and $s_H(x)=0$ if $x \in G-H$.

Definition 1.1. We say that B and b are *linked* if $s_H(Z_0(B))\, e_b \subseteq Z_0(b)$.

Let $\mathrm{Chr}(G)$ be the R-module of R-linear combinations of irreducible characters of G and $\mathrm{Chr}(B)$ its submodule of R-linear combinations of irreducible characters lying in B. Put

$$\mathrm{Chr}^0(B) = \{\theta \in \mathrm{Chr}(B);\ ht(\theta) = 0\}\ ,$$

where $ht(\theta)$ is defined as before; so $ht(\theta)=0$ if and only if $\nu(\theta(1))=\nu(|G|)-d(B)$. Let $\mathrm{Irr}^0(B)$ (resp. $\mathrm{IBr}^0(B)$) be the set of irreducible characters (resp. irreducible Brauer characters) of height 0 in B. Let $\mathrm{Bch}(G)$ be the R-linear combinations of irreducible Brauer characters of G. $\mathrm{Bch}(B)$ and $\mathrm{Bch}^0(B)$ are defined in a similar way. For $\theta \in \mathrm{Chr}(G)$ (or $\mathrm{Bch}(G)$), put $\theta^* = \sum \theta(x^{-1})\, x$, where x runs through $G_{p'}$. So $\theta^* \in ZRG_{p'}$.

The following lemma is well-known, cf. Broué [7]. Here we give a direct proof, in this special case.

Lemma 1.2. *We have*

$$(ZRG_{p'})\, e_B = \{\theta^*;\ \theta \in \mathrm{Chr}(B)\} = \{\theta^*;\ \theta \in \mathrm{Bch}(B)\}\ .$$

Proof. It suffices to show the first equality. For $\theta \in \mathrm{Chr}(G)$ and $\chi \in \mathrm{Irr}(G)$, we have $\chi(\theta^* e_B)=\chi(\theta_B^*)$, where χ is extended linearly over RG and θ_B denotes the B-component of θ. Since χ is arbitrary, we have $\theta^* e_B=\theta_B^*$. Thus the assertion follows, since $ZRG_{p'}=\{\theta^*;\ \theta \in \mathrm{Chr}(G)\}$.

The following theorem is important for our purpose. (It is a special case of [7, Proposition 3.3.4].) Here we give an alternative proof.

Theorem 1.3. *Let* $\theta \in \mathrm{Chr}(B)$ (*or* $\mathrm{Bch}(B)$)). *Then* θ *is of height* 0 *if and only if* θ^* *is of height* 0.

Proof. Let $\chi \in \mathrm{Irr}^0(B)$ and define the class function η as follows:

$$\eta(x) = \begin{cases} p^{d(B)}\, \chi(x) & \text{if } x \in G_{p'}\ , \\ 0 & \text{otherwise.} \end{cases}$$

We know that $ht(\theta)=0$ if and only if $(\theta, \eta)_G \not\equiv 0 \pmod{p}$ ([6]). On the other hand, θ^* is of height 0 if and only if $\omega_B(\theta^*) \not\equiv 0 \pmod{\pi}$. Since $\omega_B(\theta^*) \equiv \{|G|/\chi(1)\, p^{d(B)}\}\ (\theta, \eta)_G \pmod{\pi}$, the assertion follows.

Corollary 1.4. *We have*

$$Z_0(B) = \{\theta^*;\ \theta\in \mathrm{Chr}^0(B)\} = \{\theta^*:\ \theta\in \mathrm{Bch}^0(B)\}\ .$$

Proof. This follows from Lemma 1.2 and Theorem 1.3.

For $\theta\in\mathrm{Chr(G)}$ (or $\mathrm{Bch}(G)$), we denote by θ_b the b-component of θ_H. For an RG-(or kG-) module V, V_b is defined in a similar way. Also for $\theta\in\mathrm{Chr}(H)$, we denote by θ^B the B-component of θ^G.

Corollary 1.5. *The following are equivalent.*

(i) *B and b are linked.*

(ii) *For every $\theta\in\mathrm{Chr}^0(B)$, $\theta_b\in\mathrm{Chr}^0(b)$.*

(iii) *For every $\theta\in\mathrm{Bch}^0(B)$, $\theta_b\in\mathrm{Bch}^0(b)$.*

In particular, if B and b are linked, for every RG-(or kG-) module V of height 0 *in B, V_b is of height* 0.

Proof. The equivalences follow from Corollary 1.4 and the fact that $s_H(\theta^*)\,e_b=\theta_b^*$ for every $\theta\in\mathrm{Chr}(G)$ (resp. $\mathrm{Bch}(G)$). Let θ be the character (resp. Brauer character) afforded by V. If $ht(V)=0$, then $ht(\theta_b)=0$ by (ii) (resp. (iii)). This completes the proof.

The following proposition shows, in particular, that there are many examples of linked pair of blocks in block theory.

Proposition 1.6. *Assume that b^G is defined. Then B and b are linked if and only if $b^G=B$.*

Proof. Assume $b^G=B$. For $a\in Z_0(B)$, $\omega_b(s_H(a))\equiv\omega_B(a)\not\equiv 0 \pmod{\pi}$. So $s_H(a)\,\epsilon_b\in Z_0(b)$ and "if part" follows. Conversely assume that B and b are linked. We have $\omega_{b^G}(e_B)\equiv\omega_b(s_H(e_B)\,e_b)\not\equiv 0 \pmod{\pi}$, since $e_B\in Z_0(B)$. Hence $b^G=B$.

For the following, see also [3, Lemma A and Theorem B].

Corollary 1.7. *Assume that b^G is defined and equal to B. Then*

(i) *For any RG-(or kG-) module V of height* 0 *in B, V_b is of height* 0, *and*

(ii) *for $\theta\in\mathrm{Chr}(G)$ (or $Bch(G)$), $ht(\theta_B)=0$ if and only if $ht(\theta_b)=0$.*

Proof. (i) follows from Corollary 1.5 and Proposition 1.6. (ii) follows from the fact that $\omega_B(\theta_B^*)\equiv\omega_B(\theta^*)\equiv\omega_b(s_H(\theta^*)\,e_b)\equiv\omega_b(\theta_b^*) \pmod{\pi}$.

Let T_H^G denote the relative trace map when RG is considered as a G-algebra in the usual way. The following will be needed later.

Proposition 1.8. *Assume that B and b are linked and $d(b)=d(B)$. We have:*

(i) $T_H^G(Z_0(b))\, e_B \subseteq Z_0(B)$, *and*

(ii) *for any* $\xi \in \mathrm{Chr}(b)$, $ht(\xi)=0$ *if and only if* $ht(\xi^B)=0$.

Proof. Let $\chi \in \mathrm{Irr}^0(B)$. For $\xi \in \mathrm{Chr}(b)$, $(\xi^B)^* = T_H^G(\xi^*)\, e_B$. From this it follows that

$$\omega_B((\xi^B)^*) \equiv \{|G|\ \xi(1)/|H|\ \chi(1)\}\ \omega_b((\chi_b)^*) \pmod{\pi}\,.$$

Since $ht(\chi_b)=0$ by Corollary 1.5 (ii), (ii) follows. Then we get (i) by the above equality and Corollary 1.4.

The following proposition (cf. also [18, Theorem 7]) shows that our terminology is compatible with Brauer's [5]. If (P, b_P) is a Brauer pair (i.e. P is a p-subgroup of G and b_P is a block of $PC_G(P)$ with defect group P), let θ_P be the unique irreducible Brauer character in b_P and b_P^0 the block of $C_G(P)$ covered by b_P.

Proposition 1.9. *Let* (P, b_P) *and* (Q, b_Q) *be Brauer pairs such that* $P \triangleright Q$ *and that* b_Q *is* P*-invariant. Then* b_P *and* b_Q *are linked* (*in the sense of Brauer* [5]) *if and only if* b_P^0 *and* b_Q^0 *are linked in our sense.*

Proof. Put $b^* = b_Q^{PC(Q)}$, where $C(Q)=C_G(Q)$. Let ϕ be the unique irreducible Brauer character in b^*. We have $\phi_{b_P} = e\,\theta_P$, for some integer e. Since $b_P^{PC(Q)}$ is defined, Corollary 1.5 (iii) and Proposition 1.6 yield that $b_P^{PC(Q)}=b^*$ if and only if $e \not\equiv 0 \pmod{p}$. On the other hand, we must have $(\theta_Q)_{b_P^0} = e\psi$, where $\psi = (\theta_P)_{C(P)}$ is the unique irreducible Brauer character in b_P^0, since b_P is the unique block of $PC(P)$ covering b_P^0. By Corollary 1.5 (iii), b_P^0 and b_Q^0 are linked if and only if $e \not\equiv 0 \pmod{p}$. So the assertion follows.

Now we consider the case where H is normal in G. In this case linked pair has a clear meaning, as the following theorem shows; it shows also that the condition that B and b are linked does not always imply that b^G is defined (and equal to B). See also Blau [4, Theorem 2].

Theorem 1.10. *Assume that* H *is normal in* G. *The following conditions are equivalent.*

(i) B *and* b *are linked.*

(ii) $s_H(e_B)\, e_b$ *is of height* 0.

(iii) B *is weakly regular with respect to* H *and* B *covers* b.

Proof. (i)$\Rightarrow$(ii): This is obvious.
(ii)$\Rightarrow$(iii): Put $e_B=\theta^*$, $\theta \in \mathrm{Chr}^0(B)$. Since $s_H(e_B)\, e_b = \theta_b^*$, we have in particular $\theta_b \neq 0$, so B covers b. Put $s_H(e_B) = \Sigma_i\, a_i\, \hat{K}_i$, K_i being conjugacy classes of G (contained in H). We have $\omega_b(s_H(e_B)) \equiv \Sigma_i\, a_i\, \omega_b(\hat{K}_i) \equiv \Sigma_i\, a_i\, \omega_B(\hat{K}_i) \pmod{\pi}$, since B covers b. So we have $a_i\, \omega_B(\hat{K}_i) \not\equiv 0 \pmod{\pi}$ for some i, which shows that B

is weakly regular.
(iii)$\Rightarrow$(i): Let $\{B_i\}$ be the blocks of G covering b. We have $\Sigma\, e_{B_i}=\Sigma\, e_b^g(g\in G/T_b)$, so $\Sigma\, s_H(e_{B_i})\, e_b=e_b$ and $s_H(e_{B_i})\, e_b$ is of height 0 for some i. For such i, put $e_{B_i}=\theta^*$, $\theta\in \mathrm{Chr}^0(B_i)$. We have $\theta_H=\Sigma\, \theta_b^g(g\in G/T_b)$, so we get that $d(B_i)=\nu(|T_b|)-\nu(|H|)+d(b)$, because $ht(\theta_b)=0$. Let $\eta\in\mathrm{Chr}^0(B)$. Similar argument as above shows that $d(B)=\nu(|T_b|)-\nu(|H|)+d(b)-ht(\eta_b)$. So we have $ht(\eta_b)=d(B_i)-d(B)$. On the other hand, $d(B)\geqq d(B_i)$, since B is weakly regular. This proves that $ht(\eta_b)=0$, so B and b are linked (by Corollary 1.5).

The following is [9, (V.3.15)].

Corollary 1.11. *Assume that H is normal in G and that B covers b. Let B be weakly regular with respect to H. For any $\chi\in\mathrm{Irr}^0(B)$, we have $\chi_H=e\,\Sigma_i\,\xi_i$ with $e|T_b/T_{\xi_i}|\not\equiv 0 \pmod p$ and $\xi_i\in\mathrm{Irr}^0(b)$, for some i.*

Proof. By Theorem 1.10, χ_b is of height 0, so the assertion follows from the equality $\chi_H=\Sigma\,\chi_b^g(g\in G/T_b)$.

By Theorem 1.10 (and Corollary 1.5), we get that when B is weakly regular with respect to the normal subgroup H and B covers b, V_b has an indecomposable summand of height 0 for any RG-(or kG-) module V of height 0 in B. It will be proved in Theorem 4.1 that this is the case for arbitrary blocks covering b.

The rest of this section is devoted to giving alternative proofs of known facts.

Let b be a block of an arbitrary subgroup H as before. For a group X, let $B_0(X)$ be the principal block of X. The following is the Third Main Theorem (as extended by Okuyama [17]). (The present version is due to Blau [3, Corollary 1].) See also Kawai [20, Corollary 2.2].

Proposition 1.12. *Assume that there exists $\chi\in\mathrm{Irr}^0(B)$ such that χ_H is an irreducible character in b. Then*

(i) *If b_0 is a block of H for which b_0^G is defined and equal to B, then $b_0=b$.*
(ii) *If b^G is defined, then $b^G=B$ if and only if $\chi_H\in\mathrm{Irr}^0(b)$.*

Proof. (i) By Corollary 1.7 (ii), χ_{b_0} is of height 0, so $b_0=b$. (ii) "only if" part follows similarly. "if" part: Since $ht(\chi_b)=0$, $ht(\chi_{b^G})=0$ by Corollary 1.7 (ii). Hence $b^G=B$.

A result similar to the following has appeared in Robinson [19].

Proposition 1.13. *Let u be a central p-element of H. Assume that $b^G=B$. For χ, χ' in $\mathrm{Chr}(B)$, the following are evuivalent.*

(i) $ht(\chi)=ht(\chi')=0$.
(ii) $p^{d(b)}|H|^{-1}\,\Sigma\,\chi_b(us)\,\chi_b'(u^{-1}s^{-1})\not\equiv 0 \pmod \pi$,

where in the summation s runs through $H_{p'}$.

Proof. Define $\psi, \psi' \in \mathrm{Bch}(b)$ by $\psi(s)=\chi_b(us)$ and $\psi'(s)=\chi_b'(u^{-1}s)$, for $s \in H_{p'}$. Put $\chi_b=\Sigma_i n_i \xi_i$, $\xi_i \in \mathrm{Irr}(b)$. We have $\xi_i(u)=\xi_i(1)\varepsilon_i$, where ε_i is a $|u|$-th root of unity. Then $\chi_b(u)=\Sigma_i n_i(\varepsilon_i-1)\xi_i(1)+\chi_b(1)$. Since $\varepsilon_i-1\equiv 0 \pmod{\pi}$, $ht(\psi)=0 \Leftrightarrow \nu(\chi_b(u))=\nu(|H|)-d(b) \Leftrightarrow ht(\chi_b)=0 \Leftrightarrow ht(\chi)=0$. (For the last equivalence, cf. Corollary 1.7 (ii).) The same holds for ψ'(with u^{-1} in place of u). On the other hand, the number in (ii) is congruent (mod π) to $p^{d(b)}|H|^{-1}\psi'(1)\,\omega_b(\psi^*)$, so the assertion follows.

2. Block induction and normal subgroups

Let N be a normal subgroup of G and b a block of N. If B is a block of G covering b, then a defect group D of B is said to be an *inertial defect group* of B if it is a defect group of the Fong-Reynolds correspondent of B in the inertial group T_b of b in G.

In this section we shall prove the following theorem, which settles, in a special case, a question raised by Blau [2]. It has been obtained also by Fan [8, Theorem 2.3] independently. See also Blau [4, Theorem 3].

Theorem 2.1. *Let the notation be as above. The following conditions are equivalent.*

(i) *b^G is defined.*

(ii) (iia) *There exists a unique weakly regular block of G covering b, say B, and*

(iib) *for a defect group D of B, $Z(D)$ is contained in N.*

We begin with the following lemma, which is due to Berger and Knörr [1, the proof of Corollary], cf. also Fan [8, Proposition 2.1]. Another proof is included here for convenience.

Lemma 2.2. *For a block B of G covering b, let D be an inertial defect group of B and $\hat{b}$ the unique block of DN covering b. Then D is a defect group of $\hat{b}$.*

Proof. We may assume that b is G-invariant. Put $H=N_G(D)N$. Let $\tilde{B}$ be the Brauer correspondent in H of B. Take a kG-module U in B and a kH-module V in $\tilde{B}$ such that V is a direct sumand of U_H. Since b is G-invarnant, any direct summand of U_N lies in b, so the same is true for V_N. Hence $\tilde{B}$ covers b. This implies that $\tilde{B}$ covers $\hat{b}$ and a defect group of $\hat{b}$ is by Knörr's theorem (Knörr [13, Proposition 4.2], see also [20, Corollary 2.4]) $DN \cap D=D$, because DN is normal in H and $\hat{b}$ is H-invariant.

We also need the following

Lemma 2.3. (Blau [3, Lemma 2.5 (i)]) *Let H be a subgroup of G and B (resp. b) a block of G (resp. H). Let $\phi \in \mathrm{Irr}(b)$. Suppose that $\phi^G = \tau + \Sigma_{i=1}^n m_i \chi_i$, where m_i is a nonnegative integer, $\chi_i \in \mathrm{Irr}(B)$ and $\nu(m_i \chi_i(1)) \geqq \nu(\phi^G(1))$ for $1 \leqq i \leqq n$, and τ is a character of G such that for all $g \in G$, $\nu(\tau(g)) > \nu(\phi(1)) + \nu(|C_G(g)|) - \nu(|H|)$ (τ may be 0). Then b^G is defined and equals B.*

Proof of Theorem 2.1. (i)$\Rightarrow$(iia): This is Lemma 2.6 in Blau [3]. (i)$\Rightarrow$(iib): This follows from (V.1.6) (i) in Feit [9]. (ii)$\Rightarrow$(i): We may assume that D is an inertial defect group of B. Let $\hat{b}$ be the unique block of DN covering b. Since D is a defect group of $\hat{b}$ by Lemma 2.2, (iib) implies $b^{DN} = \hat{b}$. In fact, assume that $\omega_b(\hat{K}) \not\equiv 0 \pmod{\pi}$ for a conjugacy class K of DN. Let $x \in K$ and let u and s be the p-part and p'-part of x, respectively. Since $\hat{b}$ is induced from a root of it, $u \in_{DN} Z(D) \leqq N$. We get $s \in N$, since DN/N is a p-group. So $K \subseteq N$, as required. Now let ϕ be an irreducible character of height 0 in $\hat{b}$. Any irreducible constituent χ of ϕ^G lies in a block covering b. So $\nu(\chi(1)) \geqq \nu(\phi^G(1))$, and the inequality is strict if χ does not lie in B by (iia). From this it follows that $\hat{b}^G = B$ by Lemma 2.3. So $b^G = B$, completing the proof.

3. Characterization of modules of height 0

In this section we shall characterize RG-(or kG-) modules of height 0 via their vertices and sources. In the following, let $\mathfrak{o}$ denote either R or k.

Lemma 3.1. *Let T be a subgroup of G and N a normal subgroup of T such that T/N is a p'-group. Let Y be an indecomposable $\mathfrak{o}T$-module and W an indecomposable $\mathfrak{o}N$-module. If $Y_N \simeq eW$ for some integer e, then e is prime to p.*

Proof. Since k is algebraically closed, e is equal to the dimension of some projective indecomposable $k^\alpha[T/N]$-module for some $\alpha \in Z^2(T/N, k^*)$ (cf. Theorem 7.8 in [12]). Since k is algebraically closed and T/N is a p'-group, e is prime to p.

The following theorem generalizes Corollary 4.6 in Knörr [14].

Theorem 3.2. *Let U be an indecomposable $\mathfrak{o}G$-module lying in a block B with defect group D. The following are equivalent.*

(i) *$ht(U) = 0$.*

(ii) *$\mathrm{vx}(U) =_G D$ and the rank of a source of U is prime to p.*

Proof. Since $ht(U) = 0$ implies $\mathrm{vx}(U) =_G D$, it suffices to prove that for an $\mathfrak{o}G$-module U with vertex D, $ht(U) = 0$ if and only if the rank of a source is prime to p. Let V be the Green correspondent of U with respect to $(G, N_G(D), D)$. V lies in the Brauer correspondent $\tilde{B}$ of B and $ht(V) = 0$ if and only if $ht(U)=0$. Let W be an indecomposable summand of V_N, where $N=DC_G(D)$. W lies in a block b covered by $\tilde{B}$. Let T be the inertial group of W in $N_G(D)$. For some $\mathfrak{o}T$-module Y, $W \mid Y_N$ and $V=Y^{N_G(D)}$. Since Y belongs to b^T, $ht(V)=0$ if and only if $ht(Y)=0$. Put $Y_N \cong eW$. Since T/N is a p'-group, $e \not\equiv 0 \pmod{p}$ by Lemma 3.1. So $ht(Y)=0$ if and only if $ht(W)=0$. From the explicit Morita equivalence between b and $\mathfrak{o}D$ (b is, as a ring, isomorphic to a full matrix ring over $\mathfrak{o}D$), it follows that $ht(W)=0$ if and only if the rank of the corresponding $\mathfrak{o}D$-module (which is a source of U) is prime to p. This completes the proof.

REMARK 3.3. Theorem 2.1 in Kawai [20] follows (in the special case when the residue field k is algebraically closed, as we are assuming) from the above theorem and Corollary 1.7(i).

4. Normal subgroups and characters of height 0

Throughout this section, we use the following notation: N is a normal subgroup of G, B is a block of G with defect group D, and b is a block of N covered by B.

Theorem 4.1. *For any indecomposable $\mathfrak{o}G$-module U of height 0 lying in B, some indecomposable summand of U_N is a module of height 0 lying in b.*

Proof. We may assume that b is G-invariant. Let D be a defect group of B, $\tilde{B}$ the Brauer correspondent of B in $N_G(D)N$, and V the Green correspondent of U with respect to $(G, N_G(D)N, D)$. Since V lies in $\tilde{B}$, $ht(U)=0$ implies $ht(V)=0$. Let $\hat{b}$ be the unique block of DN covering b. D is a defect group of $\hat{b}$ by Lemma 2.2. Let W be an indecomposable summand of V_{DN} lying in $\hat{b}$. (Note that $\tilde{B}$ covers $\hat{b}$, cf. the proof of Lemma 2.2) Since V is DN-projective, V and W have vertex and source in common, so $ht(W)=0$ by Theorem 3.2. Since $\nu(|DN|)-d(\hat{b})=\nu(|N|)-d(b)$, some indecomposable summand of W_N is of height 0 (in b). This completes the proof.

Corollary 4.2.

(i) *For any $\chi \in \mathrm{Irr}^0(B)$, $\xi \in \mathrm{Irr}^0(b)$ for some irreducible constituent ξ of χ_N.*

(ii) (*Kawai* [20, *Corollary* 2.5]) *For any $\phi \in \mathrm{IBr}^0(B)$, $\psi \in \mathrm{IBr}^0(b)$ for some irreducible constituent ψ of ϕ_N.*

Proof. It suffices to prove (i). Let U be an R-form of a KG-module affording χ. By Theorem 4.1 some indecomposable summand V of U_N is of height 0 in b, so some irreducible constituent of $K \otimes_R V$ is of height 0.

Let $\mathrm{Irr}^0(b \backslash B)$ be the set of irreducible characters in b appearing as an irreducible constituent of χ_N for some $\chi \in \mathrm{Irr}^0(B)$. We define $\mathrm{IBr}^0(b \backslash B)$ in a similar way. To determine these sets, we need the following

Lemma 4.3. *Assume that b is G-invariant. Let D and* δ *be defect groups of B and b, respectively, such that* $\delta \leqq D$. *If* $\xi \in \mathrm{Irr}(b)$ *extends to QN for some subgroup Q with* $\delta \leqq Q \leqq D$, *then there is* $\chi \in \mathrm{Irr}(B)$ *such that* $(\chi, \xi)_N \neq 0$ *and that* $ht(\chi) \leqq d(B) - \nu(|Q|) + ht(\xi)$.

Proof. Let $\hat{\xi}$ be an extension of ξ to QN. Let $\hat{b}$ and $\tilde{B}$ be as in the proof of Theorem 4.1. Any irreducible constituent of $\hat{\xi}^{DN}$ belongs to $\hat{b}$. By the degree comparison it follows that there is $\eta \in \mathrm{Irr}(\hat{b})$ such that $(\hat{\xi}^{DN}, \eta)_{DN} \neq 0$ and that $(\eta(1))_p \leqq |DN/QN|_p(\xi(1))_p$. There is $\tilde{\chi} \in \mathrm{Irr}(\tilde{B})$ such that $(\tilde{\chi}, \eta^{N_G(D)N})_{N_G(D)N} \neq 0$. Then we have $(\tilde{\chi}(1))_p \leqq |N_G(D)N/DN|_p(\eta(1))_p$. Since $\tilde{B}$ induces B, $(\tilde{\chi}^B(1))_p = (\tilde{\chi}^G(1))_p$, cf. [9, (V.1.3)]. Thus there is $\chi \in \mathrm{Irr}(B)$ such that $(\chi(1))_p \leqq |G/N_G(D)N|_p(\tilde{\chi}(1))_p$ and that $(\tilde{\chi}^G, \chi)_G \neq 0$. Since $Q \cap N = \delta$ by Knörr's theorem, this χ is a required character.

Theorem 4.4. *With the notations as above, we have:*

(i) $\mathrm{Irr}^0(b\backslash B) = \{\xi \in \mathrm{Irr}^0(b)$; ξ *extends to DN for some inertial defect group D of B.*$\}$.

(ii) $\mathrm{IBr}^0(b\backslash B) = \{\psi \in \mathrm{IBr}^0(b)$; ψ *is D-invariant for some inertial defect group D of B.*$\}$.

Proof. We may assume that b is G-invariant. To prove (i), let $\xi \in \mathrm{Irr}^0(b\backslash B)$ and take $\chi \in \mathrm{Irr}^0(B)$ with $(\chi, \xi)_N \neq 0$. Let U be an R-form of a KG-module affording χ. As in the proof of Theorem 4.1, some indecomposable summand of U_{DN} is of height 0 in $\hat{b}$ (with $\hat{b}$ as above). So there is $\eta \in \mathrm{Irr}^0(\hat{b})$ with $(\chi, \eta)_{DN} \neq 0$. Put $\eta_N = e \sum_{i=1}^n \xi_i$. We have $\eta(1) = en\,\xi_1(1)$. Since ξ_1 is G-conjugate to ξ, $\nu(\eta(1)) = \nu(\xi(1)) = \nu(\xi_1(1))$. So $\eta_N = \xi_1$, because e and n are powers of p. If $\xi_1 = \xi^x$, $x \in G$, then ξ extends to $D^{x^{-1}}N$, as required. The reverse inclusion follows from Lemma 4.3 (with D in place of Q). (ii) It is proved in a similar way that $\mathrm{IBr}^0(b\backslash B)$ is contained in the right side. Assume that $\psi \in \mathrm{IBr}^0(b)$ is D-invariant for a defect group D of B. Let W be a kN-module affording ψ. Let $\hat{b}$ and $\tilde{B}$ be as in the proof of Theorem 4.1. Then W extends to a kDN-module $\hat{W}$. Let V be a $kN_G(D)N$-module lying in $\tilde{B}$ such that $\hat{W} \mid V_{DN}$. As in the proof of Theorem 4.1, $ht(V) = 0$. Let U be the Green correspondent of V as before, so U lies in B and $ht(U) = 0$. From the above and Mackey decomposition, U_N is a sum of G-conjugates of W. Some irreducible constituent M of U is of height 0, because $ht(U) = 0$, and we have $W \mid M_N$. This completes the proof.

Corollary 4.5. *Let* B_m *be a weakly regular block of G covering b. Then* $\mathrm{Irr}^0(b\backslash B_m) \subseteq \mathrm{Irr}^0(b\backslash B)$ *and* $\mathrm{IBr}^0(b\backslash B_m) \subseteq \mathrm{IBr}^0(b\backslash B)$. *In particular, the sets* $\mathrm{Irr}^0(b\backslash B_m)$ *and* $\mathrm{IBr}^0(b\backslash B_m)$ *do not depend on the choice of* B_m.

Proof. We may assume that b is G-invariant. Since there is a defect group of B_m containing a defect group of B, the assertion follows from Theorem 4.4.

Corollary 4.6. *Assume that B covers $B_0(N)$, then there is $\chi \in \mathrm{Irr}^0(B)$ such that $N \leqq \mathrm{Ker}(\chi)$.*

Proof. Since 1_N extends to any overgroups, this follows from Theorem 4.4 (or simply from Lemma 4.3).

REMARK 4.7. The above corollary is the same as saying that if B covers $B_0(N)$, some block of G/N dominated by B has defect group DN/N. This fact has been known for special N, cf. Chap. V, section 4 of Feit [9].

Put mod-Ker$(B) = \cap \mathrm{Ker}(\phi)$, where ϕ runs through $\mathrm{IBr}(B)$. The following corollary gives a characterization of mod-Ker(B) via the (ordinary) irreducible characters in B, which extends Theorem 2.4 in [15]. Let $\mathcal{N}(B)$ be the set of normal subgroups N of G such that $B_0(N)$ is covered by B and that for any $\chi \in \mathrm{Irr}^0(B)$, χ_N is a sum of linear characters.

Corollary 4.8. mod-Ker(B) *is the unique maximal member of* $\mathcal{N}(B)$.

Proof. Put N=mod-Ker(B). For any $\chi \in \mathrm{Irr}^0(B)$, χ_N is a sum of irreducible characters of height 0 in $B_0(N)$, by Corollary 4.2. This shows that $N \in \mathcal{N}(B)$, since N is p-nilpotent. Now conversely let $N \in \mathcal{N}(B)$. Let D be a defect group of B and $\xi \in \mathrm{Irr}^0(B_0(N))$ be D-invariant and assume that the determinantal order $o(\det \xi)$ is prime to p. Then ξ extends to DN (cf. [10]), so by Theorem 4.4 there is $\chi \in \mathrm{Irr}^0(B)$ with $(\chi, \xi)_N \neq 0$. By definition of $\mathcal{N}(B)$, ξ must be linear, and then $o(\det \xi) \not\equiv 0 \pmod p$ implies that the decomposition number $d(\xi, 1_N) = 0$ unless $\xi = 1_N$. This implies that N is p-nilpotent, cf. [15, Lemma 2.1 (ii)]. Since B covers $B_0(N)$, $N \leqq \mathrm{Ker}(\chi)$ for some $\chi \in \mathrm{Irr}(B)$. Then $O_{p'}(N) \leqq O_{p'}(G) \cap \mathrm{Ker}(\chi) = \mathrm{Ker}(B)$, so $N \leqq$ mod-Ker(B). This completes the proof.

In the rest of this section we prove the following theorem. Put $\delta = D \cap N$ for an inertial defect group D of B. (So δ is a defect group of b.)

Theorem 4.9. *Assume that $D = C_D(\delta)\,\delta$. Then we have* $\mathrm{Irr}^0(b \backslash B) = \mathrm{Irr}^0(b)$, *if one of the following conditions holds.*

(i) *$C_D(\delta)$ is abelian.*

(ii) *D is abelian.*

(iii) *There is a complement for δ in D.*

The condition (ii) above is quite natural in view of the height zero conjecture.

By Theorem 4.4, we have $\mathrm{Irr}^0(b \backslash B) = \mathrm{Irr}^0(b)$, if there is an (inertial) defect group D of B with the following properties.

(I) Every $\xi\in \mathrm{Irr}^0(b)$ is D-invariant, and

(II) every D-invariant $\xi\in \mathrm{Irr}^0(b)$ extends to DN.

We first consider the condition (II). For this purpose we may assume that $G=DN$, where D is a defect group of B and b is G-invariant. We have:

Lemma 4.10. *For a suitable root b_0 in $\delta\, C_N(\delta)$ of b, the unique block B_0 of $DC_N(\delta)$ covering b_0 has defect group D and b_0 is D-invariant.*

Proof. Let $\tilde{b}$ be the block of $N_N(\delta)$ such that $\tilde{b}^N=b$. Since $N_G(D)\subseteq N_G(\delta)$, there is a block $\tilde{B}$ of $N_G(\delta)$ such that $\tilde{B}^G=B$ and that D is a defect group of $\tilde{B}$. Since the block idempotents corresponding to B and b are the same, it follows that $\tilde{B}$ covers $\tilde{b}$. By the First Main Theorem, $\tilde{b}$ is $N_G(\delta)$-invariant. Put $C=\delta\, C_N(\delta)$ and $H=DC_N(\delta)$. Let b_1 be a block of C covered by $\tilde{b}$ and B_1 the unique block of H covering b_1. Let V be an indecomposable $kN_G(\delta)$-module in $\tilde{B}$ of height 0. It is easy to see that C is normal in $N_G(\delta)$ and that $\tilde{B}$ is a unique block of $N_G(\delta)$ covering b_1. So V_{b_1} is of height 0 by Theorem 1.10 (and Corollary 1.5). Since $V_{b_1}=(V_{B_1})_C$ and $\nu(|H|)-d(B_1)\geqq\nu(|C|)-d(b_1)$ (with equality only when b_1 is H-invariant), consideration of dimension shows that b_1 is H-invariant and that some indecomposable summand W of V_{B_1} is of height 0. Hence $\mathrm{vx}(W)$ is a defect group of B_1 and $|\mathrm{vx}(W)|=|D|$. Since $\mathrm{vx}(W)\leqq_{N_G(\delta)} D$, we get that $\mathrm{vx}(W)=D^n$ for some $n\in N_G(\delta)$. Then $n\in N_G(H)$, so $b_0=b_1^{n^{-1}}$ is the required root of b.

The following clarifies the condition (II) completely.

Proposition 4.11. *The following conditions are equivalent.*

(i) *Every D-invariant $\xi\in \mathrm{Irr}^0(b)$ extends to DN.*

(ii) *Every D-invariant linear character of δ extends to D.*

(iii) $[D,\delta]=[D,D]\cap\delta$.

Proof. Let B_0 and b_0 be chosen as in Lemma 4.10 and H, C be as in the proof of Lemma 4.10. We prove that (i) is equivalent to:

(iv) Every D-invariant $\xi_0\in \mathrm{Irr}^0(b_0)$ extends to H.

(iv)$\Rightarrow$(i): For any D-invariant $\xi\in \mathrm{Irr}^0(b)$, there is $\xi_0\in \mathrm{Irr}^0(b_0)$ such that ξ_0 is D-invariant and that $(\xi,\xi_0)_C\not\equiv 0 \pmod p$, because ξ_{b_0} is D-invariant and $ht(\xi_{b_0})=0$. Now it is easy to see that ξ extends to G if (and only if) ξ_0 extends to H. So (iv) implies (i).

(i)$\Rightarrow$(iv): For any D-invariant $\xi_0\in \mathrm{Irr}^0(b_0)$, ξ_0^b is D-invariant and of height 0, cf. Proposition 1.8, so similar argument applies.

Next we show that (ii) and (iv) are equivalent. Note that every D-invariant $\xi_0\in \mathrm{Irr}^0(b_0)$ is written as $\xi_0=\tilde{\zeta}$ for a D-invariant linear character ζ of δ (and vice versa), where $\tilde{\zeta}$ is defined as in Feit [9, (V.4.7)]. We show that ξ_0 extends to H if and only if ζ extends to D. First assume that there is an extension η of ξ_0. Since $ht(\eta)=0$, $(\eta, \lambda)_D\neq 0$ for some linear character λ of D. (Apply Theorem 3.2). Since $(\xi_0)_\delta$ is a multiple of ζ, this implies $\lambda_\delta=\zeta$. Conversely let λ be an extension of ζ. Let b_1 be a root of B_0 in $DC_H(D)$. We have $\lambda^{DC_H(D)}=\tilde{\lambda}+\theta$ for some character θ, where $\tilde{\lambda}\in \mathrm{Irr}^0(b_1)$ is defined as above. So $\zeta^C=(\lambda^H)_C=(\tilde{\lambda}^{B_0})_C+\psi$ for some character ψ. Since ζ^C is a sum of a multiple of ξ_0 and characters lying outside b_0, it follows that $(\tilde{\lambda}^{B_0})_C$ is a multiple of ξ_0. Now $ht(\tilde{\lambda}^{B_0})=0$ by Proposition 1.8, so for some irreducible constituent χ of $\tilde{\lambda}^{B_0}$, $\chi_C=\xi_0$.

The equivalence of (ii) and (iii) is obvious.

Remark 4.12. Theorem 8.26 in [10] reads: Let N be a normal subgroup of G with G/N a p-group. For a p-Sylow subgroup P of G, assume (*a*) $P\cap N\leqq Z(P)$, and (*b*) every irreducible character of $P\cap N$ extends to P. Then every G-invariant irreducible character of N extends to G.

The above proposition is related to this theorem as follows: Let $\xi\in \mathrm{Irr}(N)$ be G-invariant. Let b be the block of N (with defect group δ) containing ξ. If $ht(\xi)=0$, then (*b*) implies that ξ extends to G by Proposition 4.11. (On the other hand, δ is abelian by (*a*). So $ht(\xi)=0$ would follow from the height zero conjecture.)

To consider the condition (I), we let $T'_b=\cap I_G(\xi)$, where ξ runs through $\mathrm{Irr}(b)$. T'_b is normal in T_b. We first extend Lemma 2.2 as follows:

Lemma 4.13. *Assume that b is G-invariant. Let Q be a subgroup such that $\delta\leqq Q\leqq D$ and let $b(Q)$ be the block of QN covering b. Then Q is a defect group of $b(Q)$.*

Proof. By Lemma 2.2, D is a defect group of $b(D)$. By induction on $|D/Q|$, we may assume $|D/Q|=p$. Since $b(Q)$ is DN-invariant and covered by $b(D)$, $D\cap QN=Q$ is a defect group of $b(Q)$ by Knörr's theorem.

Lemma 4.14. *Assume that b is G-invariant. Let B_1 be a block of T'_b covered by B. Then we have*

(i) $B_1^G=B$.

(ii) $\delta\ C_D(\delta)$ *is contained in a defect group of a G-conjugate of B_1. In particular, $Z(D)\leqq T'_b$.*

Proof. Let $\xi_1\in \mathrm{Irr}(b)$ and take $\zeta_1\in \mathrm{Irr}(I_G(\xi_1)\,|\,\xi_1)$ such that $\zeta_1^G\in \mathrm{Irr}(B)\cap \mathrm{Irr}(G\,|\,\xi_1)$. If b_1 is the block containing ζ_1, then $b_1^G=B$, cf. [9, (V.1.2)]. Take another $\xi_2\in \mathrm{Irr}(b)$, if any, and take $\zeta_2\in \mathrm{Irr}(I_G(\xi_1)\cap I_G(\xi_2)\,|\,\xi_2)$ such that $\zeta_2^{I_G(\xi_1)}\in \mathrm{Irr}(b_1)\cap \mathrm{Irr}(I_G(\xi_1)\,|\,\xi_2)$. If b_2 is the block of $I_G(\xi_1)\cap I_G(\xi_2)$ containing ζ_2, then $b_2^{I_G(\xi_1)}=b_1$. Hence $b_2^G=B$. Repeating this process, we finally get a block B' of T'_b such that $B'^G=B$. Then B' is G-conjugate to B_1, so $B_1^G=B$. This implies $Z(D)\leqq T'_b$, cf. Theorem 2.1. Now for any $x\in C_D(\delta)$, put $Q=\langle x, \delta\rangle$ and let

$b(Q)$ be the block of QN covering b. By the above (with $b(Q)$, QN in place of B, G) and Lemma 4.13, we get that $x \in Z(Q) \leqq T'_b \cap QN$, so $C_D(\delta) \leqq T'_b$. Let D^x, $x \in G$, be a defect group of the Fong-Reynolds correspondent of B in the inertial group of B_1 in G. Then $\delta\, C_D(\delta) \leqq (D^x \cap T'_b)^{x^{-1}}$, which is a defect group of $B_1^{x^{-1}}$ This completes the proof.

Proposition 4.15. *Assume that b is G-invariant. Let A be a subgroup of $C_D(\delta)$ such that* (1) *A is abelian, or* (2) *δ is complemented in $A\delta$. Then for every $\xi \in \mathrm{Irr}^0(b)$,*

(i) *ξ extends to AN, and*

(ii) *there is $\chi \in \mathrm{Irr}(B)$ such that $(\chi, \xi)_N \neq 0$ and that $ht(\chi) \leqq d(B) - \nu(|A\,\delta|)$.*

Proof. (i) Put $Q = A\,\delta$ and let $b(Q)$ be as in Lemma 4.13. So Q is a defect group of $b(Q)$. In either case, the condition (ii) in Proposition 4.11 is satisfied (with Q in place of D; in case (2), use Wigner's method.) and any $\xi \in \mathrm{Irr}^0(b)$ is Q-invariant by Lemma 4.14, so the conclusion follows from Proposition 4.11. (ii) follows from (i) and Lemma 4.3.

Proof of Theorem 4.9. Since we may assume that b is G-invariant, the assertion follows from Proposition 4.15 (ii) (with $A = C_D(\delta)$).

5. A generalization of a theorem of Isaacs and Smith

In [11] Isaacs and Smith have given a characterization of groups of p-length 1 ([11], Theorem 2). Here we prove a generalization of their result.

For a block B of G, let mod-Ker(B) be as in section 4 and let $\mathrm{Ker}^0(B) = \cap\, \mathrm{Ker}(\chi)$, where χ runs through $\mathrm{Irr}^0(B)$. Let $\mathrm{Ker}(B)$ be defined in the usual way.

Lemma 5.1. *Let B be a block of G with defect group D.*

(i) *If B covers the principal block of a normal subgroup N of G, D is a p-Sylow subgroup of DN.*

(ii) $\mathrm{Ker}^0(B) \leqq \mathrm{Ker}(B)\, D$ *and* mod-Ker$(B) \leqq \mathrm{Ker}(B)\, D$.

Proof. If B covers the principal block of N, $D \cap N$ is a p-Sylow subgroup of N, by Knörr's theorem. So (i) follows. By Corollary 4.8 (or more simply, by [15, Theorem 2.3]), $\mathrm{Ker}^0(B) \leqq$ mod-Ker(B). As is well-known, (mod-Ker(B)) D is p-nilpotent and its normal p-complement is $\mathrm{Ker}(B)$. Since D is a p-Sylow subgroup of (mod-Ker(B)) D by (i), (mod-Ker(B)) $D = \mathrm{Ker}(B)\, D$. This completes the proof.

Let K be a normal subgroup of G such that B covers the principal block of K, and put $\bar{G} = G/K$ and let $\{\bar{B}_i;\ 1 \leqq i \leqq s\}$ be the blocks of $\bar{G}$ dominated by B. Put $\bar{D} = DK/K$. Then we have the following

Proposition 5.2. *Assume that there is a defect group D of B such that $\Phi(D)$ (the Frattini subgroup of D) contains a p-Sylow subgroup of K. Then for exactly one value of i, $\bar{B}_i$ has defect group $\bar{D}$.*

Proof. There is a block $\bar{B}_i$ with defect group $\bar{D}$ by Remark 4.7. Let b be the Brauer correspondent of B in $N_G(D)$. Let $\bar{b}$ be a block of $\overline{N_G(D)}$ dominated by b. (Since D is a p-Sylow subgroup of DK by Lemma 5.1, $\overline{N_G(D)}$ $=N_{\bar{G}}(\bar{D})$, by the Frattini argument.) We claim that $\bar{b}$ is unique. Let Q be a p-Sylow subgroup of K such that $Q\leqq\Phi(D)$. Put $L=N_G(D)\cap K$. Then $N_{\bar{G}}(\bar{D})\cong N_G(D)/L$. We note that b covers $B_0(L)$. In fact, there is $\chi\in\mathrm{Irr}^0(B)$ such that $\mathrm{Ker}(\chi)\geqq K$ by Corollary 4.6. Since $ht(\chi_b)=0$, $\chi_b\neq 0$. So b covers $B_0(L)$. Thus it suffices to show that b does not "decompose" in $N_G(D)/L$. We see that $L\subset\text{mod-Ker}(b)$ is p-nilpotent and that $L/L\cap\text{mod-Ker}(b)$ is a p'-group, since $Q\leqq D\leqq\text{mod-Ker}(b)$. So the claim follows from [16, Problem 9 on p. 389], since $Q\leqq\Phi(D)$. Now assume that $\bar{B}_i$ has defect group $\bar{D}$. We show that $\bar{B}_i=\bar{b}^{\bar{G}}$ with $\bar{b}$ as above, which proves the uniqueness of i. Let $\bar{U}$ be a $k\bar{G}$-module in $\bar{B}_i$ with vertex $\bar{D}$ and $\bar{V}$ the Green correspondent of $\bar{U}$ with respect to $(\bar{G}, N_{\bar{G}}(\bar{D}), \bar{D})$. Let U(resp. V) be the inflation of $\bar{U}$(resp. $\bar{V}$) to G (resp. $N_G(D)$). D is a vertex of U, since D is a p-Sylow subgroup of DK. Similarly D is a vertex of V. So V is the Green correspondent of U with respect to $(G, N_G(D), D)$. Hence V must lie in b. So $\bar{V}$ lies in $\bar{b}$, which shows that $\bar{b}$ induces $\bar{B}_i$, as required.

Theorem 5.3. *Let B be a block of G with defect group D. If every $\chi\in$ $\mathrm{Irr}^0(B)$ restricts irreducibly to $N_G(D)$, then $G=N_G(D)\,\mathrm{Ker}(B)$.*

Proof. We first consider the case where D is abelian. Let b be the Brauer correspondent of B in $N_G(D)$. For any $\xi\in\mathrm{Irr}^0(b)$, $ht(\xi^B)=0$ by Proposition 1.8, so it follows from the assumption that there is $\chi\in\mathrm{Irr}^0(B)$ such that $\chi_{N_G(D)}=\xi$. Let $I=\{\xi\in\mathrm{Irr}^0(b); D\leqq\mathrm{Ker}(\xi)\}$. For each $\xi\in I$, take $\chi(\xi)\in\mathrm{Irr}^0(B)$ whose restriction to $N_G(D)$ equals ξ and let $K=\cap\,\mathrm{Ker}\,\{\chi(\xi)\}$, where ξ runs through I. Clearly $K\cap N_G(D)\leqq\text{mod-Ker}(b)$ and, by Lemma 5.1, $\text{mod-Ker}(b)\leqq$ $\mathrm{Ker}(b)\,D$. Since $\mathrm{Ker}(b)$ is a normal p'-subgroup, $\mathrm{Ker}(b)\leqq C_G(D)$. Hence $K\cap N_G(D)\leqq C_G(D)$. On the other hand, D is a p-Sylow subgroup of K by Lemma 5.1. Hence K is p-nilpotent, by Burnside's theorem. By the Frattini argument, $G=N_G(D)\,K$. Since $K=O_{p'}(K)\,D\leqq\mathrm{Ker}(B)\,D$, we get $G=N_G(D)$ $\mathrm{Ker}(B)$, as required. For the general case, put $\bar{G}=G/\mathrm{Ker}^0(B)$. We claim that $\mathrm{Ker}^0(B)$ satisfies the assumption of Proposition 5.2 with $K=\mathrm{Ker}^0(B)$. Put $Q=$ $D\cap\mathrm{Kei}^0(B)$. Then Q is a p-Sylow subgroup of $\mathrm{Ker}^0(B)$, cf. Lemma 5.1. For any linear character λ of D, define $\tilde{\lambda}\in\mathrm{Irr}(DC_G(D))$ as in the proof of Proposition 4.11. Then $ht(\tilde{\lambda}^B)=0$, so there is $\chi\in\mathrm{Irr}^0(B)$ such that λ is an irreducible constituent of χ_D. This shows $Q\leqq\mathrm{Ker}(\lambda)$, and hence $Q\leqq[D, D]$. So the claim follows. Now let $\bar{B}$ be the block of $\bar{G}$ as in Proposition 5.2. Since every $\chi \in \mathrm{Irr}^0(B)$ comes then from $\bar{B}$, $\mathrm{Ker}^0(\bar{B})=1$. Since $\overline{N_G(D)}=N_{\bar{G}}(\bar{D})$ by the Frattini argument, $\bar{B}$ satisfies the same assumption as B. On the other hand, since (by Corollary 1.7 (ii)) $\chi_{N_G(D)} \in \mathrm{Irr}^0(b)$ for any $\chi \in \mathrm{Irr}^0(B)$, it follows that χ_D is a sum of linear characters (by Corollary 4.2 (i)). Hence $[D, D] \leqq \mathrm{Ker}^0(B)$ and $\bar{D}$ is abelian. So $\bar{G}=N_{\bar{G}}(\bar{D})\,\mathrm{Ker}(\bar{B})$, by the above. Thus $\bar{G}=N_{\bar{G}}(\bar{D})$, since $\mathrm{Ker}(\bar{B}) \leqq \mathrm{Ker}^0(\bar{B})=1$. Hence we get $G=N_G(D)\,\mathrm{Ker}^0(B)=N_G(D)\,\mathrm{Ker}(B)\,D= N_G(D)\,\mathrm{Ker}(B)$, by Lemma 5.1. This completes the proof.

6. The height zero conjecture

The following is a well-known conjecture of Brauer:

(*) Blocks with abelian defect groups contain only characters of height 0.

Berger and Knörr [1] have proved the following

Theorem 6.1. *If (*) is true for all quasi-simple groups, it is true for all finite groups.*

We prove this theorem by applying some results in section 4 and a theorem of Knörr [14, Corollary 3.7].

Lemma 6.2. *If (*) is true for all quasi-simple groups, it is true for any group H with H/C simple for a central subgroup C of H.*

Proof. The proof is done by induction on the group order. If $H=[H, H]$, then H is quasi-simple and (*) is true by assumption. If $H \neq [H, H]$, let K be such that $[H, H] \lhd K \lhd H$ with $|H/K|=q$, a prime. Let B be a block of H with abelian defect group D and let $\chi \in \mathrm{Irr}(B)$. We consider the case when $q=p$ and $\chi_K=\sum_{i=1}^{p} \zeta_i$, where all ζ_i are distinct. If b is the block of K containing ζ_1, then $b^G=B$, since $\zeta_1^G=\chi$. So D is G-conjugate to a defect group of b, cf. Theorem 2.1. Since $ht(\zeta_1)=0$ by induction, $ht(\chi)=0$. Other cases are treated similarly. This completes the proof.

Proof of Theorem 6.1. The proof is done by induction on the group order. Let B be a block of a group G with an abelian defect group D and let $\chi \in \mathrm{Irr}(B)$. Let N be a maximal normal subgroup of G. So G/N is simple. Let $\zeta \in \mathrm{Irr}(N)$ be such that $(\chi, \zeta)_N \neq 0$. Let b be the block of N containing ζ and δ a defect group of b. We may assume that b is G-invariant. Let T be the inertial group of ζ in G. If $T \neq G$, let $\eta \in \mathrm{Irr}(T|\zeta)$ be such that $\eta^G=\chi$ and let B' be the block of T to which η belongs and D' a defect group of B'. Then $D' \leqq_G D$, since $B'^G=B$. On the other hand, $D' \geqq_G Z(D)=D$. (In fact, the proof of Lemma 4.14 shows that B' is induced from a G-conjugate of B_1, B_1 being the same as in Lemma 4.14. So the assertion follows.) Hence $D'=_G D$. By induction $ht(\eta)=0$, so $ht(\chi)=0$. So we may assume ζ is G-invariant. Now take a central extension of G,

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1\,,$$

such that $f^{-1}(N)=N_1\times Z$, $N_1 \lhd \hat{G}$ and that ζ extends to a character of $\hat{G}$, say $\hat{\zeta}$, under the identification of N_1 with N through f, and that Z is a finite cyclic group. Here we note the following. Since δ is abelian, $ht(\zeta)=0$ by induction. So ζ extends to DN by Proposition 4.11, since D is abelian. Thus the above central extension may be taken so that

$$(\#)\ \text{the subextension } 1\to Z\to f^{-1}(DN) \xrightarrow{f} DN\to 1 \text{ splits.}$$

Let $\hat{\chi}$ be the inflation of χ to $\hat{G}$. Let $\hat{B}$ be the block of $\hat{G}$ to which $\hat{\chi}$ belongs. There is a unique irreducible character ψ of $\bar{G}=\hat{G}/N$ such that $\hat{\chi}=\hat{\zeta}\psi$. Let $\bar{B}$ be the block of $\bar{G}$ to which ψ belongs. Let $\hat{D}$ and $\bar{D}$ be defect groups of $\hat{B}$ and $\bar{B}$, respectively. We have

(I) $\hat{D}Z/Z=_G D$.

Proof. Since B is dominated by $\hat{B}$ and $\hat{G}$ is a central extension of G, the result follows.

(II) $\hat{D}$ is abelian.

Proof. We have $f^{-1}(DN)=\hat{D}ZN=H\times Z$ for a subgroup H by (#) and (I). So $\hat{D}Z=K\times Z$ for a subgroup K. Then $K\simeq\hat{D}Z/Z\simeq D$ is abelian, so $\hat{D}$ is abelian.

(III) $\hat{D}N/N=_{\bar{G}}\bar{D}$.

Proof. We first show $\hat{D}N/N\geqq_{\bar{G}}\bar{D}$. We have $\omega_{\hat{\chi}}(\hat{K})=\hat{\zeta}(x)\psi(x)|\hat{G}|/\hat{\zeta}(1)\psi(1)|C_{\hat{G}}(x)|$, where $x\in\hat{G}$ and K is the conjugacy class of $\hat{G}$ containing x. From this we get that $\omega_{\hat{\chi}}(\hat{K})=\omega_\psi(\hat{L})\,m_x(\hat{\zeta}(x)\,|N|/\hat{\zeta}(1)\,|C_N(x)|)$, where $m_x=|C_{\bar{G}}(\bar{x})\colon C_{\hat{G}}(x)N/N|$ and L is the conjugacy class of $\bar{G}$ containing $\bar{x}$, the image of x in $\bar{G}$. Here $\hat{\zeta}(x)\,|N|/\hat{\zeta}(1)\,|C_N(x)|$ is an integer. In fact, let A be the $\boldsymbol{Z}$-linear combinations of the N-conjugacy class sums of $\hat{G}$, where $\boldsymbol{Z}$ is the ring of rational integers. If T is a matrix representation affording $\hat{\zeta}$, then $T(A)$ is a commutative ring (with finite $\boldsymbol{Z}$-rank), since ζ_N is irreducible. If C is the N-conjugacy class containing x, $T(\hat{C})=\alpha I$, a scalar matrix, where α equals the number in question. Hence the assertion follows. Hence, if $\omega_{\hat{\chi}}(\hat{K})\not\equiv 0 \pmod{\pi}$, then $m_x\,\omega_\psi(\hat{L})\not\equiv 0 \pmod{\pi}$. This implies $\hat{D}N/N\geqq_{\bar{G}}\bar{D}$. Hence $\bar{D}$ is abelian by (II), and $ht(\psi)=0$ by assumption and Lemma 6.2. Let V(resp. W) be an R-form of $\hat{\zeta}$ (resp. ψ). Thus $V\otimes_R \operatorname{Inf} W$ is an R-form of $\hat{\chi}$. Since $ht(\psi)=0$, $\bar{D}$ is a vertex of W. So, if we let Δ be the inverse image of $\bar{D}$ in $\hat{G}$, $V\otimes_R \operatorname{Inf} W$ is Δ-projective. But $\hat{D}$ must be a vertex of it, by Knörr's theorem [14]. Hence $\hat{D}\leqq_{\hat{G}}\Delta$, and $\hat{D}N/N\leqq_{\bar{G}}\bar{D}$. This completes the proof of (III).

Now we show $ht(\chi)=0$. Since $\hat{\chi}=\hat{\zeta}\psi$, $\hat{\chi}(1)=\chi(1)$, $\hat{\zeta}(1)=\zeta(1)$, and $ht(\zeta)=ht(\psi)=0$, $ht(\chi)=d(B)-d(b)+\nu(|Z|)-d(\bar{B})$. Since $d(\bar{B})=d(\hat{B})-\nu(|\hat{D}\cap$

$N|)$ by (III), and $d(\hat{B})=d(B)+\nu(|\hat{D}\cap Z|)$ by (I), it follows that $ht(\chi)=\nu(|\hat{D}\cap N|)-d(b)+\nu(|Z|)-\nu(|\hat{D}\cap Z|)$. Since $\hat{D}\cap N$ is a defect group of b and a p-Sylow subgroup of Z is contained in $\hat{D}$, we get $ht(\chi)=0$, completing the proof.

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